

IRREDUCIBLE AND PERIODIC POSITIVE MAPS

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ABSTRACT. We extend the notions of *irreducibility* and *periodicity* of a stochastic matrix to a unital positive linear map Φ on a finite-dimensional C^* -algebra \mathcal{A} and discuss the non-commutative version of the Perron-Frobenius theorem. For a completely positive linear map Φ with $\Phi(a) = \sum_{\ell} L_{\ell}^* a L_{\ell}$, we give conditions on the L_{ℓ} 's equivalent to irreducibility or periodicity of Φ . As an example, positive linear maps on $M_2(\mathbb{C})$ are analyzed.

1. Introduction

This note is aimed at illustrating the fundamental concepts of irreducibility and periodicity of a discrete-time classical Markov chain in the algebraic language and their natural non-commutative generalizations.

A classical Markov chain with values $\{1, \dots, n\}$ can be given through the associated $n \times n$ stochastic matrix P , whose elements $P_{\ell m}$ are the probabilities of moving from ℓ to m in a unit time, and the distribution at time 0.

The concepts of irreducibility and periodicity can be formulated both in the algebraic and probabilistic languages. Indeed, in any textbook on Markov chains one reads: P is irreducible if, for any ℓ, m , there exists two non-zero natural numbers k, h such that $P_{\ell m}^k > 0$ and $P_{m \ell}^h > 0$ i.e. the probability of moving from ℓ to m and from m to ℓ in a finite amount of time is strictly positive. Then, the *period* of a ℓ is defined as the greatest common divisor of all those k such that $P_{\ell m}^k$ is strictly positive and it is shown that all the $\ell \in \{1, \dots, n\}$ of an irreducible Markov chain have the same period which is then called the period of the chain.

Thinking of P as a linear operator on the algebra $\mathcal{C} = \ell^{\infty}(\{1, \dots, n\})$ it is not hard to find the algebraic formulation of irreducibility and periodicity (Propositions 2.1, 2.3). Indeed, a Markov chain with transition matrix P is irreducible if and only if there exists no proper subset I of $\{1, \dots, n\}$ such that the subalgebra \mathcal{C}_I of \mathcal{C} of functions supported in I is P -invariant. Moreover, when P is irreducible, its period is the greatest common divisor of all the natural numbers d with the property: there exists a partition E_0, \dots, E_{d-1} of $\{1, \dots, n\}$ such that the subalgebras $\mathcal{C}_0, \dots, \mathcal{C}_{d-1}$ of functions with support in E_0, \dots, E_{d-1} satisfy $P(\mathcal{C}_k) = \mathcal{C}_{k-1}$ for $1 \leq k < d-1$ and $P(\mathcal{C}_{d-1}) = \mathcal{C}_0$.

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Noting that the algebras \mathcal{C}_I of functions with support in some subset I of $\{1, \dots, n\}$ are exactly all the *hereditary* subalgebras of \mathcal{C} , the algebraic generalization of irreducibility and periodicity is then immediate.

Starting from Definitions 2.5 and 2.6 we prove the non-commutative extensions (Theorems 3.7 and 4.3) of classical results on the spectrum of P and their consequences on the limit behaviour of transition probabilities.

Moreover, in section 5, we study irreducible and periodic linear maps enjoying a stronger positivity property called complete positivity. These maps are represented in the form $\Phi(x) = \sum_{\ell} L_{\ell}^* x L_{\ell}$ where the L_{ℓ} are suitable matrices. We characterize irreducible and periodic completely positive Φ by means of conditions on the L_{ℓ} 's (Proposition 5.2 and Theorem 5.4).

As an application we characterize irreducible and periodic positive maps on $M_2(\mathbb{C})$.

The non-commutative generalization of classical results on stochastic matrices or Markov operators to positive maps on an operator algebra is not a mere mathematical extension. Indeed, completely positive maps on operator algebras, in the applications in quantum communication and information (see e.g. [10], [12]), are called quantum communication channels and stand as a fundamental mathematical object in the investigations on the application of quantum effects to transport information.

It is worth noticing here that, as in the commutative case, the limit behaviour can be obtained by spectral theory. Evans and Høegh-Kronh [4] generalized the classical results on matrices with positive elements by Perron and Frobenius to positive operators on finite-dimensional C^* -algebras following a spectral theoretic approach. We think, however, that our approach has its own interest since it stresses the widespread probabilistic intuition on classical Markov chains. Moreover, we find the structure of Kraus' operators L_{ℓ} of a completely positive irreducible periodic map.

Infinite dimensional versions of the above concepts can be found in [1] and [6].

2. Stochastic Matrices and Operator Algebras

We start by making precise the algebraic meaning of irreducibility and periodicity of a classical Markov chain with $n \times n$ stochastic matrix P . We say that P is irreducible (resp. periodic) if the associated Markov chain is irreducible (resp. periodic).

The stochastic matrix P defines an operator on the abelian C^* -algebra $\mathcal{C} = \ell^{\infty}(\{1, \dots, n\}; \mathbb{C})$ of complex-valued functions on $\{1, \dots, n\}$ via

$$(Pf)(k) = \sum_{m=1}^n P_{km} f(m). \quad (2.1)$$

This map is clearly positive and unital (i.e. $P1 = 1$).

For each subset F of $\{1, \dots, n\}$ let \mathcal{C}_F denote the subalgebra of \mathcal{C} of functions with support in F . The following proposition is immediate.

Proposition 2.1. *An $n \times n$ stochastic matrix P is irreducible if and only if there exists no proper subset F of $\{1, \dots, n\}$ such that $P(\mathcal{C}_F) \subseteq \mathcal{C}_F$.*

As about periodicity the situation is slightly more complex. However, if P is irreducible with period d , fix a value m_0 and define

$$E_k = \{ m \in \{ 1, \dots, n \} \mid P_{m_0 m}^{sd+k} > 0 \text{ for some } s \geq 1 \} \tag{2.2}$$

for $k = 0, \dots, d - 1$. It is not difficult to prove (see e.g. Nummelin [9] Ex. 2.5 p. 21) that E_0, \dots, E_{d-1} , is a partition of $\{ 1, \dots, n \}$ and $P(\mathcal{C}_{E_k}) = \mathcal{C}_{E_{k \stackrel{d}{-} 1}}$ for $k = 0, \dots, d - 1$, where $k \stackrel{d}{-} 1$ denotes the difference $k - 1 \text{ modulo } d$.

The above partition E_0, \dots, E_{d-1} is canonical in the following sense:

Lemma 2.2. *Let P be an irreducible $n \times n$ stochastic matrix with period d . Suppose that there exists an integer d' and a partition $E'_0, \dots, E'_{d'-1}$ of $\{ 1, \dots, n \}$ into non-empty subsets such that $P(\mathcal{C}_{E'_j}) = \mathcal{C}_{E'_j \stackrel{d'}{-} 1}$ for all $j = 0, \dots, d - 1$. Then d' divides d and each E'_h is union of a finite number of the E_k 's.*

Here we omit the proof (see e.g. Nummelin [9]) because we shall demonstrate a non-commutative version (Lemma 4.2).

The collection (E_0, \dots, E_{d-1}) of disjoint not-empty subsets of $\{ 1, \dots, n \}$ is called a *d-cyclic partition* for P . If $d > 1$ the matrix P is *periodic* and d is then called the *period* of P , otherwise it is *aperiodic*. The following proposition highlights the algebraic formulation of periodicity.

Proposition 2.3. *An irreducible $n \times n$ stochastic matrix P has period d if and only if d is the greatest of all the natural numbers such that there exists a d-cyclic partition for P .*

Irreducibility and periodicity are key assumptions of the Perron-Frobenius theorem clarifying the relationship between the spectrum $\text{sp}(P)$ of a stochastic matrix P and the ergodic properties of the associated Markov chain (see, e.g. Karlin and Taylor [7] Th 2.1 p.543, Th. 2.2 p. 547).

Let \mathbb{T} denote the one-dimensional torus with the natural group structure.

Theorem 2.4. *Let P be an $n \times n$ irreducible stochastic matrix. Then*

- 1) $\text{sp}(P) \cap \mathbb{T}$ is a finite subgroup of \mathbb{T} ,
- 2) the eigenspace of each $\lambda \in \text{sp}(P) \cap \mathbb{T}$ is one-dimensional and the eigenvectors $u = (u_1, \dots, u_n)$ satisfy $|u_1|^2 = \dots = |u_n|^2$,
- 3) there exists a unique vector $\pi = (\pi_1, \dots, \pi_n)$ such that $\pi P = \pi$, $\pi_m > 0$ for all $m = 1, \dots, n$ and $\sum_m \pi_m = 1$,
- 4) for all $\ell, m \in \{ 1, \dots, n \}$ we have

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{h=1}^k P_{\ell m}^h = \pi_m$$

and (π_1, \dots, π_n) is the unique invariant distribution for the Markov chain,

- 5) P is aperiodic if and only if $\text{sp}(P) \cap \mathbb{T} = \{ 1 \}$ and, in this case,

$$\lim_{k \rightarrow \infty} P_{\ell m}^k = \pi_m.$$

In order to extend the above facts in a non-commutative framework including transition operators arising in quantum stochastic dynamics we first recall some notation and definitions.

Let \mathcal{A} be a finite-dimensional C^* -algebra acting on a (finite-dimensional) Hilbert space \mathfrak{h} with identity $\mathbf{1}$. If $\dim(\mathfrak{h}) = n$, i.e. $\mathfrak{h} = \mathbb{C}^n$, \mathcal{A} is just a subalgebra of the algebra of all complex $n \times n$ matrices (operators on \mathbb{C}^n) which is also closed under the adjoint operation $*$.

An operator $a \in \mathcal{A}$ is positive if it has positive spectrum or, in an equivalent way, it can be written in the form b^*b with $b \in \mathcal{A}$. A *projection* p in \mathcal{A} is a positive operator $p \in \mathcal{A}$ such that $p^2 = p$. A linear map Φ on \mathcal{A} is called *unital* if $\Phi(\mathbf{1}) = \mathbf{1}$, *positive* if $\Phi(x)^* = \Phi(x^*)$ for all $x \in \mathcal{A}$ and $\Phi(x)$ is positive for all positive $x \in \mathcal{A}$. A unital positive linear map is a *Schwarz map* if

$$\Phi(x^*)\Phi(x) \leq \Phi(x^*x)$$

for all $x \in \mathcal{A}$. Notice that, when \mathcal{A} is commutative, the above inequality always holds; indeed, it is an immediate consequence of the Jensen inequality $|\Phi(x)|^2 \leq \Phi(|x|^2)$. This is no longer true for a non-abelian \mathcal{A} as shows the linear positive map on $n \times n$ complex matrices defined by $x \mapsto x^T$ (transpose).

Propositions 2.1 and 2.3 allow us to find the good operator algebraic generalization of irreducibility and periodicity. These are intimately related with the concept of *hereditary subalgebra* that we now introduce.

A C^* -subalgebra \mathcal{B} of \mathcal{A} is said to be *hereditary* if, for any $y \in \mathcal{B}$ positive and $x \in \mathcal{A}$ the inequality $0 \leq x \leq y$ implies $x \in \mathcal{B}$. It can be shown that any hereditary C^* -subalgebra of \mathcal{A} is of the form $p\mathcal{A}p$ for a projection p in \mathcal{A} .

A C^* -subalgebra \mathcal{B} of \mathcal{A} is said to be *proper* if it is not either $\{0\}$ or \mathcal{A} and *non-trivial* if it is proper and it does not coincide with the algebra of scalar multiples of $\mathbf{1}$. In a similar way projections will be called *proper* if they are different from 0 and $\mathbf{1}$.

Turning back to the commutative C^* algebra \mathcal{C} it is clear that any hereditary C^* -subalgebra of \mathcal{C} is of the form \mathcal{C}_F with F subset of $\{1, \dots, n\}$ i.e. it is an ideal of \mathcal{C} . The appropriate extension to the non-commutative framework, however, is still the notion of hereditary C^* -subalgebra as it is clear from [6] and the decomposition of quantum Markov semigroups developed in [13].

This, together with Proposition 2.3, motivates the following

Definition 2.5. A positive map Φ on a C^* -algebra \mathcal{A} is *irreducible* if there exists no proper hereditary Φ -invariant C^* -subalgebra of \mathcal{A} .

Definition 2.6. Let Φ be a positive irreducible linear map on a C^* -algebra \mathcal{A} . A resolution of the identity p_0, \dots, p_{d-1} , with $p_0, \dots, p_{d-1} \in \mathcal{A}$, is called *Φ -cyclic* if $\Phi(p_k \mathcal{A} p_k) = p_{k+d-1} \mathcal{A} p_{k+d-1}$ for $k = 0, \dots, d-1$. The map Φ is called *periodic* if there exists a Φ -cyclic resolution of the identity p_0, \dots, p_{d-1} with $d \geq 2$. The biggest d for which there exists such a resolution of the identity is called the *period* of Φ .

We finish this section by introducing the notion of state which plays the role of probability in non-commutative theories.

A *state* ω is a positive linear functional on \mathcal{A} normalized by $\omega(\mathbf{1}) = 1$. A state ω is called *faithful* if $\omega(a) = 0$ for a positive $a \in \mathcal{A}$ implies $a = 0$. For every state ω on \mathcal{A} we call *support projection* of ω the smallest projection p in \mathcal{A} such that $\omega(p) = 1$. Clearly a state ω is faithful if and only if its support projection is the

identity operator $\mathbf{1}$. A state ω is Φ -invariant if $\omega(\Phi(a)) = \omega(a)$ for all $a \in \mathcal{A}$. It is well-known that the support projection p of an invariant state satisfies $\Phi(p) \geq p$.

3. Irreducible Positive Maps

In this section we discuss the notion of irreducibility for a positive map Φ on a finite-dimensional C^* -algebra \mathcal{A} (see also Fagnola and Rebolledo [6] and the references therein for the infinite-dimensional case).

Inspired by the classical theory of Markov processes we introduce the

Definition 3.1. Let Φ be a positive linear map on \mathcal{A} . A projection p in \mathcal{A} is Φ -subharmonic if $\Phi(p) \geq p$.

As we recalled at the end of Section 2, the support projection of a Φ -invariant state is Φ -subharmonic. The relationship between Φ -subharmonic projections in \mathcal{A} and irreducibility is clarified by the following

Theorem 3.2. Let Φ be a positive unital map on \mathcal{A} . Then Φ is irreducible if and only if any Φ -subharmonic projection p in \mathcal{A} is either 0 or $\mathbf{1}$.

Proof. If p is a proper subharmonic projection in \mathcal{A} , then, letting $p^\perp = \mathbf{1} - p$, we find $\Phi(p^\perp) \leq p^\perp$. Moreover, for all positive $x \in \mathcal{A}$, we have

$$\Phi(p^\perp x p^\perp) \leq \|x\| \Phi(p^\perp) \leq \|x\| p^\perp.$$

Therefore $\Phi(p^\perp x p^\perp)$ belongs to $p^\perp \mathcal{A} p^\perp$ and the hereditary C^* -subalgebra $p^\perp \mathcal{A} p^\perp$ of \mathcal{A} is Φ -invariant.

Conversely, suppose that Φ is not irreducible and let p be a proper projection such that $\Phi(p \mathcal{A} p) \subseteq p \mathcal{A} p$. This implies $p^\perp \Phi(p) p = p \Phi(p) p^\perp = p^\perp \Phi(p) p^\perp = 0$. Since $\Phi(p)$ is positive we have then $\Phi(p) = p \Phi(p) p$ (see e.g. [6] Lemma II.1) and

$$\Phi(p) = p \Phi(p) p \leq p \Phi(\mathbf{1}) p = p.$$

It follows that $\Phi(p^\perp) = \Phi(\mathbf{1} - p) \geq \mathbf{1} - p = p^\perp$, i.e. p^\perp is a proper subharmonic projection. \square

We now prove the first property of positive unital maps.

Proposition 3.3. Let Φ be a positive unital map on \mathcal{A} . Then $\|\Phi\| = 1$. In particular, the spectrum $\text{sp}(\Phi)$ of Φ is contained in the unit disk.

Proof. For all positive $a \in \mathcal{A}$ we have $a \leq \|a\| \mathbf{1}$, then

$$0 \leq \Phi(a) \leq \|a\| \Phi(\mathbf{1}) = \|a\| \mathbf{1}$$

and $\|\Phi(a)\| \leq \|a\|$. For an arbitrary $a \in \mathcal{A}$, let ω be a state on \mathcal{A} such that $\omega(\Phi(a^*) \Phi(a)) = \|\Phi(a)\|^2$ (it exists by [2] Lemma 2.3.23, p. 59). Note that $\omega(\Phi((a^* - \Phi(a^*))(a - \Phi(a)))) \geq 0$, therefore $\omega(\Phi(a^*) \Phi(a)) \leq \omega(\Phi(a^* a))$. It follows then

$$\|\Phi(a)\|^2 \leq \omega(\Phi(a^* a)) \leq \|\Phi(a^* a)\| \leq \|a^* a\| = \|a\|^2.$$

Therefore $\|\Phi(a)\| \leq \|a\|$ and $\|\Phi\| \leq 1$. Since $\Phi(\mathbf{1}) = \mathbf{1}$, it follows that Φ has unit norm. As a consequence, $\text{sp}(\Phi)$ is contained in $\{z \in \mathbb{C} \mid |z| \leq \|\Phi\|\}$. \square

Next we prove a useful property of irreducible maps.

Proposition 3.4. *A positive unital map Φ on \mathcal{A} with a non-trivial fixed point is not irreducible.*

Proof. Suppose that $x \in \mathcal{A}$ is a non-trivial fixed point of Φ . Clearly x^* is also a fixed point of Φ as well as $(x + x^*)$, $i(x^* - x)$. Moreover at least one among $(x + x^*)$, $i(x^* - x)$ is not a multiple of $\mathbf{1}$. Therefore there exists a self-adjoint $y \in \mathcal{A}$ which is a non-trivial fixed point of Φ . Adding a multiple of $\mathbf{1}$ if necessary, we find a positive $y \in \mathcal{A}$ such that $\Phi(y) = y$ with 0 as smallest eigenvalue. For all $n \geq 1$ let p_n be the spectral projection of y associated with the interval $]1/n, \|y\|]$ that clearly belongs to \mathcal{A} . The sequence $(p_n; n \geq 1)$ converges in norm to the projection p onto the orthogonal subspace of the eigenvalue 0. Since y is not a multiple of $\mathbf{1}$ all the projections p_n and p are proper. The inequalities $p_n \leq ny$ and $\Phi(p_n) \leq \mathbf{1}$ imply then

$$\Phi(p_n)^n \leq \Phi(p_n) \leq n\Phi(y) = ny.$$

We find then the inequality $\Phi(p_n) \leq n^{1/n}y^{1/n}$. The sequence $(y^{1/n}; n \geq 1)$ converges in norm to p , therefore, letting n tend to infinity we find $\Phi(p) \leq p$ and p^\perp is a proper subharmonic projection. \square

Proposition 3.5. *A positive unital irreducible linear map Φ on \mathcal{A} has a unique faithful invariant state.*

Proof. (Sketch) Let η be a state on \mathcal{A} . For all $k \geq 1$, $\eta_k = k^{-1} \sum_{h=1}^k \Phi^{*h}(\eta)$ is a state on \mathcal{A} . The sequence of states $(\eta_k; k \geq 1)$ is relatively compact in the unit ball of \mathcal{A}^* because \mathcal{A} is finite-dimensional and any convergent subsequence converges to an invariant state.

The invariant state is unique. Indeed, it belongs to $\text{Ker}(Id - \Phi^*)$ (Id being the identity map), this subspace coincides with the orthogonal of the range of $Id - \Phi$ and the dimension of this subspace is equal to the dimension of $\text{Ker}(Id - \Phi)$ which is 1 by Proposition 3.4.

Finally the unique invariant state is faithful because its support projection is subharmonic and Φ is irreducible. \square

Suppose now that Φ is a unital Schwarz map on \mathcal{A} and let $\mathcal{D} : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{C}$ be the map

$$\mathcal{D}(a, b) = \Phi(a^*b) - \Phi(a^*)\Phi(b).$$

Moreover, define

$$\mathcal{N}(\Phi) = \{x \in \mathcal{A} \mid \mathcal{D}(x, x) = 0, \mathcal{D}(x^*, x^*) = 0\}.$$

Lemma 3.6. *For all $x \in \mathcal{N}(\Phi)$ and all $a \in \mathcal{A}$ we have $\mathcal{D}(x, a) = \mathcal{D}(x^*, a) = 0$.*

Proof. Since Φ is a Schwarz map we have $\mathcal{D}(a, a) \geq 0$ for all $a \in \mathcal{A}$. Now, if $x \in \mathcal{A}$ and $\mathcal{D}(x, x) = 0$, for all $z \in \mathbb{C}$ we have $\mathcal{D}(zx + a, zx + a) \geq 0$, i.e.:

$$\bar{z} \mathcal{D}(x, a) + z \mathcal{D}(a, x) + \mathcal{D}(a, a) \geq 0.$$

Therefore, for all state ω on \mathcal{A} we have $2\Re z \omega(\mathcal{D}(a, x)) + \omega(\mathcal{D}(a, a)) \geq 0$ i.e., if $z = |z|e^{i\theta}$, $2|z|\Re e^{i\theta} \omega(\mathcal{D}(a, x)) + \omega(\mathcal{D}(a, a)) \geq 0$ for all $|z| \geq 0$ and $\theta \in \mathbb{R}$. This implies $\mathcal{D}(a, x) = 0$. \square

We are now in a position to prove the non-commutative version of the Perron-Frobenius theorem.

Theorem 3.7. *Let Φ be an irreducible Schwarz map on \mathcal{A} . Then*

- 1) $\text{sp}(\Phi) \cap \mathbb{T}$ is a finite subgroup of \mathbb{T} ,
- 2) the eigenspace of each $\lambda \in \text{sp}(\Phi) \cap \mathbb{T}$ is one-dimensional and the associated eigenvector is a multiple of a unitary element of \mathcal{A} .

Proof. Suppose $e^{i\theta} \in \text{sp}(\Phi)$ for $\theta \in \mathbb{R}$ then, since \mathcal{A} is finite-dimensional, $e^{i\theta}$ is an eigenvalue of Φ . Let $a \in \mathcal{A}$, ($a \neq 0$) such that $\Phi(a) = e^{i\theta}a$. Clearly we have also $\Phi(a^*) = e^{-i\theta}a^*$. Let ω be the unique faithful Φ -invariant state on \mathcal{A} . By the Schwarz property of Φ , we have

$$0 \leq \omega(\Phi(a^*a) - \Phi(a^*)\Phi(a)) = \omega(a^*a - a^*e^{-i\theta}e^{i\theta}a) = 0.$$

It follows that $\Phi(a^*a) = \Phi(a^*)\Phi(a)$ and, in the same way, $\Phi(aa^*) = \Phi(a)\Phi(a^*)$, i.e.: $a \in \mathcal{N}(\Phi)$. Now, by Lemma 3.6 we have then

$$\Phi(a^2) = \Phi(a)^2 = e^{2i\theta}a^2, \quad \Phi(a^3) = e^{3i\theta}a^3, \quad \dots$$

Therefore, the set $\{e^{ik\theta} \mid k \in \mathbb{N}\}$ is contained in the spectrum of Φ . This is a finite subset of the unit disk because \mathcal{A} is finite-dimensional, hence there exists a $k \in \mathbb{N}$ such that $e^{ik\theta} = 1$, i.e. $e^{i(k-1)\theta} = 1$. Taking the smallest such $k > 1$ we have then $\theta = 2\pi/(k-1)$. It follows then that, if $\text{sp}(\Phi) \cap \mathbb{T} = \{1, e^{i\theta_1}, \dots, e^{i\theta_m}\}$ with $0 < \theta_1 < \dots < \theta_m < 2\pi$, then $\theta_j = j\theta_1$ for all $j = 1, \dots, m$ and 1) is proved.

For the second part, let $a \in \mathcal{A}$ be an eigenvector $\Phi(a) = za$, with $|z| = 1$. The above argument yields $\Phi(a^*a) = a^*a$ and $\Phi(aa^*) = aa^*$. By virtue of Proposition 3.4 this implies that both a^*a and aa^* are a multiple of $\mathbf{1}$. \square

The non-commutative analog of statements 3) and 4) of Theorem 2.4 is essentially proved in Proposition 3.5. The analog of 5) will be given in the next Section.

4. Periodic Positive Maps

We start by proving a couple of auxiliary facts: the second one is the non-commutative counterpart of Lemma 2.2.

Proposition 4.1. *Let Φ be an irreducible Schwarz map on \mathcal{A} . A resolution of the identity p_0, \dots, p_{d-1} is cyclic for Φ if and only if $\Phi(p_k) = p_{k \perp 1}$ for all k .*

Proof. If $p_0, \dots, p_{k \perp 1}$ is cyclic for Φ then $\Phi(p_k)$ belongs to the algebra $p_{k \perp 1} \mathcal{A} p_{k \perp 1}$. Thus, $\Phi(p_k) \leq p_{k \perp 1}$ for all $k = 0, \dots, d-1$. Summing on k we have then

$$\mathbf{1} = \sum_{k=0}^{d-1} \Phi(p_k) \leq \sum_{k=0}^{d-1} p_{k \perp 1} = \mathbf{1}.$$

It follows that all the inequalities $\Phi(p_k) \leq p_{k \perp 1}$ are equalities.

Conversely, if $\Phi(p_k) = p_{k \perp 1}$, then p_k belongs to $\mathcal{N}(\Phi)$. Therefore, from Lemma 3.6, we find $\Phi(p_k a p_k) = p_{k \perp 1} \Phi(a) p_{k \perp 1} \in p_{k \perp 1} \mathcal{A} p_{k \perp 1}$ for all $a \in \mathcal{A}$. This completes the proof. \square

Lemma 4.2. *Let Φ be an irreducible Schwarz map on \mathcal{A} with period d . Suppose that there exists a natural number d' and a Φ -cyclic resolution of the identity $p'_0, \dots, p'_{d'-1}$, then d' divides d .*

Proof. With $p'_0, \dots, p'_{d'-1}$ we can associate the unitary operator

$$V = \sum_{j=0}^{d'-1} \exp(2\pi i j/d') p'_j. \tag{4.1}$$

Since, by Proposition 4.1, $\Phi(p'_j) = p'_{j \oplus_{d'-1} 1}$ for $j = 0, \dots, d'-1$, V is an eigenvector of Φ with eigenvalue $\exp(2\pi i/d')$ and V belongs to $\mathcal{N}(\Phi)$. It follows then from Lemma 3.6, by the same argument of the proof of Theorem 3.7, that V, V^2, V^3, \dots are all eigenvectors of Φ with eigenvalues $\exp(2\pi i/d')$, $\exp(4\pi i/d')$, $\exp(6\pi i/d')$... in $\text{sp}(\Phi) \cap \mathbb{T}$ then, d' divides d by Theorem 3.7 part 2). \square

Theorem 4.3. *Let Φ be an irreducible Schwarz map on \mathcal{A} . For all natural number $d \geq 1$ the following conditions are equivalent:*

- 1) *there exists a Φ -cyclic resolution of the identity p_0, \dots, p_{d-1} ,*
- 2) *there exists a unitary U in \mathcal{A} satisfying $U^d = \mathbf{1}$ and $\Phi(U) = e^{2\pi i/d}U$,*
- 3) *$\text{sp}(\Phi) \cap \mathbb{T}$ contains all the d -th roots of 1.*

Proof. 1) \Rightarrow 2). This follows from the argument of the proof of Lemma 4.2. Indeed, the unitary U can be defined as in (4.1).

2) \Rightarrow 3). If 2) holds, then $U \in \mathcal{N}(\Phi)$. Lemma 3.6 shows that $\Phi(U^k) = e^{2\pi i k/d}U^k$ for $k = 0, \dots, d-1$ and 3) follows.

3) \Rightarrow 2). Let a be an eigenvalue associated with the eigenvector $e^{2\pi i/d}$. By Theorem 3.7 part 2) a is a multiple of a unitary U and 2) follows.

2) \Rightarrow 1). Let $U \in \mathcal{A}$ be a unitary satisfying $U^d = \mathbf{1}$. By the spectral theorem there exists a resolution of the identity p_0, \dots, p_{d-1} (projections in \mathcal{A}) allowing us to write U in the form (4.1) (with d replacing d'). The identity $\Phi(U) = e^{2\pi i/d}U$ yields

$$\sum_{k=0}^{d-1} e^{2\pi i k/d} \Phi(p_k) = \sum_{k=0}^{d-1} e^{2\pi i (k+1)/d} p_k.$$

This implies $\Phi(p_k) = p_{k \oplus_{d-1} 1}$ for all $k = 0, \dots, d-1$. \square

Notice that Theorem 4.3 also provides a non-commutative generalization of Theorem 2.4 part 5). Indeed, if $\text{sp}(\Phi) \cap \mathbb{T} = \{1\}$, then statement on the convergence of $P_{\ell m}^k$ follows by standard spectral theory.

5. Periodic Completely Positive Maps

Most interesting positive linear maps on a C^* -algebra \mathcal{A} enjoy a property called *complete positivity* which is stronger than positivity and Schwarz property when \mathcal{A} is non-commutative. However, it coincides with positivity when \mathcal{A} is commutative. For completely positive maps we can give simpler conditions equivalent to irreducibility and periodicity based on their Kraus' representation (formula 5.2).

Let us first recall the definition and characterization of completely positive maps. A linear map Φ on \mathcal{A} it is called *n-positive* if

$$\sum_{h,k} b_h^* \Phi(a_h^* a_k) b_k$$

is positive for all $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{A}$ and it is called *completely positive* if it is *n-positive* for all n . It is well-known that a 2-positive unital map on a C^* -algebra \mathcal{A} is also a Schwarz map (see e.g. [5] Propo 2.10 p. 19) .

Theorem 5.1. *Let Φ be a completely positive linear map on a finite-dimensional C^* -algebra \mathcal{A} of operators on a finite-dimensional Hilbert space \mathfrak{h} . Then there exists an $m \in \mathbb{N}$ and an operator $L : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathbb{C}^m$ such that*

$$\Phi(x) = L^*(x \otimes \mathbf{1}_{\mathbb{C}^m})L \tag{5.1}$$

for all $x \in \mathcal{A}$. The operator L is an isometry if and only if Φ is unital. Moreover the minimality condition

$$\mathfrak{h} \otimes \mathbb{C}^m = \text{span}\{(a \otimes \mathbf{1}_{\mathbb{C}^m})Lu \mid a \in \mathcal{A}, u \in \mathfrak{h}\}$$

holds. Conversely, given an $m' \in \mathbb{N}$ and an operator $M : \mathfrak{h} \rightarrow \mathfrak{h} \otimes \mathbb{C}^{m'}$ such that $\Phi(x) = M^*(x \otimes \mathbf{1}_{\mathbb{C}^{m'}})M$, and the minimality condition $\mathfrak{h} \otimes \mathbb{C}^{m'} = \text{span}\{(a \otimes \mathbf{1}_{\mathbb{C}^{m'}})Mu \mid a \in \mathcal{A}, u \in \mathfrak{h}\}$ holds, there exists a unitary operator $V : \mathfrak{h} \otimes \mathbb{C}^m \rightarrow \mathfrak{h} \otimes \mathbb{C}^{m'}$ such that

$$M = VL, \quad V(a \otimes \mathbf{1}_{\mathbb{C}^m}) = (a \otimes \mathbf{1}_{\mathbb{C}^{m'}})V,$$

for all $a \in \mathcal{A}$. In particular $m = m'$ and V is an $m \times m$ matrix $(V_{ij})_{1 \leq i, j \leq m}$ of operators V_{ij} on \mathfrak{h} belonging to the commutant \mathcal{A}' of \mathcal{A} in $\mathcal{B}(\mathfrak{h})$.

This is an immediate consequence of Kraus' theorem on normal completely positive maps (see e.g. [5] Th. 2.20 p. 24 for a proof). Indeed, since \mathcal{A} is finite-dimensional, then it is also a von Neumann algebra and the linear map Φ enjoys the necessary continuity properties.

It is worth noticing here that, fixing an orthonormal basis e_1, \dots, e_m in \mathbb{C}^m and identifying L with an m -tuple L_1, \dots, L_m of operators on \mathfrak{h} defined by $\langle v \otimes e_\ell, Lu \rangle = \langle v, L_\ell u \rangle$ we can write the representation (5.1) as

$$\Phi(x) = \sum_{\ell=1}^m L_\ell^* x L_\ell. \tag{5.2}$$

If $\Phi(x) = \sum_{\ell=1}^{m'} M_\ell^* x M_\ell$ (M_ℓ operators on \mathfrak{h}) and also the M_ℓ 's satisfy the minimality condition, then $m = m'$ and there exists an $m \times m$ matrix $(V_{ij})_{1 \leq i, j \leq m}$ of operators in \mathcal{A}' such that the corresponding operator V on $\mathfrak{h} \otimes \mathbb{C}^m$ is unitary and $\Phi(x) = \sum_\ell M_\ell^* x M_\ell$ with $M_\ell = \sum_k V_{\ell k} L_k$ for all $x \in \mathcal{A}$.

Roughly speaking, under the minimality condition, the choice of the L_ℓ 's is unique up to some elements in the commutant of \mathcal{A} .

The characterization of completely positive irreducible maps is simple.

Proposition 5.2. *The completely positive unital linear map Φ (5.2) is irreducible if and only if the only subspaces of \mathfrak{h} which are L_ℓ -invariant for all $\ell = 1, \dots, m$ are $\{0\}$ and \mathfrak{h} .*

Proof. Indeed, if p is a Φ -subharmonic projection, we have $\Phi(p^\perp) \leq p^\perp$. It follows then

$$0 \leq p \left(\sum_{\ell=1}^m L_\ell^* p^\perp L_\ell \right) p = \sum_{\ell=1}^m |p^\perp L_\ell p|^2 \leq p p^\perp p = 0.$$

Thus $p^\perp L_\ell p$ for all $\ell = 1, \dots, m$ and the range of p is an invariant subspace for all the L_ℓ 's.

Conversely, if p is the orthogonal projection onto an L_ℓ -invariant ($\ell = 1, \dots, m$) subspace of \mathfrak{h} then $p \Phi(p^\perp) p = 0$. Therefore $p \Phi(p^\perp) p^\perp = p^\perp \Phi(p^\perp) p = 0$. It follows then ([6] Lemma II.1) that $\Phi(p^\perp) = p^\perp \Phi(p^\perp) p^\perp \leq p^\perp$ and p is a Φ -subharmonic projection. \square

We now characterize periodic completely positive maps.

Definition 5.3. For any collection $L = (L_1, \dots, L_m)$ of operators on \mathfrak{h} we will say that a resolution of the identity p_0, \dots, p_{d-1} is L -cyclic if $p_k L_\ell = L_\ell p_{k \ominus 1}$ for all $\ell = 1, \dots, m$ and all $k = 0, \dots, d - 1$.

Theorem 5.4. Let Φ be the completely positive linear map (5.2) on \mathcal{A} . A resolution of the identity p_0, \dots, p_{d-1} is Φ -cyclic if and only if it is L -cyclic. In particular Φ is periodic with period d if and only if d is the greatest of all the natural numbers such that there exists an L cyclic-invariant resolution of the identity p_0, \dots, p_{d-1} with $p_0, \dots, p_{d-1} \in \mathcal{A}$.

Proof. If p_0, \dots, p_{d-1} is Φ -cyclic, i.e. $\Phi(p_k) = p_{k \ominus 1}$ for $k = 0, \dots, d - 1$, then

$$\sum_{\ell=0} L_\ell^* p_k L_\ell = p_{k \ominus 1}$$

for all k . Multiplying by $p_{k \ominus 1}^\perp$ on the right and on the left we find

$$0 = \sum_{\ell} p_{k \ominus 1}^\perp L_\ell^* p_k L_\ell p_{k \ominus 1}^\perp = \sum_{\ell} |p_k L_\ell p_{k \ominus 1}^\perp|^2.$$

It follows that $p_k L_\ell p_{k \ominus 1}^\perp = 0$, i.e. $p_k L_\ell = L_\ell p_{k \ominus 1}$ for all ℓ and k .

The converse and the last statement are now easy. \square

6. Positive Maps on $M_2(\mathbb{C})$.

We now illustrate the notions of irreducibility and periodicity in the simplest non-commutative concrete example classifying positive unital maps on $\mathcal{A} = M_2(\mathbb{C})$, the algebra of 2×2 complex matrices. We refer to [8], [11] for more detailed information on the structures we introduce here as well as their physical importance.

In order to perform the necessary computations we introduce a convenient basis on the vector space \mathcal{A} . This is given through the Pauli's matrices $\sigma_0, \sigma_z, \sigma_x, \sigma_y$ given by :

$$\sigma_0 = \mathbf{1}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

For an $a \in M_2(\mathbb{C})$ denote $v = (v_0, v_z, v_x, v_y) \in \mathbb{C}^4$ the coordinates of a in the basis of the Pauli's matrices. Let $\mathbf{v} = (v_z, v_x, v_y)$ and write $\mathbf{v} \cdot \sigma$ to denote $v_z \sigma_z + v_x \sigma_x + v_y \sigma_y$.

The multiplication rule of an $a = v_0\mathbf{1} + \mathbf{v} \cdot \sigma$ by a $c = w_0\mathbf{1} + \mathbf{w} \cdot \sigma$ is:

$$ac = ((v_0w_0 + v_zw_z + v_xw_x + v_yw_y)\mathbf{1} + (w_0\mathbf{v} + v_0\mathbf{w} + i\mathbf{v} \wedge \mathbf{w}) \cdot \sigma) / 2$$

where $\mathbf{v} \wedge \mathbf{w}$ is the usual vector product of \mathbf{v} and \mathbf{w} .

A unital map Φ on $M_2(\mathbb{C})$ can be represented in the form

$$\Phi(a) = \left(\begin{array}{c|c} 1 & b^T \\ \hline 0 & A \end{array} \right) \begin{pmatrix} v_0 \\ v_z \\ v_x \\ v_y \end{pmatrix} \quad (6.1)$$

where (since $\Phi(a^*) = \Phi(a)^*$ for all $a \in M_2(\mathbb{C})$) $b \in \mathbb{R}^3$, $A \in M_3(\mathbb{R})$.

The map Φ is positive if and only if for all rank-one projection p in $M_2(\mathbb{C})$ we have $\Phi(p) \geq 0$. It is easy to verify that such a projection can be written in the form $p = (\mathbf{1} + \mathbf{u} \cdot \sigma)/2$ with $\mathbf{u} \in \mathbb{R}^3$ and $\|\mathbf{u}\| = 1$. Thus

$$\Phi(p) = ((1 + \langle b, \mathbf{u} \rangle)\mathbf{1} + (A\mathbf{u}) \cdot \sigma) / 2 \quad (6.2)$$

and Φ is *positive* if and only if (see e.g. [3]) $\|A\mathbf{u}\| \leq 1 + \langle b, \mathbf{u} \rangle$ for all \mathbf{u} .

Proposition 6.1. *Let Φ be a positive unital map on $M_2(\mathbb{C})$. Then*

- 1) Φ is irreducible if and only if there exists no unit vector \mathbf{u} in \mathbb{R}^3 such that $\mathbf{u} - A^*\mathbf{u} = b$,
- 2) Φ is periodic with period 2 if and only if -1 is an eigenvalue of A with an eigenvector orthogonal to b .

Proof. If Φ is not irreducible, then there exists a one-dimensional projection p such that $\Phi(p) \geq p$. The adjoint map Φ^* (this is the adjoint of the linear operator Φ on $M_2(\mathbb{C})$ endowed with the scalar product $\langle a, c \rangle = \text{trace}(a^*b)/2$ and is represented, in the Pauli matrices basis by the adjoint of the matrix in the right-hand side of 6.1) satisfies $\Phi^*(p) = p$. Indeed, for all $a \in M_2(\mathbb{C})$ we have $\text{trace}(\Phi^*(a)) = \text{trace}(a)$, thus $\text{trace}(\Phi^*(p)) = 1$. However, we have also

$$\text{trace}(p\Phi^*(p)p) = \text{trace}(\Phi^*(p)p) = \text{trace}(p\Phi(p)) = \text{trace}(p) = 1,$$

and $\Phi^*(p) = p$ follows. Conversely, by reversing the order of the above steps we can show that $\Phi^*(p) = p$ implies $\Phi(p) \geq p$. Therefore the first statement follows from the representation formula $p = (\mathbf{1} + \mathbf{u} \cdot \sigma)/2$ with $\mathbf{u} \in \mathbb{R}^3$ and $\|\mathbf{u}\| = 1$ for one-dimensional projections.

We now find conditions on A and b for Φ to be periodic. If this is not the case, then there exists a unit vector $\mathbf{u} \in \mathbb{R}^3$ such that the one-dimensional projection $p_1 = (\mathbf{1} + \mathbf{u} \cdot \sigma)/2$ and its orthogonal projection $p_2 = (\mathbf{1} - \mathbf{u} \cdot \sigma)/2$ with $\mathbf{u} \in \mathbb{R}^3$ and $\|\mathbf{u}\| = 1$ satisfy $\Phi(p_1) = p_2$ and $\Phi(p_2) = p_1$. We have then $1 + \langle b, \mathbf{u} \rangle = 1$ and $A\mathbf{u} = -\mathbf{u}$, and 2) follows. \square

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