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AN ANTICIPATIVE STOCHASTIC CALCULUS APPROACH TO PRICING IN MARKETS DRIVEN BY LÉVY PROCESS

BERNT ØKSENDAL* AND AGNÈS SULEM

Abstract. We use the Itô-Ventzell formula for forward integrals and Malliavin calculus to study the stochastic control problem associated to utility indifference pricing in a market driven by Lévy processes. This approach allows us to consider general possibly non-Markovian systems, general utility functions and possibly partial information based portfolios. In the special case of the exponential utility function $U_\alpha = -\exp(-\alpha x)$; $\alpha > 0$, we obtain asymptotics properties for vanishing α. In the special case of full information based portfolios and no jumps, we obtain a recursive formula for the optimal portfolio in a non-Markovian setting.

1. Introduction

Consider a financial market with the following investment possibilities

- A *risk free asset*, where the unit price $S_0(t)$ at time t is:

$$S_0(t) = 1 \text{ for all } t \in [0,T], \tag{1.1}$$

  where $T > 0$ is a fixed constant.
- A *risky asset*, where the unit price $S_1(t) = S(t)$ at time t is given by

$$dS(t) = S(t^-)\left[\mu(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz)\right]. \tag{1.2}$$

Here $B(t)$ is a Brownian motion and $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ is the compensated jump measure, $\tilde{N}(\cdot,\cdot)$, of an independent Lévy process

$$\eta(t) := \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(ds,dz),$$

with jump measure $N(dt,dz)$ and Lévy measure $\nu(U) = E[N([0,1],U)]$ for $U \in B(\mathbb{R}_0)$ (i.e. U is a Borel set with closure $\bar{U} \subset \mathbb{R}_0 := \mathbb{R} - \{0\}$). The underlying probability space is denoted by $(\Omega, \mathcal{F}, P)$ and the σ-algebra generated by $\{B(s)\ ;\ s \le t, \eta(s)\ ;\ s \le t\}$ is denoted by $\mathcal{F}_t$.

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The processes $\mu(t)$, $\sigma(t)$ and $\gamma(t,z)$ are assumed to be $\mathcal{F}_{t-}$ predictable and satisfying

$$\int_0^T \left\{ |\mu(t)| + \sigma^2(t) + \int_{\mathbb{R}} |\ln(1+\gamma(t,z)) - \gamma(t,z)|\nu(dz) \right\} dt < \infty \text{ a.s.} \tag{1.3}$$

and

$$\gamma(t,z) \geq -1 \text{ a.s. for all } z \in \mathbb{R}_0,\ t \in [0,T]. \tag{1.4}$$

Then, by the Itô formula for Itô-Lévy processes (see e.g. [12], Chapter 1) the solution of (1.2) is

$$S(t) = S(0)\exp\{\xi(t)\}\ ;\ t \in [0,T], \tag{1.5}$$

where

$$\xi(t) = \int_0^t \left\{ \mu(s) - \frac{1}{2}\sigma^2(s) + \int_{\mathbb{R}_0} (\ln(1+\gamma(s,z)) - \gamma(s,z))\nu(dz) \right\} ds$$
$$\int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} \ln(1+\gamma(s,z))\tilde{N}(dz,dz). \tag{1.6}$$

Let $\varphi(t) = (\varphi_0(t), \varphi_1(t))$ be an $\mathcal{F}_t$-predictable process representing a *portfolio* in this market, giving the number of units held in the risk free and the risky asset respectively, at time t. We will assume that φ is *self-financing*, in the sense that if

$$X(t) = X^{\varphi}(t) = \varphi_0(t)S_0(t) + \varphi_1(t)S_1(t) \tag{1.7}$$

is the *total value* of the investment at time t, then (since $dS_0(t) = 0$)

$$dX^{\varphi}(t) = \varphi_0(t)dS_0(t) + \varphi_1(t)dS_1(t) = \varphi_1(t)dS_1(t) \tag{1.8}$$

i.e.

$$X^{\varphi}(t) = x + \int_0^t u(s)dS(t), \qquad x = X^{\varphi}(0), \tag{1.9}$$

where $u(s) = \varphi_1(s)$.

In the following we let

$$\mathcal{E} \subseteq \mathcal{F}\ ;\ 0 \leq t \leq T$$

be a fixed subfiltration of $\{\mathcal{F}_t\}_{t\geq 0}$, representing the information available to the trader at time t. This means that we require that the portfolio $\varphi(t)$ must be $\mathcal{E}_t$-measurable for each $t \in [0,T]$. For example, we could have $\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}$, which models the situation when the trader has a delayed access to the information $\mathcal{F}_t$ from the market. This implies in particular that the control $\varphi(t)$ need not be Markovian.

If φ is self-financing and $\mathcal{E}$-adapted, and the value process $X^{\varphi}(t)$ is *lower bounded*, we say that φ is *$\mathcal{E}$-admissible*. The set of all $\mathcal{E}$-admissible controls is denoted by $\mathcal{A}_{\mathcal{E}}$.

If $\sigma \neq 0$ and $\gamma\nu \neq 0$ then it is well-known that the market is *incomplete*. This is already the case if $\mathcal{E}_t = \mathcal{F}_t$ for all $t \in [0,T]$, and even more so if $\mathcal{E}_t \subseteq \mathcal{F}_t$ for all $t \in [0,T]$. Therefore the no-arbitrage principle is not sufficient to provide a unique price for a given European T-claim $G(\omega), \omega \in \Omega$. In this paper we will apply the *utility indifference principle* of Hodges and Neuberger [7] to find the price. In short, the principle is the following:

We fix a utility function $U : \mathbb{R} \to (-\infty, \infty)$. A trader with no final payment obligations faces the problem of maximizing the expected utility of the terminal wealth $X_x^{(\varphi)}(T)$ given that the initial wealth is $X_x^{(\varphi)}(0) = x \in \mathbb{R}$:

$$V_0(x) := \sup_{\varphi \in \mathcal{A}_{\mathcal{E}}} E\left[U\left(X_x^{(\varphi)}(T)\right)\right] = E\left[U\left(X_x^{(\hat{\varphi})}(T)\right)\right], \tag{1.10}$$

where $\hat{\varphi} \in \mathcal{A}_{\mathcal{E}}$ is an optimal portfolio (if it exists).

If, on the other hand, the trader is also selling a guaranteed payoff $G(\omega)$ (a lower bounded $\mathcal{F}_T$-measurable random variable) and gets an initial payment $p > 0$ for this, the problem for the seller will be to find $V_G(x + p)$ and $u^* \in \mathcal{A}_{\mathcal{E}}$ (an optimal portfolio, if it exists), such that

$$\begin{aligned} V_G(x+p) &:= \sup_{u \in \mathcal{A}_{\mathcal{E}}} E\left[U\left(X_{x+p}^{(u)}(T) - G\right)\right] \\ &= E\left[U\left(X_{x+p}^{(u^*)}(T) - G\right)\right]. \end{aligned} \tag{1.11}$$

The *utility indifference pricing principles* states that the "right" price p of the European option with payoff G at time T is the solution p of the equation

$$V_G(x+p) = V_0(x). \tag{1.12}$$

This means that the seller is indifferent to the following two alternatives: Either (i) receiving the payment p at time 0 and paying out $G(\omega)$ at time T, or (ii) not selling the option at all, i.e. $p = G = 0$.

We see that in order to find the price p we need to solve the stochastic control problem (1.11) to find $V_G(x + p)$. Then we get $V_0(x)$ as a special case by putting $G = p = 0$.

In this paper we will use anticipative stochastic calculus (forward integrals) and Malliavin calculus to solve the problem (1.11). The motivations for our approach are the following:

- **(i):** We want a method which applies to a wide class of utility functions, not just the exponential utility $U(x) = -\mathrm{e}^{-\alpha x}$; $\alpha > 0$, which is the most studied so far.
- **(ii):** We are interested in the situation when the trader has only partial information $\mathcal{E}_t$ to her disposal. For example, if $\mathcal{E}_t = \mathcal{F}_{(t-\delta)^+}$, how does the information delay δ influence the price ?
- **(iii):** We want to allow more general payoffs $G(\omega)$ than the Markovian ones of the form $G = g(S(T))$. In particular, we want to allow path-dependent payoffs $G = g(\{S(t)\ ;\ t \le T\})$.

In Section 4 we study the exponential utility case in more detail. Under some conditions we show that if $u_\alpha^{(G)}$ is an optimal portfolio corresponding to $U(x) = -\mathrm{e}^{-\alpha x}$ and terminal payoff G, then $\tilde{u}(t) := \lim_{\alpha \to 0} \alpha u_\alpha^{(G)}(t)$ is an optimal portfolio corresponding to $\alpha = 1$ and $G = 0$ (Theorems 4.4 and 4.5). In Theorem 4.6 we obtain a recursive formula for the optimal portfolio in a non-Markovian setting if $\mathcal{E}_t = \mathcal{F}_t$ and $\nu = 0$.

For more information and results about utility indifference pricing we refer to [2], [6], [7], [8], [10] and [17], and the references therein. For more information about stochastic calculus and financial markets with Lévy processes we refer to [1], [3] and [12].

2. Some Prerequisites on Forward Integrals and Malliavin Calculus

In this section we give a brief summary of basic definitions and properties of forward integrals and Malliavin calculus for Lévy processes. General references to this section are [4], [5] and [14]. First we consider *forwards stochastic integrals*:

Definition 2.1. [14] We say that a stochastic process $\varphi(t)\ ;\ t \in [0,T]$, is *forward integrable* over the interval $[0,T]$ with respect to Brownian motion $B(\cdot)$ if there exists a process $I(t)\ ;\ t \in [0,T]$, such that:

$$\sup_{t\in[0,T]} \left(\int_0^t \varphi(s) \frac{B(s+\epsilon)-B(s)}{\epsilon} ds - I(t) \right) \to 0 \text{ as } \varepsilon \to 0 \tag{2.1}$$

in probability. If this is the case we put

$$I(t) = \int_0^t \varphi(s) d^- B(s) \tag{2.2}$$

and call $I(t)$ the *forward integral* of φ with respect to $B(\cdot)$.

The forward integral is an extension of the Itô integral, in the sense that if φ is adapted and forward integrable, then the forward integral of φ coincides with the classical Itô integral.

Example 2.2. [Simple integrands] If the process $\varphi(t)$ has the simple form

$$\varphi(t) = \sum_{j=1}^m a_j(\omega) \chi_{[t_j,t_{j+1})}(t)\ ;\ 0 \le t_j, t \le T \text{ for all } j$$

where $a_j(\omega)$ are arbitrary random variables, then φ is forward integrable and

$$\int_0^T \varphi(t) d^- B(t) = \sum_{j=1}^m a_j(\omega)(B(t_{j+1}) - B(t_j)).$$

Next we define the corresponding integral with respect to the compensated Poisson random measure $\tilde{N}(\cdot,\cdot)$:

Definition 2.3. [4] [Forward integrals with respect to $\tilde{N}(\cdot,\cdot)$.] We say that a stochastic process $\psi(t,z)$; $t \in [0,T]$, $z \in \mathbb{R}_0$ is *forward integrable* over $[0,T]$ with respect to $\tilde{N}(\cdot,\cdot)$ if there exists a process $J(t)\ ;\ t \in [0,T]$, such that

$$\sup_{t\in[0,T]} \left(\int_0^T \int_{\mathbb{R}_0} \psi(s,z) \mathbf{1}_{K_n}(z) \tilde{N}(ds,dz) - J(t) \right) \leftarrow 0 \text{ as } n \to \infty \tag{2.3}$$

in probability. Here $\{K_n\}_{n=1}^\infty$ is an increasing sequence of compact sets $K_n \subset \mathbb{R}_0$ with $\nu(K_n) < \infty$ such that $\bigcup_{n=1}^\infty K_n = \mathbb{R}_0$ and we require that $J(t)$ does not depend on the sequence $\{K_n\}_{n=1}^\infty$ chosen. If this is the case we put

$$J(t) = \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\tilde{N}(d^- s, dz) \tag{2.4}$$

and we call $J(t)$ the *forward integral of* $\psi(\cdot,\cdot)$ *with respect to* $\tilde{N}(\cdot,\cdot)$.

Also in this case the forward integral coincides with the classical Itô integral if the integrand is $\mathcal{F}_t$-predictable. We now combine the two concepts above and make the following definition:

Definition 2.4. [Generalized forward processes] A (*generalized*) *forward* (Itô-Lévy) *process* is a stochastic process $Y(t)$; $t \in [0,T]$ of the form

$$Y(t) = Y(0) + \int_0^t \alpha(s)ds + \int_0^t \varphi(s)d^- B(s) + \int_0^t \int_{\mathbb{R}_0} \psi(s,z)\tilde{N}(d^- s, dz) \tag{2.5}$$

where $Y(0)$ is an $\mathcal{F}_T$-measurable random variable and $\varphi(s)$ and $\psi(s,z)$ are forward integrable processes. A shorthand notation for this is

$$d^- Y(t) = \alpha(t)dt + \varphi(t)d^- B(t) + \int_{\mathbb{R}_0} \psi(t,z)\tilde{N}(d^- t, dz) \; ; \; t \in (0,T) \tag{2.6}$$

$$Y(0) \text{ is } \mathcal{F}_T\text{-measurable} \tag{2.7}$$

If $Y(0) = y \in \mathbb{R}$ is non-random, then the process $Y(t)$ is an Itô-Lévy process of the type discussed in [4]. The term "generalized" refers to the case when $Y(0)$ is random.

We will need an Itô formula for generalized forward processes. The following result is a slight extension of the Itô formula in [15], [16] (Brownian motion case) and [4] (Poisson random measure case). It may be regarded as a special case of the Itô-Ventzell formula given in [13]:

Theorem 2.5. [13]*[Special case of the Itô-Ventzell formula for forward processes] Let* $Y(t)$ *be a generalized forward process of the form* (2.5) *and assume that* $\psi(t,z)$ *is continuous in* z *near* $z = 0$ *for a.a.* t, ω *and that*

$$\int_0^T \int_{\mathbb{R}} \psi^2(t,z)\nu(dz)dt < \infty \;\; a.s.$$

Let $f \in C^2(\mathbb{R})$ *and define*

$$Z(t) = f(Y(t)).$$

Then $Z(t)$ *is a forward process given by*

$$\begin{aligned} d^- Z(t) = \Big[& f'(Y(t))\alpha(t) + \frac{1}{2} f''(Y(t))\varphi^2(t) \\ & + \int_{\mathbb{R}_0} \{ f(Y(t) + \psi(t,z)) - f(Y(t)) - f'(Y(t))\psi(t,z) \} \, \nu(dz) \Big] \, dt \\ & + f'(Y(t))d^- B(t) + \int_{\mathbb{R}} \big\{ f(Y(t^-) + \psi(t,z)) - f(Y(t^-)) \big\} \, \tilde{N}(d^- t, dz) \; ; \; t > 0 \end{aligned} \tag{2.8}$$

$$Z(0) = f(Y(0)). \tag{2.9}$$

Next we briefly recall the concepts and results that we need from the theory of Malliavin calculus for Lévy processes. For more information in the Brownian motion case, we refer to [11] and [15] and for the general case we refer to [4] and [5].

In the following we let D_tF denote the Malliavin derivative with respect to $B(\cdot)$ (at time t) of a given Malliavin differentiable random variable $F = F(\omega); \omega \in \Omega$. Similarly, $D_{t,z}F$ denotes the Malliavin derivative of F with respect to $\tilde{N}(\cdot,\cdot)$ (at t, z). We let $\mathbb{D}_{1,2}$ denote the set of all random variables F which are Malliavin differentiable both with respect to $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$.

The following results are useful:

- ([11],[4]) Suppose $F \in \mathbb{D}_{1,2}$ is $\mathcal{F}_s$-measurable. Then

$$D_tF = D_{t,z}F = 0 \text{ for all } t > s; z \in \mathbb{R}_0. \tag{2.10}$$

- **Chain rule** ([11], page 29)

  Suppose $F_1, \dots, F_m \in \mathbb{D}_{1,2}$ and that $\varphi : \mathbb{R}^m \to \mathbb{R}$ is C^1 with bounded partial derivatives. Then $\varphi(F_1, \dots, F_m) \in \mathbb{D}_{1,2}$ and

$$D_t\varphi(F_1, \dots, F_m) = \sum_{i=1}^{m} \frac{\partial \varphi}{\partial x_i}(F_1, \dots, F_m)D_tF_i. \tag{2.11}$$

- **Integration by parts** ([11], page 35)

  Suppose $u(t)$ is $\mathcal{F}_t$-adapted with

$$E\left[\int_0^T u^2(t)dt\right] < \infty$$

  and let $F \in \mathbb{D}_{1,2}$. Then

$$E\left[F\int_0^T u(t)dB(t)\right] = E\left[\int_0^T u(t)D_tF dt\right]. \tag{2.12}$$

- **Duality formula for forward integrals** ([15])

  Suppose $\beta(\cdot)$ is forward integrable with respect to $B(\cdot)$, $\beta(t) \in \mathbb{D}_{1,2}$ and $D_{t^+}\beta(t) := \lim_{s \to t^+} D_s\beta(t)$ exists for a.a. t with

$$E\left[\int_0^T |D_{t^+}\beta(t)|dt\right] < \infty.$$

  Then

$$E\left[\int_0^T \beta(t)d^-B(t)\right] = E\left[\int_0^T D_{t^+}\beta(t)dt\right]. \tag{2.13}$$

- **Chain rule** ([4]).

  Suppose $F_1, \dots, F_m \in \mathbb{D}_{1,2}$ and that $\varphi : \mathbb{R}^m \to \mathbb{R}$ is continuous and bounded. Then $\varphi(F_1, \dots, F_m) \in \mathbb{D}_{1,2}$ and

$$D_{t,z}\varphi(F_1, \dots, F_m) = \varphi(F_1 + D_{t,z}F_1, \dots, F_m + D_{t,z}F_m) - \varphi(F_1, \dots, F_m). \tag{2.14}$$

- **Integration by parts**([4]).
  Suppose $\psi(t,z)$ is $\mathcal{F}_t$-adapted and
$$E\left[\int_0^T\int_{\mathbb{R}_0}\psi^2(t,z)\nu(dz)dt\right]<\infty$$
and let $F\in\mathbb{D}_{1,2}$. Then
$$E\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)\tilde{N}(dt,dz)\right]=E\left[\int_0^T\int_{\mathbb{R}_0}\psi(t,z)D_{t,z}F\nu(dz)dt\right]. \tag{2.15}$$
- **Duality formula for forward integrals** ([4]).
  Suppose $\theta(t,z)$ is forward integrable with respect to $\tilde{N}$, $\theta(t,z)\in\mathbb{D}_{1,2}$ and
$$D_{t^+,z}\theta(t,z):=\lim_{s\to t^+}D_{s,z}\theta(t,z) \text{ exists for a.a. } t,z$$
with
$$E\left[\int_0^T\int_{\mathbb{R}_0}\left|D_{t^+,z}\theta(t,z)\right|\nu(dz)dt\right]<\infty.$$
Then
$$E\left[\int_0^T\int_{\mathbb{R}_0}\theta(t,z)\tilde{N}(d^-t,dz)\right]=E\left[\int_0^T\int_{\mathbb{R}_0}D_{t^+,z}\theta(t,z)\nu(dz)dt\right]. \tag{2.16}$$

3. Solving the Stochastic Control Problem

In this section we use forward integrals to solve the stochastic control problem (1.11). We assume $U\in C^3(\mathbb{R})$ and that the payoff $G=G(\omega)$ is Malliavin differentiable both with respect to $B(\cdot)$ and $\tilde{N}(\cdot,\cdot)$.

Choose $u\in\mathcal{A}_{\mathcal{E}}$, $x\in\mathbb{R}$ and consider
$$\begin{aligned}Y(t)&:=X(t)-G=X_x^{(u)}(t)-G=x-G+\int_0^t u(s)dS(s)\\&=x-G+\int_0^t\mu(s)u(s)S(s)ds+\int_0^t\sigma(s)u(s)S(s)dB(s)\\&\quad+\int_0^t\int_{\mathbb{R}}u(s)S(s^-)\gamma(s,z)\tilde{N}(ds,dz).\end{aligned} \tag{3.1}$$
By the Itô-Ventzell formula for forward integrals (Theorem 2.5) we have
$$\begin{aligned}&d(U(Y(t)))=U'(Y(t))[\mu(t)u(t)S(t)dt+\sigma(t)u(t)S(t)d^-B(t)]\\&+\frac{1}{2}U''(Y(t))\sigma^2(t)u^2(t)S^2(t)dt\\&+\int_{\mathbb{R}_0}\{U(Y(t)+u(t)S(t)\gamma(t,z))-U(Y(t))-u(t)S(t)\gamma(t,z)U'(Y(t))\}\nu(dz)dt\\&+\int_{\mathbb{R}_0}\{U(Y(t^-)+u(t)S(t^-)\gamma(t,z))-U(Y(t^-))\}\tilde{N}(d^-t,dz).\end{aligned} \tag{3.2}$$

Hence

$$U(X(T)-G) = U(x-G) + \int_0^T \alpha(t)dt + \int_0^T \beta(t)d^-B(t) + \int_0^T \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(d^-t,dz), \tag{3.3}$$

where

$$\begin{aligned}\alpha(t) &= U'(X(t)-G)u(t)S(t)\mu(t) + \frac{1}{2}U''(X(t)-G)u^2(t)S^2(t)\sigma^2(t) \\ &\quad + \int_{\mathbb{R}_0} \{U(X(t)+u(t)S(t)\gamma(t,z)-G) - U(X(t)-G) \\ &\quad - u(t)S(t)\gamma(t,z)U'(X(t)-G)\}\nu(dz), && (3.4)\\ \beta(t) &= U'(X(t)-G)u(t)S(t)\sigma(t) && (3.5)\\ \theta(t,z) &= U(X(t^-)+u(t)S(t^-)\gamma(t,z)-G) - U(X(t^-)-G). && (3.6)\end{aligned}$$

We now use the duality formulas (2.13) and (2.16). We have

$$\begin{aligned}&D_{t^+}\beta(t) = u(t)S(t)\sigma(t)U''(X(t)-G)(-D_tG)\\ &D_{t^+,z}\theta(t,z) = U(X(t^-)+u(t)S(t)\gamma(t,z)-G-D_{t,z}G)\\ &- U(X(t^-)+u(t)S(t)\gamma(t,z)-G) - U(X(t^-)-G-D_{t,z}G) + U(X(t^-)-G).\end{aligned}$$

Equation (3.3) becomes

$$\begin{aligned}&E[U(X(T)-G)]\\ &= E[U(x-G)] + E[\int_0^T \{\alpha(t) + D_{t^+}\beta(t) + \int_{\mathbb{R}_0} D_{t^+,z}\theta(t,z)\nu(dz)\}dt]\\ &= E[U(x-G)] + E[\int_0^T \{u(t)S(t)[\mu(t)U'(X(t)-G) - \sigma(t)U''(X(t)-G)D_tG]\\ &+ \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)U''(X(t)-G)\\ &+ \int_{\mathbb{R}_0} [U(X(t)+u(t)S(t)\gamma(t,z)-G) - U(X(t)-G)\\ &- u(t)S(t)\gamma(t,z)U'(X(t)-G)\\ &+ U(X(t)+u(t)S(t)\gamma(t,z)-G-D_{t,z}G) - U(X(t)+u(t)S(t)\gamma(t,z)-G)\\ &- U(X(t)-G-D_{t,z}G) + U(X(t)-G)]\nu(dz)\}dt]\\ &= E[U(x-G)] + E[\int_0^T \{u(t)S(t)[\mu(t)U'(X(t)-G) - \sigma(t)U''(X(t)-G)D_tG]\\ &\quad + \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)U''(X(t)-G)\\ &\quad + \int_{\mathbb{R}_0} [U(X(t)+u(t)S(t)\gamma(t,z)-G-D_{t,z}G) - U(X(t)-G-D_{t,z}G)\\ &- u(t)S(t)\gamma(t,z)U'(X(t)-G)]\nu(dz)\}dt]. && (3.7)\end{aligned}$$

We may insert a conditional expectation with respect to $\mathcal{F}_t$ for each t in this integral and this gives:

$$
\begin{aligned}
&E[U(X(T)-G)] = E[U(x-G)] + E[\int_0^T \{u(t)S(t)(\mu(t)E[U'(X(t)-G) \mid \mathcal{F}_t] \\
&- \sigma(t)E[U''(X(t)-G)D_tG \mid \mathcal{F}_t]) + \frac{1}{2}u^2(t)S^2(t)\sigma^2(t)E[U''(X(t)-G) \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0} E[(U(X(t)+u(t)S(t)\gamma(t,z)-G-D_{t,z}G) - U(X(t)-G-D_{t,z}G) \\
&\quad - U(t)S(t)\gamma(t,z)U'(X(t)-G)) \mid \mathcal{F}_t]\nu(dz)dt. \qquad (3.8)
\end{aligned}
$$

We conclude that our original stochastic control problem (1.11) is equivalent to a problem of the following type: Find Φ and $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ such that

$$
\Phi := \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u) = J(\hat{u}) \qquad (3.9)
$$

where

$$
J(u) = E\left[\int_0^T f(t,X(t),u(t))dt + g(X(T))\right], \qquad (3.10)
$$

with

$$
\begin{aligned}
dX(t) &= b(t,X(t),u(t))dt + c(t,X(t),u(t))dB(t) \\
&+ \int_{\mathbb{R}_0} \theta(t,X(t),u(t),z)\tilde{N}(dt,dz)\ ;\ X(0) \in \mathbb{R}. \qquad (3.11)
\end{aligned}
$$

In our case we have

$$
b(t,x,u) = b(t,x,u,\omega) = uS(t)\mu(t), \qquad (3.12)
$$

$$
c(t,x,u) = c(t,x,u,\omega) = uS(t)\sigma(t), \qquad (3.13)
$$

$$
\theta(t,x,u,z) = \theta(t,x,u,z,\omega) = uS(t)\gamma(t,z), \qquad (3.14)
$$

$$
g = 0, \qquad (3.15)
$$

$$
\begin{aligned}
f(t,x,u) &= f(t,x,u,\omega) \\
&= uS(t)(\mu(t)E[U'(x-G) \mid \mathcal{F}_t] - \sigma(t)E[U''(x-G)D_tG \mid \mathcal{F}_t]) \\
&+ \frac{1}{2}u^2S^2(t)\sigma^2(t)E[U''(x-G) \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0} E[(U(x+uS(t)\gamma(t,z)-G-D_{t,z}G) - U(x-G-D_{t,z}G) \\
&- uS(t)\gamma(t,z)U'(x-G)) \mid \mathcal{F}_t]\nu(dz). \qquad (3.16)
\end{aligned}
$$

This is a partial information stochastic control problem of the type studied in [9]. We will use the stochastic maximum principle of that paper to study this problem. From now on, we make the following assumptions:

• The functions $f(t,x,u), g(x), b(t,x,u), c(t,x,u)$ and $\theta(t,x,u,z)$ are C^1 with respect to x and u.

• For all $t, r \in (0,T)$, $t \leq r$, and all bounded $\mathcal{E}_t-$measurable random variables $\alpha = \alpha(\omega)$ the control $\beta_\alpha(s) = \alpha(\omega)\chi_{[t,r]}(s);\ s \in [0,T]$ belongs to $\mathcal{A}_{\mathcal{E}}$.

• For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded, there exists $\delta > 0$ such that $u+y\beta \in \mathcal{A}_{\mathcal{E}}$ for all $y \in (-\delta, \delta)$ and such that the family $\Big\{ \frac{\partial f}{\partial x}(t, X^{u+y\beta}(t), u(t)+y\beta(t)) \frac{d}{dy} X^{u+y\beta}(t) + \frac{\partial f}{\partial u}(t, X^{u+y\beta}(t), u(t) + y\beta(t))\beta(t) \Big\}_{y \in (-\delta,\delta)}$ is $\lambda \times P$-uniformly integrable and the family $\Big\{ g'(X^{u+y\beta}(T)) \frac{d}{dy} X^{u+y\beta}(T) \Big\}_{y \in (-\delta,\delta)}$ is P-uniformly integrable.

• For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded the process $Y(t) = Y^{(\beta)}(t) = \frac{d}{dy} X^{(u+y\beta)}(t)|_{y=0}$ exists and satisfies the equation

$$\begin{aligned} dY(t) = Y(t^-)\Big[&\frac{\partial b}{\partial x}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial x}(t, X(t), u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big] \\ &+ \beta(t^-)\Big[\frac{\partial b}{\partial u}(t, X(t), u(t))dt + \frac{\partial \sigma}{\partial u}(t, X(t), u(t))dB(t) \\ &+ \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial u}(t, X(t^-), u(t^-), z)\tilde{N}(dt, dz)\Big]; \qquad (3.17) \\ Y(0) = 0.& \end{aligned}$$

• For all $u \in \mathcal{A}_{\mathcal{E}}$, the following processes

$$K(t) := g'(X(T)) + \int_t^T \frac{\partial f}{\partial x}(s, X(s), u(s))ds,$$

$$D_t K(t) := D_t g'(X(T)) + \int_t^T D_t \frac{\partial f}{\partial x}(s, X(s), u(s))ds,$$

$$D_{t,z} K(t) := D_{t,z} g'(X(T)) + \int_t^T D_{t,z} \frac{\partial f}{\partial x}(s, X(s), u(s))ds,$$

$$H_0(s, x, u) := K(s)b(s, x, u) + D_s K(s)\sigma(s, x, u) + \int_{\mathbb{R}_0} D_{s,z} K(s)\theta(s, x, u, z)\nu(dz),$$

$$\begin{aligned} G(t, s) := \exp\Big(&\int_t^s \Big\{\frac{\partial b}{\partial x}(r, X(r), u(r), \omega) - \tfrac{1}{2}\Big(\frac{\partial \sigma}{\partial x}\Big)^2(r, X(r), u(r), \omega)\Big\}dr \\ &+ \int_t^s \frac{\partial \sigma}{\partial x}(r, X(r), u(r), \omega)dB(r) \\ &+ \int_t^s \int_{\mathbb{R}_0} \Big\{\ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega)\Big) - \frac{\partial \theta}{\partial x}(r, X(r), u(r), z, \omega)\Big\}\nu(dz)dr \\ &+ \int_t^s \int_{\mathbb{R}_0} \ln\Big(1 + \frac{\partial \theta}{\partial x}(r, X(r^-), u(r^-), z, \omega)\Big)\tilde{N}(dr, dz)\Big), \qquad (3.18) \end{aligned}$$

$$p(t) := K(t) + \int_t^T \frac{\partial H_0}{\partial x}(s, X(s), u(s))G(t, s)ds, \qquad (3.19)$$

$$q(t) := D_t p(t)\,, \qquad (3.20)$$

$$r(t, z) := D_{t,z} p(t) \qquad (3.21)$$

all exist for $0 \le t \le s \le T,\ z \in \mathbb{R}_0$.

Since $b(t,x,u) = b(t,u), \sigma(t,x,u) = \sigma(t,u)$ and $\theta(t,x,u,z) = \theta(t,u,z)$ do not depend on x this maximum principle gets a simpler form, which we now state, using the notation of (3.12)-(3.16):

Theorem 3.1. *[Stochastic maximum principle* [9] *(special case)] Suppose* b, σ *and* θ *do not depend on* x. *Put*

$$K(t) = K^{(u)}(t) = \int_t^T \frac{\partial f}{\partial x}(s, X^{(u)}(s), u(s))ds + g'(X^{(u)}(T)) \tag{3.22}$$

and define the Hamiltonian process $H : [0,T] \times \mathbb{R} \times \mathbb{R} \times \Omega \to \mathbb{R}$ *by*

$$H(t,x,u,\omega) = f(t,x,u) + K(t)b(t,u) + D_t K(t)c(t,u) + \int_{\mathbb{R}_0} D_{t,z}K(t)\theta(t,u,z)\nu(dz). \tag{3.23}$$

Suppose $u = \hat{u} \in \mathcal{A}_{\mathcal{E}}$ *is a critical point for*

$$J^{(G)}(u) := E[U(X^{(u)}(T) - G)], \tag{3.24}$$

in the sense that

$$\frac{d}{dy} J(\hat{u} + y\beta)_{y=0} = 0 \text{ for all bounded } \beta \in \mathcal{A}_{\mathcal{E}}. \tag{3.25}$$

Then $\hat{u}$ *is a conditional critical point for* H, *in the sense that*

$$E\left[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{u}(t)) \mid \mathcal{E}_t\right] = 0 \text{ for a.a.} t, \quad \omega \tag{3.26}$$

where $\hat{X}(t) = X^{(\hat{u})}(t)$, *and* H *is evaluated at*

$$K(t) = K^{(G)}(t) = \int_t^T \frac{\partial f}{\partial x}(s, \hat{X}(s), \hat{u}(s))ds + g'(\hat{X}(T)) := \hat{K}(t). \tag{3.27}$$

Conversely if (3.26) *holds then* (3.25) *holds.*

In our case we have, using (3.12)-(3.16),

$$\begin{aligned} K(t) = &\int_t^T \{u(s)S(s)(\mu(s)E[U''(X(s) - G) \mid \mathcal{F}_s] \\ &- \sigma(s)E[U'''(X(s) - G)D_s G \mid \mathcal{F}_s]) \\ &+ \frac{1}{2}u^2(s)S^2(s)\sigma^2(s)E[U'''(X(s) - G) \mid \mathcal{F}_s] \\ + &\int_{\mathbb{R}_0} E[(U'(X(s) + u(s)S(s)\gamma(s,z) - G - D_{s,z}G) - U'(X(s) - G - D_{s,z}) \\ &- u(s)S(s)\gamma(s,z)U''(X(s) - G)) \mid \mathcal{F}_s]\nu(dz)\}ds \end{aligned} \tag{3.28}$$

and, with $f(t,x,u)$ given by (3.16),

$$H(t,x,u) = f(t,x,u) + K(t)uS(t)\mu(t) + D_t K(t)uS(t)\sigma(t) + \int_{\mathbb{R}_0} D_{t,z}K(t)uS(t)\gamma(t,z)\nu(dz). \tag{3.29}$$

Therefore, if $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is optimal then by Theorem 3.1:

$$\begin{aligned}
0 &= E\left[\frac{d}{du}H(t,\hat{X}(t),u) \mid \mathcal{E}_t\right]_{u=\hat{u}(t)} \\
&= \hat{u}(t)E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G) \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)(\hat{K}(t)+U'(\hat{X}(t)-G)) + S(t)\sigma(t)(D_t\hat{K} - U''(\hat{X}(t)-G)D_tG) \\
&+ S(t)\int_{\mathbb{R}_0}\gamma(t,z)[D_{t,z}\hat{K}(t) + U'(\hat{X}(t)+\hat{u}(t)S(t)\gamma(t,z) - G - D_{t,z}G) \\
&- U'(\hat{X}(t)-G)]\nu(dz)\} \mid \mathcal{E}_t] = 0. \qquad (3.30)
\end{aligned}$$

We have proved

Theorem 3.2. *Suppose $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is optimal for the stochastic control problem* (1.11). *Then $\hat{u}(t)$ is a solution of equation* (3.30), *with $\hat{K}(t) = K^{(\hat{u})}(t)$ given by* (3.28).

In particular, we get :

Theorem 3.3. *Suppose $\mathcal{E}_t = \mathcal{F}_t$ and $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal for the problem* (1.11). *Then $\hat{u}(t)$ is a solution of the equation*

$$\begin{aligned}
&\hat{u}(s)S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t] + \mu(t)E[\{\hat{K}(t)+U'(\hat{X}(t)-G)\} \mid \mathcal{F}_t] \\
&+ \sigma(t)E[\{D_t\hat{K}(t) - U''(\hat{X}(t)-G)D_tG\} \mid \mathcal{F}_t] \\
&+ \int_{\mathbb{R}_0}\gamma(t,z)E[\{D_{t,z}\hat{K}(t) + U'(\hat{X}(t)+\hat{u}(t)S(t)\gamma(t,z) - G - D_{t,z}G) \\
&\quad - U'(\hat{X}(t)-G)\} \mid \mathcal{F}_t]\nu(dz) = 0, \qquad (3.31)
\end{aligned}$$

with $\hat{K}(t) = K^{(\hat{u})}(t)$ given by (3.28).

To illustrate these results we look at some special cases :

Corollary 3.4. *Suppose $\nu = 0$ and $\mathcal{E}_t \subseteq \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is optimal, then*

$$\begin{aligned}
\hat{u}(t) &= \frac{E[S(t)\{\mu(t)U'(\hat{X}(t)-G) - \sigma(t)U''(\hat{X}(t)-G)D_tG\} \mid \mathcal{E}_t]}{-E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G) \mid \mathcal{E}_t]} \\
&+ \frac{E[S(t)\{\mu(t)\hat{K}(t)+\sigma(t)D_t\hat{K}(t)\} \mid \mathcal{E}_t]}{-E[S^2(t)\sigma^2(t)U''(\hat{X}(t)-G) \mid \mathcal{E}_t]}. \qquad (3.32)
\end{aligned}$$

Corollary 3.5. *Suppose $\nu = 0$ and $\mathcal{E}_t = \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal, then*

$$\begin{aligned}
\hat{u}(t) &= \frac{\mu(t)E[U'(\hat{X}(t)-G) \mid \mathcal{F}_t] - \sigma(t)E[U''(\hat{X}(t)-G)D_tG \mid \mathcal{F}_t]}{-S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t]} \\
&+ \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{-S(t)\sigma^2(t)E[U''(\hat{X}(t)-G) \mid \mathcal{F}_t]}. \qquad (3.33)
\end{aligned}$$

In both (3.32) and (3.33) we have

$$\hat{K}(t) = \int_t^T \{\hat{u}(s)S(s)(\mu(s)E[U''(\hat{X}(s)-G) \mid \mathcal{F}_s] - \sigma(s)E[U'''(\hat{X}(s)-G)D_sG \mid \mathcal{F}_s]) + \frac{1}{2}\hat{u}^2(s)S^2(s)\sigma^2(s)E[U'''(X(s)-G) \mid \mathcal{F}_s]\}ds \tag{3.34}$$

(see (3.28)).

Corollary 3.6. *Suppose $\nu = G = 0$ and $\mathcal{E}_t = \mathcal{F}_t$. If $\hat{u} \in \mathcal{A}_{\mathcal{F}}$ is optimal and $\hat{X}(t) > 0$ for all $t \in [0,T]$, put*

$$\hat{\pi}(t) = \frac{\hat{u}(t)S(t)}{\hat{X}(t)} \ ; \ t \in [0,T]$$

i.e. $\hat{\pi}(t)$ represents the fraction of the total wealth invested in the risky asset. Then $\hat{\pi}(t)$ solves the equation

$$\hat{\pi}(t) = \frac{\mu(t)U'(\hat{X}(t))}{-\sigma^2(t)\hat{X}(t)U''(\hat{X}(t))} + \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{-\sigma^2(t)\hat{X}(t)U''(\hat{X}(t))} \tag{3.35}$$

where

$$\hat{K}(t) = \int_t^T \{\mu(s)\hat{\pi}(s)\hat{X}(s)U''(\hat{X}(s)) + \frac{1}{2}\sigma^2(s)\hat{\pi}^2(s)\hat{X}^2(s)U'''(\hat{X}(s))\}ds. \tag{3.36}$$

Corollary 3.7. *Suppose $\nu = G = 0$ and $\mathcal{E}_t = \mathcal{F}_t$ and that*

$$U(x) = \frac{1}{\lambda}x^\lambda \text{ for some } \lambda \in (-\infty, 1)\backslash\{0\}.$$

Then if $\hat{\pi} \in \mathcal{A}_{\mathcal{F}}$ is optimal, we have

$$\hat{\pi}(t) = \frac{\mu(t)}{(1-\lambda)\sigma^2(t)} + \frac{\mu(t)E[\hat{K}(t) \mid \mathcal{F}_t] + \sigma(t)E[D_t\hat{K}(t) \mid \mathcal{F}_t]}{(1-\lambda)\sigma^2(t)} \tag{3.37}$$

where

$$\hat{K}(t) = (\lambda - 1)\int_t^T \{\hat{\pi}(s)\hat{X}(s)^{\lambda-1}(\mu(s) + \frac{1}{2}(\lambda-2)\sigma^2(s)\hat{\pi}(s))\}ds. \tag{3.38}$$

In particular, if the coefficients $\mu(t)$ and $\sigma(t)$ are deterministic, then the last term on the right hand side of (3.37) vanishes, and the formula for $\hat{\pi}(t)$ reduces to the classical Merton formula

$$\hat{\pi}(t) = \frac{\mu(t)}{(1-\lambda)\sigma^2(t)}. \tag{3.39}$$

Thus (3.37) gives a specification of the additional term needed in the case when the coefficients $\mu(t)$ and $\sigma(t)$ are random.

4. The Exponential Utility Case

Although one of the motivations for this paper is to be able to handle a wide class of utility functions, it is nevertheless of interest to apply our general result to the widely studied exponential utility, i.e.

$$U(x) = -e^{-\alpha x}\ ;\ x \in \mathbb{R} \tag{4.1}$$

where $\alpha > 0$ is a constant.

4.1. The partial information case. We first consider the partial information case

$$\mathcal{E}_t \subseteq \mathcal{F}_t \text{ for all } t \in [0,T]. \tag{4.2}$$

For convenience we put

$$w(t) := u(t)S(t) \tag{4.3}$$

(the *amount* invested in the stock at time t). Then we get by (3.25)

$$\begin{aligned}
K(t) = &\int_t^T \{w(s)(\alpha^2\mu(s)E[U(X(s)-G) \mid \mathcal{F}_s] + \alpha^3\sigma(s)E[U(X(s)-G)D_sG \mid \mathcal{F}_s]) \\
&- \frac{1}{2}\alpha^3 w^2(s)\sigma^2(s)E[U(X(s)-G) \mid \mathcal{F}_s] \\
- \alpha &\int_{\mathbb{R}_0} E[(U(X(s)+w(s)\gamma(s,z)-G-D_{s,z}G) - U(X(s)-G-D_{s,z}G) \\
&+ \alpha w(s)\gamma(s,z)U(X(s)-G)) \mid \mathcal{F}_s]\nu(dz)\}ds \\
= -\alpha &\int_t^T \exp(-\alpha X(s))\{\alpha\mu(s)w(s)E[\exp(\alpha G) \mid \mathcal{F}_s] \\
&+ \alpha^2\sigma(s)w(s)E[\exp(\alpha G)D_sG \mid \mathcal{F}_s] \\
- &\int_{\mathbb{R}_0} (\exp(-\alpha w(s)\gamma(s,z))E[\exp(\alpha G + \alpha D_{s,z}G) \mid \mathcal{F}_s] \\
- &E[\exp(\alpha G + \alpha D_{s,z}G) \mid \mathcal{F}_s] + \alpha w(s)\gamma(s,z)E[\exp(\alpha G) \mid \mathcal{F}_s])\nu(dz)\}ds.
\end{aligned} \tag{4.4}$$

Equation (3.30) becomes:

$$\begin{aligned}
&- \alpha^2\hat{u}(t)E[S^2(t)\sigma^2(t)\exp(-\alpha\hat{X}(t)+\alpha G) \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)[\hat{K}(t) + \alpha\exp(-\alpha\hat{X}(t)+\alpha G)] \\
&+ S(t)\sigma(t)[D_t\hat{K}(t) - \alpha^2\exp(-\alpha\hat{X}(t)+\alpha G)D_tG] \\
&+ S(t)\int_{\mathbb{R}_0} \gamma(t,z)[D_{t,z}\hat{K}(t) - \alpha\exp(-\alpha\hat{X}(t) - \alpha\hat{u}(t)S(t)\gamma(t,z) + \alpha G + \alpha D_{t,z}G) \\
&+ \alpha\exp(-\alpha\hat{X}(t)+\alpha G)]\nu(dz)\} \mid \mathcal{E}_t] = 0
\end{aligned} \tag{4.5}$$

If we write

$$X(t) = y + X_0(t), \tag{4.6}$$

where

$$X_0(t) = \int_0^t u(s)dS(s) = \int_0^t w(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)] \tag{4.7}$$

we see from (4.4) that $K(t)$ has the form

$$K(t) = \exp(-\alpha y)K_0(t)$$

where $K_0(t)$ does not depend on y. Similarly we can factor out $\exp(-\alpha y)$ from the equation (4.5). This proves the following result:

Proposition 4.1. *Let $\mathcal{E}_t \subseteq \mathcal{F}_t$. Suppose there exists an optimal portfolio $\hat{u}(t)$ for Problem* (1.11)*, with $U(x) = -e^{-\alpha x}$. Then $\hat{u}(t)$ does not depend on the initial wealth $y = x + p$. Therefore*

$$V_G(x + p) = -e^{-\alpha(x+p)}V_G(0). \tag{4.8}$$

Similarly

$$V_0(x) = -e^{-\alpha x}V_0(0), \tag{4.9}$$

and hence the utility indifference price p is given by

$$p = \frac{1}{\alpha}\log\frac{V_0(0)}{V_G(0)}. \tag{4.10}$$

Remark 4.2. This result was proved in [17] under more restrictive conditions: Markovian system, Markovian payoff G and conditions necessary for the application of a Girsanov transformation. Moreover, in [17] only the full information case is considered. Proposition 4.1 holds in the general partial information case $\mathcal{E}_t \subseteq \mathcal{F}_t$.

4.2. Asymptotic behaviour of the optimal portfolio for vanishing α. Suppose an optimal portfolio $u_\alpha(t) = u_\alpha^{(G)}(t)$ exists for the problem

$$\sup_{u\in\mathcal{A}_\mathcal{E}} E[-\exp(-\alpha(\int_0^T u(t)dS(t) - G))]$$

Let $u_\alpha^{(0)}(t)$ be the corresponding optimal portfolio when $G = 0$ and $\psi_\alpha := u_\alpha^{(G)}(t) - u_\alpha^{(0)}(t)$ the difference. In the full information case ($\mathcal{E}_t = \mathcal{F}_t$), it has been proved, see e.g. [8], [17] and the references therein, that $\psi_\alpha(t)$ is itself an optimal portfolio for the problem

$$\sup_\psi E^*[-\exp(-\alpha(\int_0^T \psi(t)dS(t) - G))]$$

where E^* denotes the expectation with respect to the *minimal entropy martingale measure*. Moreover $\lim_{\alpha\to 0}\psi_\alpha(t)$ exists in some sense. It is also of interest to study the limiting behaviour of $u_\alpha^{(G)}$. We show below that, under some conditions,

$$\lim_{\alpha\to 0} \alpha u_\alpha^{(G)}(t) = u_1^{(0)}(t) \text{ a.s. } t \in [0, T],$$

where $u_1^{(0)}$ is the optimal portfolio for $\alpha = 1$ and $G = 0$. It follows that

$$|u_\alpha^{(G)}(t)| \to \infty \text{ as } \alpha \to 0.$$

This shows that $u_\alpha^{(G)}(t)$ and $u_\alpha^{(0)}(t)$ have the same singularity at $\alpha = 0$, which is cancelled by substraction. This result holds in the general non-Markovian, partial information setting. We now explain this in more detail. We use our results from the previous section to study the behaviour of the optimal portfolio $u_\alpha(t)$ corresponding to $U(x) = -\mathrm{e}^{-\alpha x}$ when $\alpha \to 0$. If we divide (4.5) by α we get

$$\begin{aligned}
&-\alpha u_\alpha(t) E[S^2(t)\sigma^2(t)\exp(-\alpha X_\alpha(t) + \alpha G) \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)[\frac{K_\alpha(t)}{\alpha} + \exp(-\alpha X_\alpha(t) + \alpha G)] \\
&+ S(t)\sigma(t)[\frac{D_t K_\alpha(t)}{\alpha} - \alpha \exp(-\alpha X_\alpha(t) + \alpha G) D_t G] \\
&+ S(t)\int_{\mathbb{R}_0} \gamma(t,z)[\frac{D_{t,z} K_\alpha(t)}{\alpha} \\
&\quad - \exp(-\alpha X_\alpha(t) - \alpha u_\alpha(t) S(t)\gamma(t,z) + \alpha G + \alpha D_{t,z} G) \\
&\quad + \exp(-\alpha X_\alpha(t) + \alpha G)]\nu(dz)\} \mid \mathcal{E}_t] = 0, \qquad (4.11)
\end{aligned}$$

where $K_\alpha(t), X_\alpha(t)$ are given by (4.4) and (4.6)-(4.7) with $u = u_\alpha$, i.e.

$$\begin{aligned}
\frac{K_\alpha(t)}{\alpha} = &\int_t^T \exp(-\alpha X_\alpha(s))\{\alpha u_\alpha(s) S(s)\mu(s) E[\mathrm{e}^{\alpha G} \mid \mathcal{F}_s] \\
&+ \alpha^2 \sigma(s) u_\alpha(s) S(s) E[\exp(\alpha G) D_s G \mid \mathcal{F}_s] \\
&- \int_{\mathbb{R}_0} (\exp(-\alpha u_\alpha(s) S(s)\gamma(s,z)) E[\exp(\alpha G + \alpha D_{s,z} G) \mid \mathcal{F}_s] \\
&- E[\exp(\alpha G + \alpha D_{s,z} G) \mid \mathcal{F}_s] \\
&+ \alpha u_\alpha(s) S(s)\gamma(s,z) E[\exp(\alpha G) \mid \mathcal{F}_s])\nu(dz)\} ds \qquad (4.12)
\end{aligned}$$

and

$$\alpha X_\alpha(t) = \alpha x + \int_0^t \alpha u_\alpha(s) S(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)] \qquad (4.13)$$

From this we deduce the following:

Lemma 4.3. *Suppose an optimal portfolio $u_\alpha(t) = u_\alpha^{(G)}(t)$ exists for all $\alpha > 0$, and that*

$$\tilde{u}(t) := \lim_{\alpha \to 0} \alpha\, u_\alpha(t) \qquad (4.14)$$

exists in $L^2(d\lambda \times dP)$, where λ denotes the Lebesgue measure on $[0,T]$. Then $\tilde{u}(t)$ is a solution of the equation

$$\begin{aligned}
&- \tilde{u}(t) E[S^2(t)\sigma^2(t) e^{-\tilde{X}(t)} \mid \mathcal{E}_t] \\
&+ E[\{S(t)\mu(t)(\tilde{K}(t) + e^{-\tilde{X}(t)}) + S(t)\sigma(t) D_t \tilde{K}(t) \\
&+ S(t)\int_{\mathbb{R}_0} \gamma(t,z)[D_{t,z}\tilde{K}(t) + e^{-\tilde{X}(t)}(1 - e^{-\tilde{u}(t)S(t)})]\nu(dz)\} \mid \mathcal{E}_t] = 0, \qquad (4.15)
\end{aligned}$$

where

$$\begin{aligned}
\tilde{K}(t) = &\int_t^T e^{-\tilde{X}(s)}\{\mu(s)\tilde{u}(s)S(s) \\
&- \int_{\mathbb{R}_0} (e^{-\tilde{u}(s)S(s)\gamma(s,z)} - 1 + \tilde{u}(s)S(s)\gamma(s,z))\nu(dz)\} ds \qquad (4.16)
\end{aligned}$$

and

$$\tilde{X}(t) = \int_0^t \tilde{u}(s)S(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0} \gamma(s,z))\tilde{N}(ds,dz)] \tag{4.17}$$

Let us now compare with the optimal portfolio $u_1^0(t)$ corresponding to $\alpha = 1$ and $X(0) = G = 0$. By (4.5) $u_1^0(t)$ is a solution of the equation

$$\begin{aligned} &- u_1^0(t)E[S^2(t)\sigma^2(t)\mathrm{e}^{-\hat{X}(t)} \mid \mathcal{E}_t] \\ &+ E[\{S(t)\mu(t)(\tilde{K}(t) + \mathrm{e}^{-\tilde{X}(t)}) + S(t)\sigma(t)D_t\hat{K}(t) \\ &\quad + S(t)\int_{\mathbb{R}_0} \gamma(t,z)[D_{t,z}\tilde{K}(t) + \mathrm{e}^{-\hat{X}(t)}(1 - \mathrm{e}^{-u_1^{(0)}S(t)\gamma(t,z)})\nu(dz)\} \mid \mathcal{E}_t] = 0, \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \hat{K}(t) = \int_t^T &\mathrm{e}^{-\tilde{X}(s)}\{\mu(s)u_1^{(0)}(s)S(s) \\ &- \int_{\mathbb{R}_0} (\mathrm{e}^{-u_1^{(0)}(s)S(s)\gamma(s,z)} - 1 + u_1^{(0)}(s)S(s)\gamma(s,z))\nu(dz)\}ds \end{aligned} \tag{4.19}$$

and

$$\hat{X}(t) = \int_0^t u_1^{(0)}(s)[\mu(s)ds + \sigma(s)dB(s) + \int_{\mathbb{R}_0} \gamma(s,z)\tilde{N}(ds,dz)]. \tag{4.20}$$

We see that the two systems of equations (4.15)-(4.17) in the unknown $\tilde{u}(t)$ and (4.18)-(4.20) in the unknown $u_1^{(0)}(t)$ are identical. Therefore we get

Theorem 4.4. *[***The limit of** $\alpha u_\alpha(t)$ **when** $\alpha \to 0$.*] Suppose an optimal portfolio* $u_\alpha(t) = u_\alpha^{(G)}(t)$ *exists for all* $\alpha > 0$ *and that*

$$\tilde{u}(t) = \lim_{\alpha \to 0} \alpha u_\alpha(t) \tag{4.21}$$

exists in $L^2(d\lambda \times dP)$*. Moreover, suppose that the system* (4.15)-(4.17) *has a unique solution* $\tilde{u}(\cdot)$*. Then* $\tilde{u}(t)$ *coincides with the optimal portfolio* $u_1^{(0)}(t)$ *corresponding to* $\alpha = 1$ *and* $G = 0$*.*

Alternatively we get

Theorem 4.5. *Suppose* (4.21) *holds. Then* $u = \tilde{u}(\cdot)$ *is a critical point for the performance functional*

$$J^{(0)}(u) := E[-\exp(-X_0^{(u)}(T))];\ u \in \mathcal{A}_{\mathcal{E}},\ X_0^{(u)}(0) = 0. \tag{4.22}$$

4.3. The complete information case $(\mathcal{E}_t = \mathcal{F}_t)$**.**

Finally, let us look at the situation when we have complete information ($\mathcal{E}_t = \mathcal{F}_t$ for all t) and exponential utility: $U(x) = -e^{-\alpha x}$; $\alpha > 0$ constant. As before let us put

$$w(t) = u(t)S(t).$$

Define

$$L(t) = K(0) - K(t).$$

Then by (4.4)

$$\begin{aligned}L(t) = \int_0^t & e^{-\alpha X(s)}\{-\alpha\mu(s)w(s)E[e^{\alpha G} \mid \mathcal{F}_s] \\ & + \alpha^2\sigma(s)w(s)E[e^{\alpha G}D_sG) \mid \mathcal{F}_s] \\ & - \int_{\mathbb{R}_0}((\exp(-\alpha w(s)\gamma(s,z)) - 1)E[e^{\alpha(G+D_{s,z}G)} \mid \mathcal{F}_s] \\ & + \alpha w(s)\gamma(s,z)E[e^{\alpha G} \mid \mathcal{F}_s])\nu(dz)\}ds. \qquad (4.23)\end{aligned}$$

Since $\mathcal{E}_t = \mathcal{F}_t$ equation (4.5) simplifies to

$$\begin{aligned}& - \alpha^2 w(t)\sigma^2(t)e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t] + \mu(t)\{E[K(t) \mid \mathcal{F}_t] + \alpha e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\} \\ & + \sigma(t)\{E[D_tK(t) \mid \mathcal{F}_t] - \alpha^2 e^{-\alpha X(t)}E[e^{\alpha G}D_tG \mid \mathcal{F}_t]\} \\ & + \int_{\mathbb{R}_0}\gamma(t,z)\{E[D_{t,z}K(t) \mid \mathcal{F}_t] - \alpha e^{-\alpha X(t)}e^{-\alpha w(t)S(t)}E[e^{\alpha(G+D_{t,z}G)} \mid \mathcal{F}_t] \\ & + \alpha e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\}\nu(dz) \\ & = 0. \qquad (4.24)\end{aligned}$$

Now assume that

$$\gamma(t,z) = 0 \text{ and } \sigma(t) \neq 0. \qquad (4.25)$$

Then (4.24) can be written

$$E[D_tK(t) \mid \mathcal{F}_t] = -a(t)E[K(t) \mid \mathcal{F}_t] + b(t)w(t) + c(t), \qquad (4.26)$$

where

$$a(t) = \frac{\mu(t)}{\sigma(t)} \qquad (4.27)$$

$$b(t) = \alpha^2\sigma(t)e^{-\alpha X(t)}E[e^{\alpha G} \mid \mathcal{F}_t] \qquad (4.28)$$

and

$$c(t) = e^{-\alpha X(t)}(\alpha^2 E[e^{\alpha G}D_tG \mid \mathcal{F}_t] - \alpha\frac{\mu(t)}{\sigma(t)}E[e^{\alpha G} \mid \mathcal{F}_t]). \qquad (4.29)$$

Then by the Clark-Ocone theorem

$$\begin{aligned}L(T) &= E[L(T)] + \int_0^T E[D_sL(T) \mid \mathcal{F}_s]dB(s) \\ &= E[L(T)] + \int_0^T E[D_sK(0) \mid \mathcal{F}_s]dB(s) \\ &= E[L(T)] + \int_0^T E[D_sK(s) \mid \mathcal{F}_s]dB(s). \qquad (4.30)\end{aligned}$$

It follows that if we define the martingale

$$M(t) = E[L(T) \mid \mathcal{F}_t] = L(t) + E[K(t) \mid \mathcal{F}_t],$$

then

$$\begin{aligned} M(t) &= E[L(T)] + \int_0^t E[D_s K(s) \mid \mathcal{F}_s] dB(s) \\ &= E[L(T)] + \int_0^t \{-a(s)E[K(s) \mid \mathcal{F}_s] + b(s)w(s) + c(s)\} dB(s) \\ &= E[L(T)] + \int_0^t \{-a(s)(E[L(T) \mid \mathcal{F}_s] - L(s)) + b(s)w(s) + c(s)\} dB(s). \end{aligned}$$

Hence $M(t)$ satisfies the equation

$$dM(t) = -a(t)M(t)dB(t) + f_w(t)dB(t) \tag{4.31}$$

where

$$f_w(t) = a(t)L(t) + b(t)w(t) + c(t). \tag{4.32}$$

Define

$$J(t) = \exp(\int_0^t a(s)dB(s) + \frac{1}{2}\int_0^t a^2(s)ds) \; ; \; t \geq 0. \tag{4.33}$$

Then

$$dJ(t) = a(t)J(t)dB(t) + J(t)a^2(t)dt$$

and hence, by (4.31)

$$\begin{aligned} d(J(t)M(t)) &= J(t)dM(t) + M(t)dJ(t) + dJ(t)dM(t) \\ &= J(t)dM(t) + M(t)J(t)[a(t)dB(t) + a^2(t)dt] \\ &\quad + J(t)[a(t)dB(t) + a^2(t)dt][-a(t)M(t)dB(t) + f_w(t)dB(t)] \\ &= J(t)dM(t) + J(t)a(t)M(t)dB(t) + J(t)a(t)f_w(t)dt \end{aligned} \tag{4.34}$$

Therefore, if we multiply (4.31) by $J(t)$ and use (4.34) we get

$$d(J(t)M(t)) = J(t)f_w(t)\{dB(t) + a(t)dt\}.$$

Integrating this we arrive at

$$M(t) = J^{-1}(t)[M(0) + \int_0^t J(s)f_w(s)\{dB(s) + a(s)ds\}] \tag{4.35}$$

where, by (4.26)

$$M(0) = E[L(T)] = E[K(0)] = E[K(0) \mid \mathcal{F}_0] = \frac{b(0)w(0) + c(0)}{a(0)} \tag{4.36}$$

Hence

$$\begin{aligned} E[K(t) \mid \mathcal{F}_t] &= M(t) - L(t) \\ &= J^{-1}(t)[E[L(T)] + \int_0^t J(s)f_w(s)\{dB(s) + a(s)ds\}] - L(t). \end{aligned} \tag{4.37}$$

This determines $E[K(t) \mid \mathcal{F}_t]$ as a function of the previous values

$$w(s) \; ; \; s \leq t$$

of our control process w. Hence

$$D_t E[K(t) \mid \mathcal{F}_t] = E[D_t K(t) \mid \mathcal{F}_t]$$

is determined by $w(s)$; $s \leq t$ also. Going back to equation (4.24), we see that we have now obtained a recursive equation for $w(t)$ in terms of previous values.Hence we have proved the following, which is one of the main results of this paper:

Theorem 4.6. *[***Optimal portfolio***]*

Suppose $\mathcal{E}_t = \mathcal{F}_t$, $\gamma(t,z) = 0$ *and* $\sigma(t) \neq 0$ *for all* $t \in [0,T]$*. Suppose* $\hat{u}(t) = \frac{\hat{w}(t)}{S(t)}$ *is an optimal portfolio for the problem*

$$\sup_{u \in \mathcal{A}} E[-\exp(-\alpha(X_u(T) - G))],$$

where

$$dX_u(t) = u(t)S(t)[\mu(t)dt + \sigma(t)dB(t)] \ ; \ X_u(0) = x.$$

Suppose $G \in \mathbb{D}^B_{1,2}$ *is* $\mathcal{F}_T$*-measurable,* $e^{\alpha G} \in L^2(P)$*. Then* $\hat{w}(t)$ *is given recursively by*

$$\begin{aligned} &\alpha^2 \hat{w}(t)\sigma^2(t)e^{-\alpha\hat{X}(t)}E[e^{\alpha G} \mid \mathcal{F}_t] \\ &= \mu(t)\{E[\hat{K}(t) \mid \mathcal{F}_t] + \alpha e^{-\alpha\hat{X}(t)}E[e^{\alpha G} \mid \mathcal{F}_t]\} \\ &+ \sigma(t)\{E[D_t\hat{K}(t) \mid \mathcal{F}_t] - \alpha^2 e^{-\alpha\hat{X}(t)}E[e^{\alpha G}D_tG \mid \mathcal{F}_t]\}, \end{aligned} \tag{4.38}$$

where $E[\hat{K}(t) \mid \mathcal{F}_t]$ *is given by* (4.36)-(4.37)*, together with* (4.27)-(4.29) *and* (4.33)*, with* $w = \hat{w}$*, and*

$$E[D_t\hat{K}(t) \mid \mathcal{F}_t] = D_tE[\hat{K}(t) \mid \mathcal{F}_t].$$

Remark 4.7. Note that we do *not* require that the terminal payoff G or the market coefficients $\mu(t)$, $\sigma(t)$ are of Markovian type.

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Bernt Øksendal: Center of Mathematics for Applications (CMA), Dept. of Mathematics, University of Oslo, P.O. Box 1053 Blindern, N–0316 Oslo and Norwegian School of Economics and Business Administration, Helleveien 30, N–5045 Bergen, Norway

E-mail address: oksendal@math.uio.no

Agnès Sulem: INRIA Paris-Rocquencourt, Project-team Mathfi, Domaine de Voluceau, Rocquencourt, BP 105, Le Chesnay Cedex, 78153, France

E-mail address: agnes.sulem@inria.fr