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CONVERSE COMPARISON THEOREMS FOR BACKWARD DOUBLY STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. Converse and general converse comparison theorems are proved for backward doubly stochastic differential equations.

1. Introduction

A new kind of backward stochastic differential equations (BSDEs in short) was introduced by Pardoux and Peng [6] in 1994, that is a class of backward doubly stochastic differential equations (BDSDEs in short) with two different directions of stochastic integrals, i.e., equations involving both a standard (forward) stochastic integral and a backward stochastic integral. That is, BDSDEs are stochastic differential equations of the form

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T h(s, Y_s, Z_s)d\overline{B}_s - \int_t^T Z_s dW_s,\ 0 \le t \le T, \quad (1.1)$$

where ξ is the terminal value, g the generator and h is a drift function.

The comparison theorem, which is an important and effective technique in the theory of BSDE, was first established for BDSDEs by [8]. It allows one to compare the solutions of two real-valued BDSDEs whenever we can compare the terminal conditions and the generators. An inverse problem is interesting: namely, if we can compare the solution of two BDSDEs with the same terminal condition, can we compare the generators?

To put our result in context, we note that if $h = 0$, the result of Chen [2] can be thought as the first step in solving this theorem and then it was further developed by Briand et al. [1], Coquet et al. [3] and Jiang [4].

On the other context, using BSDEs, Peng introduced in [7] the notion of g-expectation; he considers the function $\mathcal{E}_g$ defined on $\mathbb{L}^2(\mathcal{F}_T^W)$ with values in $\mathbb{R}$ by simply setting $\mathcal{E}_g(\xi) = Y_0$ where (Y, Z) is the unique solution of the BSDE

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T.$$

$\mathcal{E}_g(\xi)$ is called the g-expectation of ξ. Similarly, the conditional g-expectation is introduced by setting , for any stopping time τ, $\mathcal{E}_g[\xi \big| \mathcal{F}_t^W] = Y_t$ which is the unique

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$\mathcal{F}_t$-measurable and square integrable random variable η such that

$$\forall A \in \mathcal{F}_t, \quad \mathcal{E}_g(\mathbb{I}_A \eta) = \mathcal{E}_g(\mathbb{I}_A \xi).$$

In this paper, we are concerned with converse and general converse comparison theorems for BDSDEs. Let (Y^i, Z^i) be the unique square integrable and adapted solution of the BDSDE (1.1) with data (ξ, g_i, h). Under suitable conditions, we prove that for each terminal condition ξ , if $\forall\ t\ Y_t^1 \leq Y_t^2$ then $g_1 \leq g_2$. More in general, the result remain true if we suppose only that $Y_0^1 \leq Y_0^2$.

The rest of the paper is organized as follows: in Subsection 1.1, we introduce some notations and we make our main assumptions. In Section 2, we provide a priori estimate and we introduce the notion of g-expectation for BDSDEs. In Section 3, we discuss converse comparison theorem for BDSDEs whereas general comparison theorem will be proved in Section 4.

1.1. Backgrounds. More precisely, we consider two independent d-dimensional Brownian motions $(d \geq 1)$, $\{W_t,\ 0 \leq t \leq T\}$ and $\{B_t,\ 0 \leq t \leq T\}$ defined on the complete probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ respectively.

We denote

$$\mathcal{F}_{s,t}^B := \sigma\left\{B_r - B_s\,,\ s \leq r \leq t\right\}, \quad \mathcal{F}_t^W := \sigma\left\{W_r\,,\ 0 \leq r \leq t\right\}.$$

Moreover, we define $\Omega \triangleq \Omega_1 \times \Omega_2$, $\mathcal{F} \triangleq \mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathbb{P} \triangleq \mathbb{P}_1 \otimes \mathbb{P}_2$, and we put $\mathcal{F}_t \triangleq \mathcal{F}_t^W \otimes \mathcal{F}_{t,T}^B \vee \mathcal{N}$, where $\mathcal{N}$ is the collection of $\mathbb{P}$-null-sets. We notice that the family of σ-algebras $\mathrm{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is not a filtration.

Let $g : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}$ and $h : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ be two progressively measurable functions with the property that there exists constants $\kappa > 0$ and $0 < \alpha < 1$ such that the following hypothesis are satisfied

$(A.0)$ $\xi \in \mathbb{L}^2(\mathcal{F}_T)$.
$(A.1)$ The processes $(g(t,0,0))_{t\in[0,T]}$ and $(h(t,0,0))_{t\in[0,T]}$ are progressively measurable such that $\mathbb{E}\int_0^T \left(|g(t,0,0)|^2 + \|h(t,0,0)\|^2\right) dt < \infty$.
$(A.2)$ For any (y_1, z_1), $(y_2, z_2) \in \mathbb{R} \times \mathbb{R}^d$,

$$\begin{cases} |g\left(t,y_1,z_1\right) - g\left(t,y_2,z_2\right)|^2 \leq \kappa\left(|y_1 - y_2|^2 + \|z_1 - z_2\|^2\right) \\ |h\left(t,y_1,z_1\right) - h\left(t,y_2,z_2\right)|^2 \leq \kappa\,|y_1 - y_2|^2\ +\ \alpha\,\|z_1 - z_2\|^2. \end{cases}$$

$(A.3)$ $\mathbb{P}$-a.s. for all (t,y) we have $g(t,y,0) = 0$ and $h(t,y,0) = 0$.
$(A.4)$ $\mathbb{P}$-a.s. for all (y,z) the function $t \mapsto g(t,y,z)$ is continuous.

It was shown in [6] that, under the assumptions $(A.0)$, $(A.1)$ and $(A.2)$, the backward doubly stochastic differential equation (1.1) has a unique solution.

2. BDSDEs and g-expectation

In this section, we state a technical result, we define the notion of g-expectation for solution of BDSDEs and we prove a useful properties of g-expectation.

2.1. A priori estimate.

Proposition 2.1. *Let the assumptions* $(A.0)$, $(A.1)$ *and* $(A.2)$ *hold. Then, for* $\beta \geq C(\alpha,\kappa)$ *we have*

$$\mathbb{E}\left[\sup_{t\leq s\leq T} e^{\beta s}|Y_s|^2 + \int_t^T e^{\beta s}\|Z_s\|^2 ds \Big| \mathcal{F}_t\right]$$
$$\leq C\mathbb{E}\left[e^{\beta T}|\xi|^2 + \left(\int_t^T e^{(\beta/2)s}|g(s,0,0)|ds\right)^2 + \int_t^T e^{\beta s}\|h(s,0,0)\|^2 ds \Big| \mathcal{F}_t\right].$$

Proof. Itô formula yields

$$\begin{aligned} e^{\beta t}\left|Y_t\right|^2 &+ \int_t^T e^{\beta s}\|Z_s\|^2 ds + \beta \int_t^T e^{\beta s}\left|Y_s\right|^2 ds \\ &= e^{\beta T}\left|\xi\right|^2 + 2\int_t^T e^{\beta s} Y_s g\left(s,Y_s,Z_s\right) ds + \int_t^T e^{\beta s}\left\|h\left(s,Y_s,Z_s\right)\right)\|^2 ds \\ &\quad +2\int_t^T e^{\beta s} Y_s, h(s,Y_s,Z_s) d\overline{B_s} - 2\int_t^T e^{\beta s} Y_s Z_s dW_s. \end{aligned} \tag{2.1}$$

Now, using conditions $(A.2)$ and the algebraic inequality $2ab \leq \dfrac{a^2}{\varepsilon} + \varepsilon b^2,\ \varepsilon > 0$ and $(a+b)^2 \leq (1+\frac{1}{\epsilon})a^2 + (1+\epsilon)b^2,\ \epsilon > 0$ we obtain

$$\begin{aligned} 2Y_s g(s,Y_s,Z_s) &\leq 2|Y_s||g(s,0,0)| + (\varepsilon + \frac{\kappa}{\varepsilon})|Y_s|^2 + \frac{\kappa}{\varepsilon}\|Z_s\|^2 \\ \left\|h\left(s,Y_s,Z_s\right)\right)\|^2 &\leq (1+\epsilon)\|h(s,0,0)\|^2 + (1+\frac{1}{\epsilon})\kappa|Y_s|^2 + (1+\frac{1}{\epsilon})\alpha\|Z_s\|^2. \end{aligned}$$

By choosing $\varepsilon = \frac{2\kappa}{1-\alpha}$ $\epsilon = \frac{3\alpha}{1-\alpha}$ and $C(\alpha,\kappa) = (\varepsilon + \frac{\kappa}{\varepsilon}) + (1+\frac{1}{\epsilon})\kappa$ and plugging the last two inequalities in (2.1), we infer

$$\begin{aligned} e^{\beta t}|Y_t|^2 &+ \frac{1-\alpha}{6}\int_t^T e^{\beta s}\|Z_s\|^2 ds + (\beta - C(\alpha,\kappa))\int_t^T e^{\beta s}|Y_s|^2 ds \\ &\leq e^{\beta T}|\xi|^2 + 2\int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \frac{1+2\alpha}{1-\alpha}\int_t^T e^{\beta s}\left\|h(s,0,0)\right)\|^2 ds \\ &\quad +2\int_t^T e^{\beta s} Y_s h(s,Y_s,Z_s) d\overline{B_s} - 2\int_t^T e^{\beta s} Y_s Z_s dW_s. \end{aligned}$$

By taking $\beta \geq C(\alpha,\kappa)$ we obtain

$$\begin{aligned} e^{\beta t}|Y_t|^2 &+ \int_t^T e^{\beta s}\|Z_s\|^2 ds \\ &\leq C\Bigg\{e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \int_t^T e^{\beta s}\left\|h(s,0,0)\right\|^2 ds \\ &\quad + \int_t^T e^{\beta s} Y_s h(s,Y_s,Z_s) d\overline{B_s} - \int_t^T e^{\beta s} Y_s Z_s dW_s\Bigg\}. \end{aligned} \tag{2.2}$$

For fixed $w_1 \in \Omega_1$ (see the notation in Subsection 1.1), we take the conditional expectation $\mathbb{E}^{\mathcal{F}^B_{t,T}}[.] = \mathbb{E}[.|\mathcal{F}^B_{t,T}]$ to obtain

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}^B_{t,T}}\left[e^{\beta t}|Y_t|^2 + \int_t^T e^{\beta s}\|Z_s\|^2 ds\right] \\
&\le C\mathbb{E}^{\mathcal{F}^B_{t,T}}\left[e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \int_t^T e^{\beta s}\,\|h(s,0,0)\|^2\,ds \right.\\
&\qquad\qquad \left. - \int_t^T e^{\beta s} Y_s Z_s dW_s\right].
\end{aligned}$$

Similarly, taking $\mathbb{E}[.|\mathcal{F}^W_t]$ in the last inequality to get

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left[e^{\beta t}|Y_t|^2 + \int_t^T e^{\beta s}\|Z_s\|^2 ds\right] \\
&\le C\mathbb{E}^{\mathcal{F}_t}\left[e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \int_t^T e^{\beta s}\,\|h(s,0,0)\|^2\,ds\right],
\end{aligned} \tag{2.3}$$

where we have used the fact that

$$\mathbb{E}\left[.|\mathcal{F}_t\right] = \mathbb{E}\left[.|\mathcal{F}^B_{t,T} \otimes \mathcal{F}^W_t\right] = \mathbb{E}\left[\mathbb{E}[.|\mathcal{F}^B_{t,T}]|\mathcal{F}^W_t\right].$$

Coming back to the inequality (2.2) we get from Burkholder-Davis-Gundy's inequality, that

$$\begin{aligned}
&\mathbb{E}^{\mathcal{F}_t}\left[\sup_{t\le s\le T} e^{\beta s}|Y_s|^2\right] \\
&\le C\mathbb{E}^{\mathcal{F}_t}\left[e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \int_t^T e^{\beta s}\,\|h(s,0,0)\|^2\,ds \right.\\
&\left. + \left(\int_t^T e^{2\beta s}|Y_s|^2\|h(s,Y_s,Z_s)\|^2 ds\right)^{1/2} + \left(\int_t^T e^{2\beta s}|Y_s|^2\|Z_s\|^2 ds\right)^{1/2}\right].
\end{aligned}$$

By similar argument as before, we estimate the last term as follows

$$\begin{aligned}
&C\mathbb{E}^{\mathcal{F}_t}\left[\left(\int_t^T e^{2\beta s}|Y_s|^2\|h(s,Y_s,Z_s)\|^2 ds\right)^{1/2}\right] \\
&\le \frac{1}{8}\mathbb{E}^{\mathcal{F}_t}\left[\sup_{t\le s\le T} e^{\beta t}|Y_t|^2\right] + C\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta s}\|h(s,Y_s,Z_s)\|^2 ds\right] \\
&\le \frac{1}{4}\mathbb{E}^{\mathcal{F}_t}\left[\sup_{t\le s\le T} e^{\beta t}|Y_t|^2\right] + C\mathbb{E}^{\mathcal{F}_t}\left[\int_0^T e^{\beta s}\,\|h(s,0,0)\|^2\,ds + \int_t^T e^{\beta s}\|Z_s\|^2 ds\right]
\end{aligned}$$

and

$$C\mathbb{E}^{\mathcal{F}_t}\left[\left(\int_t^T e^{2\beta s}|Y_s|^2\|Z_s\|^2 ds\right)^{1/2}\right]$$
$$\leq \frac{1}{4}\mathbb{E}^{\mathcal{F}_t}\left[\sup_{t\leq s\leq T} e^{\beta t}|Y_t|^2\right] + C\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta s}\|Z_s\|^2 ds\right].$$

Using inequality (2.3) and the previous one, we deduce that

$$\mathbb{E}^{\mathcal{F}_t}\left[\sup_{t\leq s\leq T} e^{\beta t}|Y_t|^2\right]$$
$$\leq C\mathbb{E}^{\mathcal{F}_t}\left[e^{\beta T}|\xi|^2 + \int_t^T e^{\beta s}|Y_s|\,|g(s,0,0)|\,ds + \int_0^T e^{\beta s}\,\|h(s,0,0)\|^2\,ds\right].$$

Combining this with the fact that

$$C\mathbb{E}^{\mathcal{F}_t}\left[\int_t^T e^{\beta s}\,|Y_s|\,|g\,(s,0,0)|\,ds\right]$$
$$\leq \frac{1}{2}\mathbb{E}^{\mathcal{F}_t}\left[\sup_{0\leq t\leq T} e^{\beta t}\,|Y_t|^2\right] + \frac{C^2}{2}\mathbb{E}^{\mathcal{F}_t}\left[\left(\int_t^T e^{(\beta/2)s}\,|g\,(s,0,0)|\,ds\right)^2\right],$$

we obtain the desired result. □

2.2. g-expectation for BDSDEs. Assuming that the data (ξ, g, h) satisfies the assumptions $(A.0)-(A.3)$. Let us introduce the operator $\mathcal{E}_{g,h}$ as: for any $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$, denote by $\mathcal{E}_{g,h}(\xi)$ and $\mathcal{E}_{g,h}[\xi|\mathcal{F}_t]$ the initial value Y_0 and the value Y_t at time t of the solution to BDSDE (1.1) respectively. For a stopping time τ, the operator $\mathcal{E}_{g,h}[\xi|\mathcal{F}_\tau]$ can be defined in an identical way.
Note that, since Y_0 is $\mathcal{F}^B_{0,T}$-measurable, then $\mathcal{E}_{g,h}(\xi)$ is not deterministic.
N.B. For simplicity, we remove the dependence of h in the sequel.

The following properties are a direct consequence to the comparison, strict comparison theorem and the uniqueness of the solution for BDSDEs

Proposition 2.2. *Let us suppose that $\xi_1, \xi_2 \in \mathbb{L}^2(\mathcal{F}_T)$ and $\tau,\ \nu$ are two stopping times with value in $[0,T]$.*

(1) *For any $X \in \mathbb{L}^2(\mathcal{F}_0)$, $\mathcal{E}_g(X) = X$. In particular $\mathcal{E}_g(c) = c$ for any constant $c \in \mathbb{R}$.*
(2) *If $\xi_1 \leq \xi_2$ a.s. then $\mathcal{E}_g(\xi_1) \leq \mathcal{E}_g(\xi_2)$ a.s.*
(3) *If $\xi_1 \leq \xi_2$ a.s. and $\mathbb{P}(\xi_1 < \xi_2) > 0$ then $\mathcal{E}_g(\xi_1) < \mathcal{E}_g(\xi_2)$ a.s.*
(4) *If $\tau \leq \nu$, $\mathcal{E}_g\left[\mathcal{E}_g[\xi_1|\mathcal{F}_\nu]|\mathcal{F}_\tau\right] = \mathcal{E}_g[\xi_1|\mathcal{F}_\tau]$.*
In particular, $\mathcal{E}_g(\xi_1) = \mathcal{E}_g(\mathcal{E}_g[\xi_1|\mathcal{F}_\nu])$.

3. Converse Comparison Theorem for BDSDEs

Let $\sigma : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^d$ be a globally κ-Lipschitz function. It is well known that, for $(t,x) \in [0,T) \times \mathbb{R}^n$, there exists a unique process $X^{t,x}$ solution of the stochastic differential equation

$$X_s^{t,x} = x + \int_t^{t\vee s} \sigma(X_r^{t,x})dW_r, \quad 0 \le s \le T$$

which satisfies

$$\mathbb{E} \sup_{u\le s\le u'} \|X_s^{t,x}\|^2 \le C(1+\|x\|^2)|u-u'|. \tag{3.1}$$

For any fixed $(t,x,y,p) \in [0,T) \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and for any $\varepsilon > 0$, let us denote $({}^{\varepsilon}Y^{t,x,y,p}, {}^{\varepsilon}Z^{t,x,y,p}) = (Y^\varepsilon, Z^\varepsilon)$ the solution on $[0, t+\varepsilon]$ of the following BDSDE

$$\begin{aligned} Y_s^\varepsilon &= y + p\left(X_{t+\varepsilon}^{t,x} - x\right) + \int_s^{t+\varepsilon} g\left(r, Y_r^\varepsilon, Z_r^\varepsilon\right) dr \\ &\qquad + \int_s^{t+\varepsilon} h\left(r, Y_r^\varepsilon, Z_r^\varepsilon\right) d\overline{B}_r - \int_s^{t+\varepsilon} Z_r^\varepsilon dW_r. \end{aligned}$$

To introduce our converse comparison theorem, we need the following lemma.

Lemma 3.1. *Let the assumptions* $(A.0)-(A.2)$ *and* $(A.4)$ *hold and supposing that*

$$\mathbb{E}\left(\sup_{0\le t\le T} |g(t,0,0)|^2\right) < \infty \quad \textit{and} \quad h(t,0,0) = 0 \textit{ for all } t.$$

Then,

$$\frac{1}{\varepsilon}\left({}^{\varepsilon}Y_t^{t,x,y,p} - y\right) \xrightarrow[\varepsilon\to 0^+]{\mathbb{L}^2} g(t,y,\sigma(x)p).$$

Proof. For $s \in [t, t+\varepsilon]$, put $(\tilde{Y}_s^\varepsilon, \tilde{Z}_s^\varepsilon) = (Y_s^\varepsilon - (y + p^*(X_s^{t,x} - x)), Z_s^\varepsilon - \sigma(X_s^{t,x})p)$. Then $d\tilde{Y}_s^\varepsilon = dY_s^\varepsilon - pdX_s^{t,x}$, $\tilde{Y}_{t+\varepsilon}^\varepsilon = 0$. That is

$$\begin{aligned} \widetilde{Y}_s^\varepsilon = &\int_s^{t+\varepsilon} g\left(r, y + p^*(X_r^{t,x} - x) + \widetilde{Y}_r^\varepsilon, \sigma(X_r^{t,x})p + \widetilde{Z}_r^\varepsilon\right) dr \\ &+ \int_s^{t+\varepsilon} h\left(r, y + p^*(X_r^{t,x} - x) + \widetilde{Y}_r^\varepsilon, \sigma(X_r^{t,x})p + \widetilde{Z}_r^\varepsilon\right) d\overline{B}_r - \int_s^{t+\varepsilon} \widetilde{Z}_r^\varepsilon dW_r. \end{aligned}$$

Combining Proposition 2.1, Assumption $(A.2)$ and that σ is Lipschitz, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sup_{t\le s\le t+\varepsilon} |\tilde{Y}_s^\varepsilon|^2 + \int_t^{t+\varepsilon} \|\tilde{Z}_s^\varepsilon\|^2 ds \big| \mathcal{F}_t\right] \\ &\qquad \le C\mathbb{E}\left[\left|\int_t^{t+\varepsilon} g\left(r, y + p(X_r^{t,x} - x), \sigma(X_r^{t,x})p\right) dr\right|^2 \big|\mathcal{F}_t\right] \\ &\qquad \le C(\kappa, x, y, p)\varepsilon^2 \mathbb{E}\left[\sup_{t\le s\le t+\varepsilon} \left(\|X_s^{t,x}\|^2 + |g(s,0,0)|^2\right) \big|\mathcal{F}_t\right], \end{aligned}$$

which implies by taking expectation and using (3.1) that

$$\mathbb{E}\left(\sup_{t\le s\le t+\varepsilon} |\tilde{Y}_s^\varepsilon|^2 + \int_t^{t+\varepsilon} \|\tilde{Z}_s^\varepsilon\|^2 ds\right) \le C\varepsilon^2. \tag{3.2}$$

On the other hand, we have

$$
\begin{aligned}
&\frac{1}{\varepsilon}\left({}^{\varepsilon}Y_t^{t,x,y,p}-y\right)-g(t,y,\sigma(x)p)\\
&=\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\left\{g\left(r,y+p(X_r^{t,x}-x)+\tilde{Y}_r^{\varepsilon},\sigma(X_r^{t,x})p+\tilde{Z}_r^{\varepsilon}\right)\right.\right.\\
&\qquad\qquad\left.\left.-g\left(r,y+p(X_r^{t,x}-x),\sigma(X_r^{t,x})p\right)\right\}dr\Big|\mathcal{F}_t\right]\\
&\quad+\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\left\{g\left(r,y+p(X_r^{t,x}-x),\sigma(X_r^{t,x})p\right)-g(r,y,\sigma(x)p)\right\}dr\Big|\mathcal{F}_t\right]\\
&\quad+\frac{1}{\varepsilon}\mathbb{E}\left[\int_t^{t+\varepsilon}\{g(r,y,\sigma(x)p)-g(t,y,\sigma(x)p)\}dr\Big|\mathcal{F}_t\right]\\
&=T_1+T_2+T_3.
\end{aligned}
$$

First, combining Jensen and Hölder inequality we obtain

$$
\begin{aligned}
\mathbb{E}|T_1|^2 \quad\leq\quad &\frac{1}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon}\Big|g\Big(r,y+p(X_r^{t,x}-x)+\widetilde{Y}_r^{\varepsilon},\sigma(X_r^{t,x})p+\widetilde{Z}_r^{\varepsilon}\Big)\\
&\qquad\qquad-g\Big(r,y+p(X_r^{t,x}-x),\sigma(X_r^{t,x})p\Big)\Big|^2dr.
\end{aligned}
$$

From $(A.2)$ and (3.2) we have

$$
\mathbb{E}|T_1|^2\leq\frac{\kappa}{\varepsilon}\mathbb{E}\int_t^{t+\varepsilon}(|\widetilde{Y}_r^{\varepsilon}|^2+\|\widetilde{Z}_r^{\varepsilon}\|^2)dr\leq C(\varepsilon^2+\varepsilon)
$$

which show the convergence of T_1 to 0 as $\varepsilon\to 0$. On the same way, we get

$$
\mathbb{E}|T_2|^2\leq\frac{\kappa}{\varepsilon}\|p\|^2\mathbb{E}\int_t^{t+\varepsilon}\|X_r^{t,x}-x\|^2dr\leq C\mathbb{E}\int_t^{t+\varepsilon}\|\sigma(X_r^{t,x})\|^2dr\leq C\varepsilon.
$$

Consequently, $T_2\to 0$ as $\varepsilon\to 0$ in L^2.

It remains to show that T_3 converges to 0 as $\varepsilon\to 0$. Indeed, using Hölder inequality,

$$
\mathbb{E}|T_3|^2\leq\mathbb{E}\left(\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|g(r,y,\sigma(x)p)-g(t,y,\sigma(x)p)|^2dr\right)
$$

Since the process $(g(t,y,z))_{t\in[0,T]}$ is continuous, the right hand in the last inequality goes to 0 as $\varepsilon\to 0$. Moreover, we have

$$
\frac{1}{\varepsilon}\int_t^{t+\varepsilon}|g(r,y,\sigma(x)p)-g(t,y,\sigma(x)p)|^2dr\leq C\left(1+\sup_{0\leq s\leq T}|g(s,0,0)|^2\right)\in\mathbb{L}^1(\Omega).
$$

Then it follows from the dominated convergence theorem that T_3 converges in $\mathbb{L}^2(\Omega)$ to 0 which concludes the proof of the lemma. □

Now we establish a converse comparison theorem for BDSDEs. For this, let $(Y_t^i(\xi),Z_t^i(\xi));\ i=1,2$ be solution of the following BDSDE

$$
Y_t^i(\xi)=\xi+\int_t^T g_i(s,Y_s^i(\xi),Z_s^i(\xi))ds+\int_t^T h(s,Y_s^i(\xi),Z_s^i(\xi))d\overline{B}_s-\int_t^T Z_s^i(\xi)dW_s. \tag{3.3}
$$

Then we have the next theorem.

Theorem 3.2. *Suppose that* h *and* $g_i,\ i=1,2$ *verify the assumptions* $(A.2)$, $(A.3)$ *and* $(A.4)$. *If* $\forall\ \xi \in \mathbb{L}^2(\mathcal{F}_T)\ \forall\ t\in[0,T],\ Y_t^1(\xi) \le Y_t^2(\xi)$, *then*

$$\mathbb{P}-a.s.\ \forall\ t\in[0,T]\ \forall\ (y,z)\in\mathbb{R}\times\mathbb{R}^d,\ g_1(t,y,z)\le g_2(t,y,z).$$

Proof. For any fixed (t,y,z), we have for a subsequence, $\mathbb{P}$-a.s.

$$\begin{aligned} g_1(t,y,z) &= \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left\{y+Y_t^1\left(y+z(W_{t+\varepsilon}-W_t)\right)\right\} \\ &\le \lim_{\varepsilon\to 0^+}\frac{1}{\varepsilon}\left\{y+Y_t^2\left(z(W_{t+\varepsilon}-W_t)\right)\right\} = g_2(t,y,z). \end{aligned}$$

By continuity we get the desired result. □

4. General Converse Comparison Theorem for BDSDEs

Now we give our second main result, which is a general converse theorem for BDSDEs. Roughly speaking, if we can compare the solutions of two BDSDEs with data $(\xi,g_j,h),\ j=1,2$ at time $t=0$ with the same coefficient h and the same terminal condition ξ, for all terminal conditions, can we compare the generators $g_j,\ j=1,2$?

Theorem 4.1. *Suppose that* h *and* $g_j,\ j=1,2$ *verify the assumptions* $(A.2)$, $(A.3)$ *and* $(A.4)$. *Then the following two conditions are equivalent*

(i) $\forall\ \xi\in\mathbb{L}^2(\mathcal{F}_T), \mathcal{E}_{g_1}(\xi)\ge\mathcal{E}_{g_2}(\xi)$
(ii) $\mathbb{P}$-*a.s.* $\forall\ t\in[0,T]\ \forall\ (y,z)\in\mathbb{R}\times\mathbb{R}^d,\ g_1(t,y,z)\ge g_2(t,y,z)$.

Proof. By the Comparison Theorem (see [8]), it is obvious that $(ii)\Rightarrow(i)$. We need to prove that $(i)\Rightarrow(ii)$. For each $\delta>0$ and $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, let

$$\tau_\delta=\tau_\delta(y,z)\triangleq\inf\{t\ge 0;\ g_1(t,y,z)\le g_2(t,y,z)-\delta\}\wedge T.$$

If (ii) does not hold, then there exists $\delta>0,\ (y,z)\in\mathbb{R}\times\mathbb{R}^d$ such that $\mathbb{P}(\tau_\delta<T)>0$. For such a triplet (δ,y,z), we consider the following SDE defined over the interval $[\tau_\delta,T]$

$$Y_t^j=y-\int_{\tau_\delta}^t g_j(s,Y_s^j,z)ds-\int_{\tau_\delta}^t h(s,Y_s^j,z)d\overline{B}_s+\int_{\tau_\delta}^t zdW_s.$$

It is well known that the above equation (for $j=1,2$) admits a unique solution Y^j. On the other hand, let define a new stopping time

$$\nu_\delta=\nu_\delta(y,z)\triangleq\inf\{t\ge\tau_\delta;\ g_1(t,Y_t^1,z)\ge g_2(t,Y_t^2,z)-\frac{\delta}{2}\}\wedge T.$$

Thanks to the assumption $(A.4)$, it follows that $\{\tau_\delta<\nu_\delta\}=\{\tau_\delta<T\}$ and so $\mathbb{P}(\tau_\delta<\nu_\delta)>0$. Using Tanaka's formula we can write

$$\begin{aligned} \left(Y_{\nu_\delta}^1-Y_{\nu_\delta}^2\right)^+ &= -\int_{\tau_\delta}^{\nu_\delta}\mathbb{I}_{(Y_s^1-Y_s^2)}\left\{g_1(s,Y_s^1,z)-g_2(s,Y_s^2,z)\right\}ds \\ &\quad -\int_{\tau_\delta}^{\nu_\delta}\mathbb{I}_{(Y_s^1-Y_s^2)}\left\{h(s,Y_s^1,z)-h(s,Y_s^2,z)\right\}d\overline{B}_s+L_{\nu_\delta}-L_{\tau_\delta}, \end{aligned}$$

where L is the local time associated with the $Y^1 - Y^2$. Taking expectation, we conclude that

$$\begin{aligned}\mathbb{E}\left(Y^1_{\nu_\delta} - Y^2_{\nu_\delta}\right)^+ &\geq -\mathbb{E}\int_{\tau_\delta}^{\nu_\delta} \mathbb{I}_{(Y^1_s - Y^2_s)}\left\{g_1(s, Y^1_s, z) - g_2(s, Y^2_s, z)\right\} ds \\ &\geq \frac{\delta}{2}\mathbb{E}(\nu_\delta - \tau_\delta) > 0\end{aligned}$$

and consequently $Y^1_{\nu_\delta} > Y^2_{\nu_\delta}$ on $\{\tau_\delta < \nu_\delta\}$. Now, since $(Y^j, z),\ j = 1, 2$ are solutions of BDSDEs (1.1) with data (Y^j_T, g_j, h), then we obtain from Proposition 2.2 (4) that

$$\mathcal{E}_{g_1}[Y^1_{\nu_\delta}\big|\mathcal{F}_{\tau_\delta}] = \mathcal{E}_{g_1}[Y^1_T\big|\mathcal{F}_{\tau_\delta}] = y = \mathcal{E}_{g_2}[Y^2_T\big|\mathcal{F}_{\tau_\delta}] = \mathcal{E}_{g_2}[Y^2_{\nu_\delta}\big|\mathcal{F}_{\tau_\delta}]$$

and then $\mathcal{E}_{g_1}(Y^1_{\nu_\delta}) = \mathcal{E}_{g_2}(Y^2_{\nu_\delta})$. Moreover, from Proposition 2.2 (3) and the assumption (i) it follows that $\mathcal{E}_{g_1}(Y^1_{\nu_\delta}) > \mathcal{E}_{g_1}(Y^2_{\nu_\delta}) \geq \mathcal{E}_{g_2}(Y^2_{\nu_\delta})$. Which is a contradiction. □

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