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ON THE EXISTENCE OF WEAK VARIATIONAL SOLUTIONS TO STOCHASTIC DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of weak variational solutions in a Gelfand triplet of real separable Hilbert spaces, under continuity, growth, and coercivity conditions on the coefficients of the stochastic differential equation. The laws of finite dimensional approximations are proved to weakly converge to the limit which is identified as a weak solution. The solution is an H–valued continuous process in $L_2(\Omega, C([0,T], H)) \cap L_2([0,T] \times \Omega, V)$. Under the assumption of monotonicity the solution is strong and unique.

1. Introduction

The study of stochastic PDE's using extensions of techniques developped by J.P. Lions [5] for the deterministic case was first undertaken by Viot [13], who investigated variational weak solutions under the assumption of compact embedding (see also Metivier and Viot [7]). Pardoux [8], and Krylov and Rozovskii [4] considered variational strong solutions. The first, using a deterministic result of Lions [5] (see also [12]), and the latter giving a stochastic extension of Lion's result.

Kallianpur and his collaborators [3] studied stochastic differential equations in the dual of a nuclear space, and constructed generalized solutions to SPDE's. Our purpose here is to adapt the techniques in [3] to produce a function space weak solution to the variational problem, as in [13], posed in a Gelfand triplet

$$V \hookrightarrow H \hookrightarrow V^*,$$

where V and H are real separable Hilbert spaces. The space V^* is the continuous linear dual Hilbert space, and all injections are continuous and dense. The norms and scalar products are denoted by $\langle \cdot, \cdot \rangle_V$, $\| \cdot \|_V$, and similar for the spaces H and V^*. The duality on $V \times V^*$ is denoted by $\langle \cdot, \cdot \rangle$ and it agrees with the scalar product in H, i.e. $\langle v, h \rangle = \langle v, h \rangle_H$ if $h \in H$.

We use the ideas in [2] and instead of the embeddings being Hilbert–Schmidt operators as in [3], we only assume their compactness. It should be noted that the weak solution X that we construct is in $C([0,T], H)$ with $X \in L_2([0,T] \times \Omega, V)$, and it satisfies

$$E \left(\sup_{t \in [0,T]} \|X(t)\|_H^2 \right) < \infty,$$

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as in [7], [8], and [13],.

We also show that under the assumption of monotonicity, we obtain a unique strong solution. We would like to note that the deep result of Lions is used to identify the solution in $C([0,T],H)$.

Now let K be another real separable Hilbert space. Denote by $\mathcal{L}_1(K)$ the space of trace–class operators on K. Let $Q \in \mathcal{L}_1(K)$ be a symmetric non–negative definite operator and $\{W_t, t \geq 0\}$ be a K-valued Q–Wiener process defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

We can assume that the eigenvalues of Q, are all positive, $\lambda_j > 0$, $j = 1, 2, ...$, otherwise we can start with the Hilbert space ker $(Q)^\perp$ instead of K. The associated eigenvectors forming an ONB in K will be denoted by f_k.

Then the space $K_Q = Q^{1/2}K$ equipped with a scalar product

$$\langle u, v\rangle_{K_Q} = \sum_{j=1}^{\infty} \frac{1}{\lambda_j} \langle u, f_k\rangle_K \langle v, f_k\rangle_K$$

is a separable Hilbert space with an ONB $\left\{\lambda_j^{1/2} f_j\right\}_{j=1}^{\infty}$.

We denote by $\mathcal{L}_2(K_Q, H)$, the space of Hilbert–Schmidt operators from K_Q to H. If $\{\varphi_j\}_{j=1}^{\infty}$ is an ONB in H then the Hilbert–Schmidt norm of an operator $L \in \mathcal{L}_2(K_Q, H)$ is given by

$$\begin{aligned} \|L\|^2_{\mathcal{L}_2(K_Q,H)} &= \sum_{j,i=1}^{\infty} \left\langle L\left(\lambda_j^{1/2} f_j\right), \varphi_i\right\rangle_H^2 = \sum_{j,i=1}^{\infty} \left\langle LQ^{1/2} f_j, \varphi_i\right\rangle_H^2 \\ &= \|LQ^{1/2}\|^2_{\mathcal{L}_2(K,H)} = \text{tr}\,(LQL^*). \end{aligned}$$

Consider the following SDE

$$dX(t) \quad = A(t, X(t))dt + B(t, X(t))dW_t \tag{1.1}$$

with the coefficients

$$A : [0,T] \times V \to V^* \quad \text{and} \quad B : [0,T] \times V \to \mathcal{L}_2(K_Q, H),$$

and an H–valued $\mathcal{F}_0$–measurable initial condition $\xi_0 \in L^2(\Omega, H)$.

Note that $B(t,v)QB^*(t,v) \in \mathcal{L}_1(H)$. Throughout this presentation we will make the following assumption

(JC) (Joint Continuity) The mappings

$$(t,v) \to A(t,v) \in V^* \;\text{ and }\; (t,v) \to B(t,v)QB^*(t,v) \in \mathcal{L}_1(H) \tag{1.2}$$

are continuous.

Let us now define a weak variational solution to Equation (1.1).

Definition 1.1. A *weak variational solution* of Equation (1.1) is a system

$$((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\leq T}, P), W, X),$$

where W_t is a K-valued Q–Wiener process with respect to the filtration $\mathcal{F}_t$, X is an H–valued process adapted to $\mathcal{F}_t$ and satisfies the following conditions

(1) $E\int_0^T \|X(t)\|_V^2\, dt < \infty$

(2) $P\left(\int_0^T \|A(t,X(t))\|_{V^*}\,dt < \infty\right) = 1$

(3) $\int_0^t B(s,X(s))\,dW_s$ is a square integrable H-valued martingale

(4) $X(t) = X(0) + \int_0^t A(s,X(s))\,ds + \int_0^t B(s,X(s))\,dW_s$, P–a.s.

(5) $P \circ (X(0))^{-1} = \mathcal{L}(\xi_0)$.

Here the integrants $A(t,X(t))$ and $B(t,X(t))$ are evaluated at a V–valued $\mathcal{F}_t$–measurable version of $X(t)$ in $L^2([0,T]\times\Omega, V)$.

We will study the existence problem for coefficients A and B satisfying the following growth conditions

(G-A)

$$\|A(t,v)\|_{V^*}^2 \le \theta\left(1+\|v\|_H^2\right), \; v \in V \tag{1.3}$$

(G-B)

$$\operatorname{tr}\left(B(t,v)QB^*(t,v)\right) \le \theta\left(1+\|v\|_H^2\right), \; v \in V. \tag{1.4}$$

In addition we will impose the following coercivity condition on A and B

(C) There exist constants $\alpha > 0$, $\gamma, \lambda \in \mathbb{R}$ such that

$$2\langle A(t,v),v\rangle + \operatorname{tr}\left(B(t,v)QB^*(t,v)\right) \le \lambda\|v\|_H^2 - \alpha\|v\|_V^2 + \gamma. \tag{1.5}$$

2. Existence of Weak Solutions under Compact Embedding

We will make one more assumption on the initial condition of Equation (1.1),

(IC)

$$E\left\{\|\xi_0\|_H^2\left(\ln\left(3+\|\xi_0\|_H^2\right)\right)^2\right\} < c_0, \tag{2.1}$$

for some constant c_0. It will become clear that this property will be used to ensure uniform integrability of the squared norm of the approximate solutions.

We will first consider a finite dimensional SDE related to the infinite dimensional Equation (1.1). Let $\{\varphi_j\}_{j=1}^\infty \subset V$ be a complete orthonormal system in H and $\{f_k\}_{k=1}^\infty$ be a complete orthonormal system in K. Define a map $J_n : \mathbb{R}^n \to V$ by

$$J_n(x) = \sum_{j=1}^n x_j\varphi_j = u \in V,$$

and coefficients $(a^n(t,x))_j : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n$, $(b^n(t,x))_{i,j} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$, and $(\sigma^n(t,x))_{i,j} : [0,T]\times\mathbb{R}^n \to \mathbb{R}^n\times\mathbb{R}^n$, and the initial condition ξ_0^n by

$$\begin{aligned}
(a^n(t,x))_j &= \langle\varphi_j, A\left(t,J_nx\right)\rangle, \quad 1\le j\le n\\
(b^n(t,x))_{i,j} &= \left\langle Q^{1/2}B^*\left(t,J_nx\right)\varphi_i, f_j\right\rangle_K \quad 1\le i,j\le n\\
(\sigma^n(t,x))_{i,j} &= \left(b^n(t,x)\left(b^n(t,x)\right)^T\right)_{i,j}\\
(\xi_0^n)_j &= \langle\xi_0,\varphi_j\rangle_H.
\end{aligned} \tag{2.2}$$

Note that

$$\begin{aligned}(\sigma^n(t,x))_{i,j} &= \sum_{k=1}^{n} (b^n(t,x))_{i,k} (b^n(t,x))_{j,k} \\ &= \sum_{k=1}^{n} \left\langle Q^{1/2}B^*(t,J_nx)\varphi_i, f_k\right\rangle_K \left\langle Q^{1/2}B^*(t,J_nx)\varphi_j, f_k\right\rangle_K .\end{aligned}$$

The following lemma is a direct consequence of the assumptions (1.3)–(1.5) (see Section 5 in [3] for simple calculations).

Lemma 2.1. *The growth conditions* (1.3) *and* (1.4) *assumed for the coefficients A and B imply the following growth conditions on a^n and b^n,*

$$\|a^n(t,x)\|^2_{\mathbb{R}^n} \le \theta_n \left(1+\|x\|^2_{\mathbb{R}^n}\right), \tag{2.3}$$

$$\operatorname{tr}(\sigma^n(t,x)) = \operatorname{tr}\left(b^n(t,x)\,(b^n(t,x))^T\right) \le \theta\left(1+\|x\|^2_{\mathbb{R}^n}\right). \tag{2.4}$$

In addition, for $k \ge n$, and $x \in \mathbb{R}^k$, the following estimate holds true,

$$\sum_{j=1}^{n} \left(\left(a^k(t,x)\right)_j\right)^2 \le \theta_n \left(1+\|x\|^2_{\mathbb{R}^k}\right). \tag{2.5}$$

The coercivity condition (1.5) *implies that*

$$\begin{aligned}&2\langle a^n(t,x),x\rangle_{\mathbb{R}^n} + \operatorname{tr}\left(b^n(t,x)\,(b^n(t,x))^T\right)\\ &\quad\le 2\langle A(t,J_nx),J_nx\rangle + \operatorname{tr}\left(B(t,J_nx)QB^*(t,J_nx)\right)\\ &\quad\le \lambda\|J_nx\|^2_H - \alpha\|J_nx\|^2_V + \gamma.\end{aligned} \tag{2.6}$$

In particular, for a large enough value of θ, the coercivity condition (1.5) *implies that*

$$2\langle a^n(t,x),x\rangle_{\mathbb{R}^n} + \operatorname{tr}\left(b^n(t,x)\,(b^n(t,x))^T\right) \le \theta\left(1+\|x\|^2_{\mathbb{R}^n}\right). \tag{2.7}$$

The constant θ_n depends on n, but θ does not. The distribution μ_0^n of ξ_0^n on $\mathbb{R}^n$ satisfies

$$E\left\{\|\xi_0^n\|^2_{\mathbb{R}^n}\left(\ln\left(3+\|\xi_0^n\|^2_{\mathbb{R}^n}\right)\right)^2\right\} < c_0. \tag{2.8}$$

We will need the following result, Theorem V.3.10 in [1], on the existence of a weak solution. Consider the following finite dimensional SDE,

$$dX(t) \quad = a(t,X(t))dt + b(t,X(t))\,dB_t^n \tag{2.9}$$

with an $\mathbb{R}^n$–valued $\mathcal{F}_0$–measurable initial condition ξ_0^n. Here, B_t^n is a standard Brownian motion in $\mathbb{R}^n$.

Theorem 2.2. *There exists a weak solution to equation* (2.9) *if $a : [0,\infty]\times\mathbb{R}^n \to \mathbb{R}^n$, $b : [0,\infty]\times\mathbb{R}^n \to \mathbb{R}^n\otimes\mathbb{R}^n$ are continuous and satisfy the following growth condition*

$$\|b(t,x)\|^2_{L(\mathbb{R}^n)} + \langle x, a(t,x)\rangle_{\mathbb{R}^n} \le K\left(1+\|x\|^2_{\mathbb{R}^n}\right) \tag{2.10}$$

for $t \ge 0$ and $x \in \mathbb{R}^n$ and some constant K.

We will use the ideas developed in [11], Section 1.4, for proving compactness of probability measures on $C([0,T],\mathbb{R}^n)$. The method was adapted to the specific case involving linear growth and coercivity conditions in [3]. Our first step in proving the existence result in the variational problem will be establishing the existence and properties of finite dimensional Galerkin approximations in the following lemma, which relies on the techniques in [3].

Lemma 2.3. *Assume the coefficients A and B of Equation* (1.1) *satisfy the assumptions of joint continuity* (1.2)*, growth* (1.3)*,* (1.4)*, and coercivity* (1.5)*, and that the initial condition ξ_0 satisfies* (2.1)*. Let a^n, b^n, and ξ_0^n be defined as in* (2.2)*, and B_t^n be an n–dimensional standard Brownian motion. Then the finite dimensional equation*

$$dX(t) = a^n(t, X(t))dt + b^n(t, X(t))dB_t^n \tag{2.11}$$

with an initial condition ξ_0^n has a weak solution in $C\left([0,T],\mathbb{R}^n\right)$. The laws $\mu^n = P \circ X^{-1}$ have the property that for any $R > 0$,

$$\begin{aligned} \sup\nolimits_n \mu^n \Big\{ x \in C([0,T],\mathbb{R}^n): \quad \sup\nolimits_{0\le t\le T} \|x(t)\|_{\mathbb{R}^n} > R \Big\} \\ < \; 2c_0 e^{C(\theta)T} \big/ \left(1+R^2\right) \left(\ln\left(3+R^2\right)\right)^2, \end{aligned} \tag{2.12}$$

and

$$\int_{C([0,T],\mathbb{R}^n)} \sup_{0\le t\le T} \left(1+\|x(t)\|^2_{\mathbb{R}^n}\right) \ln\ln\left(3+\|x(t)\|^2_{\mathbb{R}^n}\right)\, \mu^n(dx) < C \tag{2.13}$$

for some constant C.

Proof. Since the coefficients a^n, b^n satisfy conditions (2.3) and (2.4), we can use Theorem 2.2 to construct a weak solution $X^n(t)$ to equation (2.11) for every n. Let

$$f(x) = \left(1+\|x\|^2_{\mathbb{R}^n}\right)\left(\ln\left(3+\|x\|^2_{\mathbb{R}^n}\right)\right)^2, \quad x \in \mathbb{R}^n,$$

then

$$\|f_x(x)\|^2_{\mathbb{R}^n} \le C f(x) \left(\ln\left(3+\|x\|^2_{\mathbb{R}^n}\right)\right)^2. \tag{2.14}$$

Define for $g \in C_0^2(\mathbb{R}^n)$ a differential operator

$$\left(L_t^n g\right)(x) = \sum_{i=1}^n \frac{\partial g}{\partial x_i}(x)(a^n(t,x))_i + \frac{1}{2}\sum_{i=1}^n\sum_{j=1}^n \frac{\partial^2 g}{\partial x_i \partial x_j}(x)(\sigma^n(t,x))_{i,j}.$$

It follows by straightforward calculations that the coercivity condition (2.7) implies

$$L_t^n f(x) \le C f(x). \tag{2.15}$$

Using Itô's formula for the function $f(x)$, we have

$$f\left(X^n(t)\right) = f(X^n(0)) + \int_0^t L_s^n f\left(X^n(s)\right) ds + M_t, \tag{2.16}$$

where M_t is a local martingale. Define a stopping time

$$\tau_R = \inf\left\{t : \|X^n(t)\|_{\mathbb{R}^n} > R\right\}, \qquad \text{or } T. \tag{2.17}$$

Then $M_{t\wedge\tau_R}$ is a square integrable martingale with an increasing process

$$\begin{aligned}
\langle M\rangle_{t\wedge\tau_R} &= \int_0^{t\wedge\tau_R} \|b^n(s,X^n(s))f_x(X^n(s))\|^2_{\mathbb{R}^n}\ ds \\
&\leq C\theta\int_0^{t\wedge\tau_R}\left(1+\|X^n(s)\|^2_{\mathbb{R}^n}\right) f(X^n(s))\left(\ln\left(3+\|X^n(s)\|_{\mathbb{R}^n}\right)\right)^2\ ds \\
&= C\theta\int_0^{t\wedge\tau_R} f^2(X^n(s))\,ds \\
&\leq C\theta\left(\sup_{0\leq s\leq t\wedge\tau_R} f(X^n(s))\right)\int_0^{t\wedge\tau_R} f(X^n(s))\,ds,
\end{aligned}$$

where we have applied (2.14).

Using Burkholder's inequality, we calculate

$$\begin{aligned}
&E\left(\sup_{0\leq s\leq t}|M_{t\wedge\tau_R}|\right)\leq 4E\left(\langle M\rangle_{t\wedge\tau_R}\right)^{1/2} \\
&\leq 4(C\theta)^{1/2}E\left\{\left(\sup_{0\leq s\leq t} f(X^n(s\wedge\tau_R)\right)^{1/2}\left(\int_0^{t\wedge\tau_R} f(X^n(s))\,ds\right)^{1/2}\right\} \\
&\leq E\left\{\left(\sup_{0\leq s\leq t} f(X^n(s\wedge\tau_R)\right)^{1/2}\left(16C\theta\int_0^t\sup_{0\leq r\leq s} f(X^n(r\wedge\tau_R))\,ds\right)^{1/2}\right\} \\
&\leq \frac{1}{2}E\left\{\left(\sup_{0\leq s\leq t} f(X^n(s\wedge\tau_R)\right)+\left(16C\theta\int_0^t\sup_{0\leq r\leq s} f(X^n(r\wedge\tau_R))\,ds\right)\right\}.
\end{aligned}$$

Then, by (2.15) and (2.16),

$$f\left(X^n\left(s\wedge\tau_R\right)\right)\leq f(X^n(0))+C\int_0^s f\left(X^n\left(r\wedge\tau_R\right)\right)\,dr+M_{s\wedge\tau_R},$$

hence,

$$E\sup_{0\leq s\leq t} f\left(X^n(s\wedge\tau_R)\right) \quad\leq\quad 2c_0+(2C+16C\theta)E\int_0^t\sup_{0\leq r\leq s} f(X^n(r\wedge\tau_R)),ds.$$

By applying Gronwall's inequality, we obtain a bound

$$E\sup_{0\leq s\leq t} f\left(X^n\left(s\wedge\tau_R\right)\right)\leq 2c_0e^{(2C+16C\theta)T},\quad 0\leq t\leq T.$$

Then

$$\begin{aligned}
P\left(\sup_{0\leq s\leq t}\|X^n(s)\|_{\mathbb{R}^n}>R\right) &\leq E\left(\sup_{0\leq s\leq t} f\left(X^n(s\wedge\tau_R)\right)\big/ f(R)\right) \\
&\leq 2c_0e^{C(\theta)T}\big/\left(1+R^2\right)\left(\ln\left(3+R^2\right)\right)^2,
\end{aligned}$$

proving (2.12). To prove (2.13), denote $g(r)=(1+r^2)\ln\ln(3+r^2)$, $r\geq 0$. Since g is increasing, we have

$$\begin{aligned}
&\int_{C([0,T],\mathbb{R}^n)}\sup_{0\leq t\leq T}\left(1+\|x(t)\|^2_{\mathbb{R}^n}\right)\ln\ln\left(3+\|x(t)\|^2_{\mathbb{R}^n}\right)\ \mu^n(dx) \\
&\quad=\int_{C([0,T],\mathbb{R}^n)}\sup_{0\leq t\leq T} g\left(\|x(s)\|_{\mathbb{R}^n}\right)\ \mu^n(dx)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mu^n \left(\sup_{0\le t\le T} g\left(\|x(s)\|_{\mathbb{R}^n}\right) > p \right) dp \\
&= \ln\ln 3 + \int_{\ln\ln 3}^\infty \mu^n \left(\sup_{0\le t\le T} \|x(s)\|_{\mathbb{R}^n} > g^{-1}(p) \right) dp \\
&\le \ln\ln 3 + \int_0^\infty \mu^n \left(\sup_{0\le t\le T} \|x(s)\|_{\mathbb{R}^n} > r \right) g'(r)\, dr \\
&\le \ln\ln 3 + 2c_0 e^{C(\theta)T} \int_0^\infty \frac{g'(r)}{(1+r^2)\left(\ln(3+r^2)\right)^2}\, dr < \infty,
\end{aligned}
$$

with the very last inequality being easy to verify (see [3]). □

We will need the following lemma from [3] (see also Sections 1.2–1.4 in [11]).

Lemma 2.4. *Consider the filtered probability space*

$$\left(C([0,T],\mathbb{R}^n), \mathcal{B}, \{\mathcal{C}_t\}_{0\le t\le T}, P \right),$$

where $\mathcal{B}$ is the Borel σ–field and $\mathcal{C}_t$ is the σ–field genereted by the cylinders with bases over $[0,t]$. Let the coordinate process m_t be a $\mathcal{C}_t$ square integrable martingale with its quadratic variation $\ll m \gg_t$ satisfying

$$\langle m\rangle_t - \langle m\rangle_s = \operatorname{tr}\left(\ll m \gg_t - \ll m \gg_s\right) \le \beta(t-s),$$

for some constant β and all $0\le s<t\le T$. Then for every $\varepsilon,\eta>0$, there exists $\delta>0$, depending possibly on β, ε, η, and T, but not on n, such that

$$P\left(\sup_{|t-s|<\delta} \|m_t - m_s\|_{\mathbb{R}^n} > \varepsilon \right) < \eta.$$

Now we will use the compact embedding argument, as in [2]. We point out the critical calculations in the second part of the proof adapted from [3].

Theorem 2.5. *Let the coefficients A, B of Equation* (1.1) *satisfy conditions* (1.2), (1.3), (1.4), *and* (1.5). *Consider the family of measures μ_*^n on $C([0,T],V^*)$, with support in $C([0,T],H)$, defined by*

$$\mu_*^n(Y) = \mu^n \left\{ x \in C([0,T],\mathbb{R}^n) : \sum_{i=1}^n x_i(t)\varphi_i \in Y \right\}; \quad Y \subset C([0,T],V^*),$$

where μ^n are the measures constructed in Lemma 2.3. Assume that the embedding $H \hookrightarrow V^*$ is compact. Then the family of measures $\{\mu_*^n\}_{n=1}^\infty$ is tight on $C([0,T],V^*)$.

Proof. We will use the tightness criterion in $C([0,T],V^*)$, (see e.g. Theorem 5 in [2]). Denote by $B^R_{C([0,T],H)} \subset C([0,T],H)$ the closed ball of radius R cenetred at the origin. By the definition of measures μ_*^n and Lemma 2.3, for any $\eta>0$, we can choose $R>0$, such that

$$\mu_*^n\left\{ \left(B^R_{C([0,T],H)} \right)^c \right\} \quad = \quad \mu^n \left\{ x \in C([0,T],\mathbb{R}^n) : \sup_{0\le t\le T} \left\| \sum_{i=1}^n x_i(t)\varphi_i \right\|_H > R \right\}$$

$$= \quad \mu^n \left\{ x \in C([0,T],\mathbb{R}^n) : \sup_{0\le t\le T} \|x(t)\|_{\mathbb{R}^n} > R \right\} < \eta.$$

Denote the closed ball of radius R centered at zero in H by B_H^R. Then its closure in V^*, denoted by $\overline{B_H^R}^{V^*}$, is a compact subset of V^*, and we have,

$$\mu_*^n \circ x(t)^{-1} \left(\overline{B_H^R}^{V^*} \right) \ge 1-\eta,\ 0 \le t \le T,$$

fulfilling the first condition for tightness.

Again, using the compactness of $\overline{B_H^R}^{V^*}$ in V^*, for any $\varepsilon > 0$ we can find an index $n_0 \ge 1$, such that

$$\left\| \sum_{j=n_0+1}^{\infty} x_j \varphi_j \right\|_{V^*} < \varepsilon/4, \quad \text{if } \|x\|_H \le R. \tag{2.18}$$

Since the embedding $H \hookrightarrow V^*$ is continuous and linear, we have $\|x\|_{V^*} \le C\|x\|_H$ for $x \in H$ and some constant C, independent of η and R, so that

$$\left\| \sum_{j=1}^{n_0} (x_j - y_j)\,\varphi_j \right\|_{V^*} \le C \left\| \sum_{j=1}^{n_0} (x_j - y_j)\,\varphi_j \right\|_{H}.$$

Consider the modulus of continuity of a function $x \in C([0,T],G)$ defined by

$$w_G(x,\delta) = \sup_{\substack{0\le s,t\le T\\ |t-s|<\delta}} \|x(t)-x(s)\|_G,$$

where G is a Hilbert space. Then, with $B^R_{C([0,T],\mathbb{R}^n)}$ denoting the closed ball with radius R centered at the origin in $C([0,T],\mathbb{R}^n)$, and $n > n_0$,

$$\begin{aligned}
&\mu_*^n \left\{ x \in C([0,T],V^*) : x \in B^R_{C([0,T],H)} : w_{V^*}(x,\delta) > \varepsilon \right\} \\
&\quad \le \mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : w_{V^*}\left(\sum_{j=1}^{n} (x(\cdot))_j \varphi_j, \delta \right) > \varepsilon \right\} \\
&\quad \le \mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : \sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} \left\| \sum_{j=1}^{n_0} \left((x(t))_j - (x(s))_j \right) \varphi_j \right\|_{V^*} \right. \\
&\qquad \left. + \sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} \left\| \sum_{j=n_0+1}^{n} \left((x(t))_j - (x(s))_j \right) \varphi_j \right\|_{V^*} > \varepsilon \right\} \\
&\quad \le \mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : C \sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} \left\| \sum_{j=1}^{n_0} \left((x(t))_j - (x(s))_j \right) \varphi_j \right\|_{H} + \varepsilon/4 > \varepsilon \right\}
\end{aligned}$$

$$= \mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : \sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} \left(\sum_{j=1}^{n_0} \left((x(t))_j - (x(s))_j\right)^2 \right)^{1/2} > 3\varepsilon/(4C) \right\}.$$

For the stopping time $\tau_R = \inf\left\{0 \le t \le T : x(t) \notin B^R_{C([0,T],\mathbb{R}^n)}\right\}$ or T, the $\mathbb{R}^n$–valued martingale

$$m^R_t(x) = x(t\wedge\tau_R) - \int_0^{t\wedge\tau_R} a^n(s,x(s))\,ds$$

has quadratic variation process given by

$$\ll m^R(x) \gg_t = \int_0^{t\wedge\tau_R} b^n(s,x(s))\,(b^n(s,x(s))^T\ ds,$$

with the function tr $\left(b^n(s,x(s))\,(b^n(s,x(s))^T\right)$ bounded on bounded subsets of $\mathbb{R}^n$ uniformly relative to the variable s, due to condition (2.4). Hence, for $t \ge s$,

$$\langle m^R(x)\rangle_t - \langle m^R(x)\rangle_s = \mathrm{tr}\left(\ll m^R(x) \gg_t - \ll m^R(x) \gg_s\right) \le \beta(R)(t-s),$$

with the constant $\beta(R)$ not depending on n. Now, by Lemma 2.4, we have

$$\mu^n\left(x \in C([0,T],\mathbb{R}^n) : w_{\mathbb{R}^n}(m^R(x),\delta) > \varepsilon/(2C)\right) < \eta \tag{2.19}$$

for sufficiently small δ, idependent of n.

Let $n \ge n_0$ and $\sup_{0\le t\le T}\|x(t)\|_{\mathbb{R}^n} \le R$. Using (2.5), we have

$$\left(\sum_{j=1}^{n_0}\left((a^n(t,x(t)))_j\right)^2\right)^{1/2} \le \theta_{n_0}\left(1+R^2\right),$$

hence, for a sufficiently small constant δ, we can write

$$\left(\sum_{j=1}^{n_0}\left(\int_s^t (a^n(t,x(t)))_j\right)^2\right)^{1/2} \le \varepsilon/(4C), \quad \text{whenever } |t-s|<\delta.$$

Also, whenever $\sup_{0\le t\le T}\|x(t)\|_{\mathbb{R}^n} \le R$,

$$m^n_t = x(t) - \int_0^t a^n(s,x(s))\,ds = m^R_t.$$

We can continue our calculations as follows

$$\mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : \sup_{\substack{0\le s,t\le T\\ |s-t|<\delta}} \left(\sum_{j=1}^{n_0} \left((x(t))_j - (x(s))_j\right)^2 \right)^{1/2} > 3\varepsilon/(4C) \right\}$$

$$\le \mu^n \left\{ x \in B^R_{C([0,T],\mathbb{R}^n)} : w_{\mathbb{R}^n}(m^n,\delta) + w_{\mathbb{R}^{n_0}}\left(\int_0^\cdot a^n(s,x(s))\,ds,\delta\right) > 3\varepsilon/(4C)\right\}$$

$$\leq \mu^n\left\{x\in B^R_{C([0,T],\mathbb{R}^n)}:\, w_{\mathbb{R}^n}(m^n,\delta)>\frac{\varepsilon}{2C}\right\}$$
$$\leq \mu^n\left\{w_{\mathbb{R}^n}(m^R,\delta)>\frac{\varepsilon}{2C}\right\}\leq\eta.$$

Summarizing, for any $\varepsilon,\eta>0$, and sufficiently small $\delta>0$, there exists n_0, such that for $n>n_0$,

$$\begin{aligned}&\mu^n_*\left\{x\in C([0,T],V^*):\, w_{V^*}(x,\delta)>\varepsilon\right\}\\ &\quad\leq \mu^n_*\left\{\left(B^R_{C([0,T],H)}\right)^c\right\}+\mu^n_*\left\{x\in B^R_{C([0,T],H)}:\, w_{V^*}(x,\delta)>\varepsilon\right\}\\ &\quad\leq 2\eta,\end{aligned}$$

concluding the proof. □

We will now summarize the desired properties of the measures μ^n and μ^n_*.

Corollary 2.6. *Let $X^n(t)$ be solutions to Equation (2.11), μ^n be their laws in $C([0,T],\mathbb{R}^n)$, and μ^n_* be the the measures induced in $C([0,T],V^*)$ as in Theorem 2.5. Then for some constant C independent of n,*

$$\int_{C([0,T],\mathbb{R}^n)}\sup_{0\leq t\leq T}\|x(t)\|^2_{\mathbb{R}^n}\ln\ln\left(3+\|x(t)\|^2_{\mathbb{R}^n}\right)\,\mu^n(dx)<C, \tag{2.20}$$

implying uniform integrability of $\|X^n\|^2_{\mathbb{R}^n}$. The H–norms satisfy the following properties,

$$\begin{aligned}&\int_{C([0,T],V^*)}\sup_{0\leq t\leq T}\|x(t)\|^2_H\ln\ln\left(3+\|x(t)\|^2_H\right)\,\mu^n_*(dx)\\ &\quad= E\left(\sup\nolimits_{0\leq t\leq T}\|J_nX^n(t)\|^2_H\ln\ln\left(3+\|J_nX^n(t)\|^2_H\right)\right)<C.\end{aligned} \tag{2.21}$$

For a cluster point μ_* of the tight sequence μ^n_*

$$\int_{C([0,T],V^*)}\sup_{0\leq t\leq T}\|x(t)\|^2_H\,\mu_*(dx)<C. \tag{2.22}$$

There exists a constant C such that for any $R>0$

$$\mu_*\left\{x\in C([0,T],V^*):\,\sup_{0\leq t\leq T}\|x(t)\|_H>R\right\}<C/R^2, \tag{2.23}$$

and also,

$$\mu_*\left\{x\in C([0,T],V^*):\,\sup_{0\leq t\leq T}\|x(t)\|_H<\infty\right\}=1. \tag{2.24}$$

Finally, the V–norms satisfy

$$\int_{C([0,T],V^*)}\int_0^T\|x(t)\|^2_V\,dt\,\mu^n_*(dx)<C. \tag{2.25}$$

and

$$\int_{C([0,T],V^*)}\int_0^T\|x(t)\|^2_V\,dt\,\mu_*(dx)<\infty. \tag{2.26}$$

Proof. Property (2.20) is just (2.13) and Inequality (2.21) is just a restatement of (2.20).

To prove (2.22) assume, using the Skorokhod theorem, that $J_n X_n \to X$ a.s. in $C([0,T],V^*)$. We introduce a function $\alpha_H : V^* \to \mathbb{R}$,

$$\alpha_H(u) = \sup\left\{\langle u, v\rangle,\ v \in V,\ \|v\|_H \le 1\right\}.$$

Clearly $\alpha_H(u) = \|u\|_H$ if $u \in H$, and it is a lower semicontinuous function as a supremum of continuous functions. For $u \in V^* \setminus H$, $\alpha_H(u) = +\infty$. Thus, we can extend the norm $\|\cdot\|_H$ to a lower semicontinuous function on V^*.

By the Fatou lemma and (2.20),

$$\begin{aligned}
&\int_{C([0,T],V^*)} \sup_{0\le t\le T} \|x(t)\|_H^2 \, \mu_*(dx) \\
&= E\left(\sup_{0\le t\le T} \|X(t)\|_H^2\right) \\
&\le E \liminf_{n\to\infty} \left(\sup_{0\le t\le T} \|J_n X^n(t)\|_H^2\right) \\
&\le \liminf_{n\to\infty} E \left(\sup_{0\le t\le T} \|J_n X^n(t)\|_H^2\right) \\
&= \liminf_{n\to\infty} \int_{C([0,T],V^*)} \sup_{0\le t\le T} \|x(t)\|_H^2 \, \mu_*^n(dx) < C.
\end{aligned}$$

The property (2.23) follows from the Markov inequality and (2.24) is a consequence of (2.22). To prove 2.25, we apply the Itô formula and (2.6) to obtain that

$$\begin{aligned}
E\|J_n X^n(t)\|_H^2 &= E\|J_n \xi_0^n\|_H^2 + 2E\int_0^t \langle a^n(s, X^n(s)), X^n(s)\rangle_{\mathbb{R}^n}\, ds \\
&\quad + E\int_0^t \operatorname{tr}\left(b^n(s, X^n(s))\,(b^n(s, X^n))^T\right) ds \\
&\le E\|J_n \xi_0\|_H^2 + \lambda \int_0^t E\|J_n X^n(s)\|_H^2\, ds \\
&\quad -\alpha \int_0^t E\|J_n X^n(s)\|_V^2\, ds + \gamma.
\end{aligned}$$

Using the bound in (2.21) we conclude that

$$\sup_n \int_0^T E\|J_n X^n(t)\|_V^2\, dt < \infty.$$

Finally, we can extend the norm $\|\cdot\|_V$ to a lower semicontinuous function on V^* by introducing a lower semicontinuous function

$$\alpha_V(u) = \sup\left\{\langle u, v\rangle,\ v \in V,\ \|v\|_V \le 1\right\},$$

since $\alpha_V(u) = \|u\|_V$ if $u \in V$, and for $u \in V^* \setminus V$, $\alpha_V(u) = +\infty$. Now (2.26) follows by the Fatou lemma. □

We need the following deep result, which is a stochastic extention of a lemma of Lions. This result is stated in [4], Theorem I.3.1. and a detailed proof is given in [9], Theorem 4.2.5.

Theorem 2.7. *Let $X(0) \in L_2(\Omega, \mathcal{F}_0, P, H)$ and $Y \in L_2([0,T] \times \Omega, V^*)$, $Z \in L_2([0,T] \times \Omega, \mathcal{L}_2(K_Q, H))$ be both progressively measurable. Define a continuous V^*–valued process*

$$X(t) = X(0) + \int_0^t Y(s)\, ds + \int_0^t Z(s)\, dW_s, \quad t \in [0,T].$$

If for its $dt \otimes P$–equivalence class $\hat{X}$ we have $\hat{X} \in L_2([0,T] \times \Omega, V)$, then X is an H–valued continuous $\mathcal{F}_t$–adapted process,

$$E\left(\sup_{t \in [0,T]} \|X(t)\|_H^2\right) < \infty$$

and the following Itô formula holds for the square of its H–norm P–a.s.

$$\begin{aligned} \|X(t)\|_H^2 &= \|X(0)\|_H^2 + \int_0^t \left(2\langle \bar{X}(s), Y(s)\rangle + \|Z(s)\|^2_{\mathcal{L}_2(K_Q,H)}\right) ds \\ &\quad + 2\int_0^t \langle X(s), Z(s)\, dW_s\rangle_H, \quad t \in [0,T] \end{aligned} \tag{2.27}$$

for any V–valued progressively measurable version $\bar{X}$ of $\hat{X}$.

Remark 2.8. Note that a V–valued progressively measurable version in Theorem 2.7 exists, see [9], Remark 4.2.2.

We are now ready to formulate the existence theorem.

Theorem 2.9. *Let $V \hookrightarrow H \hookrightarrow V^*$ be a Gelfand triplet with compact inclusions. Let the coefficients A, B of Equation (1.1) satisfy conditions (1.2), (1.3), (1.4), and (1.5). Let the initial condition ξ_0 be an H–valued random variable satisfying (2.1). Then Equation (1.1) has a weak solution $X(t)$ in $C([0,T], H)$, such that*

$$E\left(\sup_{0 \le t \le T} \|X(t)\|_H^2\right) < C, \tag{2.28}$$

and

$$E\int_0^T \|X(t)\|_V^2\, dt < \infty. \tag{2.29}$$

Proof. Let $X^n(t)$ be solutios to Equation (2.11), μ^n be their laws in $C([0,T], \mathbb{R}^n)$, and μ^n_* be the measures induced in $C([0,T], V^*)$ as in Theorem 2.5, with a cluster point μ_*. We need to show that μ_* is the law of a weak solution to Equation (1.1). Again, using the Skorokhod theorem, assume that $J_n X^n(t)$, $X(t)$ are processes with laws μ^n_* and μ_*, respectively, with $J_n X^n \to X$, P–a.s. By (2.21), (2.24)–(2.26) $J_n X^n$ and X are P–a.s. in $C([0,T], V^*) \cap L_\infty([0,T], H) \cap L_2([0,T], V)$.

Denote by $\{\varphi_j\}_{j=1}^\infty \subset V$ a complete orthogonal system in V, which is an ONB in H. Note that such a system always exists, since, due to the compactness of the embedding $V \hookrightarrow H$, the canonical isomorphism $I : V^* \to V$ takes a unit ball in V^* to a subset of the unit ball in V, which is relatively compact in H. Thus I restricted to H is a self–adjoint compact operator. One can choose a common orthogonal system $\{\varphi_n\}_{n=1}^{\infty}$ for V and H by selecting the unit eigenvectors of I, since

$$\langle \varphi_n, \varphi_m \rangle_H = \langle \varphi_n, \varphi_m \rangle = \langle I\varphi_n, \varphi_m \rangle_V = \lambda_n \langle \varphi_n, \varphi_m \rangle_V .$$

Then, the vectors $\psi_j = \varphi_j / \|\varphi_j\|_V$ form an ONB in V.

For $x \in C([0,T], V^*) \cap L_\infty([0,T], H) \cap L_2([0,T], V)$, consider

$$M_t(x) = x(t) - x(0) - \int_0^t A(s, x(s))\, ds.$$

Using (1.3), and (2.21), we have for any $v \in V$, and some constant C,

$$\begin{aligned} &\int \left(\langle A(s,x(s)), v\rangle^2 \ln\ln\left(3 + \|x(s)\|_H^2\right)\right) \mu_*^n(dx) \\ &\le \int \left(\|A(s,x(s))\|_{V^*}^2 \|v\|_V^2 \ln\ln\left(3 + \|x(s)\|_H^2\right)\right) \mu_*^n(dx) \\ &\le \int \theta \left(1 + \|x(s)\|_H^2\right) \ln\ln\left(3 + \|x(s)\|_H^2\right) \|v\|_V^2 \, \mu_*^n(dx) \\ &\le C\theta \|v\|_V^2, \end{aligned} \tag{2.30}$$

and, in a similar fashion, adding (2.22) to the argument,

$$\int \langle A(s,x(s)), v\rangle^2 \; \mu_*(dx) \le C\theta \|v\|_V^2. \tag{2.31}$$

Properties (2.28) and (2.29) are just restatements of (2.22) and (2.26). By involving (2.31) we conclude that the continuous process $\langle M_t(\cdot), v\rangle$ is μ_*–square integrable. We will now show that for any $v \in V$, $s \le t$ and any bounded function g_s on $C([0,T], V^*)$, which is measurable with respect to the cylindrical σ–field generated by the cylinders with bases over $[0,s]$,

$$\int \left(\langle M_t(x) - M_s(x), v\rangle \, g_s(x)\right) \mu_*(dx) = 0 \tag{2.32}$$

i.e. that $\langle M_t(\cdot), v\rangle \in \mathcal{M}_T^2(\mathbb{R})$ (continuous square integrable real–valued martingales). First, assume that g_s is continuous and extend the result to the general case by the Monotone Class Theorem (functional form).

Let for $v \in V$, $v^m = \sum_{j=1}^m \langle v, \psi_j\rangle_V \psi_j$. Then, with $m \to \infty$,

$$\int |g_s(x) \langle M_t(x), v - v^m\rangle| \; \mu_*(dx) \to 0$$

by uniform integrablility, since $|g_s(x)| \, \|M_t(x)\|_{V^*} \, \|v - v^m\|_V \to 0$. Hence,

$$\begin{aligned} \int \Big| &g_s(x) \left(\langle M_t(x), v\rangle - \langle M_s(x), v\rangle \right) \\ &- g_s(x) \left(\langle M_t(x), v^m\rangle - \langle M_s(x), v^m\rangle \right) \Big| \; \mu_*(dx) \to 0. \end{aligned} \tag{2.33}$$

By the choice of the vectors φ_j and ψ_j, we have for $x^n(t) = (x_1(t), ..., x_n(t)) \in \mathbb{R}^n$,

$$\langle J_n x^n(t), v^m \rangle = \left\langle \sum_{j=1}^{n} x_j(t)\varphi_j, v \right\rangle = \sum_{j=1}^{n\wedge m} x_j(t) \langle v, \varphi_j \rangle_H .$$

For $n \geq m$, the process

$$\begin{aligned} \langle M_t(J_n x^n), v^m \rangle &= \langle J_n x^n(t), v^m \rangle_H - \langle x(0), v^m \rangle_H - \int_0^t \langle A(s, J_n x^n(s)), v^m \rangle \, ds \\ &= \sum_{j=1}^{m} \langle v^m, \varphi_j \rangle_H \left\{ (x^n(t))_j - (x(0))_j - \int_0^t (a^n(s, x^n(s)))_j \, ds \right\} \end{aligned}$$

is a martingale relative to the measure μ^n. Hence, the above, the uniform integrability of $\langle M_t(\cdot), v \rangle$ with respect to the measure μ_* (that follows from (2.21) and (2.30)) imply that

$$\begin{aligned} &\int (g_s(x) \langle M_t(x) - M_s(x), v^m \rangle) \, \mu_*(dx) \\ &\quad = E(g_s(X) \langle M_t(X) - M_s(X), v^m \rangle) \\ &\quad = \lim_{n\to\infty} E(g_s(X^n) \langle M_t(X^n) - M_s(X^n), v^m \rangle) \\ &\quad = \lim_{n\to\infty} \int (g_s(J_n x^n) \langle M_t(J_n x^n) - M_s(J_n x^n), v^m \rangle) \, \mu^n(dx^n) = 0. \end{aligned}$$

The above conclusion, together with (2.33) ensures (2.32). Next, we find the increasing process for the martingale $\langle M_t(x), v \rangle$. We begin with some estimates. For $x, v \in V$, we have

$$\langle B(s,x)QB^*(s,x)v, v \rangle \leq \|v\|_H^2 \mathrm{tr}\,(B(s,x)QB^*(s,x)) .$$

Hence,

$$\begin{aligned} &\int \langle B(s,x(s))QB^*(s,x(s))v, v \rangle \; \mu_*^n(dx) \\ &\leq \|v\|_H^2 \int \theta \left(1 + \|x\|_H^2\right) \, \mu_*^n(dx) \\ &\leq \theta(1+C)\|v\|_H^2 \end{aligned} \tag{2.34}$$

by (2.21), and by (2.22)

$$\int \langle B(s,x(s))QB^*(s,x(s))v, v \rangle \; \mu_*(dx) \leq \theta(1+C)\|v\|_H^2 .$$

As a consequence, we obtain that

$$\Big| \int \int_s^t \big(\langle B(u,x(u))QB(u,x(u))v^m, v^m \rangle \tag{2.35}$$

$$- \langle B(u,x(u))QB(u,x(u))v, v \rangle \big) g_s(x) \, du \mu_*(dx) \Big|$$

$$\leq 2 \left| \sup_x (g_s(x)) \right| T\theta(1+C) \|v^m - v\|_H \|v\|_H < \varepsilon/2, \tag{2.36}$$

for m sufficiently large. Next, observe that

$$\begin{aligned}
&\int \left(\langle M_t(x), v^m\rangle^2 - \langle M_t(x), v\rangle^2\right) g_s(x)\,\mu_*(dx) \qquad (2.37)\\
&\quad\le \left|\sup_x (g_s(x))\right| \left(\int \langle M_t(x), v^m - v\rangle^2\ \mu_*(dx) \int \langle M_t(x), v^m + v\rangle^2\ \mu_*(dx)\right)^{1/2}\\
&\quad< \varepsilon/2,
\end{aligned}$$

since by (2.22) and (2.31) the integrals above are bounded by $D\|v^m - v\|_V^2$ and $D\|v^m + v\|_V^2$, respectively, for some constant D.

By the uniform integrability of $\langle M_t(x), v\rangle^2$ (ensured by (2.21) and (2.30)), we have

$$\begin{aligned}
&\int \left(\langle M_t(x), v^m\rangle^2 - \langle M_s(x), v^m\rangle^2\right) g_s(x)\,\mu_*(dx)\\
&\quad= E\left(\left(\langle M_t(X), v^m\rangle^2 - \langle M_s(X), v^m\rangle^2\right) g_s(X)\right)\\
&\quad= \lim_{n\to\infty} E\left(\left(\langle M_t(J_nX^n), v^m\rangle^2 - \langle M_s(J_nX^n), v^m\rangle^2\right) g_s(J_nX^n)\right)\\
&\quad= \lim_{n\to\infty} E\Bigg(\Bigg[\Big\{\sum_{j=1}^m \left(X^n(t) - \xi_0^n - \int_0^t a^n(u, X^n(u))\,du\right)_j \langle v, \varphi_j\rangle_H\Big\}^2\\
&\qquad -\Big\{\sum_{j=1}^m \left(X^n(s) - \xi_0^n - \int_0^s a^n(u, X^n(u))\,du\right)_j \langle v, \varphi_j\rangle_H\Big\}^2\Bigg] g_s(J_nX^n)\Bigg)\\
&\quad= \lim_{n\to\infty} E\int_s^t \left(\sum_{j=1}^m \left(b^n(u, X^n(u))\,(b^n(u, X^n(u)))^T\right)_{jj} \langle v, \varphi_j\rangle_H^2 g_s(J_nX^n)\right) du\\
&\quad= \lim_{n\to\infty} E\int_s^t \sum_{j=1}^m \sum_{k=1}^n \left\langle Q^{1/2}B^*(u, J_nX^n(u)))\varphi_j, f_k\right\rangle_K^2 \langle v, \varphi_j\rangle_H^2 g_s(J_nX^n)\,du.
\end{aligned}$$

Here, we have used the fact that the martingale

$$X^n(t) - \xi_0^n - \int_0^t a^n(s, X^n(s))\,ds = \int_0^t b^n(s, X^n(s))\,dB_s^n.$$

has an increasing process given by $\int_0^t \operatorname{tr}\left(b(s, X^n(s))(b(s, X^n(s))^T\right)\,ds$.

Consider the last expectation above. It is dominated by

$$\begin{aligned}
&E\int_s^t \sum_{j=1}^m \sum_{k=1}^\infty \left\langle Q^{1/2}B^*(u, J_nX^n(u)))\varphi_j, f_k\right\rangle_K^2 \langle v, \varphi_j\rangle_H^2 g_s(J_nX^n)\,du\\
&\quad= E\int_s^t \sum_{j=1}^m \left\|Q^{1/2}B^*(u, J_nX^n(u)))\varphi_j\right\|_K^2 \langle v, \varphi_j\rangle_H^2 g_s(J_nX^n)\,du\\
&\quad= E\int_s^t \sum_{j=1}^m \langle B(u, (J_nX^n(u))QB^*(u, J_nX^n(u)))\varphi_j, \varphi_j\rangle_H \langle v, \varphi_j\rangle_H^2 g_s(J_nX^n)\,du
\end{aligned}$$

$$
\begin{aligned}
&= E\int_s^t \langle B(u,(J_nX^n(u))QB^*(u,J_nX^n(u)))v^m,v^m\rangle_H\, g_s(J_nX^n)\,du\\
&\to E\int_s^t \langle B(u,(X(u))QB^*(u,X(u)))v^m,v^m\rangle_H\, g_s(X)\,du\\
&= \int\int_s^t (\langle B(u,(x(u))QB^*(u,x(u)))v^m,v^m\rangle_H\, g_s(x))\; du\mu_*(dx).
\end{aligned}
$$

using the weak convergence and uniform integrablility of the integrand ensured by (1.4) and (2.22). Hence,

$$
\begin{aligned}
\lim_{n\to\infty} E\int_0^t \sum_{j=1}^m\sum_{k=1}^n \left\langle Q^{1/2}B^*(u,J_nX^n(u)))\varphi_j,f_k\right\rangle_K^2 \langle v,\varphi_j\rangle_H^2 g_s(J_nX^n)\,du\\
\le \int\int_s^t (\langle B(u,(x(u))QB^*(u,x(u)))v^m,v^m\rangle_H\, g_s(x))\; du\mu_*(dx).
\end{aligned}
$$

To show the opposite inequality, note that if $n\ge r$, then

$$
\begin{aligned}
&\liminf_{n\to\infty}\sum_{j=1}^m\sum_{k=1}^n \left\langle Q^{1/2}B^*(u,J_nX^n(u)))\varphi_j,f_k\right\rangle_K^2 \langle v,\varphi_j\rangle_H^2 g_s(J_nX^n)\\
&\quad\ge \sum_{j=1}^m\sum_{k=1}^r \liminf_{n\to\infty}\left\langle Q^{1/2}B^*(u,J_nX^n(u)))\varphi_j,f_k\right\rangle_K^2 \langle v,\varphi_j\rangle_H^2 g_s(J_nX^n)\\
&\quad= \sum_{j=1}^m\sum_{k=1}^r \left\langle Q^{1/2}B^*(u,X(u)))\varphi_j,f_k\right\rangle_K^2 \langle v,\varphi_j\rangle_H^2 g_s(X)\\
&\quad\to \langle B(u,X(u))QB^*(u,X(u)))v^m,v^m\rangle_H\, g_s(X),
\end{aligned}
$$

and an application of the Fatou lemma gives the equality

$$
\begin{aligned}
&\int \left(\left(\langle M_t(x),v^m\rangle^2-\langle M_s(x),v^m\rangle^2\right) g_s(x)\right)\;\mu_*(dx)\\
&\quad= \int\int_s^t (\langle B(u,(x(u))QB^*(u,x(u)))v^m,v^m\rangle_H\, g_s(x))\; du\mu_*(dx).
\end{aligned}
\tag{2.38}
$$

Summarizing, calculations in (2.35), (2.37), and (2.38) prove that for $v\in V$, the process $\langle M_t(x),v\rangle$ is a square integrable continuous martingale with an increasing process given by

$$
\int_0^t \langle B(u,(x(u))QB^*(u,x(u))v,v\rangle\; du.
$$

Let $\left\{\psi_j^*\right\}_{j=1}^\infty$ be the dual orthonormal basis in V^*, defined by the duality

$$
\left\langle u,\psi_j^*\right\rangle_{V^*} = \langle\psi_j,u\rangle\,,\quad u\in V^*.
$$

Since by (1.3)

$$
\|M_t(x)\|_{V^*}^2 \le C\left(1+\sup_{0\le t\le T}\|x(t)\|_H^2\right),
$$

the martingale $M_t(x)\in\mathcal{M}_T^2(V^*)$, i.e. it is a continuous μ_*–suqare integrable V^*–valued martingale.

Denote $M_t^j(x) = \left\langle M_t(x), \psi_j^* \right\rangle_{V^*}$. Using the property of the dual basis, its increasing process is given by

$$\begin{aligned}\langle \ll M(x) \gg_t (u), v\rangle_{V^*} &= \sum_{j,k=1}^{\infty} \langle M^j(x), M^k(x)\rangle_t \langle \psi_j^*, u\rangle_{V^*} \langle \psi_k^*, v\rangle_{V^*} \\ &= \sum_{j,k=1}^{\infty} \langle M^j(x), M^k(x)\rangle_t \langle \psi_j, u\rangle \langle \psi_k, v\rangle \quad u, v \in V^*.\end{aligned}$$

Since

$$M_t^j(x) M_t^k(x) = \langle M_t(x), \psi_j\rangle \langle M_t(x), \psi_k\rangle,$$

we can write

$$\begin{aligned}\langle M^j(x), M^k(x)\rangle_t &= \langle \langle M(x), \psi_j\rangle, \langle M(x), \psi_k\rangle\rangle_t \\ &= \int_0^t \langle B(s, (x(s)) Q B^*(s, x(s))\psi_j, \psi_k\rangle \, ds.\end{aligned}$$

Define for any $0 \le t \le T$ a map $\Phi(s) : K \to V^*$ by

$$\Phi(s)(k) = \sum_{j=1}^{\infty} \left\langle Q^{1/2} B^*(s, X(s))\psi_j, f_m \right\rangle_K \psi_j^*, \quad k \in K.$$

Then

$$\Phi^*(s)(u) = \sum_{j=1}^{\infty} \langle u, \psi_j\rangle Q^{1/2} B^*(s, X(s))\psi_j, \quad u \in V^*,$$

and we have for $u, v \in V^*$

$$\begin{aligned}&\int_0^t \langle \Phi(s)\Phi^*(s)u, v\rangle_{V^*} \, ds \\ &= \int_0^t \sum_{j,k=1}^{\infty} \left\langle Q^{1/2} B^*(s, X(s))\psi_j, Q^{1/2} B^*(s, X(s)\psi_k \right\rangle_K \langle \psi_j, u\rangle \langle \psi_k, v\rangle \, ds \\ &= \int_0^t \sum_{j,k=1}^{\infty} \langle B(s, X(s)) Q B^*(s, X(s)\psi_j, \psi_k\rangle \langle \psi_j, u\rangle \langle \psi_k, v\rangle \, ds \\ &= \langle \ll M(X) \gg_t (u), v\rangle_{V^*},\end{aligned}$$

giving that

$$\ll M(X) \gg_t = \int_0^t \Phi(s)\Phi^*(s) \, ds.$$

Note that $\Phi(s) \in \mathcal{L}_2(K, V^*)$ (Hilbert–Schmidt operators from K to V^*), since

$$\begin{aligned}\sum_{m=1}^{\infty} \|\Phi(s) f_m\|_{V^*}^2 &= \sum_{m,j=1}^{\infty} \langle \psi_j^*, B(s, X(s)) Q^{1/2} f_m\rangle_{V^*}^2 \\ &= \sum_{m,j=1}^{\infty} \langle \psi_j, B(s, X(s)) Q^{1/2} f_m\rangle^2\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{m,j=1}^{\infty} \langle \varphi_j, B(s,X(s))Q^{1/2} f_m\rangle_H^2 \\
&= \sum_{m=1}^{\infty} \|B(s,X(s))Q^{1/2} f_m\|_H^2 \\
&= \|B(s,X(s))\|_{\mathcal{L}_2(K_Q,H)} < \infty,
\end{aligned}
$$

where we have used the assumption on the duality on Geldand triplet, and the fact that $\psi_j = \varphi_j/\|\varphi_j\|_V$, with the denominator at most equal to one. Consequently, the growth condition(1.4) together with (2.28) imply that

$$
E\int_0^T \|\Phi(s)\|^2_{\mathcal{L}_2(K,V^*)}\, ds < \infty.
$$

Using the cylindrical version of the Martingale Representation Theorem we can write

$$
M_t(X) = \int_0^t \Phi(s)\, d\tilde{W}_s.
$$

Define

$$
W_t = \sum_{m=1}^{\infty} \tilde{W}_t(Q^{1/2} f_m) f_m.
$$

The process W_t is a K–valued Q–Wiener process. We calculate

$$
\begin{aligned}
X(t) - X(0) - \int_0^t A(s,X(s))\, ds &= M_t(X) \\
&= \int_0^t \Phi(s)\, d\tilde{W}_s \\
&= \sum_{m=1}^{\infty} \int_0^t (\Phi(s) f_m)\ d\tilde{W}_s(f_m) \\
&= \sum_{m=1}^{\infty} \int_0^t \sum_{j=1}^{\infty} \left\langle \psi_j, B(s,X(s))Q^{1/2} f_m \right\rangle \psi_j^*\, d\tilde{W}_s(f_m) \\
&= \sum_{m=1}^{\infty} \int_0^t \sum_{j=1}^{\infty} \left\langle \psi_j^*, B(s,X(s)) f_m \right\rangle_{V^*} \psi_j^*\, d\tilde{W}_s(Q^{1/2} f_m) \\
&= \int_0^t \sum_{m=1}^{\infty} B(s,X(s)) f_m\, d\tilde{W}_s(Q^{1/2} f_m) \\
&= \int_0^t B(s,X(s))\, dW_s.
\end{aligned}
$$

We are now in a position to apply Theorem 2.7 to $X(t)$, $Y(t) = A(t,X(t))$, and $Z(t) = B(t,X(t))$ to obtain that $X \in C([0,T],H)$, completing the proof. □

Let us now address the problem of existence and the uniqueness of a strong solution using a version of Yamada and Watanabe result in infinite dimensions. The result, and its proof, follow Theorem 7.3 in [3], and are included for the reader's convenience, with an updated reference to the infinite–dimensional version of the Yamada–Watanabe Theorem.

Recall the notion of pathwise uniqueness.

Definition 2.10. If for any two H–valued weak solutions (X_1, W) and (X_2, W) of Equation (1.1) defined on the same filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\le t\le T}, P)$ and with the same Q–Wiener process W, such that $X_1(0) = X_2(0)$, P–a.s., we have that

$$P\left(X_1(t) = X_2(t),\ 0 \le t \le T\right) = 1,$$

then we say that equation (1.1) has *pathwise uniqueness property.*

Theorem 2.11. *Let the conditions of Theorem 2.9 hold true and assume the following monotonicity condition*

(M) *(Weak monotonicity)*

$$2\langle x-y, A(t,x)-A(t,y)\rangle + \|B(t,x)-B(t,y)\|_{\mathcal{L}_2(K_Q,H)} \le \theta\|x-y\|_H^2, \tag{2.39}$$

for $0 \le t \le T$*, and* $x, y \in V \hookrightarrow H$*. Then the solution to Equation* (1.1) *is pathwise unique.*

Proof. Let X_1, X_2 be two weak solutions as in Definition 2.10 and denote $Y(t) = X_1(t) - X_2(t)$, and its V–valued progressively measurable version by $\bar{Y}(t)$ (see Remark 2.8). Applying the Itô formula (2.27) and the monotonicity condition (2.39) yields

$$\begin{aligned}
e^{-\theta t}\|Y(t)\|_H^2 &= -\theta\int_0^t e^{-\theta s}\|Y(s)\|_H^2\,ds \\
&\quad + \int_0^t e^{-\theta s}\Big(2\left\langle \bar{Y}(s), A(s,X_1(s)) - A(s,X_2(s))\right\rangle \\
&\quad + \|B(s,X_1(s)) - B(s,X_2(s))\|_{\mathcal{L}_2(K_Q,H)}^2\Big)\,ds \\
&\quad +2\int_0^t e^{-\theta s}\left\langle Y_s, \left(B(s,X_1(s)) - B(s,X_2(s))\right)dW_s\right\rangle_H \\
&\le M_t,
\end{aligned}$$

where M_t is a real–valued continuous local martingale represented by the stochastic integral above. The inequaliy above also shows that $M_t \ge 0$. Hence by the Doob maximal inequality, $M_t = 0$. □

As a consequence of an infinite dimensional version of the result of Yamada and Watanabe [10] we have the following corollary.

Corollary 2.12. *Under conditions of Theorem 2.11, Equation* (1.1) *has unique strong solution.*

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