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MRM-APPLICABLE ORTHOGONAL POLYNOMIALS FOR CERTAIN HYPERGEOMETRIC FUNCTIONS

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ABSTRACT. The multiplicative renormalization method (MRM) is intorduced to obtain generating functions of orthogonal polynomials of given probability measures. Complete lists of MRM-applicable measures for MRM-factors $h(x) = e^x$ and $(1-x)^{-\kappa}$ were obtained recently. On the other hand, it is known that gamma distributions have at least two types of MRM-factors $h(x) = {}_0F_1(-;\kappa;x)$ and $h(x) = {}_1F_1(c;\kappa;x)$. The usual MRM-factor e^x is a special case of ${}_1F_1(c;\kappa;x)$ when $c = \kappa$. We first determine all MRM-applicable measures for $h(x) = {}_0F_1(-;\kappa;x)$. Then we determine all possible MRM-factors of gamma distributions.

1. MRM-applicability of Orthogonal Polynomials

A probability measure μ on $\mathbb{R}$ with density $f_\mu(x)$ is said to be applicable to *the multiplicative renormalization method for* $h(x)$ (or simply *MRM-applicable*), if there exists a suitable analytic function $\rho(t)$ around $t = 0$ with $\rho(0) = 0$, $r_1 = \rho'(0) \neq 0$ such that

$$(t,x) = \frac{h(\rho(t)x)}{\varphi(t)} \quad \text{with} \quad \varphi(t) = \theta(\rho(t)), \;\; \theta(t) = \int_{\mathbb{R}} h(tx)\, d\mu(x) \tag{1.1}$$

is a generating function of the orthogonal polynomials $\{P_n(x)\}$ in $L^2(\mu)$ with leading coefficient of one. Then there exist Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x) \tag{1.2}$$

with $\omega_0 = 1, P_{-1}(x) = 0$. Eq. (1.2) is a necessary and sufficient condition for that $\{P_n(x)\}$ are orthogonal polynomials for some signed measure (see Shohat [18]). It is known that

$$\|P_n\|^2 = \lambda_n = \omega_0\omega_1\cdots\omega_n \quad \text{for} \quad n \geq 0. \tag{1.3}$$

By Favard's Theorem [8], a set $\{P_n\}$ of polynomials with leading coefficient 1 is orthogonal for a probability measure if and only if they satisfy the recursion relation in Eq. (1.2) with $\omega_n > 0$ for all $n \geq 0$ (see [1] and [5]).

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Let us suppose that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad h_0 = 1,\ h_n \neq 0, \quad n \geq 1.$$

Then we have the expansion

$$(t, x) = \sum_{n=0}^{\infty} r_1^n h_n P_n(x) t^n. \tag{1.4}$$

In the previous papers [6] [12] [13] [14] [15], all MRM-applicable measures for $h(x) = e^x$ and $h(x) = (1-x)^{-\kappa}$ are determined.

A set of polynomials is said to be *MRM-applicable for* $h(x)$, if they are orthogonal for a probability measure and are given by the generating function in Eq. (1.1). In this article, we discuss MRM-applicability for $h(x) = {}_0F_1(-;\kappa;x)$. By Favard's Theorem, it is sufficient for the purpose to find ρ- and B-functions such that polynomials given by the following generating function in Eq. (1.5) satisfy Eq. (1.2) with $\omega_0 = 1$ and $\omega_n > 0$ for any $n \geq 0$. For convenience, we write a generating function in the form

$$(t, x) = B(t) h(\rho(t) x), \tag{1.5}$$

where $B(t) = \frac{1}{\theta(\rho(t))}$. This function (t, x) is called the *Boas–Buck generating function* (see §6 of [17]).

2. Hypergeometric Functions as MRM-factors

Hypergeometric function ${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x)$ is defined by

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; x) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n n!} x^n,$$

where

$$(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}.$$

In the special case of $p = 0$ (or $p = 1,\ q = 0$), we denote as

$${}_0F_q(-; b_1, b_2, \ldots, b_q; x) \quad (\text{respectively, } {}_1F_0(a_1; -; x)).$$

Let μ_κ be the gamma distribution with the density

$$f_\kappa(x) = \frac{1}{\Gamma(\kappa)} x^{\kappa-1} e^{-x}, \text{ for } x \geq 0. \tag{2.1}$$

Laguerre polynomials

$$L_n^{\kappa-1}(x) = \sum_{m=0}^{n} (-1)^m \frac{\Gamma(\kappa+n)}{\Gamma(m+\kappa)(n-m)!m!} x^m \tag{2.2}$$

are orthogonal for gamma distribution μ_κ and their Jacobi-Szegö parameters are

$$\alpha_n = \kappa + 2n \quad \text{and} \quad \omega_n = n(n+\kappa-1), \quad \omega_0 = 1. \tag{2.3}$$

A generating function is given as

$$(1+t)^{-\kappa} e^{\frac{t}{1+t}x} = \sum_{n=0}^{\infty} (-1)^n L_n^{\kappa-1}(x) t^n \tag{2.4}$$

(see (7) and (17) in §10.12 of [7].) There are other generating functions given by hypergeometric functions as

$$e^{-t}\,{}_0F_1(-;\kappa;tx) = \sum_{n=0}^{\infty}(-1)^n \frac{L_n^{\kappa-1}(x)}{(\kappa)_n}t^n \tag{2.5}$$

(see Eq. (1) in §113 of [17] and Eq. (8) in §19.9 of [7]) and

$$\frac{1}{(1+t)^c}\,{}_1F_1\left(c;\kappa;\frac{t}{1+t}x\right) = \sum_{n=0}^{\infty}\frac{(c)_n(-1)^n L_n^{(\kappa-1)}(x)}{(\kappa)_n}t^n \quad \text{for any} \quad c>0 \tag{2.6}$$

(see Eq. (3) in §113 of [17]). Special choices of c as $c=1$ and $c=\kappa$ give examples

$$\begin{aligned} {}_1F_1(1;\kappa;x) &= \sum_{n=0}^{\infty}\frac{\Gamma(\kappa)}{\Gamma(\kappa+n)}x^n, \\ {}_1F_1(\kappa;\kappa;x) &= \sum_{n=0}^{\infty}\frac{1}{n!}x^n = e^x. \end{aligned}$$

For convenience, we say that $h(x)$ is an *MRM-factor* of μ if μ is MRM-applicable for $h(x)$. We have the following chart.

Gamma distribution μ_κ and MRM-factor $h(x)$'s				
μ	$h(x)$	$\rho(t)$	$\varphi(t)$	(t,x)
μ_κ	e^x	$\frac{t}{1+t}$	$(1+t)^\kappa$	$(1+t)^{-\kappa}e^{\frac{tx}{1+t}}$
μ_κ	${}_0F_1(-;\kappa;x)$	t	e^t	$e^{-t}{}_0F_1(-;\kappa;tx)$
μ_κ	${}_1F_1(c;\kappa;x)$	$\frac{t}{1+t}$	$(1+t)^c$	$(1+t)^{-c}{}_1F_1(c;\kappa;\frac{tx}{1+t})$

Our purpose of this article is to find all possible probability measures (equivalently Jacobi-Szegö parameters), which are MRM-applicable for hypergeometric functions $h(x) = {}_0F_1(-;\kappa;x)$. Of course, gamma distribution is one of them as seen above.

Put

$$h(x) = {}_0F_1(-;\kappa;x) = \sum_{n=0}^{\infty}h_n x^n, \quad h_n = \frac{1}{n!(\kappa)_n}, \tag{2.7}$$

$$\rho(t) = \sum_{n=1}^{\infty}r_n t^n, \;\; r_1 = 1; \quad B(t) = \sum_{n=0}^{\infty}b_n t^n, \;\; b_0 = 1. \tag{2.8}$$

Let $\{P_n(x)\}$ be the polynomials given by

$$B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{2.9}$$

If P_n's satisfy Eq. (1.2), then parameters $\{h_n, b_n, r_n, \alpha_n, \omega_n\}$ are related. We will see relationships between them in §3. Moreover it is shown in §3 that Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$, $B(t)$ and $\rho(t)$ satisfying Eq. (1.2) are parametrized by constants $b_1,\ b_2,\ r_2$ and r_3, which are not independent. We give below some examples by using the results to be proved in §4.

We introduce a function $E(a, b; t)$ defined by

$$E(a, b; t) = \int_0^t \frac{1}{\sqrt{u(1 + bu + au^2)}} du, \tag{2.10}$$

which can be obtained by the elliptic integral of the first kind. Here for u with $u(1+bu+au^2) < 0$, we use the convention $\sqrt{u(1 + bu + au^2)} = i\sqrt{-u(1 + bu + au^2)}$. Then it is very important to see that the singularity of $\big(E(a, b; t)\big)^2$ at $t = 0$ is removable and it can be regarded analytic in t around $t = 0$. In this sense, we say that the function is analytic in t around $t = 0$ simply.

Example 2.1. If $b_2 = b_1^2,\ r_2 = 0,\ r_3 = 0$, then

$$\alpha_n = -b_1(2n + \kappa), \quad \omega_n = b_1^2 n(n + \kappa - 1), \quad B(t) = e^{b_1 t}, \quad \rho(t) = t.$$

In particular, if $b_1 = -1$, then this corresponds to Eq. (2.5).

Example 2.2. For the case of $\kappa = \frac{1}{2}$, we see

$$h(x) = {}_0F_1(-; \frac{1}{2}; x) = \begin{cases} \cosh 2\sqrt{x}, & \text{if } \ x \geq 0, \\ \cos 2\sqrt{-x}, & \text{if } \ x < 0. \end{cases}$$

By the convention $\sqrt{x} = i\sqrt{-x}$, $h(x)$ is written as $\cosh \sqrt{x}$ for both cases and it is analytic around $x = 0$. If we take as $b_1 = -\frac{1}{4},\ b_2 = \frac{1}{8},\ r_2 = -\frac{1}{3},\ r_3 = \frac{23}{120}$, then

$$\alpha_n = \frac{1}{8}(2n + 1)^2, \quad \omega_n = \frac{1}{64} n^2 (2n - 1)(2n + 1),$$

$$B(t) = \frac{4}{4 + t}, \quad \rho(t) = 4\left(\tan^{-1} \frac{\sqrt{t}}{2}\right)^2.$$

The corresponding generating function $\quad (t, x) = \frac{4}{4+t} \cosh\big(4\sqrt{x} \tan^{-1} \frac{1}{2}\sqrt{t}\big)$ gives Wilson polynomials $\{P_n(x) = \frac{(-1)^n n!}{(2n)!} W_n\big(x; 0, \frac{1}{2}, \frac{1}{2}, 1\big)\}$, which are orthogonal with respect to the probability measure with the density $\frac{16}{\pi}\sqrt{x}|\Gamma(2ix)|^2$ (cf. §3.8 of [1]). In this case such a measure is not necessarily unique (see [19]).

Example 2.3. For the case of $\kappa = \frac{3}{2}$, we see

$$h(x) = {}_0F_1(-; \frac{3}{2}; x) = \begin{cases} \dfrac{1}{2\sqrt{x}} \sinh 2\sqrt{x}, & \text{if } \ x > 0, \\ 1, & \text{if } \ x = 0, \\ \dfrac{1}{2\sqrt{-x}} \sin 2\sqrt{-x}, & \text{if } \ x < 0. \end{cases}$$

If we take as $b_1 = -\frac{1}{12},\ b_2 = \frac{1}{40},\ r_2 = -\frac{1}{3},\ r_3 = \frac{23}{120}$, then

$$\alpha_n = \frac{1}{8}(2n+1)^2, \quad \omega_n = \frac{1}{64}n^2(2n-1)(2n+1),$$
$$B(t) = \frac{2}{\sqrt{t}}\tan^{-1}\frac{\sqrt{t}}{2}, \quad \rho(t) = 4\left(\tan^{-1}\frac{\sqrt{t}}{2}\right)^2.$$

The corresponding generating function $(t,x) = \frac{1}{2\sqrt{tx}}\sinh\left(4\sqrt{x}\tan^{-1}\frac{1}{2}\sqrt{t}\right)$ gives the same Wilson polynomials $P_n(x) = \frac{(-1)^n n!}{(2n)!}W_n\left(x; 0, \frac{1}{2}, \frac{1}{2}, 1\right)$ as in Example 2.2.

3. Lemmas

By Favard's Theorem [8], a set $\{P_n\}$ of polynomials with leading coefficient 1 satisfies the recursion relation in Eq. (1.2)

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_n P_{n-1}(x), \quad n \geq 0,$$

with Jcobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying $P_{-1}(x) = 0, \alpha_{-1} = 0, \omega_0 = 1$, $\omega_n > 0$ for any $n \geq 0$, if and only if they are orthogonal polynomials with respect to a probability measure μ.

Through out this paper, we will assume that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n, \quad \rho(t) = \sum_{n=0}^{\infty} r_n t^n, \quad B(t) = \sum_{n=0}^{\infty} b_n t^n. \tag{3.1}$$

We may normalize these functions so that $h(0) = h_0 = 1,\ B(0) = b_0 = 1\ \rho(0) = r_0 = 0$, and $\rho'(0) = r_1 = 1$. For convenience, we put $h_{-1} = b_{-1} = 0$. Suppose that $(t,x) = B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$, i.e.,

$$(t,x) = B(t)h(\rho(t)x) = \sum_{n=0}^{\infty} h_n P_n(x) t^n. \tag{3.2}$$

Put

$$\theta(t) = \int_{\mathbb{R}} h(tx)\, d\mu(x), \quad \varphi(t) = \theta(\rho(t)).$$

Obviously, we have

$$B(t) = \frac{1}{\varphi(t)}. \tag{3.3}$$

Define

$$W_n(x) = P_n(x) - (x - \alpha_{n-1})P_{n-1}(x) + \omega_{n-1}P_{n-2}(x) \tag{3.4}$$

and let $W_{n,m}$ be the m-th coefficient of $W_n(x)$ for $n \geq m \geq 0$. Then

$$W_{n+1,m} = c_{n+1,m} - c_{n,m-1} + \alpha_n c_{n,m} + \omega_n c_{n-1,m} \tag{3.5}$$

for $n \geq m \geq 0$. Since $W_n(x) = 0$ must hold, all $W_{n,m}$ must vanish. Let $B_{m,k}$ be the coefficient of $B(t)\rho^m(t),\ m \geq 0$. Then $B_{m,k} = 0$ for $m > k$. Hence

$$B(t)\rho^m(t) = \sum_{k=0}^{\infty} B_{m,k}t^k = \sum_{k=m}^{\infty} B_{m,k}t^k. \tag{3.6}$$

The following Lemmas 3.1, 3.2, and 3.3 are shown in [10].

Lemma 3.1. *The following equalities hold:*

$$
\begin{aligned}
W_{n+1,0} &= \frac{b_{n+1}}{h_{n+1}} + \alpha_n \frac{b_n}{h_n} + \omega_n \frac{b_{n-1}}{h_{n-1}}, \\
W_{n+1,1} &= \frac{h_1}{h_{n+1}} r_{n+1} + \frac{h_1}{h_{n+1}} \sum_{k=1}^{n} b_{n+1-k} r_k - \frac{b_n}{h_n} \\
&\quad + \alpha_n \frac{h_1}{h_n} \sum_{k=1}^{n} b_{n-k} r_k + \omega_n \frac{h_1}{h_{n-1}} \sum_{k=1}^{n-1} b_{n-1-k} r_k, \\
W_{n+1,m} &= \frac{h_m}{h_{n+1}} B_{m,n+1} - \frac{h_{m-1}}{h_n} B_{m-1,n} + \alpha_n \frac{h_m}{h_n} B_{m,n} + \omega_n \frac{h_m}{h_{n-1}} B_{m,n-1}, \\
W_{n+1,n-1} &= \frac{h_{n-1}}{2h_{n+1}} (2b_2 + 2(n-1)(b_1 r_2 + r_3) + (n-1)(n-2) r_2^2) \\
&\quad - \frac{h_{n-2}}{2h_n} \big(2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3) r_2^2\big) \\
&\quad + \frac{h_{n-1}}{h_n} (b_1 + (n-1) r_2) \alpha_n + \omega_n, \\
W_{n+1,n} &= \frac{h_n}{h_{n+1}} (b_1 + n r_2) - \frac{h_{n-1}}{h_n} (b_1 + (n-1) r_2) + \alpha_n.
\end{aligned}
$$

Moreover, μ is MRM-applicable for $h(x)$ if and only if $W_{n,m} = 0$ for all $n-1 \geq m \geq 0$ and $\omega_n > 0$ for all $n \geq 0$.

Lemma 3.2. *For given $\{\alpha_n, \omega_n\}$ and $\{b_1, r_2, r_3\}$, the recursion formulas*

$$
\begin{aligned}
h_{n+1} &= \frac{h_n^2 (b_1 + n r_2)}{h_{n-1}\big(b_1 + (n-1) r_2\big) - h_n \alpha_n}, \\
b_{n+1} &= -\frac{h_{n+1}}{h_n} \alpha_n b_n - \frac{h_{n+1}}{h_{n-1}} \omega_n b_{n-1}, \\
r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m} r_m - \frac{h_{n+1}}{h_n} \alpha_n \sum_{m=1}^{n} b_{n-m} r_m \\
&\quad - \frac{h_{n+1}}{h_{n-1}} \omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m,
\end{aligned}
$$

hold for $n \geq 1$ if $h_{n-1}\big(b_1 + (n-1) r_2\big) - h_n \alpha_n \neq 0$ and $h_1 = -\frac{b_1}{\alpha_0}$ if $\alpha_0 \neq 0$.

Lemma 3.3. *For given $h(x)$ and fixed $\{b_1, b_2, r_2, r_3\}$, the Jacobi-Szegö parameters are uniquely determined by*

$$
\begin{aligned}
\alpha_n &= -\frac{h_n}{h_{n+1}} (b_1 + n r_2) + \frac{h_{n-1}}{h_n} \big(b_1 + (n-1) r_2\big), \quad n \geq 1, \\
\omega_n &= \frac{h_{n-1}}{2h_{n+1}} \big(2(b_1^2 - b_2 + b_1 r_2) + 2(n-1)(b_1 r_2 - r_3) + (n-1)(n+2) r_2^2\big) \\
&\quad + \frac{h_{n-2}}{2h_n} \big(2b_2 + 2(n-2)(b_1 r_2 + r_3) + (n-2)(n-3) r_2^2\big) \\
&\quad - \left(\frac{h_{n-1}}{h_n} \big(b_1 + (n-1) r_2\big) \right)^2, \quad n \geq 2,
\end{aligned}
$$

and $\alpha_0 = -\frac{b_1}{h_1}$, $\omega_0 = 1$, $\omega_1 = \frac{b_1^2 - b_2 + b_1 r_2}{h_2} - \frac{b_1^2}{h_1^2}$. *Furthermore, the recursion formulas for* $\{b_n, r_n\}$ *are given by*

$$\begin{aligned} b_{n+1} &= -\frac{h_{n+1}}{h_n}\alpha_n b_n - \frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1}, \quad n \geq 2, \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m} r_m - \frac{h_{n+1}}{h_n}\alpha_n \sum_{m=1}^{n} b_{n-m} r_m \\ &\quad -\frac{h_{n+1}}{h_{n-1}}\omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Lemma 3.4. *Suppose* $A_n = \sum_{j=0}^{n} \alpha_j \neq 0$ *for all* $n \geq 0$. *Then for given* $\{\alpha_n\ \omega_n\}$ *and* $\{b_1,\ r_2,\ r_3\}$, *we have*

$$\frac{h_{n+1}}{h_n} = -\frac{b_1 + n r_2}{\sum_{j=0}^{n} \alpha_j} \tag{3.7}$$

and

$$\begin{aligned} h_n &= (-1)^n \prod_{k=0}^{n-1} \frac{b_1 + k r_2}{A_k}, \quad n \geq 1, \\ b_{n+1} &= \frac{b_1 + n r_2}{A_n}\alpha_n b_n - \frac{(b_1 + n r_2)(b_1 + (n-1) r_2)}{A_{n-1} A_n}\omega_n b_{n-1}, \quad n \geq 1, \\ r_{n+1} &= \frac{b_1 + n r_2}{h_1 A_n} b_n - \sum_{m=1}^{n} b_{n+1-m} r_m + \frac{b_1 + n r_2}{A_n}\alpha_n \sum_{m=1}^{n} b_{n-m} r_m \\ &\quad -\frac{(b_1 + n r_2)(b_1 + (n-1) r_2)}{A_{n-1} A_n}\omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Proof. By Lemma 3.2, we have

$$(b_1 + n r_2)\frac{h_n}{h_{n+1}} = \big(b_1 + (n-1) r_2\big)\frac{h_{n-1}}{h_n} - \alpha_n.$$

Since $\frac{h_0}{h_1} b_1 = -\alpha_0$, we see that $(b_1 + n r_2)\frac{h_n}{h_{n+1}} = -\sum_{j=0}^{n} \alpha_j$, which gives Eq. (3.7). But $h_0 = 1$. Hence we have the formula for h_n. By applying Eq. (3.7) to Lemma 3.2, we obtain the formulas for b_{n+1} and r_{n+1} as stated in the lemma. □

Lemma 3.5. *If* $\alpha_n = 0$ *for all* $n \geq 0$, *then* $r_{2n} = 0$, $b_{2n+1} = 0$ *for all* $n \geq 0$. *For given* $\{\alpha_n\ \omega_n\}$ *and* $\{h_1,\ b_2,\ r_3\}$, *the following recursion formulas hold,*

$$\begin{aligned} h_{n+1} &= \frac{\big(b_2 + (n-1) r_3\big) h_{n-1} h_n}{\big(b_2 + (n-2) r_3\big) h_{n-2} - h_n \omega_n}, \quad n \geq 2 \\ b_{n+1} &= -\frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1}, \quad n \geq 2 \\ r_{n+1} &= \frac{b_n h_{n+1}}{h_1 h_n} - \sum_{m=1}^{n} b_{n+1-m} r_m - \frac{h_{n+1}}{h_{n-1}}\omega_n \sum_{m=1}^{n-1} b_{n-1-m} r_m, \quad n \geq 2. \end{aligned}$$

Furthermore, let $\Omega_n = \sum_{j=1}^{n} \omega_j$. *Then for* $n \geq 1$, *we have*

$$\frac{h_{n+1}}{h_{n-1}} = -\frac{b_2 + (n-1) r_3}{\Omega_n}, \tag{3.8}$$

$$\begin{aligned} h_{2n} &= (-1)^n \prod_{k=1}^{n} \frac{\big(b_2 + 2(k-1)r_3\big)}{\Omega_{2k-1}}, \\ h_{2n+1} &= (-1)^n h_1 \prod_{k=1}^{n} \frac{\big(b_2 + (2k-1)r_3\big)}{\Omega_{2k}}, \end{aligned} \tag{3.9}$$

$$b_{2n} = (-1)^n h_{2n} \prod_{k=1}^{n} \omega_{2k-1} = \prod_{k=1}^{n} \frac{\omega_{2k-1}\big(b_2 + 2(k-1)r_3\big)}{\Omega_{2k-1}}, \tag{3.10}$$

$$\begin{aligned} r_{2n+1} &= \prod_{k=1}^{n} \frac{\omega_{2k-1}\big(b_2 + (2k-1)r_3\big)}{\Omega_{2k}} - \sum_{m=1}^{n} b_{2(n+1-m)} r_{2m-1} \\ &\quad + \frac{\omega_{2n}\big(b_2 + (2n-1)r_3\big)}{\Omega_{2n}} \sum_{m=1}^{n} b_{2(n-m)} r_{2m-1}. \end{aligned} \tag{3.11}$$

Proof. Suppose $\alpha_n = 0$ for all $n \geq 0$. Since $W_{1,0} = \frac{b_1}{h_1} = 0$, we have $b_1 = 0$. But $W_{2,0} = \frac{b_2 + h_2\omega_1}{h_2} = 0$ and $W_{2,1} = \frac{h_1 r_2}{h_2} = 0$. Hence $r_2 = 0$ and $h_2 = -\frac{b_2}{\omega_1}$. Solving $W_{n+1,n-1} = 0$ for h_{n+1}, we obtain the first recursion formula in the lemma for h_{n+1}. Then we can rewrite it as

$$\frac{h_{n-1}}{h_{n+1}}\big(b_2 + (n-1)r_3\big) = \frac{h_{n-2}}{h_n}\big(b_2 + (n-2)r_3\big) - \omega_n,$$

which yields the equality

$$\frac{h_{n-1}}{h_{n+1}}\big(b_2 + (n-1)r_3\big) = -\sum_{j=1}^{n} \omega_j.$$

Therefore we have Eq. (3.8). The other equalities are obtained from Lemma 3.2. Since $b_1 = 0$ and $b_{n+1} = -\frac{h_{n+1}}{h_{n-1}}\omega_n b_{n-1}$, we see that $b_{2n+1} = 0$ for all $n \geq 0$. Assume that $r_{2j} = 0$ for $n \geq j \geq 0$, then

$$r_{2n+2} = \frac{b_{2n+1}h_{2n+2}}{h_1 h_{2n+1}} - \sum_{m=1}^{2n+1} b_{2n+2-m} r_m - \frac{h_{2n+2}}{h_{2n}}\omega_{2n-1} \sum_{m=1}^{2n} b_{2n-m} r_m = 0.$$

Thus by induction we have $r_{2n} = 0$ for all $n \geq 0$. □

Remark 3.6. (i) We have normalized the constants so that

$$h_0 = 1, \quad b_0 = 1, \quad r_1 = 1. \tag{3.9}$$

This does not impose essential restrictions. Since $\theta(0) = h(0)$, $B(0)h(0) = b_0 h_0 = 1$ holds. Suppose that $(t,x) = B(t)h(\rho(t)x)$ is a generating function for orthogonal polynomials. If $h_0 \neq 1$, put $\widehat{h}(x) = \frac{h(x)}{h(0)}$ and $\widehat{B}(t) = h(0)B(t)$. Then $\widehat{\ }(t,x) = \widehat{B}(t)\widehat{h}(\rho(t)x) = (t,x)$ is a generating function satisfying Eq. (3.9). On the other hand, if $r_1 \neq 1$, then put $\alpha = \frac{1}{r_1}$, $\widehat{\rho}(t) = \rho(\alpha t)$ and $\widehat{b}(t) = B(\alpha t)$. Then $\widehat{\ }(t,x) = \widehat{B}(t)h(\widehat{\rho}(t)x)$ is a generating function satisfying Eq. (3.9).

(ii) Suppose $\quad(t,x)=B(t)h(\rho(t)x)$ is a generating function of $\{P_n\}$. For given $\widetilde{h}_1,\ \widetilde{r}_1\neq 0$, the scaling transforms

$$\widetilde{h}(x)=h\Big(\frac{\widetilde{h}_1}{h_1}x\Big),\quad \widetilde{\rho}(t)=\frac{h_1}{\widetilde{h_1}}\rho\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1r_1}t\Big),\quad \widetilde{B}(t)=B\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1r_1}t\Big), \tag{3.10}$$

give a modified generating function

$$\widetilde{\ }(t,x)=\widetilde{B}(t)\widetilde{h}(\widetilde{\rho}(t)x)=\quad\Big(\frac{\widetilde{h}_1\widetilde{r}_1}{h_1r_1}t,x\Big)$$

satisfying $\widetilde{h}'(0)=\widetilde{h}_1$ and $\widetilde{r}'(0)=\widetilde{r}_1$.

Now let us apply the above lemmas to the case $h(x)={}_0F_1(-;\kappa;x)$, namely, $h_n=\frac{1}{n!(\kappa)_n}$ to obtain the Jacobi-Szegö parameters $\{\alpha_n,\ \omega_n\}$.

Lemma 3.7. *For $h_n=\frac{1}{n!(\kappa)_n}$ and given b_1,b_2,r_2,r_3, the Jacobi-Szegö parameters $\{\alpha_n,\omega_n\}$ are given by*

$$\begin{aligned}\alpha_n &= -\big(3r_2n^2+(2b_1+(2\kappa-1)r_2)\,n+\kappa b_1\big)\,,\quad n\geq 0,\\ \omega_n &= n(\kappa+n-1)\Big((8r_2^2-5r_3)n^2+\Big(2b_1^2-4b_2+6b_1r_2+2(2\kappa-7)r_2^2\\ &\quad -3(\kappa-3)r_3\Big)n+\kappa b_1^2-2(\kappa-1)b_2+2(\kappa-2)b_1r_2-2(2\kappa-3)r_2^2\\ &\quad +(3\kappa-4)r_3\Big),\quad n\geq 2,\end{aligned}$$

and $\omega_0=1$, $\omega_1=\kappa\big(-2(\kappa+1)b_2+(\kappa+2)b_1^2+2(\kappa+1)b_1r_2\big)$. Furthermore, $\widetilde{W}_{n,m}\equiv\frac{h_n}{h_m}n(n+\kappa-1)W_{n,m}$ are given by

$$\begin{aligned}\widetilde{W}_{n,m} &= \big(n(n-1)+\kappa n\big)B_{m,n}-m(m+\kappa-1)B_{m-1,n-1}\\ &\quad-\big(3r_2(n-1)(n-2)+2(b_1+(\kappa+1)r_2)(n-1)+\kappa b_1\big)B_{m,n-1}\\ &\quad+\Big((8r_2^2-5r_3)(n-2)(n-3)+\big(2b_1^2-4b_2+6b_1r_2\\ &\quad+2(2\kappa+5)r_2^2-3(\kappa+2)r_3\big)(n-2)\\ &\quad(\kappa+2)b_1^2-2(\kappa+1)(b_2-b_1r_2)\Big)B_{m,n-2}.\end{aligned}$$

4. Classification by Using Differential Equations

Put $B_m(t)=B(t)\rho(t)^m$, then by Eq. (3.6),

$$B_m(t)=\sum_{n=m}^{\infty}B_{m,n}t^n. \tag{4.1}$$

It is easily seen that

$$\sum_{n=m}^{\infty}n^pB_{m,n-q}t^n=t^q\left(t\frac{d}{dt}+q\right)^pB_m(t)$$

and

$$\sum_{n=m}^{\infty}(n-q)(n-q-1)\cdots(n+1-q-p)B_{m,n-q}t^n=t^q\left(\frac{d}{dt}\right)^pB_m(t) \tag{4.2}$$

for non-negative integers p and q.

On the other hand, multiply $n(n+\kappa-1)t^n$ to $\widetilde{W}_{n,m}$ in Lemma 3.7 and take the summation. Since $B_{m,n}=0$ for $m>n$, we get

$$\text{(i)} \qquad \sum_{n=1}^{\infty}\big(n(n-1)+\kappa n\big)B_{m,n}t^n = t^2B_m''(t)+\kappa tB_m'(t),$$

$$\text{(ii)} \qquad \sum_{n=1}^{\infty} m(m+\kappa-1)B_{m-1,n-1}t^n = m(m+\kappa-1)tB_{m-1}(t),$$

$$\begin{aligned}\text{(iii)} \qquad &\sum_{n=1}^{\infty}\big(3r_2(n-1)(n-2)+2(b_1+(\kappa+1)r_2)(n-1)+\kappa b_1\big)B_{m,n-1}t^n \\ &= 3r_2t^3B_m''(t)+2\big(b_1+(\kappa+1)r_2\big)t^2B_m'(t)+\kappa b_1tB_m(t),\end{aligned}$$

$$\begin{aligned}\text{(iv)} \qquad &\sum_{n=2}^{\infty}\Big((8r_2^2-5r_3)(n-2)(n-3)+\big(2b_1^2-4b_2+6b_1r_2+2(2\kappa+5)r_2^2 \\ &\quad -3(\kappa+2)r_3\big)(n-2)+(\kappa+2)b_1^2-2(\kappa+1)(b_2-b_1r_2)\Big)B_{m,n-2}t^n \\ &= (8r_2^2-5r_3)t^4B_m''(t)+\big(2b_1^2-4b_2+6b_1r_2+2(2\kappa+5)r_2^2 \\ &\quad -3(\kappa+2)r_3\big)t^3B_m'(t)+\big((\kappa+2)b_1^2-2(\kappa+1)(b_2-b_1r_2)\big)t^2B_m(t).\end{aligned}$$

Since $\widetilde{W}_{n,m}$ must vanish, we have a differential equation

$$\begin{aligned}&-m(m+\kappa-1)B_{m-1}(t)+\big(-\kappa b_1+\big((\kappa+2)b_1^2-2(\kappa+1)(b_2-b_1r_2)\big)t\big)\,B_m(t) \\ &+\big(\kappa-2(b_1+(\kappa+1)r_2)t+(2b_1^2-4b_2+6b_1r_2+2(2\kappa+5)r_2^2 \\ &-3(\kappa+2)r_3)t^2\big)B_m'(t)+t\big(1-3r_2t+(8r_2^2-5r_3)t^2\big)B_m''(t)=0\end{aligned}$$

by Lemma 3.3 and Eq. (4.2). Thus we have proved the next lemma.

Lemma 4.1. *Let* $c_m=-m(m+\kappa-1)$ *and*

$$\begin{aligned}p_0(t) &= -\kappa b_1+\big((\kappa+2)b_1^2-2(\kappa+1)(b_2-b_1r_2)\big)t, \\ p_1(t) &= \big(\kappa-2(b_1+(\kappa+1)r_2)t+(2b_1^2-4b_2+6b_1r_2+2(2\kappa+5)r_2^2 \\ &\qquad -3(\kappa+2)r_3)t^2\big), \\ p_2(t) &= t\big(1-3r_2t+(8r_2^2-5r_3)t^2\big).\end{aligned}$$

Then the differential equation

$$p_0(t)B_m(t)+p_1(t)B_m'(t)+p_2(t)B_m''(t)+c_mB_{m-1}(t)=0 \tag{4.3}$$

holds for all $m\ge 0$ *if and only if* $\widetilde{W}_{n,m}=0$ *holds for all* $n\ge m\ge 0$.

Now, we will first obtain conditions on parameters $b_1,\ b_2,\ r_2$ and r_3 under which all $\widetilde{W}_{n,m}$ vanish. ¿From Eq. (4.3), we have for $m=0$,

$$D_0=p_0(t)B(t)+p_1(t)B'(t)+p_2(t)B''(t)=0 \tag{4.4}$$

and for $m=1$,

$$p_0(t)B(t)\rho(t)+p_1(t)(B(t)\rho(t))'+p_2(t)(B(t)\rho(t))''+c_1B(t)=0.$$

Then we use Eq. (4.4) to show that

$$\begin{aligned}&p_0(t)B(t)\rho(t)+p_1(t)(B(t)\rho(t))'+p_2(t)(B(t)\rho(t))'' \\ &=\big(p_0(t)B(t)+p_1(t)B(t)'+p_2(t)B''(t)\big)\rho(t)\end{aligned}$$

$$
\begin{aligned}
&+\big(p_1(t)B(t)+2p_2(t)B'(t)\big)\rho'(t)+p_2(t)B(t)\rho''(t)\\
&=\big(p_1(t)B(t)+2p_2(t)B'(t)\big)\rho'(t)+p_2(t)B(t)\rho''(t)+c_1B(t),
\end{aligned}
$$

which implies that

$$D_1=\big(p_2(t)\rho''(t)+p_1(t)\rho'(t)-\kappa\big)B(t)+2p_2(t)\rho'(t)B'(t)=0. \tag{4.5}$$

For $m=2$ in Eq. (4.3), we use Eqs. (4.4) and (4.5) and the fact that $c_2+2\kappa=-2$ to show that

$$
\begin{aligned}
&p_0(t)B(t)\rho^2(t)+p_1(t)(B(t)\rho^2(t))'+p_2(t)(B(t)\rho^2(t))''+c_2B(t)\rho(t)\\
&=\big(p_0(t)B(t)+p_1(t)B(t)'+p_2(t)B''(t)\big)\rho^2(t)\\
&\quad+2\left(\big(p_2(t)\rho''(t)+p_1(t)\rho'(t)-\kappa\big)B(t)+2p_2(t)R'(t)B'(t)\right)\rho(t)\\
&\quad+2p_2(t)\rho'(t)^2B(t)+(c_2+2\kappa)\rho(t)B(t)\\
&=2\big(p_2(t)\rho'(t)^2-\rho(t)\big)B(t).
\end{aligned}
$$

Therefore

$$D_2=p_2(t)\rho'(t)^2-\rho(t)=0. \tag{4.6}$$

Since $\rho(0)=0$, the solution $\rho(t)$ of Eq. (4.6) is given by

$$\rho(t)=\left(\frac{1}{2}E(p_2;t)\right)^2,\quad E(p_2;t)=\int_0^t\frac{du}{\sqrt{p_2(u)}}, \tag{4.7}$$

where $E(p_2;t)$ can be represented by using an elliptic integral. Note that $\rho(t)$ is analytic around $t=0$ in view of Eq. (2.10). ¿From (4.6), we see that $\frac{1}{p_2(t)\rho'(t)}=\frac{\rho'(t)}{\rho(t)}$. Hence by Eq. (4.5),

$$\frac{B'(t)}{B(t)}=-\frac{1}{2}\frac{p_1(t)}{p_2(t)}-\frac{1}{2}\frac{\rho''(t)}{\rho'(t)}+\frac{\kappa}{2}\frac{\rho'(t)}{\rho(t)}.$$

Integrate both sides of this equation to obtain

$$
\begin{aligned}
B(t)&=\frac{1}{\sqrt{\rho'(t)}}\rho(t)^{\kappa/2}\exp\Big[-\frac{1}{2}\int\frac{p_1(t)}{p_2(t)}\,dt\Big]\\
&=2^{-\kappa+1/2}p_2(t)^{1/4}E(p_2;t)^{\kappa-1/2}\exp\Big[-\frac{1}{2}\int\frac{p_1(t)}{p_2(t)}\,dt\Big].
\end{aligned} \tag{4.8}
$$

Here we have used Eq. (4.7) with a suitable integral constant satisfying $B(+0)=1$. Thus $\rho(t)$ and $B(t)$ are determined explicitly. However, it is not guaranteed that $B(t)$ satisfies (4.4). We will discuss the conditions on parameters $b_1,\ b_2,\ r_2$ and r_3 later. Now we prove that the differential equations in Eq. (4.3) hold for all $k\ge 0$ if the three equations $D_0=0,\ D_1=0,\ D_2=0$ are satisfied.

Theorem 4.2. *Assume that the following differential equations hold,*

$$
\begin{aligned}
D_0&=p_0(t)B(t)+p_1(t)B'(t)+p_2(t)B''(t)=0,\\
D_1&=\big(p_2(t)\rho''(t)+p_1(t)\rho'(t)-\kappa\big)B(t)+2p_2(t)\rho'(t)B'(t)=0,\\
D_2&=p_2(t)\rho'(t)^2-\rho(t)=0.
\end{aligned}
$$

Then

$$\widetilde{D}_k=p_0(t)B_k(t)+p_1(t)B_k'(t)+p_2(t)B_k''(t)+c_kB_{k-1}(t)=0$$

holds for all $k\ge 0$.

Proof. Observe the equalities

$$\begin{aligned}
&\widetilde{D}_k - \rho^k(t)D_0 - k\rho^{k-1}(t)D_1 \\
&= p_0(t)\rho^k(t)B(t) + p_1(t)\big(\rho^k(t)B(t)\big)' + p_2(t)\big(\rho^k(t)B(t)\big)'' + c_k\rho^{k-1}(t)B(t) \\
&\quad -\big(p_0(t)\rho^k(t)B(t) + p_1(t)\rho^k(t)B'(t) + p_2(t)\rho^k(t)B''(t)\big) - k\rho^{k-1}(t)D_1 \\
&= \Big(kp_1(t)\rho^{k-1}(t)\rho'(t)B(t) + 2kp_2(t)\rho^{k-1}(t)\rho'(t)B'(t) \\
&\quad +k(k-1)p_2(t)\rho^{k-2}(t)\rho'(t)^2B(t) + kp_2(t)\rho^{k-1}(t)\rho''(t)B(t)\Big) \\
&\quad -\Big(kp_2(t)\rho^{k-1}(t)\rho''(t)B(t) + kp_1(t)\rho^{k-1}(t)\rho'(t)B(t) \\
&\quad -k\kappa\rho^{k-1}(t)B(t) + 2kp_2(t)\rho^{k-1}(t)\rho'(t)B'(t)\Big) + c_k\rho^{k-1}(t)B(t) \\
&= k(k-1)\rho^{k-1}(t)B(t)(p_2(t)\rho'(t)^2 - \rho(t)) \\
&= k(k-1)\rho^{k-1}(t)B(t)D_2 \\
&= 0.
\end{aligned}$$

Hence we see the assertion. □

Note that

$$\begin{aligned}
B'(t) &= \left(\frac{p_2'(t)}{4p_2(t)} - \frac{p_1(t)}{2p_2(t)} + \frac{2\kappa-1}{2\sqrt{p_2(t)}E(p_2;t)}\right)B(t), \\
B''(t) &= \left(\frac{p_2'(t)}{4p_2(t)} - \frac{p_1(t)}{2p_2(t)} + \frac{2\kappa-1}{2\sqrt{p_2(t)}E(p_2;t)}\right)^2 B(t) \\
&\quad + \left(\frac{p_2''(t)p_2(t) - p_2'(t)^2}{4p_2(t)^2} - \frac{p_1'(t)p_2(t) - p_1(t)p_2'(t)}{2p_2(t)^2}\right. \\
&\quad \left. - \frac{(2\kappa-1)p_2'(t)}{4(\sqrt{p_2(t)})^3E(p_2;t)} - \frac{(2\kappa-1)}{2p_2(t)E(p_2;t)^2}\right)B(t),
\end{aligned}$$

Hence from Eqs. (4.4) and (4.8), we have

$$\begin{aligned}
D_0 &= \frac{1}{16p_2(t)E(p_2;t)^2}\Big(4(2\kappa-1)(2\kappa-3)p_2(t) + \Big(-4p_1^2(t) + 16p_0(t)p_2(t) \\
&\quad -8p_2(t)p_1'(t) + 8p_1(t)p_2'(t) - 3p_2'(t)^2 + 4p_2(t)p_2''(t)\Big)E(p_2;t)^2\Big).
\end{aligned}$$

Since $p_0(t),\ p_1(t),\ p_2(t)$ are polynomials in t, the condition $D_0 = 0$ is equivalent to that both of $(2\kappa-1)(2\kappa-3) = 0$ and

$$-4p_1^2(t) + 16p_0(t)p_2(t) - 8p_2(t)p_1'(t) + p_1(t)'_2(t) - 3p_2^2(t) + 4p_2(t)p_2''(t) = 0$$

hold if $E(p_2;t)^2$ is not a polynomial in t. On the other hand, $E(p_2;t)^2$ is a polynomial if and only if $p_2(t) = t$. But the condition $p_2(t) = t\big(1 + 3r_2t + (8r_2^2 - 5r_3)t^2\big) = t$ is equivalent to $r_2 = r_3 = 0$. Therefore, we have the following classification:

(I) $r_2 = r_3 = 0$.
Then $D_0 = 0$ implies that $(b_1^2 - 2b_2)((b_1^2 - 2b_2)t - 2b_1) = 0$, which is equivalent to $b_2 = \frac{1}{2}b_1^2$.

(II) $\kappa = \frac{1}{2}$.

Then $D_0 = 0$ implies that

$$4b_1^3 - 8b_1b_2 + 3b_1^2r_2 + 6b_2r_2 - 16b_1r_2^2 + 10b_1r_3 = 0,$$
$$2b_1^4 - 8b_1^2b_2 + 8b_2^2 + 12b_1^3r_2 - 24b_1b_2r_2 - 14_1^2r_2^2 + 32b_2r_2^2 - 24b_1r_2^3 + (20b_1^2 - 20b_2 + 15b_1r_2)r_3 = 0.$$

If $b_1 \neq 0$, then

$$r_3 = \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2),$$
$$(3b_1^2 - 2b_2)(2b_1 - 3r_2)(2b_1^2 - 4b_2 + 3b_1r_2) = 0.$$

Hence the case $\kappa = \frac{1}{2}$ is classified as:

(1) $b_1 = 0$.

Then $D_0 = 0$ implies that $b_2\big(-3r_2 + 2(2b_2 + 8r_2^2 - 5r_3)t\big) = 0$. Hence either $b_2 = 0$ or $r_2 = 0,\ r_3 = \frac{2}{5}(b_2 + 4r_2^2)$ must hold. Therefore,

(i) $b_1 = b_2 = 0$,

(ii) $b_1 = 0,\ r_2 = 0,\ r_3 = \frac{2}{5}(b_2 + 4r_2^2)$.

(2) $r_3 = \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2)$.

Then $D_0 = 0$ implies that $(3b_1^2 - 2b_2)(2b_1 - 3r_2)(2b_1^2 - 4b_2 + 3b_1r_2) = 0$. Hence this case is classified as:

(i) $b_2 = \frac{3b_1^2}{2}$,

(ii) $r_2 = \frac{2b_1}{3}$,

(iii) $r_2 = -\frac{2(b_1^2 - 2b_2)}{3b_1}$.

(III) $\kappa = \frac{3}{2}$.

For this case, $D_0 = 0$ implies that

$$-4b_1^3 + 8b_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3 - 14b_1r_3 + 10r_2r_3 = 0,$$
$$2b_1^4 - 8b_1^2b_2 + 8b_2^2 + 12b_1^3r_2 - 24b_1b_2r_2 - 22b_1^2r_2^2 + 48b_2r_2^2 - 32b_1r_2^3 + 24r_2^4 + 24b_1^2r_3 - 28b_2r_3 + 17b_1r_2r_3 - 34r_2^2r_3 + 12r_3^2 = 0.$$

If $7b_1 - 5r_2 \neq 0$, then from the first equation we have

$$r_3 = \frac{1}{2(7b_1 - 5r_2)}(-4b_1^3 + 8b_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3).$$

Thus this case is classified as:

(1) $r_2 = \frac{7}{5}b_1$.

Then $D_0 = 0$ implies that either

$$b_2 = \frac{363}{250}b_1^2,\ (11271b_1^2 - 500r_3)(338b_1^2 - 125r_3) = 0$$

or

$$b_1 = 0,\ (b_2 - 3r_3)(2b_2 - r_3) = 0.$$

Therefore, this case is classified as:

(i) $b_2 = \frac{363}{250}b_1^2,\ r_3 = \frac{1127}{500}b_1^2$,

(ii) $b_2 = \frac{363}{250}b_1^2,\ r_3 = \frac{338}{125}b_1^2$,

(iii) $b_1 = 0,\ r_3 = 2b_2$,

(iv) $b_1 = 0,\ r_3 = \frac{1}{3}b_2$.

(2) $r_3 = \frac{3}{7b_1 - 5r_2}(-4b_1^3 + 8ab_1b_2 - b_1^2r_2 - 10b_2r_2 + 24b_1r_2^2 - 12r_2^3)$.
Then $D_0 = 0$ implies that

$$\begin{aligned}&(b_1 - 2r_2)(2b_1 - r_2)(10b_1^2 - 20b_2 + 15b_1r_2 - r_2^2)\\ &\times(19b_1^2 - 10b_2 - 20b_1r_2 + 12r_2^2) = 0.\end{aligned}$$

Hence this case is classified as:

(i) $r_2 = \frac{b_1}{2}$,

(ii) $r_2 = 2b_1$,

(iii) $b_2 = \frac{1}{20}(10b_1^2 + 15b_1r_2 - r_2^2)$,

(iv) $b_2 = \frac{1}{10}(19b_1^2 - 20b_1r_2 + 12r_2^2)$.

For each case, we can obtain Jacobi-Szegö parameters $\{\alpha_n,\ \omega_n\}$ and $\rho(t),\ B(t)$ by Lemma 3.3 and Eqs. (4.7) and (4.8). Applying Favard's Theorem, we can conclude the following theorems.

Theorem 4.3. *For the case (I) $r_2 = r_3 = 0$, only the followings are MRM-applicable, namely, we must have $b_2 = b_1^2 \neq 0$ and*

$$\begin{aligned}\alpha_n &= -b_1(\kappa + 2n), \quad \omega_n = b_1^2 n(\kappa - 1 + n),\\ \rho(t) &= t, \quad B(t) = e^{b_1 t}, \qquad (t, x) = e^{b_1 t}{}_0F_1(-; \kappa, tx).\end{aligned}$$

Moreover, the corresponding measure is a deformation of the gamma distribution (2.1) by the dilation τ_{1/b_1} and the corresponding orthogonal polynomials are given by $\{L_n^{\kappa-1}(x/b_1)\}$.

Proof. Since $p_2(t) = t$ by $r_2 = r_3 = 0$, we have

$$E(p_2; t) = \int_0^t \frac{du}{\sqrt{p_2(u)}} = \int_0^t \frac{du}{\sqrt{u}} = 2\sqrt{t}.$$

Hence $\rho(t) = \frac{1}{4}E(p_2; t)^2 = t$. Since $p_1(t) = \kappa - 2b_1 t$ and $\int \frac{p_1(t)}{p_2(t)}\,dt = -2b_1 + \kappa \log t$, we have $B(t) = 2^{-\kappa+1/2} t^{1/4} (2\sqrt{t})^{\kappa - 1/2} t^{-\kappa/2} e^{b_1 t} = e^{b_1 t}$. □

We have introduced the notation $E(a, b; t)$ in (2.10) as

$$E(a, b; t) = \int_0^t \frac{du}{\sqrt{u(1 + bu + au^2)}} = E(q; t), \quad q(t) = t(1 + bt + at^2),$$

where $E(q; t)$ is defined by Eq. (4.7). Let us remark that

$${}_0F_1(-; \frac{1}{2}; x) = \cosh(2\sqrt{x}), \quad {}_0F_1(-; \frac{3}{2}; x) = \frac{1}{2\sqrt{x}}\cosh(2\sqrt{x}).$$

For the case $\kappa = \frac{1}{2}$, Eq. (4.8) can be rewritten as

$$B(t) = \exp\Big[\frac{1}{2}\int_0^t \left(\frac{p_2'(u)}{2p_2(u)} - \frac{p_1(u)}{p_2(u)}\right) du\Big]. \tag{4.9}$$

Theorem 4.4. *For the case (II) $\kappa = \frac{1}{2}$, only the followings are MRM-applicable:*

(1) $b_1 = 0$. *In this case, we must have* $b_2 < 0,\ r_2 = 0,\ r_3 = \frac{2}{5}b_2$, *and*

$$\begin{aligned}
\alpha_n &= 0,\\
\omega_n &= -\frac{b_2}{2}n^2(2n-1)(2n+1),\\
\rho(t) &= \frac{1}{4}E(-2b_2, 0; t)^2,\\
B(t) &= \frac{1}{\sqrt{1-2b_2t^2}},\\
(t,x) &= \frac{1}{\sqrt{1-2b_2t^2}}\cosh(E(-2b_2,0;t)\sqrt{x}).
\end{aligned}$$

(2) $b_1 \neq 0$, $r_3 = \frac{1}{10b_1}(-4b_1^3 + 8b_1b_2 - 3b_1^2r_2 - 6b_2r_2 + 16b_1r_2^2)$. *In this case, we have the following subcases:*

(i) $b_2 = \frac{3b_1^2}{2}$, $r_3 = \frac{2}{5}(2b_1^2 - 3b_1r_2 + 4r_2^2)$ *with* $b_1(2b_1 - 3r_3) < 0$ *and*

$$\begin{aligned}
\alpha_n &= -\frac{1}{2}(6r_2n^2 + 4b_1n + b_1),\\
\omega_n &= -\frac{1}{2}b_1(2b_1 - 3r_2)n^2(2n-1)^2,\\
\rho(t) &= \frac{1}{2}E(-4b_1^2 + 6b_1r_2, -3r_2; t)^2,\\
B(t) &= \frac{1}{\sqrt{1-2b_1t}},\\
(t,x) &= \frac{\cosh(E(-4b_1^2 + 6b_1r_2, -3r_2; t)\sqrt{x})}{\sqrt{1-2b_1t}}.
\end{aligned}$$

(ii) $r_2 = \frac{2}{3}b_1,\ r_3 = \frac{1}{45}(5b_1^2 + 18b_2)$ *with* $3b_1^2 - 2b_2 > 0$ *and*

$$\begin{aligned}
\alpha_n &= -\frac{b_1}{2}(2n+1)^2,\\
\omega_n &= \frac{1}{4}(3b_1^2 - 2b_2)n^2(2n-1)(2n+1),\\
\rho(t) &= \frac{1}{4}E(3b_1^2 - 2b_2, -2b_1; t)^2,\\
B(t) &= \frac{1}{\sqrt{1 - 2b_1t + (3b_1^2 - 2b_2)t^2}},\\
(t,x) &= \frac{\cosh(E(3b_1 - b_2, -2b_1; t)\sqrt{x})}{\sqrt{1 - 2b_1t + (3b_1^2 - 2b_2)t^2}}.
\end{aligned}$$

(iii) $r_2 = -\frac{2(b_1^2 - 2b_2)}{3b_1},\ r_3 = \frac{23(b_1^2 - 2b_2)^2}{45b_1^2}$ *with* $\frac{b_1^2}{6} > b_2 > 0$ *and*

$$\begin{aligned}
\alpha_n &= \frac{1}{2b_1}\big(4(b_1^2 - 2b_2)n^2 - 4b_1^2n - b_1^2)\big),\\
\omega_n &= \frac{1}{4b_1^2}n(2n-1)\big(2(b_1^2 - 2b_2)n - 3b_1^2 + 2b_2\big)\\
&\quad \times\big((b_1^2 - 2b_2)n - 2b_1^2 + 2b_2\big),\\
\rho(t) &= \frac{b_1}{b_1^2 - 2b_2}\left(\tanh^{-1}\left(\sqrt{b_1 - 2b_2/b_1}\sqrt{t}\right)\right)^2,
\end{aligned}$$

$$\begin{aligned} B(t) &= \left(1+\frac{b_1^2-2b_2}{b_1}t\right)^{\frac{b_1^2}{b_1^2-2b_2}}. \\ (t,x) &= \left(1+\frac{b_1^2-2b_2}{b_1}t\right)^{\frac{b_1^2}{b_1^2-2b_2}} \\ &\quad \times \cosh\quad 2\sqrt{\frac{b_1}{b_1^2-2b_2}x}\ \tanh^{-1}\left(\sqrt{b_1-2b_2/b_1}\sqrt{t}\right)\Bigg). \end{aligned}$$

Proof. The proof is similar to that of Theorem 4.3. □

For the case $\kappa = \frac{3}{2}$, Eq. (4.8) becomes

$$B(t) = \frac{E(p_2;t)}{2\sqrt{t}} \exp\left[\frac{1}{2}\int_0^t \left(\frac{p_2'(u)}{2p_2(u)} - \frac{p_1(u)}{p_2(u)} + \frac{1}{u}\right) du\right]. \tag{4.10}$$

Theorem 4.5. *For the case (III) $\kappa = \frac{3}{2}$, only the followings are MRM-applicable:*

(1) *The case $r_2 = \frac{7}{5}b_1$. In this case, we have the following subcases:*

(i) $b_2 = \frac{363}{250}b_1^2,\ r_2 = \frac{7}{5}b_1,\ r_3 = \frac{1127}{500}b_1^2$ *with* $b_1 \neq 0$, *and*

$$\begin{aligned} \alpha_n &= -\frac{3}{10}b_1(14n^2+16n+5), \\ \omega_n &= \frac{9}{400}b_1^2 n(2n+1)(7n+1)(14n-5), \\ \rho(t) &= \frac{10}{21b_1}\quad \tan^{-1}\left(\sqrt{\frac{21b_1}{10}}\sqrt{t}\right)\Bigg)^2, \\ B(t) &= \frac{10^{9/14}}{\sqrt{21b_1}\sqrt{t}(10-21b_1t)^{1/7}} \tan^{-1}\left(\sqrt{\frac{21b_1}{10}}\sqrt{t}\right), \\ (t,x) &= \frac{1}{2\sqrt{10}\sqrt{t}\sqrt{x}(10-21b_1t)^{1/7}} \\ &\quad \times \sinh\quad \sqrt{\frac{10}{21b_1}}\sqrt{x}\tan^{-1}\left(\sqrt{\frac{21b_1}{10}}\sqrt{t}\right)\Bigg). \end{aligned}$$

(ii) $b_2 = \frac{363}{250}b_1^2,\ r_2 = \frac{7}{5}b_1,\ r_3 = \frac{338}{125}b_1^2$ *with* $b_1 \neq 0$, *and*

$$\begin{aligned} \alpha_n &= -\frac{3}{10}b_1(14n^2+16n+5), \\ \omega_n &= \frac{27}{50}b_1^2 n^2(2n+1)^2, \\ \rho(t) &= \frac{1}{4}E\Big(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t\Big)^2, \\ B(t) &= \frac{\sqrt{5}}{2\sqrt{t(5-3b_1t)}}E\Big(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t\Big), \\ (t,x) &= \frac{\sqrt{5}}{2\sqrt{t(5-3b_1t)}\sqrt{x}} \\ &\quad \times \sinh\left(\sqrt{x}E\Big(\frac{54}{25}b_1^2, -\frac{21}{5}b_1; t\Big)\right). \end{aligned}$$

(iii) $b_1 = 0,\ r_2 = \frac{7}{5}b_1,\ r_3 = 2b_2$ *with* $b_2 < 0$, *and*

$$\begin{aligned}
\alpha_n &= 0,\\
\omega_n &= -\frac{5b_2}{2}n^2(2n+1)(2n-1),\\
\rho(t) &= \frac{1}{4}E\big(-10b_2, 0; t\big)^2,\\
B(t) &= \frac{1}{2\sqrt{t}}E\big(-10b_2, 0; t\big),\\
(t,x) &= \frac{1}{2\sqrt{tx}}\sinh\left(E(-10b_2, 0; t)\sqrt{x}\right).
\end{aligned}$$

(iv) $b_1 = 0,\ r_2 = \frac{7}{5}b_1,\ r_3 = \frac{1}{3}b_2$ *with* $b_2 < 0$, *and*

$$\begin{aligned}
\alpha_n &= 0,\\
\omega_n &= -\frac{5b_2}{12}n(n+1)(2n+1)^2,\\
\rho(t) &= \frac{1}{4}E\Big(-\frac{5}{12}, 0; t\Big)^2,\\
B(t) &= \frac{1}{2\sqrt{t}}E\Big(-\frac{5}{12}, 0; t\Big),\\
(t,x) &= \frac{\sqrt{3}}{2\sqrt{3-5b_2t^2}\sqrt{tx}}\sinh\left(E(-10b_2, 0; t)\sqrt{x}\right).
\end{aligned}$$

(2) *The case* $r_3 = \frac{-1}{2(7b_1-5r_2)}(4b_1^3 - 8b_1b_2 + b_1^2r_2 + 10b_2r_2 - 24b_1r_2^2 + 12r_2^3)$. *In this case, we have the following subcases:*

(i) $r_2 = \frac{b_1}{2},\ r_3 = \frac{b_2}{3}$ *with* $6b_1^2 - 5b_2 > 0$, *and*

$$\begin{aligned}
\alpha_n &= -\frac{3}{2}b_1(n+1)^2,\\
\omega_n &= \frac{1}{12}(6b_1^2 - 5b_2)n(n+1)(2n+1)^2,\\
\rho(t) &= \frac{1}{4}E\Big(\frac{12b_1^2 - 5b_2}{6}, -\frac{3b_1}{2}; t\Big)^2,\\
B(t) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{t\big(6 - 9b_1t + (12b_1^2 - 5b_2)t^2\big)}}\\
&\quad\times E\Big(\frac{12b_1^2 - 5b_2}{6}, -\frac{3b_1}{2}; t\Big),\\
(t,x) &= \frac{\sqrt{3}}{\sqrt{2}\sqrt{x}\sqrt{t\big(6 - 9b_1t + (12b_1^2 - 10b_2)t^2\big)}}\\
&\quad\times \sinh E\Big(\frac{12b_1^2 - 5b_2}{6}, -\frac{3b_1}{2}; t\Big).
\end{aligned}$$

(ii) $r_2 = 2b_1,\ r_3 = b_1^2 + 2b_2$ *with* $27b_1^2 - 10b_2 > 0$, *and*

$$\begin{aligned}
\alpha_n &= -\frac{3}{2}b_1(2n+1)^2,\\
\omega_n &= \frac{1}{4}(27b_1^2 - 10b_2)n^2(2n-1)(2n+1),\\
\rho(t) &= \frac{1}{4}E\left(27b_1^2 - 10b_2, -6b_1; t\right)^2,
\end{aligned}$$

$$\begin{aligned} B(t) &= \frac{1}{2\sqrt{t}} E\left(27b_1^2 - 10b_2, -6b_1; t\right), \\ (t,x) &= \frac{1}{2\sqrt{t}\sqrt{x}} \sinh\left(\sqrt{x} E\left(27b_1^2 - 10b_2, -6b_1; t\right)\right). \end{aligned}$$

(iii) $b_2 = \dfrac{1}{20}(10b_1^2 + 15b_1r_2 - r_2^2)$, $r_3 = \dfrac{23}{20}r_2^2$ *with* $r_2(b_1 + r_2) > 0$, *and*

$$\begin{aligned} \alpha_n &= -\frac{1}{2}\big(6r_2n^2 + 4(b_1 + r_2)n + 3b_1\big), \\ \omega_n &= \frac{1}{16} n(2n+1)\big(3r_2n + 2b_1 - r_2\big)\big(6r_2n + 4b_1 - 5r_2\big), \\ \rho(t) &= \frac{1}{4} E\Big(\frac{9}{4}b_1^2, -3r_2; t\Big)^2, \\ B(t) &= \frac{1}{\sqrt{3r_2}\sqrt{t}(2 - 3r_2t)^{\frac{2b_1 - r_2}{3r_2}}} E\Big(\frac{9}{4}b_1^2, -3r_2; t\Big), \\ (t,x) &= \frac{(2 - 3r_2t)^{\frac{-2b_1 + r_2}{3r_2}}}{4\sqrt{t}\sqrt{x}} \sinh \sqrt{x} E\left(\frac{9}{4}b_1^2, -3r_2; t\right). \end{aligned}$$

(iv) $b_2 = \frac{1}{10}(19b_1^2 - 20b_1r_2 + 12r_2^2)$, $r_3 = \frac{2}{5}(2b_1^2 - 5b_1r_2 + 6r_2^2)$ *with* $(b_1 - 2r_2)(2b_1 - r_2) < 0$, *and*

$$\begin{aligned} \alpha_n &= -\frac{1}{2}\big(6r_2n^2 + 4(b_1 + r_2)n + 3b_1\big), \\ \omega_n &= -\frac{1}{2}(b_1 - 2r_2)(2b_1 - r_2)n^2(2n+1)^2, \\ \rho(t) &= \frac{1}{4} E\left(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t\right)^2, \\ B(t) &= \frac{1}{2\sqrt{t}\sqrt{1 - (2b_1 + r_2)t}} \\ &\quad \times E\left(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t\right), \\ (t,x) &= \frac{1}{2\sqrt{t}\sqrt{1 - 2(2b_1 - r_2)t}\sqrt{x}} \\ &\quad \times \sinh\left(\sqrt{x} E\left(-(4b_1^2 - 10b_1r_2 + 4r_2^2), -3r_2; t\right)\right). \end{aligned}$$

Proof. The proof is similar to that of Theorem 4.3. □

5. Wilson Polynomials and MRM-applicability

As seen from §4, the Jacobi-Szegö parameters of MRM-applicable orthogonal polynomials satisfy the condition that α_n's are at most second degree and ω_n's are at most fourth degree in n. It is known that Jacobi-Szegö parameters of Wilson polynomials are rational expressions in n. We are interested in MRM-applicable Wilson polynomials for $h(x) = {}_0F_1(-;\kappa;x)$.

Wilson polynomials $\{W_n(x;a,b,c,d)\}$ (see [20], §3.8 of [1]) are defined by

$$\begin{aligned} W_n(x^2; a, b, c, d) &= (a+b)_n(a+c)_n(a+d)_n \\ &\times {}_4F_3(-n, n+a+b+c+d-1, a+ix+a-ix; a+b, a+c, a+d; 1). \quad (5.1) \end{aligned}$$

Their normalizing orthogonal polynomials are given by

$$P_n(x) = P_n(x;a,b,c,d) = (-1)^n \frac{W_n(x;a,b,c,d;x)}{(n+a+b+c+d-1)_n}. \tag{5.2}$$

The density of the corresponding probability measure is

$$\begin{aligned} f(x;a,b,c,d) &= \frac{\Gamma(a+b+c+d)}{4\pi\sqrt{x}\,\Gamma(a+b)\Gamma(a+c)\Gamma(a+d)\Gamma(b+c)\Gamma(b+d)\Gamma(c+d)} \\ &\quad \times \left| \frac{\Gamma(a+i\sqrt{x})\Gamma(b+i\sqrt{x})\Gamma(c+i\sqrt{x})\Gamma(d+i\sqrt{x})}{\Gamma(2i\sqrt{x})} \right|^2. \end{aligned} \tag{5.3}$$

The Jacobi-Szegö parameters are given by

$$\alpha_n = B_n + C_n - a^2, \quad \omega_n = B_{n-1}C_n, \tag{5.4}$$

where

$$\begin{aligned} B_n &= \frac{(n+a+b+c+d-1)(n+a+b)(n+a+c)(n+a+d)}{(2n+a+b+c+d-1)(2n+a+b+c+d)}, \\ C_n &= \frac{n(n+b+c-1)(n+b+d-1)(n+c+d-1)}{(2n+a+b+c+d-2)(2n+a+b+c+d-1)}. \end{aligned}$$

The following is the list of parameters such that α_n's and ω_n's are polynomials of degrees 2 and 4, respectively, in n.

Proposition 5.1. *The Jacobi-Szegö parameters of Wilson polynomials are polynomials in n only for the following cases:*

(1) *The set $\{a,b,c,d\}$ is of the form $\{p, p-\frac{1}{2}, q, q-\frac{1}{2}\}$. Then*

$$\begin{aligned} \alpha_n &= \frac{1}{4}\big(2n^2 + 4(p+q-1)n + 4pq - p - q\big), \\ \omega_n &= \frac{1}{64}n(2n+4p-3)(2n+4q-3)(n+2p+2q-3). \end{aligned}$$

(2) *The set $\{a,b,c,d\}$ is of the form $\{p, p+\frac{1}{2}, q, q-\frac{1}{2}\}$. Then*

$$\begin{aligned} \alpha_n &= \frac{1}{8}\big(4n^2 + 4(2p+2q-1)n + 8pq - 2p + 2q - 1\big), \\ \omega_n &= \frac{1}{64}n(2n+4p-1)(2n+4q-3)(n+2p+2q-2). \end{aligned}$$

(3) *The set $\{a,b,c,d\}$ is of the form $\{p, p+\frac{1}{2}, q, q+\frac{1}{2}\}$. Then*

$$\begin{aligned} \alpha_n &= \frac{1}{4}\big(2n^2 + 4(p+q)n + 4pq + p + q\big), \\ \omega_n &= \frac{1}{64}n(n+2p+2q)(2n+4q-1)(n+2p+2q-1). \end{aligned}$$

(4) *The set $\{a,b,c,d\}$ is of the form $\{p, \frac{1}{2}-p, q, \frac{1}{2}-q\}$. Then*

$$\begin{aligned} \alpha_n &= \frac{1}{4}\big(2n^2 - 2p^2 - 2q^2 + p + q\big), \\ \omega_n &= \frac{1}{64}(2n+2p-2q-1)(2n-2p+2q-1)(n-p-q)(n+p+q-1). \end{aligned}$$

(5) *The set* $\{a,b,c,d\}$ *is of the form* $\{p, \frac{1}{2}-p, q, \frac{3}{2}-q\}$. *Then*

$$\begin{aligned}\alpha_n &= \frac{1}{8}\big(4n^2+4n-4p^2-4q^2+2p+6q-1\big),\\ \omega_n &= \frac{1}{64}(2n+2p-2q+1)(2n-2p+2q-1)(n+p+q-1)\\ &\quad \times(n-p-q+1).\end{aligned}$$

(6) *The set* $\{a,b,c,d\}$ *is of the form* $\{p, \frac{3}{2}-p, q, \frac{3}{2}-q\}$. *Then*

$$\begin{aligned}\alpha_n &= \frac{1}{4}\big(2n^2+4n-2p^2-2q^2+3p+3q\big),\\ \omega_n &= \frac{1}{64}(2n+2p-2q+1)(2n-2p+2q+1)(n-p-q+2)\\ &\quad \times(n+p+q-1).\end{aligned}$$

Proof. From Eq. (5.4), the remainder in deviding the numerator of α_n by its denominator is given by

$$(a+b-c-d)(a-b+c-d)(a-b-c+d)(-2+a+b+c+d).$$

Hence $d=a+b-c$, $d=a-b+c$, $d=-a+b+c$ or $d=2-a-b-c$ must hold. If $d=a+b-c$, then

$$\omega_n = \frac{\begin{array}{c} n(n+2a+2b-2)(n+2a+b-c-1)(n+a+2b-c-1)\\ \times(n+a+c-1)(n+b+c-1)\end{array}}{4(2n+2a+2b-3)(2n+2a+2b-1)}$$

with the remainder given by

$$-\big(4(a-c)^2-1\big)\big(4(b-c)^2-1\big)\big(2(a+b)-3\big)\big(2(a+b)-1\big)\big).$$

Thus $a-c=\pm\frac{1}{2}$, $b-c=\pm\frac{1}{2}$, $a+b=\frac{1}{2}$, or $a+b=\frac{3}{2}$. Hence possible sets of parameters are

$$\left(a,b,a-\frac{1}{2},b+\frac{1}{2}\right),\quad \left(a,b,a+\frac{1}{2},b-\frac{1}{2}\right),\quad \left(a,b,b-\frac{1}{2},a+\frac{1}{2}\right),$$
$$\left(a,b,b+\frac{1}{2},a-\frac{1}{2}\right),\quad \left(a,\frac{1}{2}-a,c,\frac{1}{2}-c\right),\quad \left(a,\frac{3}{2}-a,c,\frac{3}{2}-c\right).$$

Other cases can be treated similarly. □

We have seen in §2 all possible cases on the MRM-applicability for the normalized factor $h(x) = {}_0F_1(-;\kappa;x)$. By comparing the function forms in n of Jacobi-Szegö parameters in Theorems 4.3, 4.4, and 4.5 with Proposition 5.1, we see special Wilson polynomials, which can be obtained by the form in Eq. (2.9). The next theorem provides a complete list of MRM-applicable Wilson polynomials for $h(x) = {}_0F_1(-;\kappa;x)$.

For convenience, we denote by $(a,b,c,d) \sim (a',b',c',d')$ that (a,b,c,d) is a permutation of (a',b',c',d').

Theorem 5.2. *The normalized Wilson polynomials* $\{P_n(x;a,b,c,d)\}$ *defined by Eq. (5.1) are MRM-applicable if and only if either* $(a,b,c,d) \sim (p, p+\frac{1}{2}, \frac{1}{2}, 0)$ *with* $p>0$ *or* $(a,b,c,d) \sim (p, p+\frac{1}{2}, \frac{1}{2}, 1)$ *with* $p \geq 0$.

(1) *The case* $(a,b,c,d) \sim (p, p+\frac{1}{2}, \frac{1}{2}, 0)$. *In this case, we have*

$$\begin{aligned}
\alpha_n &= \frac{1}{4}\big(2n^2+4pn+p\big),\\
\omega_n &= \frac{1}{64}n(2n-1)(2n+4p-1)(n+2p-1),\\
P_n(x;a,b,c,d) &= (-16)^{-n}\big(4p\big)_{2n}\\
&\quad\times {}_4F_3(-n, n+2p, p-i\sqrt{x}, p+i\sqrt{x}; p, p+\frac{1}{2}, 2p+\frac{1}{2}; 1),\\
f(a,b,c,d;x) &= \frac{2^{4p}|\Gamma(2(p+i\sqrt{x}))|^2}{\pi\Gamma(4p)\sqrt{x}}.
\end{aligned}$$

Moreover, we have the corresponding MRM-factor $h(x)$, ρ-, B- *and   -functions,*

$$\begin{aligned}
h(x) &= {}_0F_1(-;\frac{1}{2};x) = \cosh 2\sqrt{x},\\
\rho(t) &= 4\Big(\tan^{-1}\frac{\sqrt{t}}{2}\Big)^2,\\
B(t) &= \frac{4^{2p}}{(4+t)^{2p}},\\
(t,x) &= \frac{4^{2p}}{(4+t)^{2p}}\cosh\left(4\sqrt{x}\tan^{-1}\frac{\sqrt{t}}{2}\right).
\end{aligned}$$

(2) *The case* $(a,b,c,d) \sim (p, p+\frac{1}{2}, \frac{1}{2}, 1)$. *In this case, we have*

$$\begin{aligned}
\alpha_n &= \frac{1}{8}\big(4n^2+4(2p+1)n+6p+1\big),\\
\omega_n &= \frac{1}{64}n(2n+1)(2n+4p-1)(n+2p),\\
P_n(x;a,b,c,d) &= (-16)^{-n}\big(4p+1\big)_{2n}\\
&\quad\times {}_4F_3(-n, n+2p+1, p-i\sqrt{x}, p+i\sqrt{x}; p+\frac{1}{2}, p+1, 2p+\frac{1}{2}; 1),\\
f(a,b,c,d;x) &= \frac{2^{4(p+1)}\sqrt{x}|\Gamma(2(p+i\sqrt{x}))|^2}{\pi\Gamma(4p+1)}.
\end{aligned}$$

Moreover, we have the corresponding MRM-factor $h(x)$, ρ-, B- *and   -functions,*

$$\begin{aligned}
h(x) &= {}_0F_1(-;\frac{3}{2};x) = \frac{1}{2\sqrt{x}}\sinh 2\sqrt{x},\\
\rho(t) &= 4\Big(\tan^{-1}\frac{\sqrt{t}}{2}\Big)^2,\\
B(t) &= \frac{2\cdot 4^{2p}}{(4+t)^{2p}\sqrt{t}}\tan^{-1}\frac{\sqrt{t}}{2},\\
(t,x) &= \frac{4^{2p}}{2(4+t)^{2p}\sqrt{tx}}\sinh\left(4\sqrt{x}\tan^{-1}\frac{\sqrt{t}}{2}\right).
\end{aligned}$$

Finally, let us point out a special case of Theorem 5.2. For $p = \frac{1}{2}$ in Case (1), we have $(a, b, c, d) \sim (0, \frac{1}{2}, \frac{1}{2}, 1)$. On the other hand, for $p = 0$ in Case (2), we also have $(a, b, c, d) \sim (0, \frac{1}{2}, \frac{1}{2}, 1)$. Then

$$\begin{aligned} \alpha_n &= \frac{1}{8}(2n+1)^2, \\ \omega_n &= \frac{1}{64}n^2(2n-1)(2n+1), \\ P_n(x; a, b, c, d) &= (-1)^n \frac{(2n)!}{2^{4n}} {}_4F_3(-n, n+1, -i\sqrt{x}, i\sqrt{x}; \frac{1}{2}, \frac{1}{2}, 1; 1), \\ f(a, b, c, d; x) &= \frac{2^4\sqrt{x}}{\pi}|\Gamma(2i\sqrt{x}))|^2. \end{aligned}$$

This case is MRM-applicable for both functions ${}_0F(-; \frac{1}{2}; x)$ and ${}_0F(-; \frac{3}{2}; x)$. The corresponding ρ-, B- and   -functions are stated in Examples 2.2 and 2.3.

6. MRM-factors for Given Measures

In §2, we saw that the gamma distribution μ_κ is MRM-applicable for several MRM-factors $h(x)$. It is natural to ask the question whether there are other MRM-factors. The answer is given by the next theorem.

Theorem 6.1. *All MRM-factors of the gamma distribution μ_κ are given by*

$$ {}_0F_1(-; \kappa; x) \quad \text{and} \quad {}_1F_1(c; \kappa; x) \quad (c \neq 0, -1, -2, -3, \ldots) \tag{6.1}$$

up to a scaling.

Proof. The orthogonal polynomials are Laguerre polynomials with Jacobi-Szegö parameters

$$\alpha_n = -(\kappa + 2n),\ n \geq 0; \quad \omega_n = n(\kappa + n - 1),\ n \geq 1,\ \omega_0 = 1,$$

(see [3].) Therefore,

$$\sum_{j=0}^{n-1} \alpha_j = -\sum_{j=0}^{n-1}(\kappa + 2j) = -n(\kappa + n - 1) = -\omega_n.$$

Applying Lemma 3.4, we get

$$h_n = \prod_{k=0}^{n-1} \frac{b_1 + kr_2}{(k+1)(\kappa + k)} = \begin{cases} \dfrac{b_1^n}{(\kappa)_n n!}, & \text{if } r_2 = 0, \\ r_2^n \dfrac{\left(\frac{b_1}{r_2}\right)_n}{(\kappa)_n n!}, & \text{if } r_2 \neq 0, \end{cases}$$

which yields that

$$h(x) = \sum_{n=0}^{\infty} h_n x^n = \begin{cases} {}_0F_1(-; \kappa; b_1 x), & \text{if } r_2 = 0, \\ {}_1F_1(\dfrac{b_1}{r_2}; \kappa; r_2 x), & \text{if } r_2 \neq 0. \end{cases}$$

By the condition $h_n \neq 0$, we obtain the condition on c. $\square$

We have obtained all MRM-applicable measures for $h(x) = e^x$. This leads to a new problem of finding all possible MRM-factors of these measures. The following answer will be proved in a forthcoming paper using Lemma 3.5. First, we mention that $\rho(t)$ is an odd function and $B(t)$ is an even function, if μ is symmetric or equivalently if $\alpha_n = 0$ for all $n \geq 0$. Put

$$h_{\mathrm{e}}(x) = \frac{1}{2}(h(x) + h(-x)), \quad h_{\mathrm{o}}(x) = \frac{1}{2}(h(x) - h(-x))$$

and let $h^c(x) = h_{\mathrm{e}}(x) + c\, h_{\mathrm{o}}(x)$ for $c \neq 0$. Then the triple $\big((h^c(x),\ \rho(t),\ B(t)\big)$ also gives a generating function. As mentioned already, we have a freedom of scaling. In the followings, we will state the assertion "up to a trivial deformation."

Theorem 6.2. *All MRM-factors of the standard Gaussian measure $N(0,1)$ are given by*

$$\begin{aligned} h(x) &= e^x, \\ \widetilde{h}(x) &= {}_1F_1(\frac{c}{2};\frac{1}{2};-x^2) + {}_1F_1(\frac{c}{2}+\frac{1}{2};\frac{3}{2};-x^2)\,x, \quad c \neq 0,-1,-2,-3,\dots \end{aligned}$$

up to a trivial deformation and the corresponding ρ and B-functions are given respectively by

$$\rho(t) = t,\ \ B(t) = e^{-\frac{1}{2}t^2} \quad \text{and} \quad \widetilde{\rho}(t) = \frac{t}{\sqrt{1+2t^2}},\ \ \widetilde{B}(t) = (1+2t^2)^{-\frac{1}{2}\gamma}.$$

Theorem 6.3. *Let M_κ be the Meixner distribution, whose characteristic function is $(\cosh t)^{-\kappa}$ and let $M_{\kappa,\beta}$ be the β-shift of M_κ. Then we have the following assertions:*

(i) *All MRM-factors of M_κ are unique up to a trivial deformation for $\kappa \neq 2$ and*

$$h(x) = e^x, \quad \rho(t) = \tan^{-1} t, \quad B(t) = (1+t^2)^{-\frac{\kappa}{2}}.$$

(ii) *There are only two MRM-factors of M_2 up to a trivial deformation and*

$$h(x) = e^x, \quad \rho(t) = \tan^{-1} t, \quad B(t) = (1+t^2)^{-1},$$

$$\widetilde{h}(x) = \frac{1}{x}\left(e^x - 1\right), \quad \widetilde{\rho}(t) = \tan^{-1} t, \quad \widetilde{B}(t) = \frac{1}{t}\tan^{-1} t.$$

Remark 6.4. The MRM-factors $h(x)$ in the above theorems can be represented by hypergeometric functions as follows:

$$\begin{aligned} e^x &= {}_1F_1(c;c;x), \\ \frac{e^x-1}{x} &= {}_1F_1(1;2;x). \end{aligned}$$

Our frame work can also be applied to probability measures having atoms with infinite supports. We can show that MRM-factor for the other probability measures with the MRM-factor $h(x) = e^x$ is unique. In the forthcoming paper [11], improved versions of above theorems will be shown together with other cases.

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