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SAMPLE PROPERTIES OF RANDOM FIELDS
II: CONTINUITY

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Abstract. A version of the Kolmogorov-Chentsov-argument is given which is formulated for random fields indexed by a class of metric spaces satisfying certain separability conditions. The resulting criteria for the existence of modifications which are sample (Hölder) continuous are worked out for random fields defined on open subsets of the m-dimensional euclidean space.

1. Introduction

This is the second in a series of three papers on sample properties of random fields (cf. also [16, 17]), and in the present paper the question of the existence of a continuous or Hölder continuous modification of a given random field indexed by a metric space is being discussed.

As is well-known, basically there are three different methods to conclude from statistical properties of a stochastic process or a random field defined on $\mathbb{R}^m$, $m \in \mathbb{N}$, that it has samples which are (Hölder) continuous. The first method is the one used originally by Kolmogorov for stochastic processes, as reported by Slutsky in [19], and which has been extended by Chentsov [2], i.e., the use of the Borel-Cantelli lemma as the tool to go from integral properties to sample properties. This method has been generalized in the sequel by a number of authors to the case of a random field indexed by $\mathbb{R}^m$, $m \in \mathbb{N}$, or a hypercube in $\mathbb{R}^m$, e.g., [1,3,9,11,14,18,21] and the references given there.

The second method, which is quite in spirit of Wiener's famous construction of the Wiener process and the Wiener space [22], consists in directly constructing the relevant probability measure on the space of functions which have the desired continuity property. This has been carried out by Mann [13] based on papers by Doob [4, 5].

The third method is based on the Garsia-Rodemich-Rumsey lemma [7], cf. also [20], and for fairly general result for random fields, which takes logarithmic corrections and dependence of the modulus of continuity on the direction into account the interested reader is also referred to [6].

In the present paper, a rather general result of the Kolmogorov-Chentsov type is proved for a random field indexed by a metric space having certain separability properties which generalize the hierarchy of dyadic numbers on the real line. The arguments are based on the Borel-Cantelli lemma. Furthermore it is shown that

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the concrete criteria for existence of a (Hölder) continuous modification in terms of moments or tail estimates reproduce most of the known criteria for stochastic processes or random fields on $\mathbb{R}^m$. However, the results here can also be applied directly to situations which to the best of the knowledge of the author have not been treated in the literature. For example, one may choose for the underlying indexing set "thin" subsets of of $\mathbb{R}^m$, like a grid of lines, metric graphs etc.

The paper is organized as follows. In section 2 various forms the separability property of the indexing metric space mentioned above is introduced, and the main results are given. The proof of the main results is found in section 3. Examples are worked out in section 4.

After this paper was finalized the author was informed by Professor B. Schreiber about related results in the work [8] by J. Hoffmann-Jørgensen. The setup in [8] appears to be quite different, and the relation of the results there to those of the present paper still has to be worked out.

2. Main Results

Let (M, d) be a metric space. If M is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, with the metric induced by the standard Euclidean metric on $\mathbb{R}^m$, then the set of g-adic vectors in M, i.e., those elements in M so that every cartesian component is a g-adic number, are dense in M with respect to d. In the following we shall define more general metric spaces with similar properties and analyze some of their properties.

Assume that $(D_n,\, n \in \mathbb{N})$ is an increasing sequence of finite subsets of M. For $n \in \mathbb{N}$ set

$$\delta_n^0 := \begin{cases} \min\left\{d(x,y),\, x,y \in D_n,\, x \neq y\right\}, & \text{if } |D_n| \geq 2, \\ +\infty, & \text{otherwise,} \end{cases}$$

where for a set A, $|A|$ denotes the number of elements of A. Note that δ_n^0 is either $+\infty$ or a finite strictly positive real number. Moreover — except in the uninteresting case where $|D_n| \leq 1$ for all $n \in \mathbb{N}$ — the sequence $(\delta_n^0,\, n \in \mathbb{N})$ is decreasing from a certain index on.

Suppose that we are given a decreasing sequence $(\delta_n,\, n \in \mathbb{N})$ of positive real numbers with $\delta_n^0 \leq \delta_n$ for all $n \in \mathbb{N}$, and such that

$$\limsup_n \frac{\delta_n}{\delta_n^0} < +\infty,$$

where we make the convention that $\delta_n/\delta_n^0 = 1$, if $\delta_n^0 = \delta_n = +\infty$. In particular, if $(\delta_n^0,\, n \in \mathbb{N})$ decreases to zero, then so does $(\delta_n,\, n \in \mathbb{N})$. We call a sequence $\mathcal{D} = \big((D_n,\, \delta_n),\, n \in \mathbb{N}\big)$ with these properties a *scale of* (M, d).

Definition 2.1. Let (M, d) be a metric space and let $\mathcal{D} = \big((D_n,\, \delta_n),\, n \in \mathbb{N}\big)$ be a scale of (M, d). $(M, d, \mathcal{D})$ is called a *scaled metric space* if $D := \cup_{n \in \mathbb{N}} D_n$ is dense in (M, d).

Remark 2.2. Since D is at most countable, a scaled metric space is necessarily separable.

Consider a scaled metric space $(M, d, \mathcal{D})$ with scale $\mathcal{D} = \big((D_n, \delta_n),\, n \in \mathbb{N}\big)$. Let $n \in \mathbb{N}$, $x \in D_n$. Define

$$C_n(x) := \{y \in D_n,\, d(x, y) \le \delta_n\}.$$

We call $C_n(x)$ the *clique of x in D_n*. Obviously, if $y \in C_n(x)$ then $x \in C_n(y)$. π_n denotes the set of all unordered pairs $\langle x, y\rangle$, x, $y \in D_n$, $d(x, y) \le \delta_n$, i.e., so that x and y belong to the same clique. For later purposes we mention in passing that — except for $n \in \mathbb{N}$ so that $D_n = \emptyset$ — π_n has at least one element, since $x \in D_n$ entails that $\langle x, x\rangle \in \pi_n$. Actually, in typical examples π_n grows very fast towards $+\infty$, cf. section 4.

Let $(M, d, \mathcal{D})$ be a scaled metric space, and consider the following properties it might have:

(W) Every $z \in M$ has a neighborhood V so that the following holds: For almost all $n \in \mathbb{N}$, and all x, $y \in D_{n+1} \cap V$ exist x', $y' \in D_n \cap V$ with $x' \in C_{n+1}(x) \cap V$, $y' \in C_{n+1}(y) \cap V$, and $d(x', y') \le d(x, y)$.

A global version of property (W) which will be useful below is

(U) For almost all $n \in \mathbb{N}$, and all x, $y \in D_{n+1}$ exist x', $y' \in D_n$ with $x' \in C_{n+1}(x)$, $y' \in C_{n+1}(y)$, and $d(x', y') \le d(x, y)$.

These properties state that in almost every D_n every point has — in some sense — enough points in its clique. Clearly, (U) entails (W).

A scale $\mathcal{D}\big((D_n, \delta_n),\, n \in \mathbb{N}\big)$ on a metric space (M, d) might have the following property, resembling the behavior of the g-adic numbers on the real line:

(D) There exist $\alpha > 0$, $\eta \in (0, 1)$ so that for almost all $n \in \mathbb{N}$,

$$\frac{1}{\alpha}\,\eta^n \le \delta_n \le \alpha\,\eta^n.$$

Definition 2.3. Assume that (M, d) is a metric space.

(a) If there is a scale $\mathcal{D}$ on (M, d) so that $(M, d, \mathcal{D})$ is a scaled metric space and (W) holds, then (M, d) is called *well separable (with scale $\mathcal{D}$).*

(b) If there is a scale $\mathcal{D}$ on (M, d) so that $(M, d, \mathcal{D})$ is a scaled metric space and (U) holds, then (M, d) is called *uniformly well separable (with scale $\mathcal{D}$).*

(c) Assume that (M, d) is (uniformly) well separable with scale $\mathcal{D}$. If for $\mathcal{D}$ condition (D) holds, then (M, d) is called *(uniformly) dyadically separable (with scale $\mathcal{D}$).*

Let $(\Omega, \mathcal{A}, P)$ be a probability space, let (M, d) be a metric space, and consider a random field ϕ indexed by M with real or extended real values. (For convenience, from now on we shall no longer distinguish between the sets of real numbers or of extended real numbers.)

We follow [12] and make the following

Definition 2.4. ϕ is called *sample continuous* if for all $\omega \in \Omega$ the mapping $\phi(\cdot, \omega)$ from M into the reals is continuous.

Remark 2.5. We let the words "sample continuous" be preceded by one or several of the prefixes "a.s.", "locally" or "uniformly" as needed, and the corresponding interpretations are the natural ones — the interested reader can turn this readily into a formal definition.

Similarly, we define an appropriate version of local Hölder continuity for random fields:

Definition 2.6. For $\gamma \in (0,1)$, ϕ is called *locally sample Hölder continuous of order* γ if for all $\omega \in \Omega$ the function $\phi(\cdot,\omega)$ is locally Hölder continuous of order γ on M, i.e., if for every $z \in M$ there is a neighborhood V of z, and a constant $a_{V,\gamma} > 0$ so that for all $\omega \in \Omega$,

$$\sup_{x,y \in V,\, x \neq y} \left| \frac{\phi(x,\omega) - \phi(y,\omega)}{d(x,y)^\gamma} \right| \leq a_{V,\gamma} \tag{2.1}$$

Remark 2.7. Also here we shall sometimes put "a.s." in front of "locally sample Hölder continuous" with the obvious meaning. Clearly, we could have defined a more general form of Hölder continuity than in definition 2.6 by letting the neighborhood V and the Hölder constant $a_{V,\gamma}$ depend on $\omega \in \Omega$. However, as it turns out below, this will not be necessary, and in theorem 2.9 we will even have a Hölder constant which is independent of the choice of V.

We assume from now on that we are given a scaled metric space $(M, d, \mathcal{D})$ with scale $\mathcal{D} = \big((D_n, \delta_n),\, n \in \mathbb{N}\big)$. In order to avoid trivialities, we suppose in addition, that (M, d) has at least one accumulation point. As we shall show in section 3 (cf. Lemma 3.1) — and as is almost obvious — this entails that the sequence $(\delta_n,\, n \in \mathbb{N})$ decreases to zero.

Throughout the paper we shall consider two positive, increasing functions q, r defined on an interval $[0,\rho]$, $\rho > 0$. Furthermore we consider the following conditions for q and r:

(C1) $\displaystyle\sum_{n=1}^{\infty} |\pi_n|\, q(\delta_n) < +\infty$;

(C2) $\displaystyle\sum_{n=1}^{\infty} r(\delta_n) < +\infty$;

(C3) There exist $\gamma > 0$, and $K_\gamma > 0$ so that for all $h \in [0,\rho]$,

$$r(h) \leq K_\gamma\, h^\gamma.$$

It is clear that (D) and (C3) imply (C2).

For the random field ϕ we shall consider the following condition:

(B) For all x, $y \in M$ with $d(x,y) \leq \rho$ the following inequality holds:

$$P\Big(|\phi(x) - \phi(y)| \geq r\big(d(x,y)\big)\Big) \leq q\big(d(x,y)\big).$$

Now we are ready to state our main results.

Theorem 2.8. *Let* (M,d) *be a well separable metric space, and let* ϕ *be a random field indexed by* M *so that conditions* (B), (C1), *and* (C2) *hold. Then* ϕ *has a locally uniformly sample continuous modification. If in addition* (M,d) *is uniformly well separable, the modification can be chosen such that it is uniformly sample continuous.*

Theorem 2.9. *Let (M, d) be a dyadically separable metric space, and let ϕ be a random field indexed by M so that conditions* (B)*,* (C1)*, and* (C3) *hold. Then ϕ has a modification which is locally sample Hölder continuous of order γ.*

3. Proof of the Main Results

In this section we prove theorems 2.8 and 2.9. We assume throughout this section that $(M, d, \mathcal{D})$ is a scaled metric space with scale $\mathcal{D} = \big((D_n, \delta_n),\, n \in \mathbb{N}\big)$, so that in particular

$$D = \bigcup_{n \in \mathbb{N}} D_n,$$

is dense in (M, d). Below we consider also metric spaces $(D \cap V, d)$, $V \subset M$, where for simplicity the restriction of d to $D \cap V \times D \cap V$ is denoted again by d. ϕ denotes a real valued random field on a probability space $(\Omega, \mathcal{A}, P)$ indexed by M.

The first step of the proof is the following lemma together with its corollaries which basically reduces the proof of theorems 2.8 and 2.9 to the analogous results where (M, d) is replaced by (D, d).

Lemma 3.1. *Assume that ϕ is continuous in probability, and that a.s. locally the restriction of ϕ to D is uniformly sample continuous, i.e., there exists a P-null set $N \in \mathcal{A}$ such that for every $z \in M$ and every $\omega \in N^c$ there exists a neighborhood $V(z, \omega)$ of z such that the restriction of $\phi(\cdot, \omega)$ to $V(z, \omega) \cap D$ is uniformly continuous. Then ϕ has a modification   with samples which are locally uniformly continuous, and such that $= \phi$ on $D \times N^c$.*

Proof. We construct   as follows. Without loss of generality we may assume that $N = \emptyset$, because otherwise we can set $\equiv 0$ on $M \times N$. On $D \times \Omega$ we set $:= \phi$. Let $\omega \in \Omega$, $x \in M \setminus D$, and let $V(x, \omega)$ be a neighborhood of x as in the hypothesis of lemma. Choose a sequence $(x_n,\, n \in \mathbb{N})$ in $V(x, \omega)$ which converges to x. $(x_n,\, n \in \mathbb{N})$ is Cauchy, and the restriction of $\phi(\cdot, \omega)$ to $V(x, \omega) \cap D$ is uniformly continuous. Therefore $(\phi(x_n, \omega),\, n \in \mathbb{N})$ is Cauchy, and we define $(x, \omega) := \lim_n \phi(x_n, \omega)$. Clearly,   is well-defined on $M \times \Omega$. A standard $\epsilon/3$-argument shows that   has locally uniformly continuous samples.

Finally we show that   is a modification of ϕ. To this end let $x \in M$, and let $(x_n,\, n \in \mathbb{N})$ be a sequence in D converging to x. By construction, $(\phi(x_n),\, n \in \mathbb{N})$ converges pointwise to , and by hypothesis this sequence converges in probability to ϕ. Thus $P(\ \ (x) = \phi(x)) = 1$. □

Remark 3.2. The proof of lemma 3.1 shows that if the neighborhoods $V(z, \omega)$ in the hypothesis of lemma 3.1 can be chosen independently of $\omega \in \Omega$, then the same is true for the neighborhoods of points in M on which the samples of   are uniformly continuous.

If ϕ is as in lemma 3.1 but a.s. uniformly sample continuous when restricted to $D \times \Omega$, then we get from lemma 3.1 and its proof immediately the following result:

Corollary 3.3. *Assume that ϕ is continuous in probability. Suppose furthermore that the restriction of $\phi(\cdot, \omega)$ to D is a.s. uniformly sample continuous on (D, d). Then the modification   of ϕ can be chosen in such a way that it is uniformly sample continuous.*

Corollary 3.4. *Assume that ϕ is continuous in probability. Suppose furthermore that there exist $\gamma > 0$, $a_\gamma > 0$, a P-null set $N \in \mathcal{A}$, and for every $z \in M$ a neighborhood $V(z)$, such that for every $\omega \in N^c$ and all x, $y \in D \cap V(z)$ the following inequality holds*

$$\big|\phi(x,\omega) - \phi(y,\omega)\big| \le a_\gamma \, d(x,y)^\gamma. \tag{3.1}$$

Then ϕ has a modification such that for all $\omega \in \Omega$, $z \in M$

$$\sup_{x,y \in V(z),\, x \neq y} \left| \frac{(x,\omega) - \quad (y,\omega)}{d(x,y)^\gamma} \right| \le a_\gamma \tag{3.2}$$

holds true. In particular, ϕ has a modification which is locally sample Hölder continuous of order γ.

Proof. It is clear that the assumptions of the corollary imply that ϕ satisfies the conditions of lemma 3.1. Thus we can apply lemma 3.1, and we have a modification of ϕ which is sample continuous and which coincides with ϕ on $D \times N^c$. In particular, for $(z,\omega) \in M \times N^c$, x, $y \in D \cap V(z)$, inequality (3.1) holds with ϕ replaced by . Now let x, $y \in V(z)$, $x \neq y$, and choose two sequences $(x_n,\, n \in \mathbb{N})$, $(y_n,\, n \in \mathbb{N})$ in $D \cap V(z)$ so that $x_n \to x$, $y_n \to y$ as $n \to +\infty$, and $x_n \neq y_n$ for all $n \in \mathbb{N}$. Let $\omega \in \Omega$. By the continuity of $(\cdot,\omega)$ we get

$$\begin{aligned} \left| \frac{(x,\omega) - \quad (y,\omega)}{d(x,y)^\gamma} \right| &= \lim_{n \to +\infty} \left| \frac{(x_n,\omega) - \quad (y_n,\omega)}{d(x_n,y_n)^\gamma} \right| \\ &\le a_\gamma, \end{aligned}$$

and inequality (3.2) follows. □

Now we begin to show that the conditions formulated in section 2 entail that the assumptions of lemma 3.1 and its corollaries are fulfilled.

Lemma 3.5. *If (M,d) has an accumulation point then $(\delta_n,\, n \in \mathbb{N})$ converges to zero.*

Proof. Let $x \in M$ be an accumulation point of (M,d). Then x is also an accumulation point of (D,d), where — as before — $D = \cup_n D_n$. Hence there exists a sequence $(x_n,\, n \in \mathbb{N})$ of pairwise different elements in D so that $x_n \neq x$ for all $n \in \mathbb{N}$, and $x_n \to x$ as $n \to +\infty$. In particular, $(x_n, \in \mathbb{N})$ is Cauchy with respect to d. Given $\varepsilon > 0$, we can then find $m \in \mathbb{N}$ with $0 < d(x_m, x_{m+1}) < \varepsilon$. Moreover, there exists $n_0 \in \mathbb{N}$ so that x_m, $x_{m+1} \in D_{n_0}$. It follows that $\delta^0_{n_0} < \varepsilon$. Since $(\delta^0_n, \in \mathbb{N})$ decreases, we have that for all $n \in \mathbb{N}$ with $n \ge n_0$, $\delta^0_n < \varepsilon$. Thus $(\delta^0_n, \in \mathbb{N})$ decreases to zero, and therefore $(\delta_n, \in \mathbb{N})$ decreases to zero, too. □

For the remainder of this section we assume that (M,d) has at least one accumulation point. We recall from section 2 that q and r denote two positive functions on $\mathbb{R}_+$ which are increasing on $[0,\rho]$ for some $\rho > 0$.

Lemma 3.6. *Suppose that the random field ϕ satisfies condition (B), and that $q(x)$, $r(x)$ converge to zero as $x \downarrow 0$. Then ϕ is continuous in probability.*

Proof. Let $x \in M$. We show that ϕ is continuous in probability in x. If $x \in M$ is an isolated point we have nothing to prove. Assume that x is an accumulation point of M, and let $(x_n,\, n \in \mathbb{N})$ be a sequence in M converging to x. By lemma 3.5 the sequence $(\delta_n,\, n \in \mathbb{N})$ tends to zero. By hypothesis $(q(\delta_n),\, n \in \mathbb{N})$ and $(r(\delta_n),\, n \in \mathbb{N})$ converge to zero. Given $\varepsilon > 0$, we can therefore find $n \in \mathbb{N}$ so that $r(\delta_n) < \varepsilon$, $q(\delta_n) < \varepsilon$ and $\delta_n \leq \rho$. Let $m_0 \in \mathbb{N}$ be large enough, so that for all $m \geq m_0$ we have $d(x,\, x_m) < \delta_n$. (B) implies for all $m \geq m_0$

$$\begin{aligned} P\big(|\phi(x) - \phi(x_m)| \geq \varepsilon\big) &\leq P\big(|\phi(x) - \phi(x_m)| \geq r(\delta_n)\big) \\ &\leq P\big(|\phi(x) - \phi(x_m)| \geq r(d(x, x_m))\big) \\ &\leq q\big(d(x, x_m)\big) \\ &\leq q(\delta_n) \\ &< \varepsilon, \end{aligned}$$

and the proof is finished. □

If q admits condition (C1) then this implies that $(q(\delta_n),\, n \in \mathbb{N})$ converges to zero, because $|\pi_n| \geq 1$ for all $n \in \mathbb{N}$. But since q is increasing in a neighborhood of zero, it follows that $q(x) \to 0$ as $x \downarrow 0$. Similarly, (C2) entails that $r(x) \to 0$ with $x \downarrow 0$. Hence we obtain

Corollary 3.7. *Suppose that the random field ϕ satisfies condition (B). Assume furthermore that q admits (C1), and that r satisfies (C2) or, in the case that (D) is true, r fulfills condition (C3). Then ϕ is continuous in probability.*

The following two lemmas are at the heart of the proof of theorems 2.8 and 2.9.

Lemma 3.8. *Assume that (M, d) is well separable with scale $\mathcal{D}$, ϕ admits condition (B) and q satisfies condition (C1). Then there exists a P-null set $N \in \mathcal{A}$ so that for every $\omega \in N^c$ there is $n(\omega) \in \mathbb{N}$ with*

$$\max_{\langle x,y\rangle \in \pi_n} |\phi(x,\omega) - \phi(y,\omega)| \leq r(\delta_n), \tag{3.3}$$

for all $n \in \mathbb{N}$ with $n \geq n(\omega)$.

Proof. Since $(\delta_n,\, n \in \mathbb{N})$ is decreasing to zero, there is $n_0 \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq n_0$ we have $\delta_n \leq \rho$. Let $n \in \mathbb{N}$ with $n \geq n_0$, and let $\langle x, y\rangle \in \pi_n$. Then $d(x, y) \leq \delta_n$ and condition (B) give

$$\begin{aligned} P\Big(|\phi(x) - \phi(y)| \geq r(\delta_n)\Big) &\leq P\Big(|\phi(x) - \phi(y)| \geq r\big(d(x, y)\big)\Big) \\ &\leq q\big(d(x, y)\big) \\ &\leq q(\delta_n). \end{aligned}$$

Therefore, for every $n \in \mathbb{N}$ with $n \geq n_0$ we get

$$\begin{aligned} P\Big(\max_{\langle x,y\rangle\in\pi_n} |\phi(x)-\phi(y)| \geq r(\delta_n)\Big) \\ &= P\Big(\bigcup_{\langle x,y\rangle\in\pi_n} \Big\{|\phi(x)-\phi(y)| \geq r(\delta_n)\Big\}\Big) \\ &\leq \sum_{\langle x,y\rangle\in\pi_n} P\Big(|\phi(x)-\phi(y)| \geq r(\delta_n)\Big) \\ &\leq |\pi_n|\, q(\delta_n). \end{aligned}$$

(C1) entails that the last expression is the general term of a convergent sum. An application of the Borel-Cantelli-lemma finishes the proof. □

Lemma 3.9. *Under the same conditions as in lemma 3.8 there exists a P-null set $N \in \mathcal{A}$ so that the following statements hold:*

(a) *Every $z \in M$ has a neighborhood $V(z)$ such that for all $\omega \in N^c$ there exists $n(\omega) \in \mathbb{N}$ so that for all m, $n \in \mathbb{N}$ with $m \geq n \geq n(\omega)$, and all x, $y \in D_m \cap V(z)$ with $d(x,y) \leq \delta_n$ the inequality*

$$\big|\phi(x,\omega)-\phi(y,\omega)\big| \leq 2\sum_{k=n}^{m} r(\delta_k) \tag{3.4}$$

holds.

(b) *If (M,d) is uniformly well separable with scale $\mathcal{D}$ then for every $\omega \in N^c$ there exists $n(\omega) \in \mathbb{N}$, so that for all m, $n \in \mathbb{N}$ with $m \geq n \geq n(\omega)$, and all x, $y \in D_m$ with $d(x,y) \leq \delta_n$ inequality (3.4) holds.*

Proof. Let $N \in \mathcal{A}$ be the P-null set in lemma 3.8, choose $\omega \in N^c$ and fix $n(\omega) \in \mathbb{N}$ as in lemma 3.8, so that inequality (3.3) holds for all $n \in \mathbb{N}$ with $n \geq n(\omega)$. Let $z \in M$, and let $V(z)$ be a neighborhood of z as in condition (W). Choose $n \in \mathbb{N}$ with $n \geq n(\omega)$. We prove statement (a) by induction on $m \in \mathbb{N}$, $m \geq n$.

For $m = n$, consider x, $y \in D_n \cap V(z)$ with $d(x,y) \leq \delta_n$. Then $\langle x,y\rangle \in \pi_n$, and inequality (3.4) follows from inequality (3.3).

Now suppose that the statement is true for $m-1 \in \mathbb{N}$ with $m-1 \geq n$. Let x, $y \in D_m \cap V(z)$. Condition (W) entails the existence of x', $y' \in D_{m-1} \cap V(z)$ so that $\langle x,x'\rangle$, $\langle y,y'\rangle \in \pi_m$, and $d(x',y') \leq d(x,y) \leq \delta_n$. The induction hypothesis yields the inequality

$$\big|\phi(x',\omega)-\phi(y',\omega)\big| \leq 2\sum_{k=n}^{m-1} r(\delta_k).$$

On the other hand, $\langle x,x'\rangle$, $\langle y,y'\rangle \in \pi_m$ together with inequality (3.3) gives

$$\begin{aligned} \big|\phi(x,\omega)-\phi(x',\omega)\big| &\leq r(\delta_m) \\ \big|\phi(y,\omega)-\phi(y',\omega)\big| &\leq r(\delta_m). \end{aligned}$$

Thus an application of the triangle inequality concludes the proof of (a).

For the proof of (b) we just have to choose $V(z) = M$ in the preceding argument, and use condition (U) instead of (W). □

Corollary 3.10. *Suppose that (M, d) is well separable with scale $\mathcal{D}$, and that conditions (B), (C1), and (C2) hold true. Then there is a P-null set N, and for every $z \in M$ there is a neighborhood $V(z)$ so that for every $\omega \in N^c$ the restriction of $\phi(\cdot, \omega)$ to $D \cap V(z)$ is uniformly continuous on $(D \cap V(z), d)$. If in addition (M, d) is uniformly well separable then for every $\omega \in N^c$ the restriction of $\phi(\cdot, \omega)$ to D is uniformly continuous on (D, d).*

Proof. Let $z \in M$, and let $V(z)$, N, and $n(\omega)$, $\omega \in N^c$, be as in statement (a) of lemma 3.9. Suppose that $\omega \in N^c$, and that we are given $\varepsilon > 0$. By (C2) we can choose $n_0 \in \mathbb{N}$ large enough so that we have

$$\sum_{k=n_0}^{\infty} r(\delta_k) < \frac{\varepsilon}{2}.$$

Choose

$$\delta(\omega) := \delta_{\max\{n(\omega), n_0\}}.$$

Let x, $y \in D \cap V(z)$, with $d(x, y) < \delta(\omega)$. For some $m \in \mathbb{N}$ we have x, $y \in D_m \cap V(z)$, and since $(D_n, n \in \mathbb{N})$ is increasing we may assume without loss of generality that $m \geq \max\{n(\omega), n_0\}$. Then statement (a) of lemma 3.9 implies

$$\left|\phi(x, \omega) - \phi(y, \omega)\right| < \varepsilon.$$

The second statement is proved in the same way, except that it is not necessary to localize to an appropriate neighborhood of a point in M. □

Corollary 3.11. *Suppose that (M, d) is dyadically separable and that (B), (C1) and (C3) hold. Then there exists a constant $a_\gamma > 0$, a P-null set N, and for every $z \in M$ there exists a neighborhood $V(z)$, so that for all $\omega \in N^c$ and all x, $y \in D \cap V(z)$ the following inequality holds true*

$$|\phi(x, \omega) - \phi(y, \omega)| \leq a_\gamma \, d(x, y)^\gamma. \tag{3.5}$$

Proof. Let N be the P-null set in statement (a) of lemma 3.9, choose $\omega \in N^c$ and fix $n(\omega) \in \mathbb{N}$ as there. In view of condition (C3), we may assume without loss of generality that for all $n \in \mathbb{N}$ with $n \geq n(\omega)$ we have $\delta_n \leq \rho$ — otherwise we just have to increase $n(\omega)$ appropriately. For $z \in M$ choose a neighborhood $V(z)$ of z as in part (a) of lemma 3.9. Set $\delta(\omega) := \delta_{n(\omega)} > 0$, and let x, $y \in D \cap V(z)$ with $0 < d(x, y) < \delta(\omega)$. Let n be the largest natural number so that $d(x, y) \leq \delta_n$. Then we have $n \geq n(\omega)$ and $\delta_{n+1} < d(x, y)$. Furthermore, there is $m \in \mathbb{N}$ with $m \geq n$ and such that x, $y \in D_m \cap V(z)$. By lemma 3.9 and conditions (C3), (D) we have the following estimation

$$\begin{aligned}\bigl|\phi(x,\omega)-\phi(y,\omega)\bigr| &\le 2\sum_{k=n}^{m} r(\delta_k)\\ &\le 2\,K_\gamma \sum_{k=n}^{m}\delta_k^\gamma\\ &\le 2\,K_\gamma\,\alpha^\gamma \sum_{k=n}^{m}\eta^{\gamma k}\\ &\le 2\,K_\gamma\,\frac{\alpha^\gamma}{\eta^\gamma\,(1-\eta^\gamma)}\,\eta^{\gamma(n+1)}\\ &\le 2\,K_\gamma\,\frac{\alpha^{2\gamma}}{\eta^\gamma\,(1-\eta^\gamma)}\,\delta_{n+1}^\gamma\\ &\le a_\gamma\, d(x,y)^\gamma,\end{aligned}$$

where

$$a_\gamma = 2\,K_\gamma\,\frac{\alpha^{2\gamma}}{\eta^\gamma\,(1-\eta^\gamma)}.$$

Thus we get the inequality (3.5), and the lemma is proved. □

Now we can finish the proof of theorems 2.8 and 2.9: Corollary 3.7 shows that under the hypothesis of each theorem ϕ is continuous in probability. Theorem 2.8 follows from corollaries 3.3 and 3.10, while corollaries 3.4 and 3.11 give theorem 2.9.

4. Examples

In this section we consider random fields defined on a subset M of $\mathbb{R}^m$, $m\in\mathbb{N}$, and we shall continue to use the notation from section 2.

$\mathbb{R}^m$ is endowed with the usual euclidean topology, its subsets with the relative topology. It will be convenient, however, to choose the following metric d instead of the euclidean metric:

$$d(x,y) := \max\bigl\{|x_i-y_i|,\, i=1,2,\dots,m\bigr\},\qquad x,y\in\mathbb{R}^m, \tag{4.1}$$

where x_i, y_i denote the i-th cartesian coordinates of x, y respectively. If M is a subset of $\mathbb{R}^m$ we shall denote the restriction of d to $M\times M$ again by d.

Assume that $(R_n,\, n\in\mathbb{N})$ is a sequence which increases to $+\infty$. For convenience and without loss of generality we suppose in addition that $R_1\ge 1$. For $n\in\mathbb{N}$ set

$$G_n := \left\{x\in\mathbb{R}^m,\, x=\frac{k}{2^n},\, k\in\mathbb{Z}^m\right\}, \tag{4.2}$$

and

$$H_n := G_n\cap[-R_n,R_n]^m. \tag{4.3}$$

Let M be a subset of $\mathbb{R}^m$ with non-empty interior. If M is bounded we set

$$D_n := M\cap G_n, \tag{4.4}$$

and in case that M is unbounded we define

$$D_n := M\cap H_n. \tag{4.5}$$

Thus, for all $n \in \mathbb{N}$, D_n is a finite set. Note that the assumption that M has non-empty interior implies that for almost all $n \in \mathbb{N}$, $D_n \neq \emptyset$. As in section 2, let

$$D := \bigcup_{n \in \mathbb{N}} D_n.$$

The elementary fact that

$$G := \bigcup_{n \in \mathbb{N}} G_n = \bigcup_{n \in \mathbb{N}} H_n$$

is dense in $(\mathbb{R}^m, d)$ entails that D is dense in (M, d). Clearly, there exists $n_0 \in \mathbb{N}$ so that for all $n \in \mathbb{N}$ with $n \geq n_0$ we have $|D_n| \geq 2$. (Consider first the case that M is bounded. By hypothesis, M contains a ball (with respect to d) of radius $r > 0$. Then it is easy to see that we can choose n_0 as the smallest natural number strictly larger than $\log_2(3) - \log_2(r)$. In case that M is unbounded, choose first n_0 large enough so that the intersection of M with $[-R_{n_0}, R_{n_0}]^m$ contains a ball of some strictly positive radius r. Then — if necessary — increase n_0 so that also the condition $n_0 > \log_2(3) - \log_2(r)$ holds true.) Thus for all $n \geq n_0$, we get $\delta_n^0 = 2^{-n}$. For $n \in \mathbb{N}$ with $n \geq n_0$ we set $\delta_n := \delta_n^0$, and otherwise equal to $+\infty$. Then with the scale

$$\mathcal{D} := \big((D_n, \delta_n),\, n \in \mathbb{N}\big),$$

$(M, d, \mathcal{D})$ is a scaled metric space in the sense of definition 2.1. Clearly, the scale $\mathcal{D}$ admits property (D) of section 2.

Next we consider property (U) of section 2, and specialize first to $M = [0,1]^m$. We shall show that (M, d) is uniformly dyadically separable with the above constructed scale $\mathcal{D}$. To this end, let $n \in \mathbb{N}$, x, $y \in D_{n+1}$, and denote by E_n the dyadic numbers of order $n \in \mathbb{N}$ in $[0,1]$:

$$E_n := \left\{ \xi \in [0,1],\, \xi = \frac{k}{n},\, k = 0, 1, \ldots, 2^n \right\}.$$

Let $i \in \{1, 2, \ldots, m\}$, and consider the i-th cartesian components x_i, y_i of x, y resp. We define x_i' and y_i' as follows:

$$x_i' := \begin{cases} \min\{\xi \in E_n,\, \xi \geq x_i\}, & \text{if } x_i < y_i, \\ \max\{\xi \in E_n,\, \xi \leq x_i\}, & \text{if } x_i > y_i, \end{cases} \tag{4.6}$$

$$y_i' := \begin{cases} \max\{\eta \in E_n,\, \eta \leq y_i\}, & \text{if } x_i < y_i, \\ \min\{\eta \in E_n,\, \eta \geq y_i\}, & \text{if } x_i > y_i, \end{cases} \tag{4.7}$$

$$x_i' := y_i' := \min\{\xi \in E_n,\, \xi \geq x_i\}, \qquad \text{if } x_i = y_i. \tag{4.8}$$

It is not hard to see that x', $y' \in M$, defined to have cartesian coordinates x_i', y_i' resp., $i = 1, 2, \ldots, m$, admit $d(x', y') \leq d(x, y)$, as well as $x' \in C_{n+1}(x)$ and $y' \in C_{n+1}(y)$. Thus, (U) holds for (M, d) with scale $\mathcal{D}$, and therefore $([0,1]^m, d)$ is uniformly dyadically separable with scale $\mathcal{D}$.

Let us remark that for the case $x_i = y_i$ we could have chosen

$$x_i' := y_i' := \max\{\xi \in E_n,\, \xi \le x_i\}, \qquad \text{if } x_i = y_i. \tag{4.9}$$

as an equivalent alternative to (4.8).

Consider now an arbitrary bounded interval M in $\mathbb{R}^m$ (with non-empty interior), i.e., a subset of the form

$$M = I_1 \times I_2 \times \ldots \times I_m,$$

where I_i, $i = 1, 2, \ldots, m$, is a bounded (non-empty) interval on the real axis. Let l denote the minimum of the lengths of the intervals I_i, $i = 1, 2, \ldots, m$. It is an elementary exercise to check that then for $n \in \mathbb{N}$ with $n \ge n_0$ and n_0 chosen such that $n_0 \ge \log_2(3) - \log_2(l)$, $D_n = M \cap G_n$ contains at least two elements. For $n \in \mathbb{N}$ with $n \ge n_0 + 1$ we can now make a construction as in the previous case, possibly with the exception that for points in D_n which are near to the boundary of M we have to use equation (4.9) instead of equation (4.8). As a result, we find that (U) holds for (M, d) with scale $\mathcal{D}$ in this case, too.

Finally we consider the case where M is a non-empty open subset of $\mathbb{R}^m$. Then every point $z \in M$ has a neighborhood V in M which is a bounded interval. Choose n_0 large enough so that $V \subset [-R_{n_0}, R_{n_0}]^m$. If necessary, increase $n_0 \in \mathbb{N}$ so that also the condition $n_0 \ge \log_2(3) - \log_2(l)$ holds, where l denotes the minimal side length of V. Then we can use the preceding discussion to conclude that with the scale given by $((V \cap G_n,\, 2^{-n}),\, n \in \mathbb{N})$, V is uniformly dyadically separable. Therefore (M, d) is dyadically separable with scale $\mathcal{D}$.

We collect our results in the following

Theorem 4.1. *With the scale $\mathcal{D}$ the following holds:*

(a) *If M is a bounded interval in $\mathbb{R}^m$ with non-empty interior then (M, d) is uniformly dyadically separable.*

(b) *If M is a non-empty open subset in $\mathbb{R}^m$ then (M, d) is dyadically separable.*

In view of condition (C1) of section 2 we derive next an estimate for $|\pi_n|$. Since we do not take any specific subset M of $\mathbb{R}^m$ into account here, the bound will be very rough but sufficient for the purposes below. For a specific application the interested reader might want to derive a better bound.

First consider again the situation where $M = [0, 1]^m$. Let $n \in \mathbb{N}$, and consider $x \in (0, 1)^m \cap D_n$. Then there are 3^m points in the clique $C_n(x)$ of x (with respect to the metric d, cf. (4.1)). If $x \in D_n$ belongs to the boundary of $[0, 1]^m$ then there are less than 3^m points in the clique $C_n(x)$. Thus we have for all $n \in \mathbb{N}$, $x \in D_n$, $|C_n(x)| \le 3^m$. On the other hand, there are $(2^n + 1)^m$ many points in D_n. Hence we obtain the following estimate

$$|\pi_n| \le 3^m\, (2^n + 1)^m \tag{4.10}$$

$$\le K_m\, \operatorname{diam}(M)^m\, 2^{mn}, \tag{4.11}$$

where K_m is some positive constant, and $\operatorname{diam}(M)$ denotes the diameter of M.

It is straightforward to check that for an arbitrary bounded interval $M \subset \mathbb{R}^m$ the bound (4.11) on $|\pi_n|$ remains true for all $n \in \mathbb{N}$ large enough. For an arbitrary bounded subset M of $\mathbb{R}^m$ we can then first choose an interval which contains M, and then use again the bound (4.11) for the latter.

For an unbounded set M in $\mathbb{R}^m$ we have by construction and the preceding arguments that for some constant $K_m > 0$

$$|\pi_n| \le K_m \, R_n^m \, 2^{mn} \tag{4.12}$$

holds for all $n \in \mathbb{N}$.

We have proved:

Lemma 4.2. *There is a constant $K_m > 0$ so that*

(a) *For every bounded subset M in $\mathbb{R}^m$ there exists $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $n \ge n_0$ inequality (4.11) holds;*

(b) *For every unbounded subset M in $\mathbb{R}^m$ inequality (4.12) holds for all $n \in \mathbb{N}$.*

For the sequel we make the following choice for the sequence $(R_n,\, n \in \mathbb{N})$:

$$R_n := \log_2\big(\log_2(n+3)\big), \qquad n \in \mathbb{N}. \tag{4.13}$$

Also, we define

$$\Lambda_1(h) := \log_2\big(h^{-1}\big) \tag{4.14}$$

$$\Lambda_2(h) := \log_2\Big(\log_2\big(h^{-1}\big)\Big), \tag{4.15}$$

for $h \in (0,1)$.

In view of condition (C1) of section 2 we consider the following two functions q_1, q_2. Assume that $K > 0$, $\alpha > 1$, and set

$$q_1(h) := \begin{cases} K\,\Big(\Lambda_1(h)\,\Lambda_2(h)^{\alpha}\Big)^{-1} h^m, & \text{if } h \in (0,\rho], \\ 0, & \text{if } h = 0, \end{cases} \tag{4.16}$$

$$q_2(h) := \begin{cases} K\,\Lambda_1(h)^{-\alpha}\, h^m, & \text{if } h \in (0,\rho], \\ 0, & \text{if } h = 0, \end{cases} \tag{4.17}$$

where $\rho > 0$ has to be chosen small enough so that q_i, $i = 1$,2, are positive and increasing on $[0,\rho]$, i.e., $\rho \in (0,1/2)$.

Lemma 4.3. *Assume that M is a subset of $\mathbb{R}^m$ with non-empty interior, and that D_n, $n \in \mathbb{N}$, is defined as in equation (4.4) or (4.5). Suppose furthermore that R_n, $n \in \mathbb{N}$, is defined as in equation (4.13), and that for $K > 0$, $\alpha > 1$, $\rho \in (0,1/2)$, q_1 and q_2 are given as in (4.16), (4.17) resp. Then q_1 and q_2 satisfy condition (C1).*

Proof. We only have to notice that by construction of q_1 and q_2 we have for $n \in \mathbb{N}$, $n \ge 2$,

$$2^{mn}\, q_1\left(2^{-n}\right) = K\,\frac{1}{n\,(\log_2(n))^{\alpha}},$$

$$2^{mn}\, q_2\left(2^{-n}\right) = K\,\frac{1}{n^{\alpha}}.$$

Thus for $i = 1, 2$ we have

$$\sum_n R_n^m\, 2^{mn}\, q_i(2^{-n}) < +\infty.$$

A glance at lemma 4.2 and the estimates (4.11), (4.12) finish the proof. □

Remark 4.4. With iterated logarithms of higher order for q and R_n it is possible to define functions $h \mapsto q(h)$ with somewhat weaker manner in which they converge to zero when $h \downarrow 0$ than the above q_1, q_2 (cf. also [6]). The same remark extends to the functions r_1, r_2 below. Moreover, it is easily checked that the same choices of the functions q_1, q_2 work also, if one chooses a metric equivalent to the one above, for example, the euclidean metric. In that case possibly one has to choose ρ above appropriately small. Consequently, the results below are independent of the choice of an equivalent metric, except possibly for an adjustment of ρ. The details are left to the interested reader.

For the function r appearing in conditions (B), (C2), and (C3), convenient choices are (with $\rho > 0$ as above):

$$r_1(h) := \begin{cases} \Big(\Lambda_1(h)\,\Lambda_2(h)^\beta\Big)^{-1}, & \text{if } h \in (0,\rho], \\ 0, & \text{if } h = 0, \end{cases} \tag{4.18}$$

$$r_2(h) := \begin{cases} \Lambda_1(h)^{-\beta}, & \text{if } h \in (0,\rho], \\ 0 & \text{if } h = 0, \end{cases} \tag{4.19}$$

with $\beta > 1$. In view of (C3) we shall also make use of

$$r_3(h) := h^\gamma, \qquad h \in [0,\delta], \tag{4.20}$$

with $\gamma \in (0,1)$. That condition (C2) is fulfilled by r_1 and r_2 is obvious from $(n \geq 2)$

$$\begin{aligned} r_1\left(2^{-n}\right) &= \frac{1}{n\left(\log_2(n)\right)^\beta} \\ r_2\left(2^{-n}\right) &= \frac{1}{n^\beta}. \end{aligned}$$

Now theorems 2.8, 2.9, 4.1, and lemma 4.3 give us the following result:

Theorem 4.5. *Let M be a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, with non-empty interior. Suppose that ϕ is a random field indexed by M so that there exist $\rho > 0$, $K > 0$, $\alpha > 1$, $\beta > 1$ with*

$$P\Big(|\phi(x) - \phi(y)| \geq r_i\big(d(x,y)\big)\Big) \leq q_j\big(d(x,y)\big) \tag{4.21}$$

for all x, $y \in M$ with $d(x,y) \leq \rho$, and some choice of $i \in \{1,2,3\}$, $j \in \{1,2\}$, where q_1, q_2, r_1, r_2, and r_3 are defined in equations (4.16), (4.17), (4.18), (4.19), and (4.20), respectively.

(a) *If M is a bounded interval, then ϕ has a modification which is uniformly sample continuous on M.*
(b) *If M is an open subset of $\mathbb{R}^m$, then ϕ has a modification which is locally uniformly continuous on M.*
(c) *If M is an open subset or a bounded interval and inequality (4.21) holds for $i = 3$, then ϕ has a modification which is locally sample Hölder continuous of order γ.*

We illustrate this theorem with an application to Gaussian random fields. For $\alpha > 1$, $\beta > 1$, $m \in \mathbb{N}$, $\rho > 0$, small enough, set

$$t(h) := \begin{cases} \Big(2\,\ln(2)\,\big(m\,\Lambda_1(h)^{2\beta+1} + \alpha\,\Lambda_1(h)^{2\beta}\,\Lambda_2(h)\big)\Big)^{-1}, & \text{if } h \in (0,\rho], \\ 0, & \text{if } h = 0. \end{cases} \tag{4.22}$$

Corollary 4.6. *Assume that M is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, with non-empty interior, and that ϕ is a centered Gaussian random field indexed by M, such that for all x, $y \in M$, $\sigma(x,y)^2 := \mathrm{Var}(\phi(x) - \phi(y)) > 0$. Suppose furthermore that there exist $\alpha > 1$, $\beta > 1$, $\rho > 0$ so that*

$$\sigma(x,y)^2 \le t\big(d(x,y)\big) \tag{4.23}$$

for all x, $y \in M$ with $d(x,y) \le \rho$. Then the following statements hold:

(a) *If M is a bounded interval, then ϕ has a modification which is uniformly sample continuous on M.*
(b) *If M is an open subset of $\mathbb{R}^m$, then ϕ has a modification which is locally uniformly sample continuous on M.*

Proof. For convenience, we shall work with the functions r_2, q_2 (cf. (4.19),(4.17) resp.), and check that inequality (4.21) holds. We have

$$P\Big(|\phi(x) - \phi(y)| \ge r_2\big(d(x,y)\big)\Big) = \sqrt{\frac{2}{\pi}}\int_{s(x,y)}^{+\infty} e^{-u^2/2}\,du,$$

where

$$s(x,y) := \frac{r_2\big(d(x,y)\big)}{\sigma(x,y)}.$$

Using the estimates of the error-function by Komatu-Pollak ([10], [15]) we get

$$\begin{aligned} P\Big(|\phi(x) - \phi(y)| \ge r_2\big(d(x,y)\big)\Big) & \\ &\le \sqrt{\frac{2}{\pi}}\,\frac{2}{s(x,y) + \sqrt{s(x,y)^2 + 8/\pi}}\,e^{-s(x,y)^2/2} \\ &\le e^{-s(x,y)^2/2}. \end{aligned}$$

Set $h := d(x,y)$. The assumption on $\sigma(x,y)$ gives

$$\begin{aligned} s(x,y)^2 &\ge 2\,\ln(2)\,\Lambda_1(h)^{-2\beta}\Big(m\,\Lambda_1(h)^{2\beta+1} + \alpha\,\Lambda_1(h)^{2\beta}\,\Lambda_2(h)\Big) \\ &= 2\,\Big(\ln\big(h^{-m}\big) + \ln\big(\Lambda_1(h)^{\alpha}\big)\Big), \end{aligned}$$

and therefore we find

$$\begin{aligned} e^{-s(x,y)^2/2} &\le h^m\,\Lambda_1(h)^{-\alpha} \\ &= q_2\big(d(x,y)\big) \end{aligned}$$

with $K = 1$. □

Similarly, we can formulate a simple sufficient condition for local sample Hölder continuity. Assume that $\rho \le 1/4$, and set

$$v_\gamma(h) := \begin{cases} \dfrac{h^{2\gamma}}{2\,\ln(2)\,\big(m\,\Lambda_1(h) + \alpha\,\Lambda_2(h)\big)}, & \text{if } h \in (0,\rho], \\ 0, & \text{if } h = 0. \end{cases} \tag{4.24}$$

Then, based on theorem 4.5.c the following corollary is proven in the same way as corollary 4.6:

Corollary 4.7. *Let M be an open subset of $\mathbb{R}^m$. Assume that ϕ is a centered Gaussian random field indexed by M such that for all x, $y \in M$, $x \neq y$, $\sigma(x,y)^2 := Var(\phi(x) - \phi(y)) > 0$. Suppose furthermore that there exists $\rho > 0$ so that for all x, $y \in M$ with $d(x,y) \le \rho$, $\sigma(x,y)^2 \le v_\gamma(d(x,y))$, for some $\gamma \in (0,1)$. Then ϕ has a modification   with the following property: There exists a constant a_γ such that every $z \in M$ has a neighborhood V with*

$$P\left(\frac{|\ \ (x) - \ \ (y)|}{d(x,y)^\gamma} \le a_\gamma\right) = 1, \qquad \text{for all } x,\ y \in V.$$

In particular, ϕ has a modification which is locally sample Hölder continuous of order γ.

Of course, we can combine the statements of theorem 4.5 with Chebyshev's inequality in the obvious way, in order to derive sufficient conditions in terms of moments:

Corollary 4.8. *Assume that M is a subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, with non-empty interior and that ϕ is a random field indexed by M.*

(a) *Suppose that there exist $p \ge 1$, $\rho > 0$, $\kappa \ge m$, $\lambda \ge p+1$, $\nu > p+1$ and $K > 0$ so that for all x, $y \in M$ with $d(x,y) \le \rho$*

$$\mathbb{E}\Big(|\phi(x) - \phi(y)|^p\Big) \le K\,\Lambda_1\big(d(x,y)\big)^{-\lambda}\,\Lambda_2\big(d(x,y)\big)^{-\nu}\,d(x,y)^\kappa \tag{4.25}$$

or

$$\mathbb{E}\Big(|\phi(x) - \phi(y)|^p\Big) \le K\,\Lambda_1\big(d(x,y)\big)^{-\nu}\,d(x,y)^\kappa \tag{4.26}$$

holds. If M is open, then ϕ has a modification which is locally uniformly sample continuous on M. If M is a bounded interval then ϕ has a modification which is uniformly sample continuous on M.

(b) *Suppose that there exist $p \ge 1$, $\rho > 0$, $\gamma \in (0,1)$, $\alpha > 1$, and $K > 0$ so that for all x, $y \in M$ with $d(x,y) \le \rho$*

$$\mathbb{E}\Big(|\phi(x) - \phi(y)|^p\Big) \le K\,\Lambda_1\big(d(x,y)\big)^{-1}\,\Lambda_2\big(d(x,y)\big)^{-\alpha}\,d(x,y)^{m+p\gamma} \tag{4.27}$$

or

$$\mathbb{E}\Big(|\phi(x) - \phi(y)|^p\Big) \le K\,\Lambda_1\big(d(x,y)\big)^{-\alpha}\,d(x,y)^{m+p\gamma} \tag{4.28}$$

holds. If M is open, then ϕ has a modification which is locally sample Hölder continuous of order γ (with uniform Hölder constant).

Again we illustrate the last corollary by a simple application to Gaussian random fields. Assume that M is an open subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, and that ϕ is a centered Gaussian random field indexed by M with $\sigma(x,y)^2 = \mathrm{Var}(\phi(x) - \phi(y))$, x, $y \in M$. Suppose that there exist $\rho > 0$, $\eta \in (0,1)$, and a constant $C > 0$ so that

$$\sigma(x,y)^2 \leq C\, d(x,y)^\eta \tag{4.29}$$

for all x, $y \in M$, $d(x,y) \leq \rho$. For $n \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}\Big(\big(\phi(x) - \phi(y)\big)^{2n}\Big) &= K_n \left(\sigma(x,y)^2\right)^n \\ &\leq K_n\, C\, d(x,y)^{n\eta}\end{aligned}$$

with $K_n = (2n-1)!!$. Let $\gamma \in (0, \eta/2)$ and set $\varepsilon_1 = \eta/2 - \gamma > 0$. Next choose $n \in \mathbb{N}$ large enough so that $2n\varepsilon_1 > m$, say, $2n\varepsilon_1 = m + \varepsilon_2$ with $\varepsilon_2 > 0$. Let $\alpha > 1$. Then there is a constant C' so that

$$d(x,y)^{n\eta} \leq C'\, \Lambda_1\big(d(x,y)\big)^{-\alpha}\, d(x,y)^{m+2n\gamma}.$$

Now set $K = K_n\, C\, C'$, and we have an estimate like in inequality (4.28). Consequently, for every $\gamma < \eta/2$ the Gaussian random field ϕ has a modification which is locally sample Hölder continuous of order γ.

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