Communications on Stochastic Analysis
Vol. 4, No. 1 (2010) 91-113

Serials Publications
www.serialspublications.com

SOME ASYMPTOTIC RESULTS FOR NEAR CRITICAL BRANCHING PROCESSES*

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Abstract. Near critical single type Bienaymé-Galton-Watson (BGW) processes are considered. It is shown that, under appropriate conditions, Yaglom distributions of suitably scaled BGW processes converge to that of the corresponding diffusion approximation. Convergences of stationary distributions for Q-processes and models with immigration to the corresponding distributions of the associated diffusion approximations are established as well. Although most of the work is concerned with the single type case, similar results for multitype settings can be obtained. As an illustration, convergence of Yaglom distributions of suitably scaled multitype subcritical BGW processes to that of the associated diffusion model is established.

1. Introduction and Main Results

Consider a population consisting of k types of particles whose evolution is described in terms of a discrete time multitype (k-type) Bienaymé-Galton-Watson (k-BGW) process – such a process is a Markov chain $\{Z_p\}_{p\in\mathbb{N}_0}$ on $\mathbb{N}_0^k$, with the vector Z_p representing the number of particles of each type in generation p. We are interested in the long time behavior of the scaled process $\frac{1}{p}Z_{\lfloor pt\rfloor}$, $t \geq 0$, when the k-BGW process is close to criticality. More precisely, we consider a sequence of BGW processes $\{Z_p^{(n)}, p \in \mathbb{N}_0\}_{n\in\mathbb{N}}$ such that, as n becomes large, the processes approach criticality. It is well known (see [3], [8]) that, under suitable conditions, the process $X_t^{(n)} = \frac{1}{n}Z_{\lfloor nt\rfloor}^{(n)}$, $t \geq 0$, converges weakly to a diffusion ξ. Such a result implies convergence of finite time statistics of $X^{(n)}$ to those of ξ, but does not provide any information on relationships between the time asymptotic behaviors of $X^{(n)}$ and ξ. The main goal of this work is to make such relationships mathematically precise. In particular, we show that, under appropriate assumptions, the time asymptotic distribution of $X_t^{(n)}$ with suitable conditioning converges to that of ξ_t with a similar conditioning, as $n \to \infty$ (see Theorems 1.5 and 1.8). An analogous result for models with immigration (where no conditioning is required) is also established (Theorem 1.11). The results say that the long time behavior of

Received 2010-1-11; Communicated by the editors.
2000 *Mathematics Subject Classification.* Primary 60J80; Secondary 60F05.
Key words and phrases. Branching processes, quasi stationary distributions, Yaglom distributions, near critical regime, diffusion approximations, Q-processes.
* This research is partially supported by the Army Research Office (Grant W911NF-0-1-0080), NSF Emerging Frontiers in Research and Innovation (EFRI) (Grant CBE0736007) and the US-Israel Binational Science Foundation (Grant 2008466).

a BGW process is well approximated by that of the corresponding diffusion limit ξ. Most of the results in this work are for single type BGW processes, namely for the case $k = 1$. Similar results can be obtained in multitype settings and we consider one such result in Theorem 1.20.

When $k = 1$, the transition probabilities of a BGW process $\{Z_p\}$ can be written as

$$p(i,j) = P(Z_{p+1} = j | Z_p = i) = \begin{cases} p_j^{*i} & \text{if } i \geq 1, \quad j \geq 0, \\ \delta_{0j} & \text{if } i = 0, \quad j \geq 0, \end{cases} \tag{1.1}$$

where $\{p_l\}_{l \in \mathbb{N}_0}$ is a given probability function – the offspring distribution of each particle – and $\{p_l^{*i}\}_{l \in \mathbb{N}_0}$ is the i-fold convolution of $\{p_l\}_{l \in \mathbb{N}_0}$. The process starts with Z_0 particles; each of the Z_n particles alive at time n lives for one unit of time and then dies, giving rise to l offspring particles with probability p_l, $l \in \mathbb{N}_0$. The particles behave independently of each other and of the past.

Depending on the mean m of the offspring distribution, BGW processes can be divided into three cases: subcritical, critical, and supercritical, according to whether $m < 1$, $m = 1$, or $m > 1$, respectively.

Consider a sequence of processes $Z^{(n)}$ described as follows. If $Z_0^{(n)} = 1$, then $Z_1^{(n)}$ has the probability generating function (pgf)

$$F^{(n)}(s) = \sum_{l=0}^{\infty} p_l^{(n)} s^l, \quad s \in [0,1], \tag{1.2}$$

with mean m_n and variance σ_n^2, where $\{p_l^{(n)}\}_{l \in \mathbb{N}_0}$ is the offspring distribution of $Z^{(n)}$. We denote the p^{th} iterate of $F^{(n)}$ by $F_p^{(n)}$, i.e. for $s \in [0,1]$ and $p \geq 0$

$$F_0^{(n)}(s) = s, \quad F_{p+1}^{(n)}(s) = F^{(n)}(F_p^{(n)}(s)).$$

Let q_n be the extinction probability of $Z^{(n)}$ starting from a single particle, i.e. $q_n = P(Z_p^{(n)} = 0 \text{ for some } p \in \mathbb{N} | Z_0^{(n)} = 1)$. Let

$$X_t^{(n)} := \frac{1}{n} Z_{\lfloor nt \rfloor}^{(n)}, \quad t \in \mathbb{R}_+. \tag{1.3}$$

Then $\{X_t^{(n)}\}_{t \in \mathbb{R}_+}$ is an $\mathbb{S}_n := \{\frac{l}{n} | l \in \mathbb{N}_0\}$ valued (time inhomogeneous) Markov process with sample paths in $D(\mathbb{R}_+ : \mathbb{S}_n)$, the space of càdlàg functions from $\mathbb{R}_+ := [0,\infty)$ to $\mathbb{S}_n$. Throughout, $\mathbb{S}_n$ is endowed with the discrete topology and, given a metric space S, $D(\mathbb{R}_+ : S)$ is endowed with the usual Skorohod topology. Space of probability measures on a metric space S will be denoted by $\mathcal{P}(S)$.

Condition 1.1. *(i) For each n, $p_0^{(n)} > 0$, $p_0^{(n)} + p_1^{(n)} < q_n$, $m_n = 1 + \frac{c_n}{n}, c_n \in (-n,\infty) \setminus \{0\}$, and $\sigma_n^2 < \infty$. (ii) As $n \to \infty$, $c_n \to c \in \mathbb{R} \setminus \{0\}$ and $\sigma_n^2 \to \sigma^2 \in (0,\infty)$. (iii) The family of functions $\{F^{(n)''}\}_{n \in \mathbb{N}}$ is equicontinuous at 1. (iv) As $n \to \infty$, $\sum_{l: l > \epsilon\sqrt{n}} (l - m_n)^2 p_l^{(n)} \to 0$, and $X_0^{(n)}$ converges in distribution to some $\mu \in \mathcal{P}(\mathbb{R}_+)$.*

Condition 1.1 (i) ensures that, as $n \to \infty$, $m_n \to 1$, and thus the processes approach criticality without being critical. The case where $c < 0$ will be referred to as the subcritical case while $c > 0$ corresponds to the supercritical case. Condition

1.1 (iii) will be used in the study of the supercritical case in Theorem 1.5. Condition 1.1 (iv) is needed for the diffusion approximation result in Theorem 1.2.

We now recall a well known weak convergence result for $X^{(n)}$ (see [5], [8, Theorem 4.2.2]), which describes the asymptotic behavior of $X^{(n)}$, as $n \to \infty$, over any fixed finite time horizon. Here we only give the result in a one dimensional setting. The multidimensional result will be presented later in this section.

Theorem 1.2. *Assume Condition 1.1. Then $X^{(n)}$ converges weakly in $D(\mathbb{R}_+ : \mathbb{R}_+)$ to the unique (in law) diffusion process ξ with generator*

$$(Lf)(x) = xcf'(x) + \frac{1}{2}x\sigma^2 f''(x), \quad f \in C^2(\mathbb{R}_+), \quad x \in \mathbb{R}_+, \tag{1.4}$$

and initial distribution (i.e. probability law of ξ_0) equal to μ.

We are concerned with the study of relationships between the steady state behavior of $X^{(n)}$ and that of ξ. However, one needs to suitably interpret the term "steady state" since, as is well known, as $t \to \infty$, for $m_n > 1$, $X_t^{(n)}$ tends to infinity on the set of non-extinction, and for $m_n \leq 1$, $X_t^{(n)}$ eventually becomes extinct (see [1]). There are two well studied approaches for formulating time asymptotic questions in the subcritical case. The first is to condition the processes $X^{(n)}$ on non-extinction, where, loosely speaking, the conditioning can either be on non-extinction at the present time or in the distant future. The state process $X^{(n)}$ under these two conditionings has different limiting distributions as $t \to \infty$. The first is called the Yaglom distribution of $X^{(n)}$, while the second is the stationary distribution of the Q-process associated with $X^{(n)}$ (see Section I.14 of [1]). The second approach for obtaining a nontrivial time asymptotic behavior is to introduce an immigration component. Namely, in each generation a (random) number of particles that are indistinguishable from the original set of particles is added to the population. The immigration component in particular ensures that the resulting scaled state process, denoted by $V^{(n)}$, has a non-degenerate stationary distribution. For the supercritical case, a common approach is to reduce the problem to that of a subcritical setting by conditioning on the event of eventual extinction. The so conditioned state process $X^{(n)}$ has the same law as the state process corresponding to a certain subcritical BGW process. In this work we will show that the time asymptotic distribution of $X_t^{(n)}$ (in both subcritical and supercritical settings), under suitable conditioning, converges to that of ξ_t under a similar conditioning, as $n \to \infty$. For models with immigration we will prove convergence of stationary distributions.

We begin by describing results for models without immigration. For a Markov process $\{Y_t\}_{t\in\mathbb{R}_+}$ with initial value $Y_0 = y$, we write $P(Y_t \in \cdot)$ as $P_y(Y_t \in \cdot)$. Similarly, when the distribution of Y_0 is μ, we write $P(Y_t \in \cdot)$ as $P_\mu(Y_t \in \cdot)$. Similar notations will be used for conditional expectations. Let $\mathbb{S}$ be a subset of $\mathbb{R}_+^k$, for some $k \in \mathbb{N}$. When $\mathbb{S}$ is endowed with a topology, we will denote by $\mathcal{B}(\mathbb{S})$ the σ-field generated by the open sets of $\mathbb{S}$. Let $Y \equiv \{Y_t\}_{t\in\mathbb{R}_+}$ be an $\mathbb{S}$-valued Markov process such that $\mathbf{0} \in \mathbb{S}$ is an absorbing state.

Definition 1.3. (i) A *quasi-stationary distribution* (qsd) for Y is a probability distribution μ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$ such that $P_\mu(Y_t \in B | t < T_Y < \infty) = \mu(B)$ for all

$B \in \mathcal{B}(\mathbb{S})$ and $t \geq 0$, where $T_Y := \inf\{t | Y_t = \mathbf{0}\}$.
(ii) If for all $\mathbf{y} \in \mathbb{S} \setminus \{\mathbf{0}\}$, as $t \to \infty$, $P_{\mathbf{y}}(Y_t \in \cdot | t < T_Y < \infty)$ converges weakly to some probability measure μ on $(\mathbb{S}, \mathcal{B}(\mathbb{S}))$, then μ is called the *Yaglom distribution* of Y.

The following result follows from [9] and Proposition 2.3.2.1 of [10].

Theorem 1.4. *The Yaglom distribution of ξ exists and is Exponential with density*

$$f(x) = \frac{2|c|}{\sigma^2} \exp\left(-\frac{2|c|}{\sigma^2} x\right), \quad x \geq 0. \tag{1.5}$$

Our first result, Theorem 1.5 below, says that the Yaglom distribution of $X^{(n)}$ approaches that of ξ, as $n \to \infty$.

Theorem 1.5. *Assume Condition 1.1. For each n, $X^{(n)}$ has a Yaglom distribution $\nu^{(n)}$. This distribution is also a qsd, and it converges weakly to the Yaglom distribution ν of ξ.*

We now consider the second form of conditioning where one conditions the process on not being extinct in the "distant future". We will see that in this case a somewhat different asymptotic behavior emerges. For this result we restrict ourselves to the subcritical case (i.e. $c_n < 0$). We begin with the definition of a Q-process (see [1], [9]).

Let $\hat{\Omega} = D(\mathbb{R}_+ : \mathbb{R}_+)$ and $\hat{\mathcal{F}}$ be the corresponding Borel σ-field (with the usual Skorohod topology). Denote by $\{\mathcal{F}_t\}_{t \in \mathbb{R}_+}$ the canonical filtration on $(\hat{\Omega}, \hat{\mathcal{F}})$, i.e. $\mathcal{F}_t = \sigma(\pi_s : s \leq t)$, where $\pi_s(x) = x_s$ for $x \in \hat{\Omega}$. We denote by $\hat{P}^{(n)}_\mu$ the measure induced by $X^{(n)}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ when $Z^{(n)}_0$ has distribution μ (supported on $\mathbb{N}$). Let $T := \inf\{t | \pi_t = 0\}$.

By Lemma 4.3 in the appendix, there is a probability measure $P^{(n)\uparrow}_\mu$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ such that, as $s \to \infty$, $\hat{P}^{(n)}_\mu(\Theta | T > s) \to P^{(n)\uparrow}_\mu(\Theta)$, for all $\Theta \in \mathcal{F}_t$, $t \in \mathbb{R}_+$. Furthermore if $\{Z^{(n)\uparrow}_k\}_{k \in \mathbb{N}_0}$ is a Markov chain with state space $\mathbb{N}$, l-step transition function

$$p^{(n)\uparrow}_l(i,j) = P(Z^{(n)}_l = j | Z^{(n)}_0 = i) \frac{j}{i} m_n^{-l}, \quad i, j \in \mathbb{N},$$

and initial distribution μ, then $P^{(n)\uparrow}_\mu$ is the law of $\{X^{(n)\uparrow}_t\}_{t \in \mathbb{R}_+}$, where $X^{(n)\uparrow}_t := \frac{1}{n} Z^{(n)\uparrow}_{\lfloor nt \rfloor}$, $t \in \mathbb{R}_+$. The process $Z^{(n)\uparrow}$ [respectively $X^{(n)\uparrow}$] is called the Q-process associated with $Z^{(n)}$ [respectively $X^{(n)}$]. Q-processes associated with branching processes can be interpreted as branching processes conditioned on being never extinct.

Next, we introduce the Q-process associated with the diffusion ξ from Theorem 1.2. Denote by $P_{\xi,x}$ the measure induced by ξ on $(\hat{\Omega}, \hat{\mathcal{F}})$, where $\xi(0) = x > 0$. The following theorem is contained in [9].

Theorem 1.6. *There is a probability measure $P^{\uparrow}_{\xi,x}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$, such that for all $t \in \mathbb{R}_+$ and $\Theta \in \mathcal{F}_t$, $P_{\xi,x}(\Theta | T > s)$ converges to $P^{\uparrow}_{\xi,x}(\Theta)$, as $s \to \infty$. Let $\xi^{\uparrow}$ be the* *unique weak solution of the SDE*

$$d\xi_t^{\uparrow} = c\xi_t^{\uparrow} dt + \sqrt{\sigma^2 \xi_t^{\uparrow}} dB_t + \sigma^2 dt, \quad \xi_0^{\uparrow} = x,$$

where B is a standard Brownian motion. Then $P^{\uparrow}_{\xi,x}$ equals the measure induced by $\xi^{\uparrow}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$.

The process $\xi^{\uparrow}$ is referred to as the Q-process associated with ξ. The following result (see [9], Section 5.2) says that the process $\xi^{\uparrow}$ has a unique stationary distribution, $\nu^{\uparrow}$, which is given as the convolution of two copies of the exponential distribution ν with density as in (1.5).

Theorem 1.7. *Assume $c < 0$. As $t \to \infty$, for every initial condition x, $\xi_t^{\uparrow}$ converges in distribution to a random variable $\xi_\infty^{\uparrow}$, whose distribution, denoted by $\nu^{\uparrow}$, is the convolution of two copies of the Yaglom distribution ν. In particular, $\nu^{\uparrow}$ has density*

$$f(x) = \left(\frac{2c}{\sigma^2}\right)^2 x \exp\left(\frac{2c}{\sigma^2} x\right), \quad x \geq 0. \tag{1.6}$$

Our next result shows that the time asymptotic behavior of the Q-process associated with $X^{(n)}$ can be well approximated by that of the Q-process associated with the diffusion approximation of $X^{(n)}$.

Theorem 1.8. *Assume Condition 1.1 and that $c_n < 0$ for all $n \in \mathbb{N}$. For each n, $X_t^{(n)\uparrow}$ converges in distribution, as $t \to \infty$, to a random variable $X_\infty^{(n)\uparrow}$. The distribution $\nu^{(n)\uparrow}$ of $X_\infty^{(n)\uparrow}$ is the unique stationary distribution of the $\mathbb{S}_n$ valued Markov process $X^{(n)\uparrow}$. As $n \to \infty$, $\nu^{(n)\uparrow}$ converges weakly to $\nu^{\uparrow}$.*

We now describe the results for BGW processes with immigration. Let F and G be pgf's of $\mathbb{N}_0$ valued random variables. A Bienaymé-Galton-Watson branching process with immigration corresponding to (F,G) (referred to as a DBI(F,G) process), is a Markov chain $\{Y_n\}$ with state-space $\mathbb{N}_0$ and transition probability function described in terms of the corresponding pgf: Given $Y_0 = i \in \mathbb{N}$, the pgf $H(i,\cdot)$ of Y_1 is $H(i,s) = \sum_{j=0}^{\infty} P(Y_1 = j | Y_0 = i) s^j = F(s)^i G(s)$, $s \in [0,1]$.

Let $G^{(n)}$ be a sequence of pgf's, and consider a sequence of DBI$(F^{(n)}, G^{(n)})$ processes $Y^{(n)}$.

Condition 1.9. *(i) There are $\iota_0, \kappa_0 \in (0,\infty)$ such that, for all $n \in \mathbb{N}$, $G^{(n)'}(1) = \iota_n \geq \iota_0$ and $G^{(n)''}(1) = \kappa_n \leq \kappa_0$. (ii) As $n \to \infty$, $\iota_n \to \iota$. (iii) There is a $\tau_0 \in [0,\infty)$ such that, for all $n \in \mathbb{N}$, $F^{(n)'''}(1) = \tau_n < \tau_0$.*

Let $V_t^{(n)} := \frac{1}{n} Y_{\lfloor nt \rfloor}^{(n)}$, $t \in \mathbb{R}_+$. The proof of the following theorem is easy to establish using [9] and [11, Theorem 2.1].

Theorem 1.10. *Assume Conditions 1.1 and 1.9 and that $c < 0$. Suppose that $V_0^{(n)}$ converges in distribution to some $\mu \in \mathcal{P}(\mathbb{R}_+)$. Then $V^{(n)}$ converges weakly in $D(\mathbb{R}_+ : \mathbb{R}_+)$ to the process ζ which is the unique weak solution of*

$$d\zeta_t = c\zeta_t dt + \sqrt{\sigma^2 \zeta_t} dB_t + \iota dt, \quad t \geq 0,$$

where ζ_0 has distribution μ. The Markov process ζ has a unique stationary distribution η, which is a gamma distribution with parameters $2\iota/\sigma^2$ and $\sigma^2/(2|c|)$, i.e., η has density g given as

$$g(x) = x^{\frac{2\iota}{\sigma^2}-1} \frac{\exp\left(-x\frac{2|c|}{\sigma^2}\right)}{\left(\frac{\sigma^2}{2|c|}\right)^{\frac{2\iota}{\sigma^2}} \Gamma\left(\frac{2\iota}{\sigma^2}\right)}, \quad x > 0.$$

We are interested in the long time behavior of the scaled processes $V^{(n)}$ as they approach criticality. Our main result is the following.

Theorem 1.11. *Assume Conditions 1.1 and 1.9 and that $c_n < 0$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, $V^{(n)}$ has a unique stationary distribution $\eta^{(n)}$, and as $n \to \infty$, $\eta^{(n)}$ converges weakly to η.*

As noted earlier in the introduction, results similar to Theorems 1.5, 1.7, and 1.11 can be established for multitype settings as well. To illustrate the key ideas involved, we only discuss one case in detail, namely the convergence of the Yaglom distribution in the setting of a subcritical multitype process. We begin with some notation and definitions. Let $\{Z_j^{(n)}, j \in \mathbb{N}_0\}_{n\in\mathbb{N}}$ be a sequence of k-BGW processes with transition mechanism described below. Let $C := [0,1]^k$, $\mathbf{e}_\alpha := (\delta_{1\alpha}, \dots, \delta_{k\alpha})'$ be the α^{th} canonical basis vector, and $\mathbf{s}^{\mathbf{i}} := \prod_{\alpha=1}^k s_\alpha^{i_\alpha}$, for $\mathbf{i} = (i_1, \dots, i_k)' \in \mathbb{N}_0^k$ and $\mathbf{s} = (s_1, \dots, s_k)' \in \mathbb{R}_+^k$. Similar to the single type case, the evolution of $Z_j^{(n)} = (Z_{j,1}^{(n)}, \cdots Z_{j,k}^{(n)})'$ is described as follows. For any $\alpha = 1, \dots, k$, each of the $Z_{j,\alpha}^{(n)}$ type α particles alive at time j (if any) lives for one unit of time and then dies, giving rise to a number of offspring particles, represented by $\mathbf{l} = (l_1, \dots, l_k)$, l_β being the number of type β offspring, with probability $p^{(n)}(\mathbf{e}_\alpha, \mathbf{l})$. The particles behave independently of each other and of the past. The probability law of $Z^{(n)}$ is given in terms of the pgf $F^{(n)}(\mathbf{s}) := (F_{(1)}^{(n)}(\mathbf{s}), \dots, F_{(k)}^{(n)}(\mathbf{s}))$, $\mathbf{s} \in C$, where $F_{(\alpha)}^{(n)}(\mathbf{s}) := \sum_{\mathbf{j}\in\mathbb{N}_0^k} p^{(n)}(\mathbf{e}_\alpha, \mathbf{j})\mathbf{s}^{\mathbf{j}}$, $1 \le \alpha \le k$, $\mathbf{s} \in C$. Let $m_{\alpha\beta}^{(n)} = E_{\mathbf{e}_\alpha} Z_{1,\beta}^{(n)}$ be the expected number of type β offspring from a single particle of type α in one generation. Then the $k \times k$ matrix $\mathbf{M}^{(n)} = (m_{\alpha\beta}^{(n)})_{\alpha,\beta=1,\dots,k}$ is called the *mean matrix* of $Z^{(n)}$. Note that $m_{\alpha\beta}^{(n)} = \frac{\partial F_{(\alpha)}^{(n)}}{\partial s_\beta}(\mathbf{1})$, where the partial derivative is understood to be the left hand derivative. The processes $Z^{(n)}$ will be assumed to have a *uniformly strictly positive* mean matrix $\mathbf{M}^{(n)}$, by which we mean that there exist $U \in \mathbb{N}$ and $a \in (0,\infty)$ such that for every $n \ge 1$ $((\mathbf{M}^{(n)})^U)_{\alpha,\beta} \ge a$ for all $1 \le \alpha, \beta \le k$. From the Perron-Frobenius Theorem it then follows that $\mathbf{M}^{(n)}$ has a real, positive maximal eigenvalue ρ_n with associated positive left and right eigenvectors $\mathbf{v}^{(n)}$ and $\mathbf{u}^{(n)}$, respectively, which, without loss of generality, are normalized so that $\mathbf{u}^{(n)'}\mathbf{v}^{(n)} = 1$ and $\mathbf{u}^{(n)'}\mathbf{1} = 1$ (see [1]). The maximal eigenvalue ρ_n plays a similar role in the classification of the k-BGW process as the mean played in classifying the (single type) BGW process. The k-BGW process is called subcritical, critical, or supercritical, according to whether $\rho_n < 1$, $\rho_n = 1$, or $\rho_n > 1$, respectively. We will consider the subcritical case, namely for all $n \ge 1$ $\rho_n \in (0,1)$, and study the behavior of quasi-stationary and Yaglom distributions of the scaled process

$X^{(n)}(t) = \frac{Z^{(n)}_{\lfloor nt \rfloor}}{n}$, $t \geq 0$, as $\rho_n \to 1$. The existence of the Yaglom distribution of $X^{(n)}$ is assured by the following result, which is proved in Section 3.

Condition 1.12. *For each* $n \geq 1$, $E_{\mathbf{1}}(||Z_1^{(n)}|| \log ||Z_1^{(n)}||) < \infty$.

Theorem 1.13. *Assume Condition 1.12. For each* $n \in \mathbb{N}$, $X^{(n)}$ *has a Yaglom distribution* $\nu^{(n)}$. *This distribution is also a qsd.*

Condition 1.14. *There exist* $b, d \in (0, \infty)$ *such that for all* $n \in \mathbb{N}$ *(i)* $\sum_{\alpha\beta\gamma} \partial^2 F^{(n)}_{(\alpha)}(\mathbf{1})/\partial s_\beta \partial s_\gamma \geq b$, *and (ii)* $\sum_{\alpha,\beta,\gamma,\delta} \partial^3 F^{(n)}_{(\alpha)}(\mathbf{1})/\partial s_\beta \partial s_\gamma \partial s_\delta \leq d$, *where* $\alpha, \beta, \gamma, \delta$ *in the above sums vary over* $\{1, \ldots, k\}$.

Part (i) of the assumption can be interpreted as a non-degeneracy condition, and part (ii) says that the third moments of the offspring distributions are uniformly bounded in n.

The assumption on convergence of means translates into the following requirement in the multitype setting.

Condition 1.15. *For some strictly positive matrix* $\mathbf{M}$ *and each* $n \in \mathbb{N}$, $\mathbf{M}^{(n)} = \mathbf{M} + \frac{\mathbf{C}^{(n)}}{n}$, *and* $\lim_{n\to\infty} \mathbf{C}^{(n)} = \mathbf{C}$. *The maximal eigenvalues* ρ_n *of* $\mathbf{M}^{(n)}$ *are of the form* $\rho_n = 1 + \frac{c_n}{n}$, *with* $c_n \in (-n, 0)$ *and* $\lim_{n\to\infty} c_n = c \in (-\infty, 0)$. *Moreover,* $\mathbf{M}$ *has maximal eigenvalue* 1 *with corresponding eigenvectors* $\mathbf{v} = \lim \mathbf{v}^{(n)}$ *and* $\mathbf{u} = \lim \mathbf{u}^{(n)}$. *Finally,* $\mathbf{v}'\mathbf{C}\mathbf{u} = c$.

Example 1.16. Let $\mathbf{C}^{(n)} = c_n \mathbf{I}$, where $\mathbf{I}$ is the identity matrix and $c_n \in (-n, 0)$ such that $c_n \to c \in (0, \infty)$. Let $\mathbf{M}$ be a strictly positive matrix with maximal eigenvalue equal to 1. Then $\mathbf{M}^{(n)} = \mathbf{M} - \frac{\mathbf{C}^{(n)}}{n}$ satisfies Condition 1.15.

Let

$$\sigma^{(n)}_{i,j}(l) = \sum_{\mathbf{r}\in\mathbb{N}_0^k} (r_i - m^{(n)}_{li})(r_j - m^{(n)}_{lj}) p^{(n)}(\mathbf{e}_l, \mathbf{r}).$$

The following condition is analogous to the assumption on convergence of variances in the single type case.

Condition 1.17. *As* $n \to \infty$, $\sigma^{(n)}_{i,j}(l) \to \sigma_{i,j}(l)$ *for all* $1 \leq i, j, l \leq k$ *and* $Q := \frac{1}{2}\sum_{l=1}^k v_l \mathbf{u}'\sigma(l)\mathbf{u} > 0$, *where* $\sigma(l)$ *is the matrix with* $(i,j)^{th}$ *entry* $\sigma_{i,j}(l)$.

The following diffusion approximation result can be established along the lines of Theorem 4.3.1 of [8] and Theorem 9.2.1 of [3]. We provide a sketch in Section 3.

Theorem 1.18. *Assume Conditions 1.14, 1.15, and 1.17. Suppose that the distribution of* $X^{(n)}(0)$ *converges to some* $\mu \in \mathcal{P}(\mathbb{R}^k_+)$. *Let* $\mu_1 \in \mathcal{P}(\mathbb{R}_+)$ *be given as*

$$\mu_1(A) = \mu\{\mathbf{x} \in \mathbb{R}^k_+ | \mathbf{x}'\mathbf{u} \in A\}, \quad A \in \mathcal{B}(\mathbb{R}_+). \tag{1.7}$$

Let $\zeta^{(n)} = X^{(n)\prime}\mathbf{u}^{(n)}$. *Then* $\zeta^{(n)}$ *converges weakly in* $D(\mathbb{R}_+ : \mathbb{R}_+)$ *to the unique (in law) diffusion* ζ *with initial distribution* μ_1 *and generator* $\tilde{L}$ *given as*

$$(\tilde{L}f)(x) = cxf'(x) + Qxf''(x), \quad f \in C_c^\infty(\mathbb{R}_+), \quad x \in \mathbb{R}_+. \tag{1.8}$$

Furthermore, for any $t_0 \in (0, \infty)$, the process $X^{(n,0)}$, defined by $X^{(n,0)}(t) = X^{(n)}(t_0 + t)$, $t \geq 0$, converges weakly to $X^{(0)} = \mathbf{v}\zeta^{(0)}$, where $\zeta^{(0)}(t) = \zeta(t_0 + t)$, $t \geq 0$.

The process $X^{(0)}$ is a Markov process with state space $S_{\mathbf{v}} = \{\theta\mathbf{v} | \theta \geq 0\}$ and can be formally regarded as the limit of $X^{(n)}$. Indeed, if the support of μ is contained in $S_{\mathbf{v}}$, then, noting that $\mathbf{u}'\mathbf{v} = 1$, we see that the law of $\mathbf{v}\zeta(0)$ equals μ, and that in fact $X^{(n)}$ converges weakly to $\mathbf{v}\zeta$, where ζ is as in Theorem 1.18. We will be concerned with the Yaglom distribution of the $S_{\mathbf{v}}$ valued Markov process $X^{(0)}$ and its relation to the Yaglom distribution of $X^{(n)}$. For that it will be convenient to regard a probability measure on $S_{\mathbf{v}}$ as one on $\mathbb{R}^k_+$. Denote by $\tilde{\nu}$ the Exponential distribution with density $f(x) = |c|Q^{-1}\exp(-|c|Q^{-1}x)$, $x \geq 0$. Theorem 1.4 says that the Yaglom distribution of $\zeta^{(0)}$ is given by $\tilde{\nu}$. Since $X^{(0)} = \mathbf{v}\zeta^{(0)}$, the Yaglom distribution of $X^{(0)}$ exists as well and equals the distribution of $\mathbf{v}Y$, where Y has distribution $\tilde{\nu}$. Thus, we have the following:

Theorem 1.19. *Assume Conditions 1.14, 1.15, and 1.17. The Yaglom distribution of $\zeta^{(0)}$ exists and equals $\tilde{\nu}$. Furthermore, the Yaglom distribution of $X^{(0)}$, denoted by $\bar{\nu}$, exists and equals the distribution of $\mathbf{v}Y$, where Y has distribution $\tilde{\nu}$.*

The following is our main result that relates the qsd's and Yaglom distributions of $X^{(n)}$ to that of its "diffusion limit" $X^{(0)}$. Probability distributions similar to $\bar{\nu}$ have previously been noted in the study of qsd's of multitype BGW processes. In [1] (p. 191), a single critical BGW process Z (rather than a sequence of near critical BGW processes) is considered and it is shown that Z_n/n conditioned on non-extinction converges to a random variable that is concentrated on the ray $\{x\mathbf{v}_Z | x \geq 0\}$, where $\mathbf{v}_Z$ is the left eigenvector of the mean matrix of Z corresponding to the eigenvalue 1. In [13] (see Theorem 3 therein) the case where Z is near critical and a somewhat differently (component wise) scaled process Z^* is considered. The asymptotic behavior of Z^*_n conditioned on non-extinction, as $n \to \infty$ and the offspring distribution approaches criticality, is related to the limiting distributions considered here. We remark that none of these results concern the setting of diffusion approximation, where time and space are scaled and one starts with a large number of particles.

Theorem 1.20. *Assume Conditions 1.14, 1.15, and 1.17. The Yaglom distribution $\nu^{(n)}$ of $X^{(n)}$ converges weakly to the Yaglom distribution $\bar{\nu}$ of $X^{(0)}$.*

2. Proofs: Single Type Case

In this section we give proofs of Theorems 1.5, 1.8, and 1.11. We begin with Theorem 1.5.

Proof of Theorem 1.5. Proof of the fact that $X^{(n)}$ has a Yaglom distribution $\nu^{(n)}$ that is also a qsd is an immediate consequence of Theorem 1.13, the proof of which is given in Section 3. We now show that $\nu^{(n)}$ converges weakly to ν. The first step is to establish the representation for the Laplace transform of $\nu^{(n)}$ given in Lemma 2.1 below. In the subcritical case, define

$$Q^{(n)}_k(s) := m_n^{-k}(F^{(n)}_k(s) - 1), \quad s \in [0, 1]. \tag{2.1}$$

Then $Q_k^{(n)}$ converges pointwise over $[0,1]$, as $k \to \infty$, to a continuous function $Q^{(n)}$ that is positive on $[0,1)$ (see [1], p. 40, Corollary I.11.1), i.e.

$$\lim_{k\to\infty} Q_k^{(n)}(s) =: Q^{(n)}(s), \quad s \in [0,1], \tag{2.2}$$

where $Q^{(n)}(s) > 0$ for $s \in [0,1)$. The function $Q^{(n)}$ will determine the Laplace transform of $\nu^{(n)}$ in the subcritical case. In the supercritical case, we proceed as follows. Note that, since $p_0^{(n)} > 0$, we have that $q_n > 0$. Also since $m_n > 1$, we have $q_n \in (0,1)$ and that q_n is the smallest root of $F^{(n)}(t) = t$ (see [1], Theorem I.5.1). Define $\tilde{F}^{(n)}(s) := q_n^{-1} F^{(n)}(q_n s)$, $s \in [0,1]$. Since $F^{(n)}(q_n) = q_n$, each $\tilde{F}^{(n)}$ is again a pgf and thus has a representation $\tilde{F}^{(n)}(s) = \sum_{l=0}^{\infty} \tilde{p}_l^{(n)} s^l$, $s \in [0,1]$, with $\sum_{l=0}^{\infty} \tilde{p}_l^{(n)} = 1$. In fact, $\tilde{p}_l^{(n)} = p_l^{(n)} q_n^{l-1}$, $l \in \mathbb{N}_0$. The probability distribution $\{\tilde{p}_l^{(n)}\}$ has mean $\tilde{m}_n = q_n^{-1} F^{(n)\prime}(q_n) q_n = F^{(n)\prime}(q_n) < 1$ and variance $\tilde{\sigma}_n^2 = \tilde{F}^{(n)\prime\prime}(1) - \tilde{m}_n^2 + \tilde{m}_n = q_n F^{(n)\prime\prime}(q_n) - \tilde{m}_n^2 + \tilde{m}_n$. That $F^{(n)\prime}(q_n) < 1$ is a consequence of $F^{(n)\prime}(1) > 1$, $F^{(n)}(q_n) = q_n$, and the strict convexity of $F^{(n)}$ on $[0,1]$. The latter follows from the assumption that $p_0^{(n)} + p_1^{(n)} < q_n$. Let $\tilde{Q}_k^{(n)}(s) := \tilde{m}_n^{-k}(\tilde{F}_k^{(n)}(s) - 1)$, $s \in [0,1]$. Then

$$\lim_{k\to\infty} \tilde{Q}_k^{(n)}(s) =: \tilde{Q}^{(n)}(s), \quad s \in [0,1], \tag{2.3}$$

and $\tilde{Q}^{(n)}$ has the same properties as those of $Q^{(n)}$ in the subcritical case noted earlier.

Lemma 2.1. *The Laplace transform of $\nu^{(n)}$, $\int_{[0,\infty)} e^{-\alpha x}\nu^{(n)}(dx)$, in the subcritical case, is given as $[Q^{(n)}(0) - Q^{(n)}(e^{-\alpha/n})]/(Q^{(n)}(0))$ and, in the supercritical case as $[\tilde{Q}^{(n)}(0) - \tilde{Q}^{(n)}(e^{-\alpha/n})]/(\tilde{Q}^{(n)}(0))$.*

Proof. Consider first the subcritical case. Since $T_{X^{(n)}} < \infty$ *a.s.*, it suffices to show that, for each $\alpha \geq 0$,

$$\lim_{t\to\infty} E_{\frac{i}{n}}\left(e^{-\alpha X_t^{(n)}} | X_t^{(n)} > 0\right) = \frac{Q^{(n)}(0) - Q^{(n)}(e^{-\alpha/n})}{Q^{(n)}(0)}. \tag{2.4}$$

Elementary calculations give

$$E_{\frac{i}{n}}\left(e^{-\alpha X_t^{(n)}} | X_t^{(n)} > 0\right) = 1 - \frac{A_{n,t}(e^{-\alpha/n})}{A_{n,t}(0)}, \tag{2.5}$$

where, for $\theta \in [0,1]$, $A_{n,t}(\theta) = m_n^{-\lfloor nt \rfloor}\left(1 - \left[F_{\lfloor nt \rfloor}^{(n)}(\theta)\right]^i\right)$. Next,

$$\begin{aligned}\lim_{t\to\infty} A_{n,t}(e^{-\alpha/n}) &= \lim_{t\to\infty} m_n^{-\lfloor nt \rfloor}\left(1 - \sum_{k=0}^{i}\binom{i}{k}\left(F_{\lfloor nt \rfloor}^{(n)}(e^{-\alpha/n}) - 1\right)^k\right)\\ &= -i\lim_{t\to\infty} m_n^{-\lfloor nt \rfloor}\left(F_{\lfloor nt \rfloor}^{(n)}(e^{-\alpha/n}) - 1\right) = -iQ^{(n)}(e^{-\alpha/n}),\end{aligned} \tag{2.6}$$

where the second and third equalities follow from (2.2). In exactly the same way one sees that $\lim_{t\to\infty} A_{n,t}(0) = -iQ^{(n)}(0)$. Combining the above observations we have (2.4), which proves the lemma for the subcritical case.

Consider now the supercritical case. Similar to the subcritical case

$$E_{\frac{i}{n}}(e^{-\alpha X_t^{(n)}}|t < T_{X^{(n)}} < \infty) = \frac{\left[F_{\lfloor nt \rfloor}^{(n)}(q_n e^{-\alpha/n})\right]^i - \left[F_{\lfloor nt \rfloor}^{(n)}(0)\right]^i}{\left[F_{\lfloor nt \rfloor}^{(n)}(q_n)\right]^i - \left[F_{\lfloor nt \rfloor}^{(n)}(0)\right]^i} \equiv 1 - \frac{\tilde{A}_{n,t}(e^{-\frac{\alpha}{n}})}{\tilde{A}_{n,t}(0)}$$

where, for $\theta \in [0,1]$, $\tilde{A}_{n,t}(\theta) = \tilde{m}_n^{-\lfloor nt \rfloor}\left(1 - \left[\tilde{F}_{\lfloor nt \rfloor}^{(n)}(\theta)\right]^i\right)$. This says in particular that

$$E_{\frac{i}{n}}(e^{-\alpha X_t^{(n)}}|t < T_{X^{(n)}} < \infty) = E_{\frac{i}{n}}(e^{-\alpha \tilde{X}_t^{(n)}}|\tilde{X}_t^{(n)} > 0), \tag{2.7}$$

where $\tilde{X}_t^{(n)} := \frac{1}{n}\tilde{Z}_{\lfloor nt \rfloor}^{(n)}$, $t \in \mathbb{R}_+$, and $\tilde{Z}^{(n)}$ is a BGW process with pgf $\tilde{F}^{(n)}$. Making now use of (2.3) instead of (2.2), the proof for the supercritical case is completed exactly as for the subcritical case. □

We continue with the proof of Theorem 1.5, which is based on the fact that the Laplace transform of ν is $G(\alpha) = (1 + \frac{\alpha\sigma^2}{2|c|})^{-1}$, $\alpha \geq 0$. First, we show that $\nu^{(n)}$ converges to ν for a special subcritical model where the pgf is of the so-called linear fractional form (see [1], pp. 6-7, [7], pp. 9-10). We then establish a comparison lemma which allows us to prove the general subcritical result by an approximation argument.

Lemma 2.2. *Assume Condition 1.1 and that $c_n < 0$ for all n. Let, for each n, $F^{(n)}$ be of the linear fractional form:*

$$F^{(n)}(s) = 1 - \frac{b^{(n)}}{1 - p^{(n)}} + \frac{b^{(n)}s}{1 - p^{(n)}s}, \quad s \in [0,1], \tag{2.8}$$

where $b^{(n)}, p^{(n)} \in (0,1)$ and $b^{(n)} < 1 - p^{(n)}$. Then $\nu^{(n)}$ converges weakly to ν.

We note that Condition 1.1 imposes certain restrictions on $b^{(n)}$ and $p^{(n)}$ which are not made explicit in the statement of the lemma. See Lemma 4.2 for a precise relationship between the parameters $b^{(n)}$, $p^{(n)}$, and the mean and variance of $Z_1^{(n)}$.

Proof. With $A_{n,t}$ as in the proof of Lemma 2.1, we have

$$E_{\frac{i}{n}}\left(e^{-\alpha X_t^{(n)}}|X_t^{(n)} > 0\right) = 1 - \frac{A_{n,t}(e^{-\alpha/n})}{A_{n,t}(0)}.$$

In order to prove the lemma it suffices to show that

$$\lim_{n\to\infty}\lim_{t\to\infty}\frac{n}{i}A_{n,t}(0) = -\frac{2c}{\sigma^2} \text{ and } \lim_{n\to\infty}\lim_{t\to\infty}\frac{n}{i}A_{n,t}(e^{-\alpha/n}) = \frac{2c\alpha}{2c - \alpha\sigma^2}. \tag{2.9}$$

Since $m_n < 1$ for each n, we get (see [1], p. 7) for each $l \geq 1$

$$\begin{aligned} F_l^{(n)}(s) &= 1 - m_n^l\left(\frac{1 - s_{n,0}}{m_n^l - s_{n,0}}\right) + \frac{m_n^l\left(\frac{1-s_{n,0}}{m_n^l - s_{n,0}}\right)^2 s}{1 - \left(\frac{m_n^l - 1}{m_n^l - s_{n,0}}\right)s} \\ &= 1 - m_n^l a_{n,l} + \frac{m_n^l a_{n,l}^2 s}{1 - b_{n,l}s}, \end{aligned} \tag{2.10}$$

where $a_{n,l} = \frac{1-s_{n,0}}{m_n^l - s_{n,0}}$, $b_{n,l} = \frac{m_n^l - 1}{m_n^l - s_{n,0}}$, and $s_{n,0}$ is the unique root of $F^{(n)}(s) = s$ that is strictly greater than 1. Note that both $a_{n,l}$ and $b_{n,l}$ converge as $l \to \infty$. We get, by using (2.10) in the definition of $A_{n,t}$, $\lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) = n\frac{s_{n,0}-1}{s_{n,0}}$. From the explicit form of $F^{(n)}$ (see [1], p. 6) we have $\frac{1-p^{(n)}}{1-p^{(n)}s_{n,0}} = \frac{1}{m_n}$, and thus $s_{n,0} = \frac{1-m_n(1-p^{(n)})}{p^{(n)}}$. As a consequence of Condition 1.1, we have that

$$s_{n,0} \to 1,\ p^{(n)} \to p, \text{ and } \sigma^2 = \frac{2p}{1-p}. \tag{2.11}$$

Combining these observations we obtain

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) = \lim_{n\to\infty} n\frac{s_{n,0}-1}{s_{n,0}} = \lim_{n\to\infty} n\frac{(1-p^{(n)})(1-m_n)}{p^{(n)}s_{n,0}} = -\frac{2c}{\sigma^2},$$

which proves the first equality in (2.9). Similarly one can show that

$$\lim_{t\to\infty} \frac{n}{i} A_{n,t}(e^{-\alpha/n}) = n\frac{s_{n,0}-1}{s_{n,0}} - n\frac{(s_{n,0}-1)^2 e^{-\alpha/n}}{(s_{n,0}-e^{-\alpha/n})s_{n,0}}.$$

Using (2.11) and the above display, we now have

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(e^{-\alpha/n}) = -\frac{2c}{\sigma^2} + \frac{2c}{\sigma^2} \lim_{n\to\infty} \frac{s_{n,0}-1}{s_{n,0}-e^{-\alpha/n}} = -\frac{2c}{\sigma^2}\left(1 - \frac{1}{1-\frac{\alpha\sigma^2}{2c}}\right),$$

which proves the second identity in (2.9). □

We will next treat the general case and begin with the following comparison lemma, which extends a result due to Spitzer (see [1], p. 22). The latter is concerned with pgf's with mean 1. The lemma given below extends Spitzer's result to a setting where the two pgf's have the same mean m which may be strictly less than 1.

Lemma 2.3. *Let $f^{(1)}$ and $f^{(2)}$ be pgf's of two $\mathbb{N}_0$ valued random variables having the same mean $m \in (0,1]$ and variances $\sigma_1^2 < \sigma_2^2 \le \infty$. Then there exist integers n_i, $i = 1,2$, such that for all $n \ge 0$*

$$f^{(1)}_{n+n_1}(t) \le f^{(2)}_{n+n_2}(t), \quad \textit{for } t \in [0,1]. \tag{2.12}$$

Proof. The proof is adapted from [1]. Using L'Hospital's rule, we get for $f = f^{(1)}, f^{(2)}$ and $\sigma^2 = \sigma_1^2, \sigma_2^2$

$$\lim_{t\to 1} \frac{f(t) - mt - (1-m)}{(1-t)^2} = \lim_{t\to 1} \frac{f'(t) - m}{2(t-1)} = \frac{\sigma^2 + m^2 - m}{2} =: a. \tag{2.13}$$

Note that $a \in (0,\infty]$. Define $\epsilon(t) := \frac{f(t)-mt-(1-m)}{(1-t)^2}$. We are interested in $\epsilon'(t)$ for t close to 1, $t \in (0,1]$. Once more by L'Hospital's rule, $\lim_{t\to 1} \epsilon'(t) = \lim_{t\to 1} \frac{f'''(t)}{6} \in [0,\infty]$. Thus $\epsilon(t)$ is non-decreasing in a (left) neighborhood of 1 and it converges to a. We define for $f^{(i)}$, $i = 1,2$, a_i and ϵ_i analogous to a, ϵ, by replacing f by $f^{(i)}$ and σ^2 by σ_i^2. Since $\sigma_1^2 < \sigma_2^2$ and the means of $f^{(1)}$ and $f^{(2)}$ are equal, we have that $a_1 < a_2$. Thus, from (2.13) and the monotonicity of ϵ_i near 1, there exists a

$\delta \in (0,1]$, such that $f^{(1)}(t) \le f^{(2)}(t)$ for all $t \in [1-\delta, 1]$. Using the monotonicity of $f^{(i)}$ we now have, for all $n \ge 0$ and $n_1 \le n_2$,

$$f^{(1)}_{n+n_1}(t) \le f^{(2)}_{n+n_2}(t) \quad \text{for } t \in [1-\delta, 1]. \tag{2.14}$$

To show that (2.12) holds, it remains to consider $t \in [0, 1-\delta]$. We can choose n_1 and $n_2 > n_1$, such that $f^{(1)}_{n_1}(0) \in [1-\delta, 1]$ and $f^{(1)}_{n_1}(1-\delta) \le f^{(2)}_{n_2}(0)$, and thus

$$1-\delta \le f^{(1)}_{n_1}(0) \le f^{(1)}_{n_1}(t) \le f^{(1)}_{n_1}(1-\delta) \le f^{(2)}_{n_2}(0) \le f^{(2)}_{n_2}(t) < 1.$$

Since $1-\delta \le f^{(1)}_{n_1}(t) \le f^{(2)}_{n_2}(t)$, we get, using the monotonicity of $f^{(i)}$, that for $n \ge 0$, $f^{(1)}_{n+n_1}(t) \le f^{(2)}_{n+n_2}(t)$, for $t \in [0, 1-\delta]$. Combining this with (2.14) we have (2.12). □

Continuing the proof of Theorem 1.5, we now establish the convergence of the Yaglom distribution of $X^{(n)}$ to that of ξ in the general setting.

Consider first the subcritical case. From Lemma 4.2 in the appendix, it follows that for all $\epsilon > 0$ and $n \in \mathbb{N}$ we can find pgf's of the linear fractional form, $f^{(n,1)}$ and $f^{(n,2)}$, such that their means are m_n and variances are $\sigma^2_{n,1} = \sigma^2_n - \epsilon$ and $\sigma^2_{n,2} = \sigma^2_n + \epsilon$, respectively.

By Lemma 2.3, for all $n, i \in \mathbb{N}$, there exist an l_n and a $t_0 := t_0(n)$, such that for all $t \ge t_0$ and all $r \in [0,1]$

$$[f^{(n,1)}_{\lfloor nt \rfloor - l_n}(r)]^i \le [F^{(n)}_{\lfloor nt \rfloor}(r)]^i \le [f^{(n,2)}_{\lfloor nt \rfloor + l_n}(r)]^i,$$

where $f^{(n,j)}_l$ denotes the l^{th} iterate of $f^{(n,j)}$. Thus, with $A_{n,t}$ as before, for all $t \ge t_0$,

$$m_n^{-\lfloor nt \rfloor}\left(1 - \left[f^{(n,2)}_{\lfloor nt \rfloor + l_n}(0)\right]^i\right) \le A_{n,t}(0) \le m_n^{-\lfloor nt \rfloor}\left(1 - \left[f^{(n,1)}_{\lfloor nt \rfloor - l_n}(0)\right]^i\right). \tag{2.15}$$

Denote by $s_0^{(n,j)}$ the root of $f^{(n,j)}(r) = r$ that is greater 1. Then, for all $n \ge 1$,

$$\frac{n(s_0^{(n,1)} - 1)}{s_0^{(n,1)}} \ge \lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) \ge \frac{n(s_0^{(n,2)} - 1)}{s_0^{(n,2)}}.$$

Similar to the calculation below (2.11), we now have, on letting $n \to \infty$ in the above display,

$$-\frac{2c}{\sigma^2 - \epsilon} \ge \limsup_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) \ge \liminf_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) \ge -\frac{2c}{\sigma^2 + \epsilon}.$$

Letting $\epsilon \to 0$, we have $\lim_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(0) = -\frac{2c}{\sigma^2}$. Similarly, it is seen that

$$\lim_{n\to\infty} \lim_{t\to\infty} \frac{n}{i} A_{n,t}(e^{-\alpha/n}) = -\frac{2c}{\sigma^2}\left(1 - \frac{1}{1 - \frac{\alpha\sigma^2}{2c}}\right). \tag{2.16}$$

Combining the above observations, we have

$$\lim_{n\to\infty} \lim_{t\to\infty} E_{\frac{i}{n}}\left(e^{-\alpha X_t^{(n)}} | X_t^{(n)} > 0\right) = \frac{2c}{2c - \alpha\sigma^2} = \left(1 + \frac{\alpha\sigma^2}{2|c|}\right)^{-1},$$

and this proves Theorem 1.5 for the subcritical case.

We now consider the supercritical case. From (2.7) it follows that the Yaglom distribution $\nu^{(n)}$ of $X^{(n)}$ is the same as the Yaglom distribution $\tilde{\nu}^{(n)}$ of $\tilde{X}^{(n)}$. Thus it suffices, in view of the result for the subcritical case, to show that $\lim_{n\to\infty} n(\tilde{m}_n - 1) = -c$ and $\lim_{n\to\infty} \tilde{\sigma}_n^2 = \sigma^2$.

We begin by showing that $q_n \to 1$ as $n \to \infty$. We argue via contradiction. Suppose $\liminf_{n\to\infty} q_n = q < 1$. Let $\epsilon \in (0, \sigma^2/2)$. By the equicontinuity assumption in Condition 1.1, there exist a $\delta \in (0, 1-q)$ and an n_δ such that for $n \geq n_\delta$

$$|F^{(n)''}(1-\delta) - \sigma^2| \leq |F^{(n)''}(1-\delta) - F^{(n)''}(1)| + |F^{(n)''}(1) - \sigma^2| \leq 2\epsilon < \sigma^2.$$

Since $F^{(n)''}$ is nondecreasing, we have

$$F^{(n)}(1-\delta) \geq F^{(n)}(1) - \delta F^{(n)'}(1) + \frac{\delta^2}{2} F^{(n)''}(1-\delta) \geq 1 - \delta - \delta\frac{c_n}{n} + \frac{\delta^2}{2}(\sigma^2 - 2\epsilon).$$

Choose n large enough so that $q_n < 1-\delta$ and $\frac{\delta^2}{2}(\sigma^2 - 2\epsilon) > \delta\frac{c_n}{n}$. Then $F^{(n)}(1-\delta) > 1-\delta$. Since $q_n < 1-\delta$, we arrive at a contradiction because $F^{(n)}(x) < x$ for all $x \in (q_n, 1)$. The convergence of q_n to 1 and equicontinuity of $F^{(n)''}$ now immediately yield the convergence of $\tilde{\sigma}_n^2$ to σ^2.

We next establish the convergence of $n(\tilde{m}_n - 1)$. Observe that $\tilde{m}_n - m_n = F^{(n)'}(q_n) - F^{(n)'}(1) = -\int_{q_n}^1 F^{(n)''}(u)du$ and thus

$$n(\tilde{m}_n - 1) = n(m_n - 1) - n(1-q_n)\left(\int_{q_n}^1 F^{(n)''}(u)\frac{1}{1-q_n}du\right). \tag{2.17}$$

Moreover,

$$\begin{aligned} 1 - q_n = 1 - F^{(n)}(q_n) &= \int_{q_n}^1 (F^{(n)'}(u) - F^{(n)'}(1))du + (1-q_n)m_n \\ &= -\int_{q_n}^1 \int_u^1 F^{(n)''}(v)dvdu + (1-q_n)m_n. \end{aligned}$$

Rearranging terms gives

$$(1-q_n)(m_n - 1) = \frac{(1-q_n)^2}{2}\left(\int_{q_n}^1 F^{(n)''}(v)\frac{v - q_n}{(1-q_n)^2/2}dv\right).$$

Thus

$$n(1-q_n) = 2n(m_n - 1)\left(\int_{q_n}^1 F^{(n)''}(v)\frac{v - q_n}{(1-q_n)^2/2}dv\right)^{-1}. \tag{2.18}$$

Combining equations (2.17) and (2.18), we get

$$n(\tilde{m}_n - 1) = n(m_n - 1)\left(1 - 2\frac{\int_{q_n}^1 F^{(n)''}(u)g_{n,1}(u)du}{\int_{q_n}^1 F^{(n)''}(v)g_{n,2}(v)dv}\right),$$

where $g_{n,1}(u) = \frac{1}{1-q_n}$ and $g_{n,2}(v) = \frac{v-q_n}{(1-q_n)^2/2}$. To complete the proof, we will now show that the ratio of integrals in the last display converges to 1, as $n \to \infty$. In fact, we will show that each integral converges to σ^2. Observing that $\int_{q_n}^1 g_{n,i}(u)du = 1$, $i = 1, 2$, and using the monotinicity of $F^{(n)''}$, we get, for $i = 1, 2$,

$$\begin{aligned}\limsup_{n\to\infty} \int_{q_n}^1 F^{(n)''}(u) g_{n,i}(u) du &\leq \limsup_{n\to\infty} \int_{q_n}^1 F^{(n)''}(1) g_{n,i}(u) du \\ &= \limsup_{n\to\infty} (\sigma_n^2 + m_n^2 - m_n) = \sigma^2. \qquad (2.19)\end{aligned}$$

Similarly,

$$\begin{aligned}\liminf_{n\to\infty} \int_{q_n}^1 F^{(n)''}(u) g_{n,i}(u) du &\geq \liminf_{n\to\infty} \int_{q_n}^1 F^{(n)''}(q_n) g_{n,i}(u) du \\ &= \liminf_{n\to\infty} F^{(n)''}(q_n) = \sigma^2. \qquad (2.20)\end{aligned}$$

This proves $n(\tilde{m}_n - 1) \to -c$ and as argued earlier this proves Theorem 1.5 for the supercritical case.

Proof of Theorem 1.8. We will show that for all $i \in \mathbb{N}$

$$\lim_{t\to\infty} E_{\frac{i}{n}} \left(e^{-\alpha X_t^{(n)\uparrow}}\right) = Q^{(n)'}(e^{-\alpha/n}) e^{-\alpha/n}, \quad \alpha \geq 0. \qquad (2.21)$$

Since $Q^{(n)'}$ is continuous at 1 (see [1], p. 40), this will show that $h_n(\alpha)$ defined by the right side of (2.21) is a Laplace transform of some random variable $X_\infty^{(n)\uparrow}$ with probability law $\nu^{(n)\uparrow}$. Similar to the calculation in [1], pp. 59-60, we have for $\alpha > 0$

$$E_{\frac{i}{n}} \exp\left(-\alpha X_t^{(n)\uparrow}\right) = \frac{\partial}{\partial \alpha} \left(\frac{n}{i} \left[m_n^{-\lfloor nt \rfloor} \left(1 - \left[F_{\lfloor nt \rfloor}^{(n)}(e^{-\alpha/n}) \right]^i \right) \right] \right).$$

Taking the limit, as $t \to \infty$, we get

$$\begin{aligned}\lim_{t\to\infty} E_{\frac{i}{n}} \exp\left(-\alpha X_t^{(n)\uparrow}\right) &= \lim_{t\to\infty} \left(\left[F_{\lfloor nt \rfloor}^{(n)}(e^{-\alpha/n}) \right]^{i-1} Q_{\lfloor nt \rfloor}^{(n)'}(e^{-\alpha/n}) e^{-\alpha/n} \right) \\ &= Q^{(n)'}(e^{-\alpha/n}) e^{-\alpha/n}. \qquad (2.22)\end{aligned}$$

This proves (2.21) and thus $X_t^{(n)\uparrow}$ converges in distribution, as $t \to \infty$, to $X_\infty^{(n)\uparrow}$. It is easily checked that $\nu^{(n)\uparrow}$ is a stationary distribution.

We now show that, as $n \to \infty$, $\nu^{(n)\uparrow}$ converges weakly to $\nu^\uparrow$. For this it suffices to show that

$$\lim_{n\to\infty} \lim_{t\to\infty} E_{\frac{i}{n}} \exp\left(-\alpha X_t^{(n)\uparrow}\right) = \left(\frac{1}{1 - \frac{\alpha\sigma^2}{2c}} \right)^2, \quad \alpha \in (0, \infty). \qquad (2.23)$$

From (2.22) we have

$$\lim_{n\to\infty} \lim_{t\to\infty} E_{\frac{i}{n}} \exp\left(-\alpha X_t^{(n)\uparrow}\right) = \lim_{n\to\infty} \frac{\partial}{\partial \alpha} \left(-n Q^{(n)}(e^{-\alpha/n}) \right).$$

We next show that for $\alpha \in (0, \infty)$

$$\lim_{n\to\infty} \frac{\partial}{\partial \alpha} \left(-n Q^{(n)}(e^{-\alpha/n}) \right) = \frac{\partial}{\partial \alpha} \lim_{n\to\infty} \left(-n Q^{(n)}(e^{-\alpha/n}) \right). \qquad (2.24)$$

Define $g_n(\alpha) := -nQ^{(n)}(e^{-\alpha/n})$. Note that $Q^{(n)}(s) = \sum_{j=0}^{\infty} v_j^{(n)} s^j$, $0 \le s < 1$, for some $\{v_j^{(n)}\}_{j\in\mathbb{N}_0}$ with $v_0^{(n)} < 0$ and $v_j^{(n)} > 0$, for $j \ge 1$, and $\lim_{s\nearrow 1} Q^{(n)\prime}(s) = 1$ (see [1], pp. 40-41); in particular, $Q^{(n)}$ is convex. Next note that $|g_n'(\alpha)| \le \sup_{s\in(0,1)} \left\{|Q^{(n)\prime}(s)s|\right\} = 1$, which implies that $\{g_n\}_{n\in\mathbb{N}}$ is equicontinuous on $[0,\infty)$. From (2.6) and (2.16) we have that g_n converges pointwise to g, where $g(\alpha) = \frac{2c\alpha}{2c-\alpha\sigma^2}$, $\alpha \ge 0$. Thus, by equicontinuity and uniform boundedness on compacts of $\{g_n\}$, we have that for every interval $[a,b]$, $0 < a < b < \infty$, there exists a subsequence $\{g_{n_k}\}$ which convergences to g uniformly on $[a,b]$. Thus, by [2], (9.12.1), p. 229, g is analytic on $(0,\infty)$ and $\lim_{k\to\infty} \frac{\partial}{\partial\alpha} g_{n_k}(\alpha) = \frac{\partial}{\partial\alpha} g(\alpha)$, for $\alpha \in (0,\infty)$. This proves equation (2.24). Equation (2.23) is now immediate on combining the above two displays.

Proof of Theorem 1.11. Let $H_l^{(n)}(i,\cdot)$ be the l^{th} iterate of the pgf $H^{(n)}(i,\cdot)$ of $Y_1^{(n)}$ given $Y_0^{(n)} = i$. Then, for each $n \in \mathbb{N}$ and $s \in [0,1]$, $H_l^{(n)}(i,s) = [F_l^{(n)}(s)]^i \prod_{r=0}^{l-1} G^{(n)}(F_r^{(n)}(s))$ and $H_l^{(n)}(i,\cdot)$ converges, as $l \to \infty$, to the pgf $\tilde{\Pi}^{(n)}$ given as $\tilde{\Pi}^{(n)}(s) = \prod_{r=0}^{\infty} G^{(n)}(F_r^{(n)}(s))$ (see [12]). This shows that, for each $n \in \mathbb{N}$, $V^{(n)}$ has a unique stationary distribution $\eta^{(n)}$, which is characterized through its pgf $\Pi^{(n)}(s) = \prod_{r=0}^{\infty} G^{(n)}(F_r^{(n)}(s^{1/n}))$. We now show that, as $n \to \infty$, $\eta^{(n)}$ converges weakly to η. Let $\alpha(n,l) = \frac{(m_n^l - 1)F^{(n)''}(1)}{2(m_n-1)m_n}$. Then $V^{(n)}(t) = W_{\lfloor nt\rfloor}^{(n)} \frac{\alpha(n,\lfloor nt\rfloor)}{n}$, where $W_l^{(n)} = \frac{Y_l^{(n)}}{\alpha(n,l)}$. Theorem 3 of [4] gives the weak convergence, as $t \to \infty$ and $n \to \infty$, of $W_{\lfloor nt\rfloor}^{(n)}$ to W, where W has a $\Gamma\left(\frac{2\iota}{\sigma^2}, 1\right)$ distribution. The result now follows on observing that

$$\lim_{n\to\infty}\lim_{t\to\infty} \frac{\alpha(m_n,\lfloor nt\rfloor)}{n} = \lim_{n\to\infty} \frac{-F^{(n)''}(1)}{2n(m_n-1)m_n} = -\frac{\sigma^2}{2c}.$$

3. Proofs: Multitype Case

In this section we prove Theorems 1.13, 1.18 and 1.20.

Proof of Theorem 1.13. Denote by $F_p^{(n)} = (F_{p,(1)}^{(n)}, \ldots, F_{p,(k)}^{(n)})$ the p^{th} iterate of $F^{(n)}$, i.e. for $\mathbf{s} \in C$ and $p \in \mathbb{N}_0$, $F_{p+1}^{(n)}(\mathbf{s}) = F^{(n)}(F_p^{(n)}(\mathbf{s}))$, where $F_0^{(n)}(\mathbf{s}) = \mathbf{s}$. Let $\gamma^{(n)}(\mathbf{s}) := \lim_{p\to\infty} \frac{\mathbf{v}^{(n)\prime}[\mathbf{1} - F_p^{(n)}(\mathbf{s})]}{\rho_n^p}$, $\mathbf{s} \in C$. The latter limit exists and defines a positive function on $C \setminus \{\mathbf{1}\}$ that is continuous at $\mathbf{1}$ (see [1, Theorems V.4.1]). We will next show that, for each $\mathbf{s} \in C$,

$$\lim_{t\to\infty} E_{\frac{\mathbf{i}}{n}}(e^{-\mathbf{s}' X_t^{(n)}} | X_t^{(n)} \neq \mathbf{0}) = \frac{\gamma^{(n)}(\mathbf{0}) - \gamma^{(n)}(\mathbf{r}_n)}{\gamma^{(n)}(\mathbf{0})}, \tag{3.1}$$

where $\mathbf{r}_n = (e^{-s_1/n}, \cdots, e^{-s_k/n})'$ and $\mathbf{s} = (s_1, \cdots s_k)'$. Denoting by $\nu^{(n)}$ the probability law corresponding to the Laplace transform on the right hand side of the above display, we will then have that $\nu^{(n)}$ is the Yaglom distribution of $X^{(n)}$. The fact that $\nu^{(n)}$ is also a qsd is a consequence of Lemma 4.1 in the appendix. We now prove (3.1).

Elementary calculations give

$$E_{\frac{\mathbf{i}}{n}}(e^{-\mathbf{s}'X_t^{(n)}}|X_t^{(n)} \neq \mathbf{0}) = E(e^{-\frac{\mathbf{s}'}{n}Z_{\lfloor nt\rfloor}^{(n)}}|Z_0^{(n)} = \mathbf{i}, Z_{\lfloor nt\rfloor}^{(n)} \neq \mathbf{0}) = 1 - \frac{A_{n,t}(\mathbf{r}_n)}{A_{n,t}(0)},$$

where, for $\theta \in C$, $A_{n,t}(\theta) = \rho_n^{-\lfloor nt\rfloor}\left(1 - (F_{\lfloor nt\rfloor}^{(n)}(\theta))^{\mathbf{i}}\right)$. Next note that

$$\begin{aligned}(F_{\lfloor nt\rfloor}^{(n)}(\mathbf{r}_n))^{\mathbf{i}} &= \prod_{\alpha=1, i_\alpha \neq 0}^{k} \sum_{r=1}^{i_\alpha} \binom{i_\alpha}{r} 1^{i_\alpha - r}\left(F_{\lfloor nt\rfloor,(\alpha)}^{(n)}(\mathbf{r}_n) - 1\right)^r \\ &= 1 - \mathbf{i}'\left(\mathbf{1} - F_{\lfloor nt\rfloor}^{(n)}(\mathbf{r}_n)\right) + \tilde{R}_{n,t}, \qquad (3.2)\end{aligned}$$

where the term $\tilde{R}_{n,t}$ is a linear combination of terms of the form $\left(\mathbf{1} - F_{\lfloor nt\rfloor}^{(n)}(\mathbf{r}_n)\right)^{\mathbf{d}}$, where $\mathbf{d} = (d_1, \ldots, d_k)$ and $\sum_{j=1}^k d_j > 1$. Since $E_{\mathbf{1}}(||Z_1^{(n)}|| \log ||Z_1^{(n)}||) < \infty$, we have

$$\lim_{t\to\infty} \rho_n^{-\lfloor nt\rfloor}(\mathbf{1} - F_{\lfloor nt\rfloor}^{(n)}(\mathbf{r}_n)) = \gamma^{(n)}(\mathbf{r}_n)\mathbf{u}^{(n)} \qquad (3.3)$$

(see [1, Theorems V.4.1]), and thus

$$\lim_{t\to\infty} \rho_n^{-\lfloor nt\rfloor}\tilde{R}_{n,t} = 0. \qquad (3.4)$$

This implies $\lim_{t\to\infty} A_{n,t}(\mathbf{r}_n) = \gamma^{(n)}(\mathbf{r}_n)\mathbf{i}'\mathbf{u}^{(n)}$. In exactly the same way, we see that $\lim_{t\to\infty} A_{n,t}(0) = \gamma^{(n)}(\mathbf{0})\mathbf{i}'\mathbf{u}^{(n)}$. Combining the above observations, we now have (3.1) and the result follows.

Proof of Theorem 1.18. The proof is similar to that of Theorem 4.3.1 of [8] and thus only a sketch is provided. Let

$$Y^{(n)}(t) := \mathbf{y}^{(n)} + n\int_0^t (\mathbf{M}' - \mathbf{I})X^{(n)}(\tau-)dA_n(\tau),$$

where $A_n(\tau) = \frac{\lfloor n\tau\rfloor}{n}$, $\tau \geq 0$. Define a Markov chain $\{(\check{X}^{(n)}(k), \check{Y}^{(n)}(k))\}_{k\in\mathbb{N}_0}$ as

$$(\check{X}^{(n)}(k), \check{Y}^{(n)}(k)) = (X^{(n)}(k/n), Y^{(n)}(k/n)), \quad k \in \mathbb{N}_0.$$

This chain has transition probabilities given by

$$\check{P}^{(n)}(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}) = \check{Q}^{(n)}(\mathbf{x}, \tilde{\mathbf{x}})1_{\tilde{\mathbf{y}}=\mathbf{y}+(\mathbf{M}'-\mathbf{I})\mathbf{x}},$$

where $\check{Q}^{(n)}(\cdot,\cdot)$ is the transition probability of the process $\check{X}^{(n)}$. Let

$$(\check{L}^{(n)}f)(\mathbf{x}, \mathbf{y}) = \sum_{\tilde{\mathbf{x}},\tilde{\mathbf{y}}} \check{P}^{(n)}(\mathbf{x}, \mathbf{y}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}})[f(\tilde{\mathbf{x}}, \tilde{\mathbf{y}}) - f(\mathbf{x}, \mathbf{y})]$$

and $L^{(n)} = n\check{L}^{(n)}$. Then we have that for each smooth test function f

$$f(X^{(n)}(t), Y^{(n)}(t)) - \int_0^t (L^{(n)}f)(X^{(n)}(\tau-), Y^{(n)}(\tau-))dA^{(n)}(\tau)$$

is a martingale (with respect to the filtration generated by $(X^{(n)}, Y^{(n)})$). Let $f(\mathbf{x}, \mathbf{y}) = \phi(\mathbf{x} - \mathbf{y})$ with $\phi \in C_c^\infty(\mathbb{R}^k)$. Taking $\mathbf{y}^{(n)} = \mathbf{0}$ and using a Taylor series expansion about $(X^{(n)}(0), Y^{(n)}(0))$, we have that

$$\begin{aligned}&\phi(X^{(n)}(t) - Y^{(n)}(t)) - \phi(X^{(n)}(0)) + E_n(t)\\&\quad - \int_0^t \sum_{i=1}^k (\mathbf{C}^{(n)\prime} X^{(n)}(\tau-))_i \frac{\partial \phi}{\partial s_i}(X^{(n)}(\tau-) - Y^{(n)}(\tau-))dA^{(n)}(\tau)\\&\quad - \frac{1}{2}\int_0^t \sum_{i,j,l=1}^k (X^{(n)}(\tau-))_l \sigma_{i,j}^{(n)}(l)\frac{\partial^2\phi}{\partial s_i \partial s_j}(X^{(n)}(\tau-) - Y^{(n)}(\tau-))dA^{(n)}(\tau)\end{aligned}$$

is a martingale, where the remainder $E_n(t)$ is such that $\sup_{0\le t\le T}|E_n(t)| \to 0$, in probability for all $T \in \mathbb{R}_+$. From Condition 1.15 and the Perron-Frobenius Theorem it follows (see Remark 4.3.2 in [8]) that with $P = \mathbf{u}\mathbf{v}'$

$$(\mathbf{I} - P')X^{(n,0)} \longrightarrow \mathbf{0} \ \text{ in probability,} \tag{3.5}$$

uniformly on compacts for all $t_0 > 0$.

Also, using the fact that $P'(M' - I) = 0$, we have $P'Y^{(n)}(t) = 0$ for all $t \ge 0$. Using these observations, it can be shown that, for all $t \in \mathbb{R}_+$,

$$\lim_{n\to\infty}\int_0^t E|(\hat{L}^{(n)}\phi)(X^{(n)}(\tau-), \xi^{(n)}(\tau-)) - (L\phi)(\xi^{(n)}(\tau-))|dA_\tau^{(n)} = 0, \tag{3.6}$$

where $\xi^{(n)} = X^{(n)} - Y^{(n)}$, and for $(\mathbf{x}, \mathbf{z}) \in \mathbb{R}_+^k \times \mathbb{R}^k$,

$$(\hat{L}^{(n)}\phi)(\mathbf{x}, \mathbf{z}) = \sum_{i=1}^k ((\mathbf{C}^{(n)})'\mathbf{x})_i \frac{\partial\phi}{\partial s_i}(\mathbf{z}) + \frac{1}{2}\sum_{l=1}^k \sum_{i,j=1}^k \mathbf{x}_l \sigma_{i,j}(l)\frac{\partial^2\phi}{\partial s_i\partial s_j}(\mathbf{z}).$$

and

$$(L\phi)(\mathbf{z}) = \sum_{i=1}^k (\mathbf{C}'\mathbf{P}'\mathbf{z})_i \frac{\partial\phi}{\partial s_i}(\mathbf{z}) + \frac{1}{2}\sum_{l=1}^k\sum_{i,j=1}^k (\mathbf{P}'\mathbf{z})_l \sigma_{i,j}(l)\frac{\partial^2\phi}{\partial s_i \partial s_j}(\mathbf{z}).$$

Following [8], one can show that $\xi^{(n)}$ is a tight sequence in $D(\mathbb{R}_+ : \mathbb{R}^k)$, and, using (3.6), it follows that if ξ is any weak limit of $\xi^{(n)}$, then

$$\phi(\xi(t)) - \phi(\xi(0)) - \int_0^t (L\phi)(\xi(s))ds$$

is an $\mathcal{F}_t^\xi := \sigma(\xi(s) : s \le t)$ martingale. Thus $\xi^{(n)}$ converges weakly to the diffusion ξ with generator L and initial condition μ. Next note that $\zeta^{(n)} = X^{(n)\prime}\mathbf{u}^{(n)} = \xi^{(n)\prime}\mathbf{u}^{(n)}$. The weak convergence of $\xi^{(n)}$ to ξ shows that $\zeta^{(n)}$ converges in distribution to $\xi'\mathbf{u} \equiv \zeta$. Let $g \in C_c^\infty(\mathbb{R}_+)$ and define $\phi \in C_c^\infty(\mathbb{R}_+^k)$ as $\phi(\mathbf{z}) = g(\mathbf{z}'\mathbf{u})$, $\mathbf{z} \in \mathbb{R}_+^k$. Then

$$\begin{aligned}(Lg)(\mathbf{z}'\mathbf{u}) &= \sum_{i=1}^k (\mathbf{z}'\mathbf{P}\mathbf{C})_i' g'(\mathbf{z}'\mathbf{u})u_i + \frac{1}{2}\sum_{l=1}^k\sum_{i,j=1}^k (\mathbf{z}'\mathbf{P})_l' g''(\mathbf{z}'\mathbf{u})u_i u_j \sigma_{i,j}^{(n)}(l)\\&= (\mathbf{z}'\mathbf{u}\mathbf{v}'\mathbf{C}\mathbf{u})g'(\mathbf{z}'\mathbf{u}) + \frac{1}{2}\sum_{l=1}^k (\mathbf{z}'\mathbf{u}\mathbf{v}')_l' \mathbf{u}'\sigma(l)\mathbf{u} g''(\mathbf{z}'\mathbf{u}).\end{aligned}$$

Since $\mathbf{v}'\mathbf{C}\mathbf{u} = c$, we see that ζ is a Markov process with generator

$$(\tilde{L}g)(x) = cxg'(x) + Qxg''(x), \quad x \in \mathbb{R}_+.$$

This proves the first part of the theorem.

Next noting that $P'X^{(n)} = P'\xi^{(n)}$ and recalling (3.5) we see that $X^{(n,0)}$ converges weakly to $P'\xi^{(0)}$, where $\xi^{(0)}(t) = \xi(t+t_0)$, $t \geq 0$. Finally, since $P = \mathbf{u}\mathbf{v}'$ and $\zeta = \xi' u$ we have that $P'\xi^{(0)} = \mathbf{v}\zeta^{(0)} = X^{(0)}$ and the result follows.

Proof of Theorem 1.20. We begin with some preliminary results. For each $n \in \mathbb{N}, \mathbf{s} \in C$, and $\alpha = 1, \ldots, k$, define $q_\alpha^{(n)}[\mathbf{s}] = \frac{1}{2} \sum_{\beta\gamma} \frac{\partial^2 F_{(\alpha)}^{(n)}(\mathbf{1})}{\partial s_\beta \partial s_\gamma} s_\beta s_\gamma$, $Q_n[\mathbf{s}] = \sum_\alpha v_\alpha^{(n)} q_\alpha^{(n)}[\mathbf{s}]$, $Q_n = Q_n[\mathbf{u}^{(n)}]$. Let

$$\pi_{n,p} = \begin{cases} \sum_{r=1}^p \rho_n^{r-2} & \text{for } p = 1, 2, \ldots \\ 0 & \text{for } p = 0 \end{cases} \tag{3.7}$$

and

$$h_{n,p}(\mathbf{s}) = \rho_n^p \mathbf{v}^{(n)'} \mathbf{s} / (1 + \pi_{n,p} Q_n \mathbf{v}^{(n)'} \mathbf{s}), \quad \mathbf{s} \in C. \tag{3.8}$$

In what follows, $\{o(p,n) | p, n \in \mathbb{N}\}$ will denote a collection of functions from C to $\mathbb{R}^k$ satisfying the property that for every $\epsilon > 0$, there exist $N, P \in \mathbb{N}$, such that for $n \geq N$ and $p \geq P$ we have $\sup_{\mathbf{s} \in C} ||o(p,n)(\mathbf{s})|| < \epsilon$.

Proposition 3.1. *Assume Conditions 1.14, 1.15, and 1.17. For each $n, p \in \mathbb{N}$ and $\mathbf{s} \in C$*

$$\mathbf{1} - F_p^{(n)}(\mathbf{s}) = h_{n,p}(\mathbf{1} - \mathbf{s})\{\mathbf{u}^{(n)} + o(p,n)(\mathbf{s})\}. \tag{3.9}$$

The proof of the proposition is immediate from Theorem 1 of [13] (see equation (2.3) therein) and is therefore omitted. The following corollary facilitates the proof of the main result.

Corollary 3.2. *Assume Conditions 1.14, 1.15, and 1.17. For any convergent sequence $\{\mathbf{r}_n\} \subset C$,*

$$\lim_{n\to\infty} \lim_{t\to\infty} \rho_n^{-\lfloor nt \rfloor} \left(\mathbf{1} - F_{\lfloor nt \rfloor}^{(n)}(\mathbf{r}_n)\right) = \lim_{n\to\infty} \lim_{t\to\infty} \rho_n^{-\lfloor nt \rfloor} h_{n,\lfloor nt \rfloor}(\mathbf{1} - \mathbf{r}_n)\mathbf{u}^{(n)}. \tag{3.10}$$

Proof. Let $a(n,t) = \rho_n^{-\lfloor nt \rfloor} h_{n,\lfloor nt \rfloor}(\mathbf{1} - \mathbf{r}_n)\mathbf{u}^{(n)}$ and with $o(p,n)$ as in Proposition 3.1, let $b(n,t) = \rho_n^{-\lfloor nt \rfloor} h_{n,\lfloor nt \rfloor}(\mathbf{1} - \mathbf{r}_n) o(\lfloor nt \rfloor, n)(\mathbf{r}_n)$. Note that for each n we have from (3.3) and (3.9)

$$\lim_{t\to\infty} \rho_n^{-\lfloor nt \rfloor} \left(\mathbf{1} - F_{\lfloor nt \rfloor}^{(n)}(\mathbf{r}_n)\right) = \lim_{t\to\infty} (a(n,t) + b(n,t)) = \gamma^{(n)}(\mathbf{r}_n)\mathbf{u}^{(n)}.$$

Moreover, $\lim_{t\to\infty} a(n,t)$ and $\lim_{t\to\infty} o(\lfloor nt \rfloor, n)(\mathbf{r}_n)$ exist. Denoting the latter limit by $o(\infty, n)(\mathbf{r}_n)$, we have

$$\lim_{t\to\infty} b(n,t) = \frac{\mathbf{v}^{(n)'}(\mathbf{1} - \mathbf{r}_n) o(\infty, n)(\mathbf{r}_n)}{1 + \pi_{n,\infty} Q_n \mathbf{v}^{(n)'}(\mathbf{1} - \mathbf{r}_n)} =: d(n),$$

where $\pi_{n,\infty} := \frac{\rho_n^{-1}}{1-\rho_n}$. Since $\lim_{n\to\infty} o(\infty, n)(\mathbf{r}_n) = 0$, we get that $\lim_{n\to\infty} d(n) = 0$, and thus $\limsup_{n\to\infty} || \lim_{t\to\infty} b(n,t)|| = \limsup_{n\to\infty} ||d(n)|| = 0$. The result follows. □

We now prove Theorem 1.20. In view of Theorem 1.13, it suffices to show that, for $\mathbf{s} \in C$,

$$\lim_{n\to\infty} \lim_{t\to\infty} E_{\frac{\mathbf{i}}{n}}(e^{-\mathbf{s}' X_t^{(n)}} | X_t^{(n)} \neq \mathbf{0}) = \frac{1}{1 - \frac{Q}{c}\mathbf{v}'\mathbf{s}}. \tag{3.11}$$

With $A_{n,t}$ and $\mathbf{r}_n$ as in the proof of Theorem 1.13, we have

$$E_{\frac{\mathbf{i}}{n}}(e^{-\mathbf{s}' X_t^{(n)}} | X_t^{(n)} \neq \mathbf{0}) = 1 - \frac{A_{n,t}(\mathbf{r}_n)}{A_{n,t}(0)}.$$

Using Proposition 3.1, Corollary 3.2, and equations (3.2) and (3.4), we get

$$\lim_{n\to\infty} \lim_{t\to\infty} nA_{n,t}(\mathbf{r}_n) = \lim_{n\to\infty} n\mathbf{i}'\mathbf{u}^{(n)} \left(\frac{\mathbf{v}^{(n)'}(\mathbf{1} - \mathbf{r}_n)}{1 + \pi_{n,\infty} Q_n \mathbf{v}^{(n)'}(\mathbf{1} - \mathbf{r}_n)} \right) = \frac{\mathbf{i}'\mathbf{u}}{(\mathbf{v}'\mathbf{s})^{-1} - \frac{Q}{c}},$$

where the last equality follows on noting that $n\mathbf{v}^{(n)'}(\mathbf{1} - \mathbf{r}_n) \to \mathbf{v}'\mathbf{s}$, $\frac{\pi_{n,\infty}}{n} \to -\frac{1}{c}$, and using Lemma 4.4. Setting $\mathbf{r}_n = 0$ in the above display, we have

$$\lim_{n\to\infty} \lim_{t\to\infty} nA_{n,t}(0) = -\frac{c\mathbf{i}'\mathbf{u}}{Q}.$$

Combining the above observations we have (3.11) and the result follows.

4. Appendix

Lemma 4.1. *Let $\mathbb{S}_n^k = \{\mathbf{x} \in \mathbb{R}_+^k | n\mathbf{x} \in \mathbb{N}_0^k\}$ and $\{X_t\}_{t\in\mathbb{R}_+}$ be an $\mathbb{S}_n^k$ valued Markov process with $\mathbf{0}$ as an absorbing state, such that $X_t = X_{\lfloor nt \rfloor/n}$, $t \geq 0$. Suppose for some $\nu \in \mathcal{P}(\mathbb{S}_n^k)$, $P_{\mathbf{y}}(X_t \in \cdot | X_t \neq \mathbf{0})$ converges weakly, as $t \to \infty$, to ν for all $\mathbf{y} \in \mathbb{S}_n^k$. Then ν is a qsd for $\{X_t\}_{t\in\mathbb{R}_+}$.*

Proof. We need to show that for each $A \subseteq \mathbb{S}_n^k$ and $t \geq 0$

$$P_\nu(X_t \in A | X_t \neq \mathbf{0}) = \nu(A). \tag{4.1}$$

The left hand side of (4.1) equals

$$\frac{P_\nu(X_t \in A, X_t \neq \mathbf{0})}{P_\nu(X_t \neq \mathbf{0})}. \tag{4.2}$$

Letting $A^\circ := A \setminus \{\mathbf{0}\}$, and denoting the measure $P_{\mathbf{y}}(X_t \in \cdot | X_t \neq \mathbf{0})$ by $\nu_t^{\mathbf{y}}$,

$$\begin{aligned} P_\nu(X_t \in A, X_t \neq \mathbf{0}) &= \int P(X_t \in A^\circ | X_0 = \mathbf{x})\nu(d\mathbf{x}) \\ &= \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} \int P(X_t \in A^\circ | X_0 = \mathbf{x})\nu_s^{\mathbf{y}}(d\mathbf{x}) \\ &= \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} \int P(X_{t+s} \in A^\circ | X_s = \mathbf{x})\nu_s^{\mathbf{y}}(d\mathbf{x}) \\ &= \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} \frac{1}{P_{\mathbf{y}}(X_s \neq \mathbf{0})} P_{\mathbf{y}}(X_{t+s} \in A, X_{t+s} \neq \mathbf{0}), \end{aligned}$$

where $\mathbb{S}_n = \{\frac{j}{n} | j \in \mathbb{N}_0\}$, the second equality follows from the assumption in the lemma while the third and fourth use the Markov property of X and the observation that $P(X_{t+s} \in A^\circ | X_s = \mathbf{0}) = 0$.

Setting $A = \mathbb{S}_n$, we have

$$P_\nu(X_t \neq \mathbf{0}) = \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} \frac{1}{P_{\mathbf{y}}(X_s \neq \mathbf{0})} P_{\mathbf{y}}(X_{t+s} \neq \mathbf{0}).$$

Combining the above, we have

$$\begin{aligned} P_\nu(X_t \in A | X_t \neq \mathbf{0}) &= \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} \frac{P_{\mathbf{y}}(X_{t+s} \in A, X_{t+s} \neq \mathbf{0})}{P_{\mathbf{y}}(X_{t+s} \neq \mathbf{0})} \\ &= \lim_{\substack{s\to\infty \\ s\in\mathbb{S}_n}} P_{\mathbf{y}}(X_{t+s} \in A | X_{t+s} \neq \mathbf{0}) = \nu(A), \end{aligned}$$

from which the result follows. □

Lemma 4.2. *Let $m \in (0,1)$ and $\sigma^2 \in (0,\infty)$. Then there exists a pgf f of linear fractional form,*

$$f(s) = 1 - \frac{b}{1-p} + \frac{bs}{1-ps}, \quad s \in [0,1], \tag{4.3}$$

with $b, p \in (0,1)$, $b < 1-p$, such that the corresponding probability distribution has mean m and variance σ^2. Specifically,

$$p = \frac{\sigma^2/m + m - 1}{2 + \sigma^2/m + m - 1} \quad \text{and} \quad b = m\left(1 - \frac{\sigma^2/m + m - 1}{2 + \sigma^2/m + m - 1}\right)^2. \tag{4.4}$$

Proof. Fix $b, p \in (0,1)$, $b < 1-p$. Define f by (4.3). Then $f'(s) = \frac{b}{(1-ps)^2}$ and $f''(s) = \frac{2bp}{(1-ps)^3}$. The mean $\bar{m}$ and the variance $\bar{\sigma}^2$ of the probability distribution corresponding to f is given as $\bar{m} = f'(1) = \frac{b}{(1-p)^2}$ and

$$\begin{aligned} \bar{\sigma}^2 &= f''(1) - [f'(1)]^2 + [f'(1)] = \frac{2bp(1-p) - b^2 + b(1-p)^2}{(1-p)^4} \\ &= \bar{m}\left(2\frac{p}{(1-p)} - \bar{m} + 1\right). \end{aligned}$$

Solving the last two equations for p and b, we get (4.4) with $m = \bar{m}$ and $\sigma^2 = \bar{\sigma}^2$. □

Recall the notation introduced below Theorem 1.5.

Lemma 4.3. *Assume that $Z_0^{(n)}$ has distribution μ (supported on $\mathbb{N}$). Then there exist a probability measure $P_\mu^{(n)\uparrow}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ such that as $s \to \infty$*

$$\hat{P}_\mu^{(n)}(\Theta | T > s) \to P_\mu^{(n)\uparrow}(\Theta), \quad \text{for all } \Theta \in \mathcal{F}_t, \quad t \in \mathbb{R}_+.$$

Furthermore if $\{Z_k^{(n)\uparrow}\}_{k\in\mathbb{N}_0}$ is a Markov chain with state space $\mathbb{N}$, l-step transition function

$$p_l^{(n)\uparrow}(i,j) = P(Z_l^{(n)} = j | Z_0^{(n)} = i)\frac{j}{i} m_n^{-l}, \tag{4.5}$$

and initial distribution μ*, then* $P_\mu^{(n)\uparrow}$ *is the law of* $\{X_t^{(n)\uparrow}\}_{t\in\mathbb{R}_+}$*, where*

$$X_t^{(n)\uparrow} := \frac{1}{n} Z_{\lfloor nt \rfloor}^{(n)\uparrow}, \quad t \in \mathbb{R}_+.$$

Proof. The proof is along the lines of [1], p. 58. Fix $\alpha \in \mathbb{N}$ and $0 \le t_1 < \cdots < t_\alpha < t_\alpha + s$. Let $k_l = \lfloor nt_l \rfloor$, $l = 1, \ldots, \alpha$, and $\tilde{k} = \lfloor n(t_\alpha + s) \rfloor$. First assume $Z_0 = i$. Then

$$\begin{aligned}
&P_{i/n}\left(X_{t_1}^{(n)} = \frac{i_1}{n}, \ldots, X_{t_\alpha}^{(n)} = \frac{i_\alpha}{n} | t_\alpha + s < T_{X^{(n)}} < \infty\right) \\
&= P_i(Z_{k_1}^{(n)} = i_1, \ldots, Z_{k_\alpha}^{(n)} = i_\alpha | \tilde{k} < T_{Z^{(n)}} < \infty) \\
&= P_i(Z_{k_1}^{(n)} = i_1, \ldots, Z_{k_\alpha}^{(n)} = i_\alpha) \frac{\sum_{j=1}^\infty P_{\tilde{k}-k_\alpha}(i_\alpha, j)}{\sum_{j=1}^\infty P_{\tilde{k}}(i,j)} \\
&= P_i(Z_{k_1}^{(n)} = i_1, \ldots, Z_{k_\alpha}^{(n)} = i_\alpha) \frac{P_{\tilde{k}-k_\alpha}(1,1)}{P_{\tilde{k}}(1,1)} \frac{\sum_{j=1}^\infty \frac{P_{\tilde{k}-k_\alpha}(i_\alpha, j)}{P_{\tilde{k}-k_\alpha}(1,1)}}{\sum_{j=1}^\infty \frac{P_{\tilde{k}}(i,j)}{P_{\tilde{k}}(1,1)}}. \qquad (4.6)
\end{aligned}$$

Using Theorem I.7.4 of [1], we get that the right hand side of (4.6) converges, as $\tilde{k} \to \infty$, to

$$\begin{aligned}
&P_{i/n}\left(X_{t_1}^{(n)} = \frac{i_1}{n}, \ldots, X_{t_\alpha}^{(n)} = \frac{i_\alpha}{n}\right) m^{-k_\alpha} \frac{i_\alpha}{i} \\
&=: P_{i/n}^{(n)\uparrow}\left(\pi_{t_1} = \frac{i_1}{n}, \ldots, \pi_{t_\alpha} = \frac{i_\alpha}{n}\right),
\end{aligned}$$

where $\pi_t(x) = x_t$ for $x \in \hat{\Omega}$ and $t \in \mathbb{R}_+$. The right hand side of the last display determines a probability measure $P_i^{(n)\uparrow}$ on $\bigcup_{t\ge 0} \sigma\{\pi_t\}$, which extends uniquely to a measure $P_i^{(n)\uparrow}$ on $\hat{\mathcal{F}}$. The measure $P_\mu^{(n)\uparrow}$ on $(\hat{\Omega}, \hat{\mathcal{F}})$ for a general initial distribution μ of $Z_0^{(n)}$ is defined as $\sum_{i=1}^\infty \mu(i) P_i^{(n)\uparrow}$. Let $Z^{(n)\uparrow}$ be a Markov chain on a probability space $(\tilde{\Omega}^{(n)}, \tilde{\mathcal{F}}^{(n)}, \tilde{P}^{(n)})$ as in the statement of the lemma. Then

$$\tilde{P}^{(n)}\left(X_{t_1}^{(n)\uparrow} = \frac{i_1}{n}, \ldots, X_{t_\alpha}^{(n)\uparrow} = \frac{i_\alpha}{n}\right) = \sum_{i=1}^\infty P_{i/n}^{(n)\uparrow}\left(\pi_{t_1} = \frac{i_1}{n}, \ldots, \pi_{t_\alpha} = \frac{i_\alpha}{n}\right) \mu(i),$$

which implies that $P_\mu^{(n)\uparrow}$ is the law of $X^{(n)\uparrow}$. □

Lemma 4.4. *Assume Conditions 1.14, 1.15, 1.17. Let* Q_n *be as introduced above* (3.7)*. Then* $Q_n \to Q$*.*

Proof. For $l = 1, \ldots, k$ and $n \in \mathbb{N}$, let $\{\gamma_{l,j}^{(n)}\}_{1 \le j \le k}$ denote a random variable representing the offspring count (in a single generation) of a particle of type l for the n-th BGW process. Then, since $\mathbf{u}^{(n)} = (u_1^{(n)}, \ldots, u_k^{(n)})'$ and $\mathbf{v}^{(n)} = (v_1^{(n)}, \ldots, v_k^{(n)})'$ are the right and left eigenvectors of $\mathbf{M}^{(n)}$, and $m_{l,j}^{(n)} = E(\gamma_{l,j}^{(n)})$, we get

$$\begin{aligned}
\sum_{l=1}^{k} v_l^{(n)} \mathbf{u}^{(n)\prime} \sigma^{(n)}(l) \mathbf{u}^{(n)} - 2Q_n &= \sum_{l=1}^{k} v_l^{(n)} \mathbf{u}^{(n)\prime} \sigma^{(n)}(l) \mathbf{u}^{(n)} - \sum_{l=1}^{k} v_l^{(n)} q_{n,i}[\mathbf{u}^{(n)}] \\
&= \sum_{l=1}^{k} v_l^{(n)} \left(\sum_{i,j=1}^{k} u_i^{(n)} E(\gamma_{l,i}^{(n)} \gamma_{l,j}^{(n)}) u_j^{(n)} - \sum_{i,j=1}^{k} u_i^{(n)} E(\gamma_{l,i}^{(n)}) E(\gamma_{l,j}^{(n)}) u_j^{(n)} \right) \\
&\quad - \sum_{l=1}^{k} v_l^{(n)} \left(\sum_{i,j=1}^{k} u_i^{(n)} E(\gamma_{l,i}^{(n)} \gamma_{l,j}^{(n)}) u_j^{(n)} - \sum_{i=1}^{k} \left(u_i^{(n)}\right)^2 E(\gamma_{l,i}^{(n)}) \right) \\
&= \sum_{i=1}^{k} \left(u_i^{(n)}\right)^2 \left(1 + \frac{c_n}{n}\right) v_i^{(n)} - \sum_{l=1}^{k} v_l^{(n)} \left(1 + \frac{c_n}{n}\right)^2 \left(u_l^{(n)}\right)^2 \\
&= -\frac{c_n}{n} \left(1 + \frac{c_n}{n}\right) \sum_{i=1}^{k} \left(u_i^{(n)}\right)^2 v_i^{(n)}.
\end{aligned}$$

The result follows on sending $n \to \infty$ in the above display. □

Acknowledgment. The proof of Theorem 1.5 for the supercritical case is based on an idea communicated to us by Professor K. B. Athreya. We gratefully acknowledge his help in the proof.

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