Communications on Stochastic Analysis
Vol. 4, No. 2 (2010) 143-160

Serials Publications
www.serialspublications.com

WHAT IS A GAUSSIAN STATE?

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Abstract. Stimulated by a remark of J. L. Doob at the beginning of Appendix I to Kai Lai Chung's English translation "Limit Distributions for Sums of Independent Variables" of the Russian classic by B. V. Gnedenko and A. N. Kolmogorov [5] we highlight the somewhat nonprobabilistic importance of characteristic functions and their positive definiteness property in a pedagogical attempt to introduce the notion of a quantum Gaussian state and its properties to a classical probabilist. Such a presentation leads to some natural open problems on symmetry transformation properties of Gaussian states.

1. Introduction

In Appendix I to Kai Lai Chung's English translation 'Limit Distributions for Sums of Independent Random Variables' of the Russian classic by B. V. Gnedenko and A. N. Kolmogorov [5], J. L. Doob writes : "In this book, since much of it is concerned only with characteristic functions, it would even be possible to phrase much of the material entirely in terms of characteristic functions, omitting reference both to random variables and distribution functions. Although this type of treatment is not uncommon in distribution theory, it would be undesirable in a large work of the present kind ..." However, it is necessary to keep in view Bochner's theorem that a characteristic function is characterized by its continuity and the key property of positive definiteness. Thanks to the Gelfand-Naimark-Segal (GNS) principle [10] positive definiteness carries with it the scalar product structure in a Hilbert space which is at the heart of the computation of all probability distributions and expectation values of observables in quantum theory. In order to highlight this particular feature for the benefit of classical probabilists in this short pedagogical essay we focus our attention on one particular class of quantum states known as *Gaussian states.* They constitute a natural extension of the idea of Gaussian or normal distributions in classical probability.

We shall restrict ourselves to Gaussian states in the complex Hilbert space $L^2(\mathbb{R}^n)$ of a quantum system with n degrees of freedom. In all the Hilbert spaces that we come across we shall use the Dirac notation with the scalar product denoted as $\langle\cdot|\cdot\rangle$ where $\langle u|v\rangle$ is linear in v and conjugate linear in u. For any operator A in $\mathcal{H}$ and elements u, v in $\mathcal{H}$ we write $\langle u|A|v\rangle = \langle u|Av\rangle = \langle A^\dagger u|v\rangle$ whenever v is

Received 2010-3-25; Communicated by the editors.
2000 *Mathematics Subject Classification.* 81S25; 60B15, 42A82, 81R30.
Key words and phrases. Gaussian state, canonical commutation relations, Weyl operators, quantum Bochner's theorem, positive definite kernel, creation and annihilation operators, symplectic group.

in the domain of A or u is in the domain of the adjoint $A^\dagger$ of A, provided, they are well-defined. Let $S(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ be the Schwarz subspace of rapidly decreasing C^∞-functions on $\mathbb{R}^n$. In $L^2(\mathbb{R}^n)$ one has the fundamental momentum and position observables p_j and q_j, $1 \leq j \leq n$ which are selfadjoint operators with $S(\mathbb{R}^n)$ as core satisfying the following properties: (i) They form an irreducible family; (ii) The operators p_j, $1 \leq j \leq n$ commute among themselves. So do the operators q_j, $1 \leq j \leq n$. In $S(\mathbb{R}^n)$

$$[q_r, p_s] = i\delta_{rs} \quad \forall \quad r, s \in \{1, 2, \ldots, n\}.$$

These are known as the *canonical commutation relations* (CCR) in the sense of Heisenberg.

A bounded hermitian operator A in a Hilbert space is said to be *positive,* $A \geq 0$ in symbols, if $\langle u|A|u\rangle \geq 0 \; \forall \; u$. If A, B are two hermitian operators we say that $A \geq B$ if $A - B \geq 0$. By a *state* ρ in a Hilbert space $\mathcal{H}$ we mean a positive operator of unit trace. A selfadjoint operator X in $\mathcal{H}$ is called an *observable.* If X has the spectral resolution

$$X = \int_{\mathbb{R}} \lambda P^X(d\lambda)$$

with spectral measure P^X on the Borel σ-algebra $\mathcal{B}_{\mathbb{R}}$ of $\mathbb{R}$ and ρ is a state then the probability measure $\mu(E) = \mu^{\rho, X}(E) = \operatorname{Tr} \rho P^X(E)$, $E \in \mathcal{B}_{\mathbb{R}}$, is called the distribution of the observable X in the state E. If $X\rho$ and $X^2\rho$ are defined as trace class operators then the mean and variance of μ are respectively equal to $\mathrm{Tr}X\rho$ and $\mathrm{Tr}X^2\rho - (\mathrm{Tr}X\rho)^2$. For any scalar c we denote the operator cI by c itself. If the mean $\mathrm{Tr}X\rho$ is denoted by m then the variance can also be expressed as $\mathrm{Tr}(X - m)^2\rho$. If $X_1, X_2, \ldots, X_k$ are selfadjoint operators (or observables) with means $m_1, m_2, \ldots, m_k$ respectively in the state ρ then the covariance matrix

$$((\mathrm{Cov}(X_i, X_j))), i, j \in \{1, 2, \ldots, k\}$$

in the state ρ is defined by

$$\mathrm{Cov}(X_i, X_j) = \operatorname{Tr} \frac{1}{2} \left\{(X_i - m_i)(X_j - m_j) + (X_j - m_j)(X_i - m_i)\right\} \rho.$$

The fundamental momentum and position observables p_j, q_j, $1 \leq j \leq n$ in $L^2(\mathbb{R}^n)$ have the property that any real linear combination of the form $\sum_{j=1}^n (x_j p_j + y_j q_j)$ is uniquely defined by closure as a selfadjoint operator with $S(\mathbb{R}^n)$ as a core and so becomes an observable. We denote this observable by the same symbol $\sum_{j=1}^n (x_j p_j + y_j q_j)$. Now we say that a state ρ in $L^2(\mathbb{R}^n)$ is *Gaussian* if and only if, for every $x_j, y_j \in \mathbb{R}$, $1 \leq j \leq n$, the observable $\sum_j (x_j p_j + y_j q_j)$ has a normal distribution on the line in the state ρ. This at once leads to the means of p_j, q_j as well as the covariance matrix of $p_1, \ldots, p_n, q_1, \ldots, q_n$ in the state ρ. Starting from this definition of a Gaussian state we arrive at a criterion for the existence of ρ in terms of the covariance matrix by using the quantum Bochner's theorem and elementary properties of infinitely divisible positive definite kernels in the theory outlined in K. R. Parthasarathy and K. Schmidt [11]. Then we investigate the structure of a Gaussian state in terms of the operators p_j, q_j, $1 \leq j \leq n$ and some natural symmetry operations of quantum theory. In the process there seem to arise some interesting open problems calling for further investigation.

The importance of the notion of positive definiteness and conditional positive definiteness in arriving at models in quantum probability is also highlighted in [6], [7], [8] and [10]. For approaches to Gaussian states through quantum central limit theorems we refer to [2], [4], [8] and [9]. The importance and role of Gaussian states in quantum statistics is emphasized in [6].

2. The Weyl Operators and the Quantum Bochner's Theorem

We shall introduce the Weyl commutation relations and the position and momentum observables in $L^2(\mathbb{R}^n)$ starting from the standard normal distribution $N(\mathbf{0}, I)$ in $\mathbb{R}^n$ with the density function $\varphi(\mathbf{x}) = (2\pi)^{-n/2} \exp -\frac{1}{2}\|\mathbf{x}\|^2$ and the associated probability measure μ. In $L^2(\mu)$ consider the exponential random variables

$$e_g(\mathbf{z})(\mathbf{x}) = \exp\left\{\sum_{j=1}^n z_j x_j - \frac{1}{2}\sum_{j=1}^n z_j^2\right\}, \mathbf{z} = (z_1, \dots, z_n)^\tau \in \mathbb{C}^n. \tag{2.1}$$

The set $\{e_g(\mathbf{z}) | \mathbf{z} \in \mathbb{C}^n\}$ of random variables enjoys the following important properties: (i) Their closed linear span is the whole of $L^2(\mu)$. One says that they are *total* in $L^2(\mu)$; (ii) In $L^2(\mu)$, $\langle e_g(\mathbf{z}) | e_g(\mathbf{z}')\rangle = \exp\langle \mathbf{z}|\mathbf{z}'\rangle$ for all $\mathbf{z}, \mathbf{z}' \in \mathbb{C}^n$ where we view $\mathbb{C}^n$ itself as an n-dimensional Hilbert space; (iii) For any finite set $\left\{\mathbf{z}^{(i)}, 1 \le i \le m\right\} \subset \mathbb{C}^n$ the set $\left\{e_g(\mathbf{z}^{(i)}), 1 \le i \le m\right\} \subset L^2(\mu)$ consists of linearly independent elements. All these properties are carried over to the standard complex Hilbert space $L^2(\mathbb{R}^n)$ of a quantum system with n degrees of freedom through the natural isomorphism

$$f(\mathbf{x}) \to (2\pi)^{-n/4} f(\mathbf{x}) \exp -\frac{1}{4}\|\mathbf{x}\|^2, \quad f \in L^2(\mu)$$

from $L^2(\mu)$ onto $L^2(\mathbb{R}^n)$. Thus we define the *exponential vectors* $e(\mathbf{z})$, $\mathbf{z} \in \mathbb{C}^n$ in $L^2(\mathbb{R}^n)$ by

$$e(\mathbf{z})(\mathbf{x}) = (2\pi)^{-n/4} \exp\left(\sum_{j=1}^n z_j x_j - \frac{1}{2}\sum_{j=1}^n z_j^2 - \frac{1}{4}\sum_{j=1}^n x_j^2\right) \tag{2.2}$$

with the important property

$$\langle e(\mathbf{z}) | e(\mathbf{z}')\rangle = \exp\langle \mathbf{z}|\mathbf{z}'\rangle, \quad \forall\ \mathbf{z}, \mathbf{z}' \in \mathbb{C}^n \tag{2.3}$$

which, in turn, implies the other two properties (i) and (iii) described after (2.1) with $L^2(\mu)$ replaced by $L^2(\mathbb{R}^n)$. Equation (2.2) enables us to define the *Weyl operators* $W(\boldsymbol{\alpha})$, $\boldsymbol{\alpha} \in \mathbb{C}^n$ by putting

$$W(\boldsymbol{\alpha}) e(\mathbf{z}) = e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2 - \langle \boldsymbol{\alpha}|\mathbf{z}\rangle} e(\mathbf{z} + \boldsymbol{\alpha}), \quad \forall\ \mathbf{z} \in \mathbb{C}^n \tag{2.4}$$

and observing that

$$\langle W(\boldsymbol{\alpha}) e(\mathbf{z}) | W(\boldsymbol{\alpha}) e(\mathbf{z}')\rangle = \langle e(\mathbf{z}) | e(\mathbf{z}')\rangle.$$

Thus $W(\boldsymbol{\alpha})$ is scalar product preserving on the total set of all exponential vectors and therefore extends uniquely to a *unitary* operator denoted again as $W(\boldsymbol{\alpha})$. Furthermore, the following product relation follows from (2.4) by elementary algebra:

$$W(\boldsymbol{\alpha})W(\boldsymbol{\beta}) = e^{-i\ \mathrm{Im}\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle} W(\boldsymbol{\alpha}+\boldsymbol{\beta}), \quad \forall\ \boldsymbol{\alpha},\boldsymbol{\beta} \in \mathbb{C}^n. \tag{2.5}$$

This implies

$$W(\boldsymbol{\alpha})W(\boldsymbol{\beta})W(\boldsymbol{\alpha})^{-1} = e^{-2i\ \mathrm{Im}\,\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle} W(\boldsymbol{\beta}). \tag{2.6}$$

Equation (2.5) together with the fact that the map $\boldsymbol{\alpha} \to W(\boldsymbol{\alpha})$ is strongly continuous means that $W(\cdot)$ is a projective unitary representation of $\mathbb{C}^n$ in $L^2(\mathbb{R}^n)$. It is irreducible in the sense that the commutant $\{W(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathbb{C}^n\}'$ in the von Neumann algebra $\mathcal{B}(L^2(\mathbb{R}^n))$ of all bounded operators in $L^2(\mathbb{R}^n)$ is $\mathbb{C}I$. According to the Stone-von Neumann theorem, if $\boldsymbol{\alpha} \to \widetilde{W}(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathbb{C}^n$ is a strongly continuous projective unitary representation in a complex separable Hilbert space $\mathcal{H}$ satisfying the product relation (2.5) with W replaced by $\widetilde{W}$ then $\widetilde{W}$ is unitarily equivalent to a projective unitary representation of the form $W \otimes id$, i.e.,

$$\widetilde{W}(\boldsymbol{\alpha}) = U\left(W(\boldsymbol{\alpha}) \otimes I\right)U^{-1}, \quad \forall\ \boldsymbol{\alpha} \in \mathbb{C}^n$$

where U is a Hilbert space isomorphism between $\mathcal{H}$ and $L^2(\mathbb{R}^n) \otimes k$ with k being another Hilbert space.

Equation (2.5) implies that $\{W(t\boldsymbol{\alpha}), t \in \mathbb{R}\}$ is a strongly continuous one parameter unitary group and therefore by Stone's theorem there exists a selfadjoint operator $p(\boldsymbol{\alpha})$ such that

$$W(t\boldsymbol{\alpha}) = e^{-it\ p(\boldsymbol{\alpha})}, \quad \boldsymbol{\alpha} \in \mathbb{C}. \tag{2.7}$$

All the $p(\boldsymbol{\alpha})$'s have the Schwarz space $S(\mathbb{R}^n)$ as a common core. Let $\mathbf{e}_j = (0,0,\dots,0,1,0,\dots,0)^\tau$, $1 \le j \le n$ be the canonical orthonormal basis in $\mathbb{C}^n$, τ indicating transpose. Define

$$p_j = 2^{-1/2}p(\mathbf{e}_j), \quad q_j = 2^{-1/2}p(i\mathbf{e}_j), \tag{2.8}$$

$$a_j = \frac{q_j + ip_j}{\sqrt{2}}, \qquad a_j^\dagger = \frac{q_j - ip_j}{\sqrt{2}} \tag{2.9}$$

where we use the same symbols to denote the operators with domain $S(\mathbb{R}^n)$ and their closures whenever such operators are closable. Equation (2.5) implies that $\{p_j, 1 \le j \le n\}$ commute among themselves and so do $\{q_j, 1 \le j \le n\}$. Furthermore on $S(\mathbb{R}^n)$

$$[q_r,\ p_s] = i\ \delta_{rs} \tag{2.10}$$

or equivalently,

$$\left[a_r,\ a_s^\dagger\right] = \delta_{rs} \tag{2.11}$$

with $\{a_r,\ 1 \le r \le n\}$, $\left\{a_r^\dagger,\ 1 \le s \le n\right\}$ commuting among themselves in $S(\mathbb{R}^n)$. Equations (2.5) are called *Weyl commutation relations* whereas their infinitesimal versions (2.10), (2.11) are called the *canonical commutation relations* in the sense of Heisenberg. One can express the Weyl operators $W(\boldsymbol{\alpha})$ as

$$\begin{aligned} W(\boldsymbol{\alpha}) &= e^{\sum\limits_{j=1}^{n}(\alpha_j a_j^\dagger - \bar{\alpha}_j a_j)} \\ &= e^{-i\sqrt{2}\sum\limits_{j=1}^{n}(x_j p_j - y_j q_j)}, \quad \alpha_j = x_j + iy_j, \quad \forall\ j. \end{aligned} \tag{2.12}$$

We call p_j, q_j as the j-th *momentum* and *position* observable for each $1 \leq j \leq n$ and $a_j, a_j^\dagger$ the j-th *annihilation* and *creation* operator. For a comprehensive account of Weyl operators and the role of momentum, position, annihilation and creation operators in quantum probability and stochastic calculus we refer to [10]. Now we introduce the notion of a quantum characteristic function.

Definition 2.1. Let ρ be a state in $L^2(\mathbb{R}^n)$, i.e., a positive operator of unit trace. Then the (possibly) complex-valued function $\hat{\rho}$ on $\mathbb{C}^n$ defined by

$$\hat{\rho}(\boldsymbol{\alpha}) = \mathrm{Tr}\ \rho\, W(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \mathbb{C}^n \tag{2.13}$$

where $W(\boldsymbol{\alpha})$ denotes the Weyl operator corresponding to $\boldsymbol{\alpha}$, is called the *quantum characteristic function* of the state ρ.

Clearly, $\hat{\rho}(\mathbf{0}) = \mathrm{Tr}\,\rho = 1$ and the map $\boldsymbol{\alpha} \to \hat{\rho}(\boldsymbol{\alpha})$ is continuous. Suppose c_j, $1 \leq j \leq m$ are complex scalars and $\boldsymbol{\alpha}_j$, $1 \leq j \leq m$ are elements in $\mathbb{C}^n$. Define the bounded operator

$$X = \sum_{j=1}^{m} c_j\, W(\boldsymbol{\alpha}_j).$$

Then the relations (2.5) imply

$$\begin{aligned} 0 &\leq \mathrm{Tr}\ \rho\ X^\dagger X \\ &= \sum_{r,s} \bar{c}_r c_s\, \mathrm{Tr}\,\rho W(-\boldsymbol{\alpha}_r) W(\boldsymbol{\alpha}_s) \\ &= \sum_{r,s} \bar{c}_r c_s\ e^{i \mathrm{Im}\langle \boldsymbol{\alpha}_r | \boldsymbol{\alpha}_s \rangle} \hat{\rho}(\boldsymbol{\alpha}_s - \boldsymbol{\alpha}_r). \end{aligned} \tag{2.14}$$

To formalize this inequality we introduce a definition.

Definition 2.2. Let $\mathcal{X}$ be any set. A complex-valued function $K(x, y)$, $x, y \in \mathcal{X}$ is called a *positive definite kernel* on $\mathcal{X}$ if for any finite set $\{x_i, 1 \leq i \leq m\} \subset \mathcal{X}$ the matrix $((K(x_i, x_j)))\,,\ i, j \in \{1, 2, \ldots, m\}$ is nonnegative definite.

In terms of this definition it follows from (2.14) that the kernel K on $\mathbb{C}^n$ defined by $K_\rho(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e^{i\,\mathrm{Im}\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle}\ \hat{\rho}(\boldsymbol{\beta} - \boldsymbol{\alpha})$ is positive definite for any state ρ. Our next theorem is a converse of this property.

Theorem 2.3 (Quantum Bochner's theorem in $L^2(\mathbb{R}^n)$)**.** *A complex-valued function f on $\mathbb{C}^n$ is the quantum characteristic function $\hat{\rho}$ of a state ρ in $L^2(\mathbb{R}^n)$ if and only if the following conditions are fulfilled:*

(i) $f(0) = 1$ *and f is continuous;*
(ii) *The kernel $K(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e^{i\,\mathrm{Im}\,\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle} f(\boldsymbol{\beta} - \boldsymbol{\alpha})$ on $\mathbb{C}^n$ is positive definite.*

Proof. Necessity is already proved in the discussion after Definition 2.1. To prove sufficiency we apply the GNS (Gelfand-Naimark-Segal) principle as outlined, for example, in [10] and construct a Hilbert space $\mathcal{H}$ and a map $\lambda : \mathbb{C}^n \to \mathcal{H}$ satisfying the following:

(i) The set $\{\lambda(\boldsymbol{\alpha}), \boldsymbol{\alpha} \in \mathbb{C}^n\}$ is total in $\mathcal{H}$;
(ii) $\langle \lambda(\boldsymbol{\alpha}) | \lambda(\boldsymbol{\beta}) \rangle = e^{i \operatorname{Im} \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle} f(\boldsymbol{\beta} - \boldsymbol{\alpha})$ for all $\boldsymbol{\alpha}, \boldsymbol{\beta}$;
(iii) The pair $(\mathcal{H}, \lambda)$ with properties (i) and (ii) is unique up to a unitary isomorphism.

Define, for any $\boldsymbol{\beta} \in \mathbb{C}^n$, a unitary operator $\Gamma(\boldsymbol{\beta})$ in $\mathcal{H}$ by putting

$$\Gamma(\boldsymbol{\beta})\, \lambda(\boldsymbol{\alpha}) = e^{-i \operatorname{Im} \langle \boldsymbol{\beta} | \boldsymbol{\alpha} \rangle} \lambda(\boldsymbol{\alpha} + \boldsymbol{\beta}), \quad \forall\, \boldsymbol{\alpha}$$

and observing that

$$\langle \Gamma(\boldsymbol{\beta}) \lambda(\boldsymbol{\alpha}) | \Gamma(\boldsymbol{\beta}) \lambda(\boldsymbol{\alpha}') \rangle = \langle \lambda(\boldsymbol{\alpha}) | \lambda(\boldsymbol{\alpha}') \rangle, \quad \forall\, \boldsymbol{\alpha}, \boldsymbol{\alpha}'.$$

Simple algebra shows that

$$\Gamma(\boldsymbol{\beta}) \Gamma(\boldsymbol{\beta}') = e^{-i \operatorname{Im} \langle \boldsymbol{\beta} | \boldsymbol{\beta}' \rangle} \Gamma(\boldsymbol{\beta} + \boldsymbol{\beta}')$$

which is the same kind of identity as (2.5). Since f is continuous it follows that the map λ is continuous and hence the correspondence $\boldsymbol{\beta} \to \Gamma(\boldsymbol{\beta})$ is strongly continuous. Hence by the Stone-von Neumann theorem $\Gamma(\cdot)$ is unitarily equivalent to a projective representation of the form $\boldsymbol{\beta} \to W(\boldsymbol{\beta}) \otimes I$ where I is the identity operator in some separable Hilbert space k.

In view of the properties (i)-(iii) at the beginning of this proof we may assume, without loss of generality, that $\mathcal{H} = L^2(\mathbb{R}^n) \otimes k$ and $\Gamma(\boldsymbol{\beta}) = W(\boldsymbol{\beta}) \otimes I$. From property (ii) we get

$$\begin{aligned} f(\boldsymbol{\beta}) &= \langle \lambda(\mathbf{0}) | \lambda(\boldsymbol{\beta}) \rangle \\ &= \langle \lambda(\mathbf{0}) | \Gamma(\boldsymbol{\beta}) | \lambda(\mathbf{0}) \rangle \\ &= \operatorname{Tr}\ |\lambda(\mathbf{0})\rangle\langle\lambda(\mathbf{0})|\, \Gamma(\boldsymbol{\beta}) \\ &= \operatorname{Tr}\ (\operatorname{Tr}_k |\lambda(\mathbf{0})\rangle\langle\lambda(\mathbf{0})|)\, W(\boldsymbol{\beta}). \end{aligned}$$

Since $\lambda(\mathbf{0})$ is a unit vector, $|\lambda(\mathbf{0})\rangle\langle\lambda(\mathbf{0})|$ is a pure state and hence its relative trace over k, $\operatorname{Tr}_k |\lambda(\mathbf{0})\rangle\langle\lambda(\mathbf{0})| = \rho$ is a state with the property $f(\boldsymbol{\beta}) = \hat{\rho}(\boldsymbol{\beta})$ for all $\boldsymbol{\beta}$. □

Proposition 2.4. *The correspondence $\rho \to \hat{\rho}$ in Theorem 2.3 is bijective.*

Proof. If ρ' is another state such that $f(\boldsymbol{\beta}) = \hat{\rho}(\boldsymbol{\beta}) = \hat{\rho}'(\boldsymbol{\beta})$ then $\operatorname{Tr}(\rho - \rho') W(\boldsymbol{\beta}) = 0$ for all $\boldsymbol{\beta}$. Since $\{W(\boldsymbol{\beta}), \boldsymbol{\beta} \in \mathbb{C}^n\}'' = \mathcal{B}(\mathbb{R}^n)$ it follows that $\operatorname{Tr}(\rho - \rho') X = 0$ for every bounded operator X in $L^2(\mathbb{R}^n)$ and hence $\rho = \rho'$. □

We shall now describe some elementary properties of quantum characteristic functions which will be used in the next section.

Proposition 2.5. *For any unitary operator U in $\mathbb{C}^n$ let $\Gamma(U)$ denote the unique unitary operator in $L^2(\mathbb{R}^n)$ satisfying the property*

$$\Gamma(U) e(\boldsymbol{\alpha}) = e(U \boldsymbol{\alpha}), \quad \forall\, \boldsymbol{\alpha} \in \mathbb{C}^n$$

where $e(\boldsymbol{\alpha})$ *is the exponential vector defined by* (2.2). *Then for any state* ρ *the state* $\Gamma(U)\rho\Gamma(U)^{-1}$ *has the quantum characteristic function given by*

$$\left\{\Gamma(U)\rho\Gamma(U)^{-1}\right\}^{\wedge}(\boldsymbol{\beta}) = \hat{\rho}(U^{-1}\boldsymbol{\beta}).$$

If $\boldsymbol{\alpha} \in \mathbb{C}^n$, *the quantum characteristic function of* $W(\boldsymbol{\alpha})\rho W(\boldsymbol{\alpha})^{-1}$ *is given by*

$$\left\{W(\boldsymbol{\alpha})\rho W(\boldsymbol{\alpha})^{-1}\right\}^{\wedge}(\boldsymbol{\beta}) = \hat{\rho}(\boldsymbol{\beta})e^{2i\,\mathrm{Im}\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}$$

Proof. That $\Gamma(U)$ is a well-defined unitary operator in $L^2(\mathbb{R}^n)$ follows from the basic properties of exponential vectors described in the beginning of this section and the fact that $\Gamma(U)$ preserves scalar products between exponential vectors. We have for $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{C}^n$, by (2.4),

$$\begin{aligned}
&\langle e(\boldsymbol{\beta})|\Gamma(U)W(\boldsymbol{\alpha})\Gamma(U)^{-1}|e(\boldsymbol{\gamma})\rangle \\
&= \langle e(U^{-1}\boldsymbol{\beta})|W(\boldsymbol{\alpha})|e(U^{-1}\boldsymbol{\gamma})\rangle \\
&= e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2 - \langle\boldsymbol{\alpha}|U^{-1}\boldsymbol{\gamma}\rangle}\langle e(U^{-1}\boldsymbol{\beta})|e(U^{-1}\boldsymbol{\gamma}+\boldsymbol{\alpha})\rangle \\
&= e^{-\frac{1}{2}\|U\boldsymbol{\alpha}\|^2 - \langle U\boldsymbol{\alpha}|\boldsymbol{\gamma}\rangle}\langle e(\boldsymbol{\beta})|e(\boldsymbol{\gamma}+U\boldsymbol{\alpha})\rangle \\
&= \langle e(\boldsymbol{\beta})|W(U\boldsymbol{\alpha})|e(\boldsymbol{\gamma})\rangle.
\end{aligned}$$

Thus

$$\Gamma(U)W(\boldsymbol{\alpha})\Gamma(U)^{-1} = W(U\boldsymbol{\alpha}).$$

Hence

$$\begin{aligned}
&\mathrm{Tr}\,\Gamma(U)\,\rho\,\Gamma(U)^{-1}\,W(\boldsymbol{\beta}) \\
&= \mathrm{Tr}\,\rho\,\Gamma(U)^{-1}\,W(\boldsymbol{\beta})\,\Gamma(U) \\
&= \mathrm{Tr}\,\rho\,W(U^{-1}\boldsymbol{\beta}) \\
&= \hat{\rho}(U^{-1}\boldsymbol{\beta}).
\end{aligned}$$

This proves the first part. To prove the second part we use (2.6) and observe that

$$\begin{aligned}
&\left\{W(\boldsymbol{\alpha})\rho W(\boldsymbol{\alpha})^{-1}\right\}^{\wedge}(\boldsymbol{\beta}) \\
&= \mathrm{Tr}\,\rho\,W(\boldsymbol{\alpha})^{-1}\,W(\boldsymbol{\beta})\,W(\boldsymbol{\alpha}) \\
&= e^{2i\,\mathrm{Im}\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}\,\mathrm{Tr}\,\rho\,W(\boldsymbol{\beta}) \\
&= e^{2i\,\mathrm{Im}\langle\boldsymbol{\alpha}|\boldsymbol{\beta}\rangle}\,\hat{\rho}(\boldsymbol{\beta}).
\end{aligned}$$

□

In $\mathbb{R}^{2n}$ consider linear transformations of the form

$$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}$$

where L_{ij} are linear transformations in $\mathbb{R}^n$ and L acts in $\mathbb{R}^{2n}$ by

$$L\begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} L_{11}\mathbf{u} + L_{12}\mathbf{v} \\ L_{21}\mathbf{u} + L_{22}\mathbf{v} \end{bmatrix}, \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Let

$$J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix} \tag{2.15}$$

where I denotes the identity operator in $\mathbb{R}^n$. Denote by Sp_{2n} the group of all linear transformations L in $\mathbb{R}^{2n} = \mathbb{R}^n \oplus \mathbb{R}^n$ described above satisfying the property

$$L^\tau J L = J. \tag{2.16}$$

The group Sp_{2n} is the symplectic group of order n. In particular $Sp_2 = SL(2,\mathbb{R})$. Any $L \in Sp_{2n}$ defines an additive map $\widetilde{L}$ in $\mathbb{C}^n$ as follows: if $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then

$$\widetilde{L}\mathbf{z} = \mathbf{x}' + i\,\mathbf{y}' \tag{2.17}$$

where

$$\begin{pmatrix} \mathbf{x}' \\ \mathbf{y}' \end{pmatrix} = L \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}. \tag{2.18}$$

Then L has the property

$$\text{Im } \langle \widetilde{L}\boldsymbol{\alpha} | \widetilde{L}\boldsymbol{\beta} \rangle = \text{Im } \langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle, \quad \forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^n$$

and conversely any additive transformation of $\mathbb{C}^n$ preserving the imaginary part of scalar products can be realized as $\widetilde{L}$ for some $L \in Sp_{2n}$. For such a transformation $\widetilde{L}$ we have

$$W(\widetilde{L}\boldsymbol{\alpha})\; W(\widetilde{L}\boldsymbol{\beta}) = e^{-i\,\text{Im}\,\langle \boldsymbol{\alpha}|\boldsymbol{\beta}\rangle} W(\widetilde{L}(\boldsymbol{\alpha} + \boldsymbol{\beta})), \quad \forall\, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{C}^n.$$

Thus, once again by the Stone-von Neumann theorem there exists a unitary transformation $\Gamma(L)$, unique up to multiplication by a scalar of modulus unity, satisfying

$$W(\widetilde{L}\boldsymbol{\alpha}) = \Gamma(L) W(\boldsymbol{\alpha}) \Gamma(L)^{-1}, \quad \boldsymbol{\alpha} \in \mathbb{C}^n, \quad L \in Sp_{2n}. \tag{2.19}$$

It is possible to choose the transformations $\Gamma(L)$ such that the correspondence $L \to \Gamma(L)$ is a strongly continuous projective representation of Sp_{2n} in which the multiplier assumes the values ± 1 [1]. With this notation we have the following proposition.

Proposition 2.6. *For any state ρ in $L^2(\mathbb{R}^n)$ and any element $L \in Sp_{2n}$ the state $\Gamma(L)\rho\Gamma(L)^{-1}$ has the quantum characteristic function*

$$\left\{\Gamma(L)\rho\Gamma(L)^{-1}\right\}^\wedge(\boldsymbol{\alpha}) = \hat{\rho}(\widetilde{L}^{-1}\boldsymbol{\alpha}), \quad \forall\, \boldsymbol{\alpha} \in \mathbb{C}^n.$$

Proof. This is exactly along the same lines as the proof of the first part of Proposition 2.5 but using (2.19). □

Remark 2.7. Note that every unitary transformation U of $\mathbb{C}^n$ can be realized as $\widetilde{L}$ where $L \in Sp_{2n} \cap O_{2n}$. Thus Proposition 2.6 includes the first part of Proposition 2.5.

Proposition 2.8. *If ρ_i is a state in $L^2(\mathbb{R}^{n_i})$, $i =, 1, 2$ then the quantum characteristic function of the product state $\rho_1 \otimes \rho_2$ in $L^2(\mathbb{R}^{n_1+n_2})$ is given by*

$$\begin{aligned} &(\rho_1 \otimes \rho_2)^\wedge \left((\alpha_1, \alpha_2, \ldots, \alpha_{n_1+n_2})^\tau\right) \\ &\quad = \quad \hat{\rho}_1\left((\alpha_1\alpha_2 \ldots, \alpha_{n_1})^\tau\right) \hat{\rho}_2\left((\alpha_{n_1+1}\alpha_{n_1+2} \ldots, \alpha_{n_1+n_2})^\tau\right), \quad \forall\, \boldsymbol{\alpha} \in \mathbb{C}^{n_1+n_2} \end{aligned}$$

If ρ is a state in $L^2(\mathbb{R}^{n_1+n_2}) = L^2(\mathbb{R}^{n_1}) \otimes L^2(\mathbb{R}^{n_2})$ and the marginal state ρ_1 in $L^2(\mathbb{R}^{n_1})$ is obtained by

$$\rho_1 = \text{Tr}_2\; \rho$$

with Tr_2 *being the relative trace over the second factor* $L^2(\mathbb{R}^{n_2})$ *in the tensor product then*

$$\hat{\rho}_1\left((\alpha_1, \alpha_2, \ldots, \alpha_{n_1})^\tau\right) = \hat{\rho}\left((\alpha_1, \alpha_2, \ldots, \alpha_{n_1}, 0, 0, \ldots, 0)^\tau\right)$$

where 0 *is repeated* n_2 *times on the right hand side.*

Proof. These are straightforward from the product property of Weyl operators:

$$\begin{aligned} & W\left((\alpha_1, \alpha_2, \ldots, \alpha_{n_1+n_2})^\tau\right) \\ = \quad & W\left((\alpha_1, \alpha_2, \ldots, \alpha_{n_1})^\tau\right) \otimes W\left((\alpha_{n_1+1}, \alpha_{n_1+2}, \ldots, \alpha_{n_1+n_2})^\tau\right) \end{aligned}$$

in the factorization

$$L^2(\mathbb{R}^{n_1+n_2}) = L^2(\mathbb{R}^{n_1}) \otimes L^2(\mathbb{R}^{n_2}).$$

□

We now introduce the normalized exponential vectors

$$\psi(\boldsymbol{\alpha}) = e^{-\frac{1}{2}\|\boldsymbol{\alpha}\|^2} e(\boldsymbol{\alpha}), \quad \boldsymbol{\alpha} \in \mathbb{C}^n$$

in $L^2(\mathbb{R}^n)$ and call the pure state $|\psi(\boldsymbol{\alpha})\rangle\langle\psi(\boldsymbol{\alpha})|$ the *coherent state* associated with $\boldsymbol{\alpha}$. Then we have the following:

Proposition 2.9. *The quantum characteristic function of the coherent state* $|\psi(\boldsymbol{\alpha})\rangle\langle\psi(\boldsymbol{\alpha})|$ *is given by*

$$\left(|\psi(\boldsymbol{\alpha})\rangle\langle\psi(\boldsymbol{\alpha})|\right)^\wedge(\mathbf{z}) = \exp -\frac{1}{2}\left(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2\right) - 2i\ \left(\mathbf{v}^\tau\mathbf{x} - \mathbf{u}^\tau\mathbf{y}\right) \tag{2.20}$$

where $\boldsymbol{\alpha} = \mathbf{u} + i\mathbf{v}$, $\mathbf{z} = \mathbf{x} + i\mathbf{y}$ *with* $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

Proof. This is immediate from the fact that the left hand side of (2.20) is equal to

$$\begin{aligned} \langle\psi(\boldsymbol{\alpha})|W(\mathbf{z})|\psi(\boldsymbol{\alpha})\rangle &= e^{-\|\boldsymbol{\alpha}\|^2}\langle e(\boldsymbol{\alpha})|W(\mathbf{z})|e(\boldsymbol{\alpha})\rangle \\ &= e^{-\|\boldsymbol{\alpha}\|^2 - \frac{1}{2}\|\mathbf{z}\|^2 - \langle\mathbf{z}|\boldsymbol{\alpha}\rangle + \langle\boldsymbol{\alpha}|\boldsymbol{\alpha}+\mathbf{z}\rangle} \\ &= \exp\left\{-\frac{1}{2}\|\mathbf{z}\|^2 - 2i\ \mathrm{Im}\,\langle\mathbf{z}|\boldsymbol{\alpha}\rangle\right\}. \end{aligned}$$

□

Remark 2.10. Equation (2.12) and (2.20) imply that in the coherent state $|\psi(\boldsymbol{\alpha})\rangle\langle\psi(\boldsymbol{\alpha})|$, the observable $\sum_j(x_jp_j - y_jq_j)$ has a normal distribution with variance $\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ for every $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$. In other words it is an example of a pure Gaussian state as described in the introduction. That this happens in a pure state is one of the most remarkable features of quantum probability.

Proposition 2.11. *In* $L^2(\mathbb{R}^n)$

$$\frac{1}{\pi^n}\int_{\mathbb{R}^{2n}} |\psi(\mathbf{z})\rangle\langle\psi(\mathbf{z})| \prod_{j=1}^n dx_j\ dy_j = I, \tag{2.21}$$

where $\mathbf{z} = (z_1, z_2, \ldots, z_n)^\tau$, $z_j = x_j + iy_j$ *for all* j *and the left hand side is weak operator integral. For any state* ρ *in* $L^2(\mathbb{R}^n)$

$$\hat{\rho}(\alpha) = \frac{1}{\pi^n}\int_{\mathbb{R}^{2n}} \langle\psi(\mathbf{z})|\rho W(\boldsymbol{\alpha})|\psi(\mathbf{z})\rangle \prod_{j=1}^n dx_j\ dy_j. \tag{2.22}$$

Proof. We shall give the proof when $n = 1$. The general case follows along the same lines. For any $\alpha = s + it$, $\beta = u + iv$, $z = x + iy$ we have

$$\begin{aligned}
&\langle e(\alpha)|\frac{1}{\pi}\int_{\mathbb{R}^2} |\psi(z)\rangle\,\langle\psi(z)|\;dx\,dy|e(\beta)\rangle \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-(x^2+y^2)+x(\bar\alpha+\beta)+iy(\bar\alpha-\beta)}dx\,dy \\
&= e^{\left(\frac{\bar\alpha+\beta}{2}\right)^2-\left(\frac{\bar\alpha-\beta}{2}\right)^2} \\
&= e^{\bar\alpha\beta} \\
&= \langle e(\alpha)|e(\beta)\rangle.
\end{aligned}$$

Since exponential vectors are total we have proved (2.21) when $n = 1$.

From (2.21) for $n = 1$ we have

$$\begin{aligned}
\hat\rho(\alpha) &= \mathrm{Tr}\ \rho\ W(\alpha) \\
&= \mathrm{Tr}\ \rho\,W(\alpha)\frac{1}{\pi}\int_{\mathbb{R}^2} |\psi(z)\rangle\langle\psi(z)|\;dx\,dy \\
&= \frac{1}{\pi}\int_{\mathbb{R}^2}\langle\psi(z)|\rho W(\alpha)|\psi(z)\rangle\,dx\,dy.
\end{aligned}$$

□

As an application of (2.22) in Proposition 2.11 we shall evaluate the quantum characteristic functions of a class of states arising from exponentials of operators of the form $-s\,a_j^\dagger\,a_j$ where $s > 0$ and a_j's are the annihilation operators defined in (2.9). We first begin with the case $n = 1$, put $a_j = a$ and consider the positive selfadjoint number operator $a^\dagger a = \frac{1}{2}(p^2 + q^2 - 1)$. Expanding the exponential vector $e(z)$ in power series

$$|e(z)\rangle = \sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}\,|\psi_n\rangle$$

observe that $\{|\psi_n\rangle, n = 0, 1, 2, \ldots\}$ is a complete orthonormal basis for $L^2(\mathbb{R})$,

$$\begin{aligned}
a\,|e(z)\rangle &= z\,|e(z)\rangle, \\
a^\dagger a\,|\psi_n\rangle &= n\,|\psi_n\rangle, n = 0, 1, 2, \ldots
\end{aligned}$$

This shows that

$$\mathrm{Tr}\,e^{-s\,a^\dagger a} = (1 - e^{-s})^{-1}, \quad \forall\ s > 0$$

and therefore the states

$$\rho_s = (1 - e^{-s})e^{-sa^\dagger a},\ s > 0 \tag{2.23}$$

are well-defined. We may write (2.23) as

$$\rho_s = 2\sinh\frac{s}{2}e^{-\frac{s}{2}(p^2+q^2)},\ s > 0 \tag{2.24}$$

and note that ρ_s is the normalized exponential of $-\frac{s}{2}(p^2 + q^2)$ where $p^2 + q^2$ is a quadratic expression in the noncommuting variables p, q. Our next proposition shows that ρ_s is, indeed, a Gaussian state.

Proposition 2.12. *The state ρ_s defined by* (2.23) *(or* (2.24)*) has the quantum characteristic function*

$$\hat{\rho}_s(\alpha) = \exp -\frac{1}{2}(\coth\frac{s}{2})\,|\alpha|^2,\ s>0,\ \alpha\in\mathbb{C}. \tag{2.25}$$

Proof. We have from Proposition 2.11 in the case $n=1$

$$\begin{aligned}
\hat{\rho}_s(\alpha) &= \frac{1}{\pi}\int_{\mathbb{R}^2} e^{-|z|^2}\langle e(z)|\rho_s W(\alpha)|e(z)\rangle\,dx\,dy\\
&= \frac{1-e^{-s}}{\pi}\int_{\mathbb{R}^2} e^{-|z|^2}\langle e(e^{-s}z)|e(z+\alpha)\rangle e^{-\frac{1}{2}|\alpha|^2-\bar{\alpha}z}dx\,dy\\
&= \frac{1-e^{-s}}{\pi}\int_{\mathbb{R}^2}\exp\left\{-(1-e^{-s})(x^2+y^2)-\frac{1}{2}|\alpha|^2+e^{-s}\bar{z}\alpha-\bar{\alpha}z\right\}\,dx\,dy\\
&= \frac{1-e^{-s}}{\pi}e^{-\frac{1}{2}|\alpha|^2}\int_{\mathbb{R}} e^{-(1-e^{-s})x^2+x(\alpha e^{-s}-\bar{\alpha})}dx\\
&\qquad\qquad\times\int_{\mathbb{R}} e^{-(1-e^{-s})y^2-iy(\alpha e^{-s}-\bar{\alpha})}dy.
\end{aligned}$$

Now the evaluation of these two Gaussian integrals shows that (2.25) obtains. □

Remark 2.13. Equation (2.25) shows that every real linear combination of p and q has a Gaussian distribution in the state ρ_s. More generally, for $s_j>0$ the state

$$\rho_{s_1}\otimes\rho_{s_2}\otimes\cdots\otimes\rho_{s_n} = \Pi_1^n(1-e^{-s_j})e^{-\sum\limits_{j=1}^{n} s_j a_j^\dagger a_j}$$

is Gaussian in $L^2(\mathbb{R})$.

Remark 2.14. Let

$$L_t = \begin{bmatrix} e^{-t/2} & 0\\ 0 & e^{t/2}\end{bmatrix},\quad t\in\mathbb{R}.$$

Then L_t can be viewed as an element of the group $Sp_2 = SL(2,\mathbb{R})$. Following the discussion preceding Proposition 2.6 define

$$\begin{aligned}
\widetilde{L}_t(u+iv) &= e^{-t/2}u + ie^{t/2}v,\\
\rho_{s,t} &= \Gamma(L_t)\,\rho_s\,\Gamma(L_t)^{-1},
\end{aligned} \tag{2.26}$$

as in Proposition 2.6. Then

$$\hat{\rho}_{s,t}(u+iv) = \exp -\frac{1}{2}(\coth\frac{s}{2})(e^t u^2 + e^{-t}v^2). \tag{2.27}$$

It is not very difficult to show that

$$\Gamma(\widetilde{L}_t) = e^{\frac{1}{2}t(a^{\dagger 2}-a^2)}$$

where $a^{\dagger 2}-a^2$ is the skew adjoint operator with core $S(\mathbb{R})$.

Now consider the unitary operator $\Gamma(e^{i\theta})$ in $L^2(\mathbb{R})$ satisfying

$$\Gamma(e^{i\theta})e(z) = e(e^{i\theta}z),\quad \forall\, z\in\mathbb{C}$$

where $0\le\theta<2\pi$. Then $\Gamma(e^{i\theta})$ is of the form $\Gamma(U)$ in Proposition 2.5 when $n=1$. Define the state

$$\rho_{s,t,\theta} = \Gamma(e^{i\theta})\rho_{s,t}\Gamma(e^{-i\theta}),\ s>0,\ t\in\mathbb{R},\ 0\le\theta<2\pi \tag{2.28}$$

where $\rho_{s,t}$ is given by (2.26). Then for all $\alpha = u + iv$ we have from (2.27) and Proposition 2.5

$$\hat{\rho}_{s,t,\theta}(\alpha) = \exp -\frac{1}{2}(u,v)S\begin{pmatrix} u \\ v \end{pmatrix}, \quad \alpha = u + iv, \tag{2.29}$$

where

$$S = (\coth\frac{s}{2})\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} e^{t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}. \tag{2.30}$$

As s, t, θ vary in (2.28) the matrix S in (2.30) varies over all second order real positive matrices of determinant > 1.

Now consider a real positive second order matrix S of determinant unity. Then so is $S^{-\frac{1}{2}}$ which can now be looked upon as an element of the symplectic group Sp_2. Consider now the pure state

$$\rho = \Gamma(S^{-\frac{1}{2}})|e(0)\rangle\langle e(0)|\ \Gamma(S^{-\frac{1}{2}})^{-1},$$

where $\Gamma(S^{-\frac{1}{2}})$ is defined by Proposition 2.6. Since $|e(0)\rangle\langle e(0)|$ is a coherent state it follows from Proposition 2.8 that

$$\hat{\rho}(u+iv) = \exp -\frac{1}{2}(u,v)\,S\begin{pmatrix} u \\ v \end{pmatrix}, \quad u, v \in \mathbb{R}.$$

Thus we have constructed for every real positive second order matrix S of determinant ≥ 1 a unique state ρ_S in $L^2(\mathbb{R})$ such that

$$\hat{\rho}_S(u+iv) = \exp -\frac{1}{2}(u,v)S\begin{pmatrix} u \\ v \end{pmatrix}.$$

When $\det S = 1$, ρ_S is conjugate to the vacuum state and when $\det S > 1$, ρ_S is conjugate to $(1 - e^{-s})\, e^{-s\, a^\dagger a}$ for $s > 0$. The conjugation is always achieved by a $\Gamma(L)$ where $L \in SL(2, \mathbb{R})$.

3. Gaussian States

We shall now analyse the structure of a general Gaussian state in $L^2(\mathbb{R}^n)$. Recall that a state ρ in $L^2(\mathbb{R}^n)$ is Gaussian if for all $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$ the observable $\sum_{j=1}^n (x_j p_j - y_j q_j)$ has a normal distribution on $\mathbb{R}$ in the state ρ where p_j, q_j are the j-th momentum and position observables for each $1 \leq j \leq n$ described in terms of the Weyl operators by (2.7)-(2.9). If the mean and variance of the observable $\sum_{j=1}^n (x_j p_j - y_j q_j)$ in the state ρ are denoted by $m(\mathbf{x}, \mathbf{y})$ and $\sigma(\mathbf{x}, \mathbf{y})$ respectively then the Gaussian property of ρ can be expressed in terms of characteristic functions as

$$\mathrm{Tr}\,\rho\, e^{-it \sum\limits_{j=1}^{n} (x_j p_j - y_j q_j)} = e^{-it\, m(\mathbf{x},\mathbf{y}) - \frac{1}{2} t^2 \sigma(\mathbf{x},\mathbf{y})}, \quad \forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \tag{3.1}$$

Since $X = \sum_{j=1}^n (x_j p_j - y_j q_j)$ is a selfadjoint operator and the right hand side of (3.1) is an analytic function of t it follows that

$$\begin{aligned} \mathrm{Tr}\, X\, \rho &= m(\mathbf{x}, \mathbf{y}), \\ \mathrm{Tr}\, X^2\, \rho &= \sigma(\mathbf{x}, \mathbf{y}) + m(\mathbf{x}, \mathbf{y})^2 \end{aligned}$$

and therefore $m(.,.)$ and $\sigma(.,.)$ can be expressed as

$$m(\mathbf{x},\mathbf{y}) = \boldsymbol{\ell}^\tau \mathbf{x} - \mathbf{m}^\tau \mathbf{y}, \tag{3.2}$$

$$\sigma(\mathbf{x},\mathbf{y}) = (\mathbf{x}^\tau, \mathbf{y}^\tau) \begin{pmatrix} A & C \\ C^\tau & B \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \tag{3.3}$$

for all $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ where $\boldsymbol{\ell}$, $\mathbf{m} \in \mathbb{R}^n$, A, B, C are real $n \times n$ matrices such that

$$S = \begin{pmatrix} A & C \\ C^\tau & B \end{pmatrix} \geq 0 \tag{3.4}$$

in the sense of positive definiteness.

From (2.12) we have

$$e^{-it \sum\limits_{j=1}^{n} (x_j p_j - y_j q_j)} = W(2^{-\frac{1}{2}} t\mathbf{z}), z_j = x_j + iy_j \tag{3.5}$$

and therefore (3.1) is equivalent to

$$\hat{\rho}(\mathbf{z}) = \exp -i\sqrt{2}(\boldsymbol{\ell}^\tau \mathbf{x} - \mathbf{m}^\tau \mathbf{y}) - (\mathbf{x}^\tau, \mathbf{y}^\tau) S \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \ \mathbf{z} = \mathbf{x} + i\mathbf{y}. \tag{3.6}$$

Thus we have shown that for a state in ρ to be Gaussian it is necessary that its characteristic function $\hat{\rho}$ has the form (3.6) for some $\boldsymbol{\ell}, \mathbf{m} \in \mathbb{R}^n$ and a real positive definite matrix S of order $2n$.

Thus we can make a formal definition: A state ρ in $L^2(\mathbb{R}^n)$ is *Gaussian* with *mean momentum* and *positive vectors* $\boldsymbol{\ell}, \mathbf{m}$ and *momentum-position covariance matrix* S as in (3.4) if its quantum characteristic function $\hat{\rho}$ is given by (3.6). However, for a given triple $\boldsymbol{\ell}, \mathbf{m}, S$ there need not exist a state ρ for which (3.6) holds. Our next result gives a necessary and sufficient condition for the existence of such a ρ.

Theorem 3.1. *Let* $\boldsymbol{\ell}$, $\mathbf{m} \in \mathbb{R}^n$ *and let* $S = \begin{pmatrix} A & C \\ C^\tau & B \end{pmatrix}$ *be a real symmetric matrix of order* $2n$ *where* A, B, C *are matrices of order* $n \times n$. *Define*

$$f(\mathbf{z}) = \exp -i\sqrt{2}(\boldsymbol{\ell}^\tau \mathbf{x} - \mathbf{m}^\tau \mathbf{y}) - (\mathbf{x}^\tau, \mathbf{y}^\tau)\, S \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}, \ \mathbf{z} = \mathbf{x} + i\mathbf{y}. \tag{3.7}$$

Then f *is the quantum characteristic function* $\hat{\rho}$ *of a state* ρ *in* $L^2(\mathbb{R}^n)$ *if and only if the following matrix inequality*

$$2S - \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \geq 0 \tag{3.8}$$

holds in the sense of positive definiteness, $0, I$ *being the null and identity matrices of order* n *in the left hand side.*

Proof. Define the kernel

$$K(\boldsymbol{\alpha}, \boldsymbol{\beta}) = e^{i \operatorname{Im}\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle} f(\boldsymbol{\beta} - \boldsymbol{\alpha}), \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^n.$$

By Theorem 2.3 f is a quantum characteristic function if and only if K is positive definite. If $\boldsymbol{\alpha} = \mathbf{x} + i\mathbf{y}$, $\boldsymbol{\beta} = \mathbf{u} + i\mathbf{v}$ where $\mathbf{x}, \mathbf{y}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ then $\operatorname{Im}\langle \boldsymbol{\alpha} | \boldsymbol{\beta} \rangle = \mathbf{x}^\tau \mathbf{v} - \mathbf{y}^\tau \mathbf{u}$

and the positive definiteness of K in $\mathbb{C}^n$ reduces to the positive definiteness of the kernel L in $\mathbb{R}^{2n}$ where

$$\begin{aligned} L\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) &= \exp\Big\{i(\mathbf{x}^\tau\mathbf{v}-\mathbf{y}^\tau\mathbf{u}) - i\sqrt{2}\left(\boldsymbol{\ell}^\tau(\mathbf{u}-\mathbf{x})-\mathbf{m}^\tau(\mathbf{v}-\mathbf{y})\right) \\ &\qquad - \left((\mathbf{u}-\mathbf{x})^\tau,(\mathbf{v}-\mathbf{y})^\tau\right) S \begin{pmatrix} \mathbf{u}-\mathbf{x} \\ \mathbf{v}-\mathbf{y} \end{pmatrix}\Big\}. \end{aligned}$$

Since the kernels

$$\kappa_\pm\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) = \exp \pm i\sqrt{2}\left(\boldsymbol{\ell}^\tau(\mathbf{u}-\mathbf{x})-\mathbf{m}^\tau(\mathbf{v}-\mathbf{y})\right)$$

are already positive definite in $\mathbb{R}^{2n}$ it follows that K is positive definite on $\mathbb{R}^n$ if and only if the kernel M defined by

$$M\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) = \exp\left\{i(\mathbf{x}^\tau\mathbf{v}-\mathbf{y}^\tau\mathbf{u}) - \left((\mathbf{u}-\mathbf{x})^\tau,(\mathbf{v}-\mathbf{y})^\tau\right) S \begin{pmatrix} \mathbf{u}-\mathbf{x} \\ \mathbf{v}-\mathbf{y} \end{pmatrix}\right\}$$

is positive definite on $\mathbb{R}^{2n}$. This is equivalent to the positive definiteness of

$$M_t\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) = M\left(\sqrt{t}(\mathbf{x},\mathbf{y}),\sqrt{t}(\mathbf{u},\mathbf{v})\right), \quad \forall\, t \geq 0.$$

But $\{M_t\}$ is a one parameter multiplicative semigroup of kernels on $\mathbb{R}^{2n}$. By the elementary properties of infinitely divisible positive definite kernels as described in [11], Section 1 the positive definiteness of M reduces to the conditional positive definiteness of

$$N\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) = i(\mathbf{x}^\tau\mathbf{v}-\mathbf{y}^\tau\mathbf{u}) - \left((\mathbf{u}-\mathbf{x})^\tau,(\mathbf{v}-\mathbf{y})^\tau\right) S \begin{pmatrix} \mathbf{u}-\mathbf{x} \\ \mathbf{v}-\mathbf{y} \end{pmatrix}$$

or equivalently, the positive definiteness of

$$\begin{aligned} &N\left((\mathbf{x},\mathbf{y}),(\mathbf{u},\mathbf{v})\right) - N\left((\mathbf{x},\mathbf{y}),(\mathbf{0},\mathbf{0})\right) - N\left((\mathbf{0},\mathbf{0}),(\mathbf{x},\mathbf{y})\right) \\ &= (\mathbf{x}^\tau,\mathbf{y}^\tau)\left\{2S - \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}\right\}\begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \end{aligned}$$

on $\mathbb{R}^{2n}$. This is the same as inequality (3.8). □

Remark 3.2. For any triple $(\boldsymbol{\ell},\mathbf{m},S)$ as in Theorem 3.1 denote the unique Gaussian state ρ with $\hat{\rho} = f$ in (3.7) by $\rho_g(\boldsymbol{\ell},\mathbf{m},S)$. Inequality (3.8) implies, in particular, by going to real parts that $S \geq 0$. From (3.8) and the expression for S in terms of A, B, C we have the matrix inequalities

$$\begin{pmatrix} 2a_{jj} & 2c_{jj}+i \\ 2c_{jj}-i & 2b_{jj} \end{pmatrix} \geq 0, \text{ where } A = ((a_{ij})), \ldots$$

In particular,

$$a_{jj}b_{jj} \geq \frac{1}{4} + c_{jj}^2$$

where a_{jj} and b_{jj} are the variances of the j-th momentum and position observables. This is the uncertainty principle for p_j, q_j. Thus (3.8) can be interpreted as the *complete uncertainty principle* for all the momentum and position observables in the Gaussian state $\rho_g(\boldsymbol{\ell},\mathbf{m},S)$.

Corollary 3.3. *Let* $\boldsymbol{\alpha} = \mathbf{u} + i\mathbf{v}$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. *Then*

$$W(\boldsymbol{\alpha})\rho_g(\boldsymbol{\ell}, \mathbf{m}, S)W(\boldsymbol{\alpha})^{-1} = \rho_g(\boldsymbol{\ell} + \sqrt{2}\mathbf{v}, \mathbf{m} + \sqrt{2}\mathbf{u}, S).$$

In particular

$$W\left(-2^{-\frac{1}{2}}(\mathbf{m} + i\boldsymbol{\ell})\right)\rho_g(\boldsymbol{\ell}, \mathbf{m}, S)W\left(-2^{-\frac{1}{2}}(\mathbf{m} + i\boldsymbol{\ell})\right)^{-1} = \rho_g(\mathbf{0}, \mathbf{0}, S).$$

Proof. Immediate from (3.7) and Proposition 2.5, second part. □

Remark 3.4. By a suitable conjugation with a Weyl operator the momentum and position means of a Gaussian state can be made zero.

Corollary 3.5. *Let* $L \in Sp_{2n}$ *and let* $\Gamma(L)$ *be the unitary operator satisfying* (2.19). *Then*

$$\Gamma(L)\rho_g(\boldsymbol{\ell}, \mathbf{m}, S)\Gamma(L)^{-1} = \rho_g(\boldsymbol{\ell}', \mathbf{m}', S'),$$

where

$$\begin{aligned} \begin{pmatrix} \boldsymbol{\ell}' \\ -\mathbf{m}' \end{pmatrix} &= (L^{-1})^\tau \begin{pmatrix} \boldsymbol{\ell} \\ -\mathbf{m} \end{pmatrix} \\ S' &= (L^{-1})^\tau S L^{-1}. \end{aligned}$$

Proof. Immediate from Proposition 2.6 and (3.7). □

Corollary 3.6. *Let* $\boldsymbol{\ell}, \mathbf{m} \in \mathbb{R}^n$, $\boldsymbol{\ell}', \mathbf{m}' \in \mathbb{R}^{n'}$ *and let*

$$S = \begin{bmatrix} A & C \\ C^\tau & B \end{bmatrix}, \qquad S' = \begin{bmatrix} A' & C' \\ C^\tau & B' \end{bmatrix}$$

satisfy condition (3.8). *Then*

$$\rho_g(\boldsymbol{\ell}, \mathbf{m}, S) \otimes \rho_g(\boldsymbol{\ell}', \mathbf{m}', S') = \rho_g(\boldsymbol{\ell} \oplus \boldsymbol{\ell}', \mathbf{m} \oplus \mathbf{m}', S \oplus S'),$$

where

$$S \oplus S' = \begin{bmatrix} A \oplus A' & C \oplus C' \\ C^\tau \oplus C'^\tau & B \oplus B' \end{bmatrix}$$

with $X \oplus Y$ *in the right hand side defined by*

$$X \oplus Y = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$$

Proof. Immediate from Proposition 2.8. □

Corollary 3.7. *If* $n = d + d'$ *then the marginal in* $L^2(\mathbb{R}^d)$ *of a Gaussian state* $\rho_g(\boldsymbol{\ell}, \mathbf{m}, S)$ *with* $\boldsymbol{\ell}, \mathbf{m} \in \mathbb{R}^n$ *and* $S = \begin{pmatrix} A & C \\ C^\tau & B \end{pmatrix}$ *in* $L^2(\mathbb{R}^n)$ *is the Gaussian state* $\rho_g(\boldsymbol{\ell}', \mathbf{m}', S')$ *where* $\boldsymbol{\ell}' = (\ell_1, \ell_2, \ldots, \ell_d)^\tau$, $\mathbf{m}' = (\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_d)^\tau$ *and* $S' = \begin{pmatrix} A' & C' \\ C'^\tau & B' \end{pmatrix}$ *with* $A' = ((a_{ij})), B' = ((b_{ij})), C' = ((c_{ij}))$ *where* $i, j \in \{1, 2, \ldots, d\}$.

Proof. Immediate from (3.7) and Proposition 2.8, second part. □

Proposition 3.8. *Let* $\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}$ *be a* $2n$*-dimensional Gaussian random vector with mean vector* $\begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix}$ *and covariance matrix*

$$\Sigma = \begin{pmatrix} K & M \\ M^\tau & L \end{pmatrix}.$$

Define the state

$$\bar{\rho} = \mathbb{E}\ W(\boldsymbol{\xi} + i\boldsymbol{\eta})\, \rho_g(\boldsymbol{\ell}, \mathbf{m}, S)\, W(\boldsymbol{\xi} + i\boldsymbol{\eta})^{-1}$$

where the expectation is taken with respect to the distribution of $\begin{pmatrix} \boldsymbol{\xi} \\ \boldsymbol{\eta} \end{pmatrix}$. *Then*

$$\bar{\rho} = \rho_g\left(\boldsymbol{\ell} + \sqrt{2}\mathbf{d},\, \mathbf{m} + \sqrt{2}\mathbf{c},\, S + 2J^\tau \Sigma J\right),$$

where J *is given by* (2.15).

Proof. From Corollary 3.3 to Theorem 3.1 we have

$$\begin{aligned}
\hat{\bar{\rho}}(\mathbf{z}) &= \mathbb{E}\ \hat{\rho}_g\left(\boldsymbol{\ell} + \sqrt{2}\boldsymbol{\eta},\, \mathbf{m} + \sqrt{2}\boldsymbol{\xi}, S\right)(\mathbf{z}) \\
&= \mathbb{E}\exp\left\{-i\sqrt{2}\left((\boldsymbol{\ell} + \sqrt{2}\boldsymbol{\eta})^\tau \mathbf{x} - (\mathbf{m} + \sqrt{2}\boldsymbol{\xi})^\tau \mathbf{y}\right) - (\mathbf{x}^\tau, \mathbf{y}^\tau) S \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\right\} \\
&= \exp\Bigg\{ - i\sqrt{2}\left((\boldsymbol{\ell} + \sqrt{2}\mathbf{d})^\tau \mathbf{x} - (\mathbf{m} + \sqrt{2}\mathbf{c}))^\tau \mathbf{y}\right) \\
&\qquad\qquad - (\mathbf{x}^\tau, \mathbf{y}^\tau)(S + 2J^\tau \Sigma J)\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix}\Bigg\}.
\end{aligned}$$

□

Remark 3.9. Proposition 3.8 shows that whenever the density operator of a Gaussian state $\rho_g(\mathbf{0}, \mathbf{0}, S)$ with momentum-position covariance matrix is known then any Gaussian state $\rho_g(\boldsymbol{\ell}, \mathbf{m}, S')$ with $S' \geq S$ can be obtained by a classical expectation after conjugation by a random Weyl operator from $\rho_g(\mathbf{0}, \mathbf{0}, S)$.

Proposition 3.10. *Let* $L \in Sp_{2n}$ *and let* $\Gamma(L)$ *denote the unitary operator in* $L^2(\mathbb{R}^n)$ *satisfying* (2.19). *Define*

$$|\psi_L\rangle = \Gamma(L^{-1})\ |e(\mathbf{0})\rangle.$$

Then

$$|\psi_L\rangle\langle\psi_L| = \rho_g(\mathbf{0}, \mathbf{0}, \frac{1}{2}L^\dagger L).$$

Proof. This is immediate from Proposition 2.6 and Proposition 2.9. □

Proposition 3.11. *Consider two Gaussian states in* $L^2(\mathbb{R}^n)$

$$\rho_i = \rho_g(\mathbf{0}, \mathbf{0}, S_i), \quad i = 1, 2.$$

For any $0 \leq \theta < 2\pi$ *define the unitary operator* Γ_θ *in* $L^2(\mathbb{R}^{2n}) = L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$ *by the relations*

$$\Gamma_\theta e(\mathbf{z} \oplus \boldsymbol{\xi}) = e(\mathbf{z}' \oplus \boldsymbol{\xi}'),$$

where

$$\mathbf{z}' = (\cos\theta)\mathbf{z} - (\sin\theta)\boldsymbol{\xi}, \quad \boldsymbol{\xi}' = (\sin\theta)\mathbf{z} + (\cos\theta)\boldsymbol{\xi}, \quad \forall\ \mathbf{z}, \boldsymbol{\xi} \in \mathbb{C}^n.$$

Then

$$\mathrm{Tr}_2\Gamma_\theta(\rho_1 \otimes \rho_2)\Gamma_\theta^{-1} = \rho_g(\mathbf{0}, \mathbf{0}, (\cos^2\theta)S_1 + (\sin^2\theta)S_2),$$

where Tr_2 *indicates relative trace over the second factor in* $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$.

Proof. Immediate from Proposition 2.5, Proposition 2.8 and (3.7). □

Remark 3.12. More generally, if $\rho_i = \rho_g(\mathbf{0}, \mathbf{0}, S_i)$, $1 \leq i \leq k$ are Gaussian states in $L^2(\mathbb{R}^n)$, and $S = s_1S_1 + s_2S_2 + \cdots + s_kS_k$ where $s_i > 0$, $\sum s_i = 1$ then one can construct an orthogonal matrix R whose first row is $(\sqrt{s_1}, \sqrt{s_2}, \ldots, \sqrt{s_k})$ and a unitary operator Γ_R in the product $L^2(\mathbb{R}^n) \otimes \cdots, \otimes L^2(\mathbb{R}^n)$ with k copies so that

$$\rho_g(\mathbf{0}, \mathbf{0}, S) = \mathrm{Tr}_{2,3,\ldots,k}\Gamma_R\ \rho_1 \otimes \rho_2 \otimes \cdots \otimes \rho_k\Gamma_R^{-1}$$

where the relative trace is taken over the last $k-1$ factors.

As a special case consider elements $L_1, L_2, \ldots, L_k$ in Sp_{2n} and the vectors $|\psi_{L_i}\rangle$ as in Proposition 3.10. Let

$$S = \frac{1}{2}\sum_{i=1}^{k} s_i L_i^\tau L_i$$

where $s_i > 0$ and $\sum s_i = 1$. Then the Gaussian state $\rho_g(\mathbf{0}, \mathbf{0}, S)$ can be realized as

$$\rho_g(\mathbf{0}, \mathbf{0}, S) = \mathrm{Tr}_{2,\ldots,k}\Gamma_R\ |\psi_{L_1} \otimes \psi_{L_2} \otimes \cdots \otimes \psi_{L_k}\rangle\langle\psi_{L_1} \otimes \psi_{L_2} \otimes \cdots \otimes \psi_{L_k}|\Gamma_R^{-1}.$$

In other words the Gaussian state $\rho_g(\mathbf{0}, \mathbf{0}, S)$ can be purified to a Gaussian state. However, we do not know whether every Gaussian state enjoys this property. The answer is in the affirmative if every positive matrix $\geq iJ$ is in the convex hull of $\{L^\tau L, L \in Sp_{2n}\}$. Here J is given by (2.15).

It would be interesting to know whether every pure Gaussian state with momentum and position means zero is of the form $|\psi_L\rangle\langle\psi_L|$ for some $L \in Sp_{2n}$.

We conclude with a problem on the nature of symmetries of the class of all Gaussian states. Let $\mathcal{G}_n$ denote the set of all Gaussian states in $L^2(\mathbb{R}^n)$. Define $\mathcal{U}_g(n)$ to be the group of all unitary operators in $L^2(\mathbb{R}^n)$ with the property $U \in \mathcal{U}_g(n)$ if and only if $U\rho U^{-1} \in \mathcal{G}_n$ whenever $\rho \in \mathcal{G}_n$. Is it true that $\mathcal{U}_g(n)$ is generated by all operators of the form $W(\boldsymbol{\alpha})$, $\Gamma(L)$ and λI where $\boldsymbol{\alpha} \in \mathbb{C}^n$, $L \in Sp_{2n}$ and λ varies over all scalars of modulus unity?

Acknowledgment. I thank Mr. Anil Shukla for his efficient Texing of the manuscript.

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