

STOCHASTIC MAGNETO-HYDRODYNAMIC SYSTEM PERTURBED BY GENERAL NOISE

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ABSTRACT. The existence and uniqueness of solutions to the two-dimensional stochastic magneto-hydrodynamic system is established in the presence of either a multiplicative noise or an additive fractional Brownian noise. The method of monotonicity is employed when the noise is multiplicative. In the case of a fractional Brownian noise, a unique mild solution of the system is established under suitable conditions.

1. Introduction

Magneto-hydrodynamics refers to the study of motion of an electrically conductive fluid in the presence of a magnetic field. Magneto-hydrodynamic (MHD) system consists of the Navier-Stokes equations coupled in a certain manner with the Maxwell equations. The MHD system has been studied in a number of articles (see [2], [3], [17]). The stochastic MHD system was studied by us [14] using the jump-Markov process approximation scheme.

In the first part of this paper, our goal is to prove the existence and uniqueness of solutions to the two-dimensional MHD system perturbed by a multiplicative noise. Local monotonicity of the coefficients was established by Menaldi and Sritharan [8] which led to the solvability of the two-dimensional Navier-Stokes system with additive noise. The Minty-Browder argument was used by us [15] to establish the existence and uniqueness of the two-dimensional Navier-Stokes system with a small multiplicative noise. In our earlier works such as [15] and [7], the objective was to prove the Freidlin-Wentzell type large deviations result for solutions of stochastic Navier-Stokes equations and certain shell models of turbulence.

In this article, the method of monotonicity is employed to obtain a strong solution (in the sense of partial differential equations as well as stochastic analysis) of the stochastic MHD system perturbed by a suitable multiplicative force driven by a space-time Wiener process.

In the second part of this paper, the objective is to establish the solvability of the two-dimensional MHD system in the presence of an additive fractional Brownian noise (fBn) term. Fractional Brownian motions are processes with long-range memory, and are not semimartingales. Therefore the usual notions of stochastic

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integrals, energy equality, and the ensuing martingale methods are not applicable in this context.

Tindel, Tudor and Viens [18] have studied stochastic heat equations with a fBn. Subsequently, we [6] have built the tools to establish the existence and uniqueness of mild solutions of the two-dimensional Navier-Stokes equations perturbed by a fBn. Our method is illustrated in the context of the two-dimensional MHD system.

The paper is organized as follows: The MHD system is cast as an evolution equation in section 2. The noise term driven by a Wiener process is also described in it. Local monotonicity of the coefficients, and a priori estimates for solutions are given in Section 3. Section 4 contains the proof of existence and uniqueness of strong solutions. Stochastic integration with respect to an infinite-dimensional fractional Brownian motion is discussed in section 5. A unique mild solution is shown to exist for the MHD system under suitable conditions.

2. The Stochastic MHD Problem

Suppose that $G \subset \mathbf{R}^2$ is a bounded, open, simply connected domain with a smooth boundary ∂G . The stochastic MHD system (see Temam [17]) in the non-dimensional form is given by

$$\frac{\partial v}{\partial t} + (v \cdot \nabla)v - \frac{1}{R_e} \Delta v - S(B \cdot \nabla)B + \nabla(p + \frac{S|B|^2}{2}) = \sigma_1(v, B) \partial W_1(t), \quad (2.1)$$

where v denotes the velocity, B , the magnetic field, p , the pressure field, and R_e , the Reynolds number. Let $S := \frac{M^2}{R_e R_m}$, where M is the Hartman number and R_m is the magnetic Reynolds number. The Maxwell equation is

$$\frac{\partial B}{\partial t} + (v \cdot \nabla)B + \frac{1}{R_m} \text{curl}(\text{curl } B) - (B \cdot \nabla)v = \sigma_2(v, B) \partial W_2(t), \quad (2.2)$$

where $\nabla \cdot v = 0$ and $\nabla \cdot B = 0$. The processes W_1 and W_2 are independent Wiener processes taking values in a divergence-free function space.

It is worthwhile to remember that for a vector-valued function $u = (u_1, u_2)$ defined on G ,

$$\text{curl } u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}.$$

If ϕ is a scalar-valued function defined on G ,

$$\text{curl } \phi = \left(\frac{\partial \phi}{\partial y}, -\frac{\partial \phi}{\partial x} \right).$$

Equations (2.1), (2.2) are equipped with the following boundary conditions: $v = 0$ on ∂G ; $B \cdot \mathbf{n} = 0$ and $\text{curl } B = 0$ on ∂G , where \mathbf{n} is the unit outer normal on ∂G . The initial conditions are $v(x, 0) = v_0(x)$ and $B(x, 0) = B_0(x)$ for $x \in G$.

The system needs to be written as an abstract evolution equation which in the integral form reads is given by

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t [-\mathbf{A}\mathbf{y}(s) - \mathbf{B}(\mathbf{y}(s))] ds + \int_0^t \sigma(\mathbf{y}(s)) dW(s), \quad (2.3)$$

where \mathbf{y} denotes the transpose of (v, B) . The noise term σdW is the transpose of $\{\sigma_1 dW_1, \sigma_2 dW_2\}$. As in Sermange and Temam [12], we will begin with the choice of function spaces.

Set $H = H_1 \times H_2$ and $V = V_1 \times V_2$, where

$$\begin{aligned} H_1 &= \{\phi \in L^2(G); \nabla \cdot \phi = 0, \phi \cdot \mathbf{n}|_{\partial G} = 0\} \\ H_2 &= H_1 \\ V_1 &= \{\phi \in H_0^1(G) : \nabla \cdot \phi = 0\} \\ V_2 &= \{\phi \in H^1(G) : \nabla \cdot \phi = 0, \phi \cdot \mathbf{n}|_{\partial G} = 0\} \end{aligned}$$

where

$$\begin{aligned} H^1 &= \{u \in L^2(G) : \nabla u \in L^2(G)\} \\ H_0^1 &= \{u \in L^2(G) : \nabla u \in L^2(G) \text{ and } u|_{\partial G} = 0\} \end{aligned}$$

H_1 and H_2 are equipped with the $L^2(G)$ norm. Define the inner product on H by $[\mathbf{y}_1, \mathbf{y}_2] = (v_1, v_2)_{H_1} + S(B_1, B_2)_{H_2}$ where $\mathbf{y}_i = \{v_i, B_i\}$. The above inner product is equivalent to $(\mathbf{y}_1, \mathbf{y}_2)_H = (v_1, v_2)_{H_1} + (B_1, B_2)_{H_2}$. The norm on H is given by $\|\mathbf{y}\|_H = \sqrt{(y, y)_H}$. The space V_1 is endowed with the inner product $[[\phi, \psi]]_{V_1} = (\nabla \phi, \nabla \psi)_{L^2(G)}$. The norm on V_1 is given by $\|\phi\|_{V_1} = \sqrt{[[\phi, \phi]]_{V_1}}$. Note that the V_1 norm given here is equivalent to the usual $H^1(G)$ norm since $\|\nabla \phi\|_{L^2(G)} \geq C\|\phi\|_{L^2(G)}$ for a suitable C by the Poincaré inequality.

Let V_2 be endowed with the inner product $[[\phi, \psi]]_{V_2} = (\text{curl } \phi, \text{curl } \psi)_{L^2(G)}$ so that the norm on V_2 is given by $\|\phi\|_{V_2} = \sqrt{[[\phi, \phi]]_{V_2}}$. From Proposition 1.8 and Lemma 1.6 of [16], we can conclude that the norm given by $\{\|\phi\|_{L^2(G)}^2 + \|\text{curl } \phi\|_{L^2(G)}^2 + \|\phi \cdot \mathbf{n}\|_{H^{1/2}(\partial G)}^2\}^{1/2}$ is equivalent to the $H^1(G)$ norm. Thus on V_2 , the norm defined above is equivalent to the $H^1(G)$ norm. We thus endow the space V with the scalar product:

$$[[\mathbf{y}_1, \mathbf{y}_2]] = [[v_1, v_2]]_{V_1} + S[[B_1, B_2]]_{V_2}.$$

Having defined the spaces H and V , we have the dense, continuous and compact embedding:

$$V \hookrightarrow H = H' \hookrightarrow V'.$$

The next step consists in defining the operators \mathbf{A} and \mathbf{B} that appear in (2.3). \mathbf{A} is defined through a bilinear coercive form, and \mathbf{B} by means of a trilinear continuous form.

Define a function $a : V \times V \rightarrow \mathbf{R}$ as follows:

$$a(\mathbf{y}_1, \mathbf{y}_2) = \frac{1}{R_e} [[v_1, v_2]]_{V_1} + \frac{S}{R_m} [[B_1, B_2]]_{V_2} \quad (2.4)$$

Proposition 2.1. *The function a defined by (2.4) is continuous and coercive.*

Proof. Let R_e and R_m be equal to 1 without loss of generality. $\|\cdot\|_{L^2(G)}$ will simply be denoted $\|\cdot\|$ from now on. There exists a constant k such that

$$\begin{aligned} |a(\mathbf{y}_1, \mathbf{y}_2)| &= \|\nabla v_1\| \|\nabla v_2\| + S \|\text{curl } B_1\| \|\text{curl } B_2\| \\ &\leq k(\|v_1\|_{H^1(G)} \|v_2\|_{H^1(G)} + S \|B_1\|_{H^1(G)} \|B_2\|_{H^1(G)}) \\ &\leq k(\|v_1\|_{H^1(G)}^2 + \|B_1\|_{H^1(G)}^2)^{1/2} (\|v_2\|_{H^1(G)}^2 + \|B_2\|_{H^1(G)}^2)^{1/2}. \end{aligned}$$

Hence $|a(\mathbf{y}_1, \mathbf{y}_2)| \leq C \|\mathbf{y}_1\|_V \|\mathbf{y}_2\|_V$ for $\mathbf{y}_1, \mathbf{y}_2 \in V$.

To establish the coercive property, consider

$$\begin{aligned} a(\mathbf{y}, \mathbf{y}) &= \|\nabla v\|^2 + S \|\operatorname{curl} B\|^2 \\ &\geq c(\|v\|_{H^1(G)}^2 + \|B\|_{H^1(G)}^2) \end{aligned}$$

for a constant c so that $a(\mathbf{y}, \mathbf{y}) \geq c \|\mathbf{y}\|_V^2$. \square

By the Lax-Milgram lemma, there exists an operator $\mathbf{A} : V \rightarrow V'$ such that $a(\mathbf{y}, \mathbf{z}) = \langle \mathbf{A}\mathbf{y}, \mathbf{z} \rangle_{V' \times V}$ for all $\mathbf{y}, \mathbf{z} \in V$.

Therefore $\mathbf{A} : V \rightarrow V'$ can be restricted to a self-adjoint operator $\mathbf{A} : \mathcal{D}(\mathbf{A}) \rightarrow H$.

We can write $\mathcal{D}(\mathbf{A}) = \mathcal{D}(\mathbf{A}_1) \times \mathcal{D}(\mathbf{A}_2)$ where $\mathbf{A}_1, \mathbf{A}_2, \mathcal{D}(\mathbf{A}_1)$, and $\mathcal{D}(\mathbf{A}_2)$ are obtained as follows:

Consider the ‘‘Stokes’’ problem in G :

$$-\frac{1}{R_e} \Delta v + \nabla p = g \quad \text{with} \quad \nabla \cdot v = 0, \quad \text{and} \quad v|_{\partial G} = 0.$$

This can be written as $\frac{1}{R_e} [v, \phi]_{V_1} = (g, \phi)$ for $\phi \in V_1$.

Define $\mathbf{A}_1 v = \frac{1}{R_e} [v, \cdot]_{V_1} = g \in V_1'$. By the Cattabriga regularity theorem, we can conclude that $\mathcal{D}(\mathbf{A}_1) = H^2(G) \cap V_1$ whenever $g \in H_1$.

To define \mathbf{A}_2 , consider the elliptic problem in G :

$$\frac{1}{R_m} \operatorname{curl} \operatorname{curl} B = g \quad \text{with} \quad \nabla \cdot B = 0; \quad B \cdot \mathbf{n}|_{\partial G} = 0; \quad \operatorname{curl} B|_{\partial G} = 0.$$

so that $\mathbf{A}_2 B = \frac{1}{R_m} \operatorname{curl} \operatorname{curl} B = g$. It is shown in [12] that if $g \in H$, then

$$\mathcal{D}(\mathbf{A}_2) = \{u \in H^2(G) : \nabla \cdot u = 0, u \cdot \mathbf{n}|_{\partial G} = 0 \text{ and } \operatorname{curl} u|_{\partial G} = 0\}$$

The next step consists in defining \mathbf{B} that figures in (2.3). Consider the trilinear form $b(\cdot, \cdot, \cdot) : V \times V \times V \rightarrow \mathbf{R}$ by

$$b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \tilde{b}(v_1, v_2, v_3) - S\tilde{b}(B_1, B_2, v_3) + S\tilde{b}(v_1, B_2, B_3) - S\tilde{b}(B_1, v_2, B_3) \quad (2.5)$$

for all $\mathbf{y}_i = (v_i, B_i) \in V$ where $\tilde{b}(\cdot, \cdot, \cdot) : (H^1(G))^{\otimes 3} \rightarrow \mathbf{R}$ is defined by

$$\tilde{b}(\phi, \psi, \theta) = \sum_{i,j=1}^2 \int_G \phi_i \frac{\partial \psi_j}{\partial x_i} \theta_j \, dx.$$

Proposition 2.2. *The function b defined by (2.5) is continuous.*

Proof. It suffices to show the continuity of \tilde{b} . Towards this note that

$$\begin{aligned} |\tilde{b}(\phi, \psi, \theta)| &= \left| \sum_{i,j} \int_G \phi_i \frac{\partial \psi_j}{\partial x_i} \theta_j \, dx \right| \\ &\leq \|\phi\|_{L^4(G)} \|\nabla \psi\|_{L^2(G)} \|\theta\|_{L^4(G)} \end{aligned}$$

using the Hölder’s inequality.

$H^1(G) \subset L^4(G)$ by the Sobolev embedding theorem, using which we get

$$|\tilde{b}(\phi, \psi, \theta)| \leq c \|\phi\|_{H^1} \|\psi\|_{H^1} \|\theta\|_{H^1}$$

so that

$$\begin{aligned} |b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3)| &\leq c(\|v_1\|_{H^1} \|v_2\|_{H^1} \|v_3\|_{H^1} + \|B_1\|_{H^1} \|B_2\|_{H^1} \|v_3\|_{H^1} \\ &\quad + \|v_1\|_{H^1} \|B_2\|_{H^1} \|B_3\|_{H^1} + \|B_1\|_{H^1} \|v_2\|_{H^1} \|B_3\|_{H^1}) \\ &\leq c\|\mathbf{y}_1\|_V \|\mathbf{y}_2\|_V \|\mathbf{y}_3\|_V \end{aligned}$$

since $\|\mathbf{y}\|_V$ is equivalent to $(\|v\|_{H^1}^2 + \|B\|_{H^1}^2)^{1/2}$. \square

Note that for all $\phi \in H^1(G)$ and $\psi \in V_1$,

$$\begin{aligned} \tilde{b}(\phi, \psi, \psi) &= \sum_{i,j} \int_G \phi_i \frac{\partial \psi_j}{\partial x_i} \psi_j dx \\ &= 0 \end{aligned}$$

by integration by parts. Also, $\tilde{b}(\phi, \psi, \theta) = -\tilde{b}(\phi, \theta, \psi)$ if ψ or $\theta \in V_1$ and $\phi \in H^1(G)$. Therefore $b(\mathbf{y}, \mathbf{z}, \mathbf{z}) = 0$.

We now define $\mathbf{B}: V \times V \rightarrow V'$ as the continuous bilinear operator such that $b(\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3) = \langle \mathbf{B}(\mathbf{y}_1, \mathbf{y}_2), \mathbf{y}_3 \rangle$ for all $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3 \in V$. The existence of such an operator is guaranteed by the Riesz representation theorem. Note that $\mathbf{B}(\mathbf{y})$ will denote $\mathbf{B}(\mathbf{y}, \mathbf{y})$.

The noise term in the stochastic MHD system in its integral form is given by $\int_0^t \sigma(r, \mathbf{y}(r)) dW(r)$ where W is an H -valued Wiener process with a nuclear covariance form Q . In order to state the conditions on the noise coefficient σ , we develop the following:

Notation: From now on, we will use the notation $|y|$ to denote the H -norm of \mathbf{y} , and $\|\mathbf{y}\|$ to denote the V -norm of \mathbf{y} .

Let $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space with the inner product

$$(\mathbf{y}, \mathbf{z})_0 = (Q^{-1/2}\mathbf{y}, Q^{-1/2}\mathbf{z}) \quad \forall \mathbf{y}, \mathbf{z} \in H_0 \quad (2.6)$$

Let $|\cdot|_0$ denote the norm in H_0 . Clearly, the imbedding of H_0 in H is Hilbert-Schmidt since Q is a trace class operator.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H . Define the norm on the space L_Q by $|S|_{L_Q}^2 = \text{tr}(SQS^*)$. The noise coefficient $\sigma: [0, T] \times V \rightarrow L_Q(H_0; H)$ is such that it satisfies the following hypotheses :

- (1) A.1. The function $\sigma \in C([0, T] \times V; L_Q(H_0; H))$
- (2) A.2. For all $t \in (0, T)$, there exists a positive constant $K < 1$ such that $|\sigma(t, \mathbf{y})|_{L_Q}^2 \leq K(1 + \|\mathbf{y}\|^2)$.
- (3) A.3. For all $t \in (0, T)$, there exists a positive constant $L < 1$ such that for all $\mathbf{y}, \mathbf{v} \in V$, $|\sigma(t, \mathbf{y}) - \sigma(t, \mathbf{v})|_{L_Q}^2 \leq L\|\mathbf{y} - \mathbf{v}\|^2$.

Thus we are able to write the stochastic system in the form we can write the stochastic MHD system as

$$d\mathbf{y} + [\mathbf{A}\mathbf{y} + \mathbf{B}(\mathbf{y})] dt = \sigma(t, \mathbf{y}) dW(t). \quad (2.7)$$

with $\mathbf{y}(0) \in H$.

Physically, the two-dimensional MHD system pertains to the case when the domain is a cylinder $G \times \mathbf{R}$ with G as a domain in \mathbf{R}^2 , and all the quantities are independent of z . Thus v and B are parallel to the xy -plane.

3. A Priori Estimates

It is well-known that for any $\mathbf{y} \in V$,

$$\|\mathbf{B}(\mathbf{y})\|_{V'} \leq C \|\mathbf{y}\| \|\mathbf{y}\| \quad (3.1)$$

where C is a constant that depends only on G . Since such constants do not play a crucial role in this paper, we will set $C = 1$. First, we prove a result on local monotonicity.

Proposition 3.1. *For a given $r > 0$, let B_r denote the $L^4(G)$ ball in V , i.e., $B_r = \{\mathbf{z} \in V : \|\mathbf{z}\|_{L^4(G)} \leq r\}$. Define the nonlinear operator F on V by $F(\mathbf{y}) := -\mathbf{A}\mathbf{y} - \mathbf{B}(\mathbf{y})$. Then, the pair (F, σ) is monotone in B_r , i.e., for any $\mathbf{y} \in V$ and $\mathbf{z} \in B_r$, if \mathbf{w} denotes $\mathbf{y} - \mathbf{z}$,*

$$\langle F(\mathbf{y}) - F(\mathbf{z}), \mathbf{w} \rangle - Cr^4 \|\mathbf{w}\|^2 + \|\sigma(t, \mathbf{y}) - \sigma(t, \mathbf{z})\|_{L^Q}^2 \leq 0 \quad (3.2)$$

for a suitable constant $C > 0$.

Proof. First, it is clear that $\langle \mathbf{A}\mathbf{w}, \mathbf{w} \rangle = \|\mathbf{w}\|^2$. Using the bilinearity of the operator B , it follows that

$$\langle \mathbf{B}(\mathbf{y}), \mathbf{w} \rangle = -\langle \mathbf{B}(\mathbf{y}, \mathbf{w}), \mathbf{z} \rangle$$

Likewise, $\langle \mathbf{B}(\mathbf{z}), \mathbf{w} \rangle = -\langle \mathbf{B}(\mathbf{z}, \mathbf{w}), \mathbf{v} \rangle$. Using the two equations above, one obtains

$$\langle \mathbf{B}(\mathbf{y}) - \mathbf{B}(\mathbf{z}), \mathbf{w} \rangle = -\langle \mathbf{B}(\mathbf{w}), \mathbf{z} \rangle.$$

Using the Hölder inequality, and Sobolev embedding,

$$|\langle \mathbf{B}(\mathbf{y}) - \mathbf{B}(\mathbf{z}), \mathbf{w} \rangle| \leq \frac{\epsilon}{2} \|\mathbf{w}\|^2 + C_\epsilon \|\mathbf{w}\|^2 \|\mathbf{z}\|_{L^4(G)}^4$$

for any $\epsilon > 0$. Here $C_\epsilon > 0$ is a constant that depends on ϵ . Using the definition of the operator F yields

$$\langle F(\mathbf{y}) - F(\mathbf{z}), \mathbf{w} \rangle \leq C_\epsilon r^4 \|\mathbf{w}\|^2 \quad (3.3)$$

since $\|\mathbf{v}\|_{L^4(G)} \leq r$. The proof is finished upon using condition (A.3) and choosing $\epsilon < 1 - L$. \square

Next, we define the Galerkin approximations of the MHD system. Let $H_n := \text{span}\{e_1, e_2, \dots, e_n\}$ where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in D(\mathbf{A})$. Let P_n denote the orthogonal projection of H to H_n . Define $W_n = P_n W$. Let $\sigma_n = P_n \sigma$. Define \mathbf{y}_n as the solution of the following stochastic differential equation: For each $\mathbf{v} \in H_n$,

$$d(\mathbf{y}_n(t), \mathbf{v}) = (F(\mathbf{y}_n(t)), \mathbf{v})dt + (\sigma_n(t, \mathbf{y}_n(t)) dW_n(t), \mathbf{v}) \quad (3.4)$$

with $\mathbf{y}_n(0) = P_n \mathbf{y}(0)$.

Proposition 3.2. *Let $E(|\mathbf{y}(0)|^2) < \infty$. Let \mathbf{y}_n denote the unique strong solution of the finite system of equations (3.4) in $C([0, T] : H_n)$. Then, with K as in condition (A.2), the following estimates hold:*

(1) For all $0 \leq t \leq T$,

$$E(|\mathbf{y}_n(t)|^2) + \int_0^t E\|\mathbf{y}_n(s)\|^2 ds \leq E|\mathbf{y}(0)|^2 + KT \tag{3.5}$$

and

$$E\left(\sup_{0 \leq t \leq T} |\mathbf{y}_n(t)|^2 + \int_0^T \|\mathbf{y}_n(s)\|^2 ds\right) \leq C \tag{3.6}$$

where C is a constant that depends on $E|\mathbf{y}(0)|^2$ and T .

(2) Fix any $\delta > 0$. Then

$$E|\mathbf{y}_n(t)|^2 e^{-\delta t} + \int_0^t E\|\mathbf{y}_n(s)\|^2 e^{-\delta s} ds \leq E|\mathbf{y}(0)|^2 + \frac{K}{\delta} \tag{3.7}$$

(3) If $E|\mathbf{y}(0)|^4 < \infty$, then for all $0 \leq t \leq T$,

$$\begin{aligned} E|\mathbf{y}_n(t)|^4 e^{-\delta t} + 3 \int_0^t E\|\mathbf{y}_n(s)\|^2 |\mathbf{y}_n(s)|^2 e^{-\delta s} ds \\ \leq E|\mathbf{y}(0)|^4 + \frac{4K}{\delta} \sup_{0 \leq t \leq T} E(|\mathbf{y}_n(s)|^2) \end{aligned} \tag{3.8}$$

Proof. To prove (3.5), we use the Itô Lemma to obtain

$$\begin{aligned} (d|\mathbf{y}_n(t)|^2 + 2\nu \|\mathbf{y}_n(t)\|^2 dt) = \text{tr}(\sigma_n(t, \mathbf{y}_n(t))Q\sigma_n(t, \mathbf{y}_n(t)))dt \\ + 2(\sigma_n(t, \mathbf{y}_n(t))dW_n(t), \mathbf{y}_n(t)). \end{aligned} \tag{3.9}$$

Stopping (3.9) at $\tau_N = \inf\{t : |\mathbf{y}_n(t)|^2 + \int_0^t \|\mathbf{y}_n(s)\|^2 ds > N\}$, one can take expectation, and use the hypothesis (A.2). A Gronwall argument allows us to get (3.5) till $t \wedge \tau_N$.

The proof of (3.6) requires us to take supremum in (3.9) on both sides upto $T \wedge \tau_N$, and then take expectation using the Burkholder-Davis-Gundy inequality. It also shows that $T \wedge \tau_N$ increases to T a.s. as $N \rightarrow \infty$. Taking the limit as $N \rightarrow \infty$ gives the inequality (3.6).

By applying the Itô Lemma to each of the functions $g(t, \xi) = e^{-\delta t}|\xi|^2$ and $h(t, \xi) = e^{-\delta t}|\xi|^4$, and proceeding as above, (3.7) and (3.8) are obtained. \square

Proposition 3.3. *Let $E|\mathbf{y}(0)|^4 < \infty$. Then following inequality holds:*

$$E\left\{\sup_{0 \leq t \leq T} |\mathbf{y}_n(t)|^4 e^{-\delta t} + \int_0^T \|\mathbf{y}_n(t)\|^2 |\mathbf{y}_n(t)|^2 e^{-\delta t} dt\right\} \leq E|\mathbf{y}_n(0)|^4 + \frac{1}{\delta}$$

Proof. The proof follows by a use of the Burkholder-Davis-Gundy inequality, and the estimates given in the previous proposition. \square

4. Existence and Uniqueness of Solutions

The method of monotonicity to prove existence of strong solutions of SPDEs was initiated by Pardoux [11] (also see Metivier [9]). Existence of strong solutions for Navier-Stokes equations with additive noise was first shown using local-monotonicity method by Menaldi and Sritharan [8]. Since a multiplicative noise appears in the system studied in this paper, the proof of existence and uniqueness of solutions is given in full.

Theorem 4.1. *Let $E|\mathbf{y}_0|^4 < \infty$. Under the conditions (A.1)-(A.3) on σ , there exists a strong solution of the following stochastic Navier-Stokes equation:*

$$d\mathbf{y} + [\mathbf{A}\mathbf{y} + \mathbf{B}(\mathbf{y})] dt = \sigma(t, \mathbf{y}) dW(t). \tag{4.1}$$

in $L^2(\Omega : C(0, T; H)) \cap L^2(\Omega \times (0, T); V)$. The solution is pathwise unique.

Proof. Let $\Omega_T := \Omega \times [0, T]$. Using the estimates given in the previous section, it follows from the Banach-Alaoglu theorem that along a subsequence, the Galerkin approximations $\{\mathbf{y}_n\}$ have the following limits:

$$\mathbf{y}_n \rightarrow \mathbf{y} \text{ weakly in } L^2(\Omega_T, V)$$

$$\mathbf{y}_n \rightarrow \mathbf{y} \text{ weak star in } L^4(\Omega; L^\infty(0, T; H))$$

and

$$\mathbf{y}_n(T) \rightarrow \eta \text{ weakly in } L^2(\Omega; H).$$

Recall that $F(\mathbf{y}) := -\mathbf{A}\mathbf{y} - \mathbf{B}(\mathbf{y})$. Since $F(\mathbf{y}_n)$ is bounded in $L^2(\Omega_T, V')$,

$$F(\mathbf{y}_n) \rightarrow F_0 \text{ weakly in } L^2(\Omega_T, V')$$

and likewise

$$\sigma_n(\cdot, \mathbf{y}_n) \rightarrow S \text{ weakly in } L^2(\Omega_T, L_Q).$$

The assertion of the last statement holds since σ has linear growth (condition A.2) and \mathbf{y}_n is bounded in $L^2(0, T; V)$ uniformly in n by the a-priori estimates.

Extend the equation (3.4) to an open interval $(-\delta, T + \delta)$ by setting the terms equal to 0 outside of the interval $[0, T]$. Let $\phi(t)$ be a function in $H^1(-\delta, T + \delta)$ with $\phi(0) = 1$. Define for all integers $j \geq 1$, $e_j(t) = \phi(t)e_j$ where $\{e_j\}$ is the fixed orthonormal sequence for H .

Using the Itô formula for the function $(\mathbf{y}_n(t), e_j(t))$, and letting $n \rightarrow \infty$, one obtains

$$\begin{aligned} - \int_0^T (\mathbf{y}(s), e'_j(s)) ds &= (\mathbf{y}_0, e_j) + \int_0^T \langle F_0(s), e_j \rangle \phi(s) ds + \\ &\int_0^T \phi(s) (S(s) dW(s), e_j) - (\eta, e_j) \phi(T) \end{aligned} \tag{4.2}$$

Choose a sequence of functions $\{\phi_k\}$ in the place of ϕ such that $\phi_k \rightarrow 1_{[0, T]}$ and the time derivative of ϕ_k converges to δ_t weakly as $k \rightarrow \infty$. Using ϕ_k in (4.2) in the place of ϕ and then letting $k \rightarrow \infty$,

$$\mathbf{y}(t) = \mathbf{y}(0) + \int_0^t F_0(s) ds + \int_0^t S(s) dW(s) \tag{4.3}$$

with $\mathbf{y}(T) = \eta$. Let $\mathbf{v} \in L^\infty(\Omega_T; H_m)$ for $m \leq n$. Define

$$r(t) = \int_0^t \|\mathbf{v}(s)\|_{L^4(G)}^4 ds.$$

Note that by monotonicity,

$$\begin{aligned}
X_n(T) &:= 2E \int_0^T \langle F(\mathbf{y}_n(s)) - F(\mathbf{v}(s)), \mathbf{y}_n(s) - \mathbf{v}(s) \rangle e^{-r(s)} ds \\
&\quad + 2E \int_0^T r'(s) e^{-r(s)} |\mathbf{y}_n(s) - \mathbf{v}(s)|^2 ds \\
&\quad + E \int_0^T e^{-r(s)} |\sigma_n(s, \mathbf{y}_n(s)) - \sigma_n(s, \mathbf{v}(s))|_{L_Q}^2 ds \\
&\leq 0.
\end{aligned} \tag{4.4}$$

Let $X_n(T)$ be written as $Y_n + Z_n$ where

$$\begin{aligned}
Y_n &= 2(E \int_0^T (\langle F(\mathbf{y}_n(s)), \mathbf{y}_n(s) \rangle + r'(s) |\mathbf{y}_n(s)|^2) e^{-r(s)} ds) \\
&\quad + E \int_0^T e^{-r(s)} |\sigma_n(s, \mathbf{y}_n(s))|_{L_Q}^2 ds
\end{aligned} \tag{4.5}$$

and

$$\begin{aligned}
Z_n &= 2E \int_0^T (\langle F(\mathbf{v}(s)), \mathbf{v}(s) - \mathbf{y}_n(s) \rangle - \langle F(\mathbf{y}_n(s)), \mathbf{v}(s) \rangle) e^{-r(s)} ds \\
&\quad + 2E \int_0^T r'(s) e^{-r(s)} (|\mathbf{v}(s)|^2 - 2\langle \mathbf{y}_n(s), \mathbf{v}(s) \rangle) ds \\
&\quad + E \int_0^T e^{-r(s)} \{ |\sigma_n(s, \mathbf{v}(s))|_{L_Q}^2 - 2(\sigma_n(\mathbf{y}_n(s), \sigma_n(\mathbf{v}(s)))_{L_Q}) \} ds.
\end{aligned} \tag{4.6}$$

By the Itô formula,

$$\begin{aligned}
Y_n &= E(e^{-r(T)} |\mathbf{y}_n(T)|^2 - |\mathbf{y}_n(0)|^2) \\
&\geq E(e^{-r(T)} |\mathbf{y}_n(T)|^2 - |\mathbf{y}_0|^2)
\end{aligned} \tag{4.7}$$

Therefore,

$$\begin{aligned}
\liminf_{n \rightarrow \infty} Y_n &\geq E e^{-r(T)} |\mathbf{y}_n(T)|^2 - E |\mathbf{y}(0)|^2 \\
&= 2E \int_0^T e^{-r(s)} \langle F_0(s), \mathbf{y}(s) \rangle ds + 2E \int_0^T r'(s) e^{-r(s)} |\mathbf{y}(s)|^2 ds \\
&\quad + E \int_0^T e^{-r(s)} \|S(s)\|_{L_Q}^2 ds + E \int_0^T e^{-r(s)} (S(s) dW(s), \mathbf{y}(s))
\end{aligned}$$

In Z_n , each term has a limit so that we can conclude that

$$\begin{aligned}
&2E \int_0^T e^{-r(s)} \langle F_0(s) - F(\mathbf{v}(s)), \mathbf{y}(s) - \mathbf{v}(s) \rangle ds \\
&\quad + E \int_0^T (r'(s) |\mathbf{y}(s) - \mathbf{v}(s)|^2 + \|S(s) - \sigma(s, \mathbf{v}(s))\|_{L_Q}^2) e^{-r(s)} ds \\
&\leq \liminf_{n \rightarrow \infty} X_n \\
&\leq 0.
\end{aligned}$$

Take $\mathbf{v} = \mathbf{y}$ to see that $Z(s) = \sigma(s, \mathbf{y}(s))$. Take $\mathbf{v} = \mathbf{y} - \lambda \mathbf{w}$ with $\lambda > 0$. Then,

$$\lambda E \int_0^T \langle F_0(s) - F(\mathbf{y}(s) + \lambda \mathbf{w}(s)), \mathbf{w}(s) \rangle ds + \lambda^2 E \int_0^T r'(s) e^{-r(s)} |\mathbf{w}(s)|^2 ds$$

is less than or equal to 0. Dividing by λ on both sides of the inequality above, and letting $\lambda \rightarrow 0$, one obtains

$$E \int_0^T \langle F_0(s) - F(\mathbf{y}(s)), \mathbf{w}(s) \rangle ds \leq 0.$$

Since \mathbf{w} is arbitrary, one can show that F_0 can be identified with $F(\mathbf{y}(s))$. Thus the existence of a strong solution has been proved.

In what follows, the proof of pathwise uniqueness is sketched. Let \mathbf{y}_1 and \mathbf{y}_2 be two solutions of the SNSE (2.7). For $i = 1, 2$, and $N > 0$, define $\tau_N^i := \inf\{t \leq T : |\mathbf{y}_i(t)| \geq N\}$. Let $\tau_N = \min\{\tau_N^1, \tau_N^2\}$.

$$P\{\tau_N < T\} \leq P\{\max\{\sup_{t \in [0, T]} |\mathbf{y}_1(t)|, \sup_{t \in [0, T]} |\mathbf{y}_2(t)|\} \geq N\} \leq \frac{C}{N^2} \tag{4.8}$$

by Chebyshev inequality. Thus $\lim_{N \rightarrow \infty} \tau_N = T$ a.s.

Let $\mathbf{w} = \mathbf{y}_1 - \mathbf{y}_2$, and $\sigma_{12} = \sigma(\mathbf{y}_1) - \sigma(\mathbf{y}_2)$. By the energy equality,

$$\begin{aligned} & |\mathbf{w}(t \wedge \tau_N)|^2 + 2 \int_0^{t \wedge \tau_N} \|\mathbf{w}(s)\|^2 ds \\ & \leq -2 \int_0^{t \wedge \tau_N} b(\mathbf{w}(s), \mathbf{y}_1(s), \mathbf{w}(s)) ds + \int_0^{t \wedge \tau_N} \text{tr}(\sigma_{12}(s) Q \sigma_{12}(s)) ds \\ & \quad + 2 \int_0^{t \wedge \tau_N} (\mathbf{w}, \sigma_{12}(s) dW(s)). \end{aligned}$$

Let $k > 0$ be any constant. Define $\xi(t) := e^{-k \int_0^{t \wedge \tau_N} \|\mathbf{y}_1(r)\|^2 dr}$ for all t . By applying the Itô formula for the function $\xi(t) |\mathbf{w}(t \wedge \tau_N)|^2$, and taking expectation, one obtains

$$\begin{aligned} & E e^{-k \int_0^{t \wedge \tau_N} \|\mathbf{y}_1(r)\|^2 dr} |\mathbf{w}(t \wedge \tau_N)|^2 \\ & \leq L \text{tr} Q E \int_0^{t \wedge \tau_N} e^{-k \int_0^{s \wedge \tau_N} \|\mathbf{y}_1(r)\|^2 dr} |\mathbf{w}(s)|^2 ds \end{aligned}$$

An application of Gronwall's Lemma implies that $\mathbf{w}(t \wedge \tau_N) = 0$ a.s. Since $\tau_N \rightarrow T$ as $N \rightarrow \infty$, we get that $\mathbf{y}_1 = \mathbf{y}_2$ for all $t \in [0, T]$ a.s. \square

Remark 4.2. :

- (1) An additional external force $\mathbf{f}(t)dt$ can be introduced on the right side of (2.3) where $\mathbf{f} \in L^2([0, T] : H)$. The above proof carries over with very slight modifications.
- (2) For the stochastic MHD system, a Freidlin-Wentzell type large deviations principle can be proved exactly as in our paper [15].

5. Fractional Brownian Motion

In this section, a quick introduction to Wiener integrals with respect to fractional Brownian motions is presented. A space-time fractional Brownian integral is built in order to study the MHD system perturbed by such a term. The symbol H that appears as a superscript in the following definition should not be confused with the Hilbert space H introduced earlier.

Definition 5.1. A centered Gaussian process $\{\beta_t^H\}$ is called a fractional Brownian motion with Hurst parameter $H \in (0, 1)$ if its covariance function is given by

$$E [(\beta_t^H \beta_s^H)] = R_H(t, s) := \frac{1}{2} (s^{2H} + t^{2H} - |t - s|^{2H}). \tag{5.1}$$

Let S be the set of all step functions on $[0, T]$. If $\phi = \sum_{j=0}^{n-1} a_j 1_{(t_j, t_{j+1}]}$, we define $\int_0^T \phi(s) d\beta_s^H := \sum_{j=0}^{n-1} a_j (\beta_{t_{j+1}}^H - \beta_{t_j}^H)$. Let \mathcal{H} be the closure of S with respect to the inner product $\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} := R_H(t, s)$. Then $1_{[0,t]} \rightarrow \beta_t^H$ extends to an isometry between \mathcal{H} and the $L^2(\Omega)$ -closure of the linear span of $\{\beta_t^H : t \in [0, T]\}$. This extension is called the Wiener integral with respect to β^H , and is denoted by $\mathcal{H}\phi := \int_0^T \phi(s) d\beta^H(s) \in L^2(\Omega)$.

It should be noted that the Wiener integral of any function $\phi \in \mathcal{H}$ w.r.t. β^H is a centered Gaussian random variable, and that for $\phi, \psi \in \mathcal{H}$ we have that $\int_0^T \phi d\beta^H$ and $\int_0^T \psi d\beta^H$ are jointly Gaussian with covariance equal to $\langle \phi, \psi \rangle_{\mathcal{H}}$, thereby extending the Wiener integral for standard Brownian motion.

There is a connection between the standard Wiener process and fractional Brownian motions. One begins by noting that R_H is, by definition, a non-negative definite kernel, which means that there exists a kernel function K_H such that $R_H(t, s) = \int_0^{t \wedge s} K_H(t, u) K_H(s, u) du$. In fact, its expression is explicit (see [10]):

$$K_H(t, s) = c_H \left(\frac{t}{s}\right) (t - s)^{H-1/2} + s^{1/2-H} F\left(\frac{t}{s}\right)$$

where $F(z) = c_H(1/2 - H) \int_0^{z-1} r^{H-3/2} (1 - (1+r)^{H-1/2}) dr$. Using these facts, one proves that there exists a standard Brownian motion W such that

$$\beta_t^H = \int_0^t K_H(t, s) dW_s.$$

For $s < T$, if we define the adjoint operator K^* on a possible subset of $L^2([0, T])$ by

$$(K_T^* \phi)(s) = K(T, s) \phi(s) + \int_s^T (\phi(r) - \phi(s)) \frac{\partial K}{\partial r}(r, s) dr,$$

a result of Alos, Mazet, Nualart [1] then guarantees that K_T^* is an isometry between \mathcal{H} and $L^2[0, T]$, and that the Wiener integral w.r.t β^H can be represented in the following convenient way: for all $\phi \in \mathcal{H}$, $K_T^* \phi \in L^2[0, T]$ and

$$\int_0^T \phi(s) d\beta^H(s) = \int_0^T (K_T^* \phi)(s) dW_s$$

where the last integral is a Wiener integral w.r.t. standard Brownian motion. It is easy to check that $K_T^*[\phi 1_{[0,t]}] = K_t^*[\phi]1_{[0,t]}$. Therefore,

$$\int_0^t \phi(s) d\beta^H(s) = \int_0^t (K_t^* \phi)(s) dW_s. \tag{5.2}$$

We will now define integrals of the form $\int_0^t \phi(s) d\mathbf{W}^H(s)$ where \mathbf{W}^H is a cylindrical H -valued fBm. This \mathbf{W}^H is an infinite-dimensional stochastic process with a fBm behavior in time, taking values in the Hilbert space H , with equal weights on all directions of H .

The operators $\mathbf{A}_i, i = 1, 2$, introduced in Section 2 have inverses from H_i into itself. The inverses are compact, self-adjoint with discrete eigenvalues, each with finite multiplicity and can accumulate only at infinity. The eigenvalues λ_n grow like n . Also, the set of eigenfunctions is complete. Let $\{e_n\}$ denote the complete orthonormal basis in H , formed by the eigenfunctions of the operator \mathbf{A} on G . Define

$$\mathbf{W}^H(t) = \sum_{j=1}^{\infty} e_n \beta_n^H(t),$$

where $\{\beta_n^H\}_n$ is a family of IID scalar fBm's. Strictly speaking, $\mathbf{W}^H(t)$ is not a member of $L^2(\Omega, H)$, since its norm is infinite, but it will be easy to guarantee that an integral w.r.t. \mathbf{W}^H will be.

Indeed, let $\{\phi(s) : s \in [0, T]\}$ be a deterministic measurable function such that for every $s, \phi(s) \in H$. So we can write $\phi(s) e_n = \sum_m (\phi(s) e_n, e_m) e_m$. We may now define

$$\begin{aligned} \int_0^t \phi(s) d\mathbf{W}^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \phi(s) e_n d\beta_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t (K^*(\phi(\cdot) e_n))(s) dW_n(s) \end{aligned}$$

which follows from the representation (5.2), in which W_n is the standard Brownian motion that represents β_n^H .

Since all the terms in the last expression above are independent centered Gaussian r.v.'s, we can immediately give a necessary and sufficient condition for the above integral to exist: it is a Gaussian random variable in $L^2(\Omega)$ if and only if

$$E \left[\left\| \int_0^t \phi(s) d\mathbf{W}^H(s) \right\|^2 \right] = \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \int_0^t |[K^*((\phi(\cdot) e_n, e_j))](s)|^2 ds \tag{5.3}$$

is finite.

6. Mild Solutions

Consider the stochastic MHD equation

$$dy(t) + [\mathbf{A} y(t) + \mathbf{B}(y(t))] dt = \Phi d\mathbf{W}^H(t) \tag{6.1}$$

which in the integral form reads as

$$\mathbf{y}(t) = \mathbf{y}(0) - \nu \int_0^t \mathbf{A} \mathbf{y}(s) ds - \int_0^t \mathbf{B}(\mathbf{y}(s)) ds + \int_0^t \Phi d\mathbf{W}^H(s).$$

To write this in its evolution form, we will need $S(t)$ the semigroup generated by \mathbf{A} . In mild form,

$$\mathbf{y}(t) = S(t) \mathbf{y}(0) - \int_0^t S(t-s) \mathbf{B}(\mathbf{y}(s)) ds + \int_0^t [S(t-s) \Phi] d\mathbf{W}^H(s). \tag{6.2}$$

The existence and uniqueness of mild solutions of stochastic evolution systems have been studied by a number of authors (cf. Da Prato and Zabczyk [4], Sohr [13], Temam [17]). Solvability of the stochastic MHD system consists in breaking up the system (6.2) into a linear stochastic system and a nonlinear partial differential equation.

In order to prove the main theorem, we need the following Lemmas to ensure that the convolution space-time stochastic integral that appears in (6.2) takes values in $L^4([0, T] \times G)$ a.s. The first Lemma is taken from [6], and is an improvement of a result in [18].

Lemma 6.1. *For any $\lambda, t \geq 0$, there is a constant $c_{t,H}$ such that*

$$\mathbf{E} \left[\left(\int_0^t e^{-\lambda(t-s)} d\beta^H(s) \right)^2 \right] = |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|_{\mathcal{H}}^2 \leq c_{t,H} \lambda^{-2H}.$$

In fact, there exists a constant $C(H)$ depending only on H such that $c_{t,H}$ is given as follows:

$$\begin{cases} c_{t,H} \leq C(H) \text{ for all } H > 1/2, \\ c_{t,H} \leq C(H)(1 + t^{2H-1}) \text{ for all } H < 1/2. \end{cases}$$

Proof. Case 1: $H > 1/2$. Using the notation in [18] (see equation (23) therein), we have for any $\lambda \geq 0$,

$$\begin{aligned} A(\lambda, t) &:= |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|_{\mathcal{H}}^2 = \int_0^{\lambda t} v^{2H-2} e^{-v} [1 - e^{-2(t\lambda-v)}] dv \\ &\leq \int_0^{\infty} v^{2H-2} e^{-v} dv =: C_0(H). \end{aligned}$$

Case 2: $H < 1/2$. Using the notation in [18] (see the calculation immediately preceding equation (26) therein), we have for all $\lambda \geq 0$,

$$A(\lambda, t) := |\mathbf{1}_{[0,t]} e^{-\lambda(t-\cdot)}|_{\mathcal{H}}^2 \leq B_1(\lambda, t) + B_2(\lambda, t)$$

where

$$B_1(\lambda, t) := \lambda^{-2H} c(H) \int_0^{2\lambda t} e^{-v} v^{2H-1} (t - v/(2\lambda))^{2H-1} dv$$

with the well-known constant $c(H)$ defined for instance in [5], Theorem 3.2, and

$$B_2(\lambda, t) := C(H) \int_0^t e^{-2\lambda s} \left(\int_0^s (e^{\lambda r} - 1) r^{H-3/2} dr \right)^2 ds$$

where $C(H) := c(H)(H - 1/2)$.

By a linear change of variables, and then using Lemma 2 in [18] with the constant K_A defined therein, we get

$$\begin{aligned} B_2(\lambda, t) &= C(H)t^{2H} \int_0^1 e^{-2\lambda ts} \left(\int_0^s (e^{\lambda tr} - 1)r^{H-3/2} \right) \\ &\leq C(H)K_{H-1/2} t^{2H}(\lambda t)^{-2H} \\ &=: C_2(H)\lambda^{-2H}. \end{aligned}$$

Now for the term B_1 , splitting the integral up at the midpoint λt , and changing the variable for the second half of the interval, we can write,

$$\begin{aligned} B_1(t, \lambda) &\leq c(H)\lambda^{-2H}[(t/2)^{2H-1} \int_0^\infty e^{-v}v^{2H-1}dv \\ &\quad + (\lambda t)^{2H-1} \int_0^{\lambda t} e^{-(2\lambda t-v)}(v/(2\lambda))^{2H-1}dv] \\ &\leq c(H)\lambda^{-2H}(t/2)^{2H-1}[\int_0^\infty e^{-v}v^{2H-1}dv + e^{-\lambda t}(\lambda t)^{2H}/(2H)]. \end{aligned}$$

The function $x \mapsto e^{-x}x^{2H}$ attains its maximum value of $e^{-2H}(2H)^{2H}$ on \mathbf{R}_+ at the point $x = 2H$. Therefore we can write

$$B_1(t, \lambda) \leq \lambda^{-2H}c(H)(t/2)^{2H-1}d(H)$$

where $d(H) := \int_0^\infty e^{-v}v^{2H-1}dv + e^{-2H}(2H)^{2H}$. We now have for all $\lambda, t \geq 0$, when $H < 1/2$, with $c(H)$, $d(H)$, and $C_2(H)$ defined above,

$$A(\lambda, t) \leq \lambda^{-2H}(2^{1-2H}c(H)d(H)t^{2H-1} + C_2(H)).$$

Gathering our results, the lemma now follows, with

$$C(H) = \max(C_0(H), 2^{1-2H}c(H)d(H), C_2(H)).$$

□

Using the above Lemma, one can establish the following Lemmas. The proofs are long and hence omitted. The interested reader can find full proofs of these results in [6].

Lemma 6.2. *Assume $H > 1/4$. The process $\{\int_0^t S(t-s)\Phi d\mathbf{W}_s^H : t \in [0, T]\}$, takes values in $L^4(\Omega \times [0, T] \times G)$ provided that*

$$\sum_n (\sum_j \lambda_j^{-H+1/4}(\Phi e_n, e_j))^2 < \infty, \tag{6.3}$$

where $(\lambda_j, e_j)_j$ are the eigen-elements of \mathbf{A} .

Lemma 6.3. *Assume $H \in (1/8; 1/4]$. The process $\{\int_0^t S(t-s)\Phi d\mathbf{W}_s^H ; t \in [0, T]\}$, takes values in $L^4(\Omega \times [0, T] \times G)$ provided that*

$$\sum_n \left(\sum_i |(\Phi e_n, e_i)| \lambda_i^{1/4} \right)^2 < \infty.$$

In Lemma 6.2, it is understood that the Hurst parameter H is not equal to $1/2$. The main theorem on existence and uniqueness of solutions to the MHD system perturbed by a fractional Brownian noise is given. Full details of the proof is given in [6].

Theorem 6.4. *Let $\{e_n : n \in \mathbf{N}\}$ be the orthonormal basis in the Hilbert space H of eigenfunctions of the operator \mathbf{A} . Under the following two conditions, there exists a unique mild solution of the stochastic MHD system, i.e. \mathbf{P} -almost surely, there is a unique solution in $L^4([0, T] \times G)$ to equation (6.2) driven by the infinite-dimensional fractional Brownian noise $\Phi \mathbf{W}^H$:*

1. $H > 1/4$, and $\sum_n (\sum_j j^{1/4-H} |(\Phi e_n, e_j)|)^2 < \infty$, or
2. $1/8 < H \leq 1/4$, and $\sum_n (\sum_j j^{1/4} |(\Phi e_n, e_j)|)^2 < \infty$.

Proof. STEP 1: Consider the system

$$d\mathbf{y} + [\nu \mathbf{A} \mathbf{y} + \mathbf{B}(\mathbf{y})] dt = \Phi d\mathbf{W}_t^H.$$

In order to find the solution \mathbf{y} , we will use the previous theorems, which tell us how to find the unique evolution (mild) solution $\mathbf{z}(t)$ of

$$d\mathbf{z}(t) + \mathbf{A} \mathbf{z} dt = \Phi d\mathbf{W}_t^H,$$

with $\mathbf{z}(0) = 0$. If \mathbf{y} existed, say in a strong sense, we would denote $\mathbf{v} := \mathbf{y} - \mathbf{z}$, and notice that

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} &= \frac{\partial \mathbf{y}}{\partial t} - \frac{\partial \mathbf{z}}{\partial t} \\ &= (-\mathbf{A} \mathbf{y} - \mathbf{B}(\mathbf{y}) + \Phi \frac{d\mathbf{W}^H}{dt}) - (-\mathbf{A} \mathbf{z} + \Phi \frac{d\mathbf{W}^H}{dt}) \\ &= -\mathbf{A}(\mathbf{y} - \mathbf{z}) - \mathbf{B}(\mathbf{y}) = -\mathbf{A} \mathbf{v} - \mathbf{B}(\mathbf{v} + \mathbf{z}) \end{aligned}$$

Therefore, with \mathbf{z} given, solving for \mathbf{y} in (6.2) would be equivalent to solving for \mathbf{v} in

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{A} \mathbf{v} + \mathbf{B}(\mathbf{v} + \mathbf{z}) = 0 \tag{6.4}$$

with initial data $\mathbf{v}(0) = \mathbf{y}_0 \in H$.

It can be shown that the hypotheses on Φ guarantee the existence (and uniqueness) in $L^4(\Omega \times [0, T] \times G)$ of \mathbf{z} as a mild solution of

$$\mathbf{z}(t) - \mathbf{z}(0) - \int_0^t \mathbf{A} \mathbf{z}(s) ds = \Phi \mathbf{W}^H(t) \tag{6.5}$$

which is given by the formula

$$\mathbf{z}(t) := \int_0^t S(t-s) \Phi d\mathbf{W}_s^H \tag{6.6}$$

Therefore the evolution equation (6.2) has a unique solution mild in that same space (starting from \mathbf{y}_0) if the evolution (mild) version of equation (6.4) admits a solution in $L^4(\Omega \times [0, T] \times G)$ as well. This evolution solution \mathbf{v} , when it exists in that space, satisfies

$$\mathbf{v}(t) = S(t) \mathbf{y}_0 - \int_0^t S(t-s) \mathbf{B}(\mathbf{v}(s) + \mathbf{z}(s)) ds \tag{6.7}$$

where $S(t) = e^{-t\mathbf{A}}$ is the semigroup generated by the operator \mathbf{A} . Let us introduce notation meant to signify that equation (6.7) is a fixed point problem:

$$\Lambda(\mathbf{w}) := S(t)\mathbf{y}_0 - \int_0^t S(t-s)\mathbf{B}(\mathbf{w}(s) + \mathbf{z}(s))ds.$$

Studying the properties of this operator Λ is the main object of the next step.

STEP 2: Let $\mathbf{w} \in L^4([0, T] \times G) \cap V$. It easily follows that $\mathbf{B}(\mathbf{w} + \mathbf{z}) \in L^2(0, T; V')$.

Define the space $Y := L^\infty(0, T; H) \cap L^2(0, T; V)$. Using the Sobolev embedding theorem and the interpolation theorem, one obtains $\Lambda(\mathbf{w}) \in Y$, and in fact,

$$\|\Lambda(\mathbf{w})\|_Y \leq \|\mathbf{B}(\mathbf{w} + \mathbf{z})\|_{L^2(0, T; V')}.$$

Let L^4 denote $L^4([0, T] \times G) = L^4(0, T; L^4(G))$. We claim that for any $\mathbf{w}_1, \mathbf{w}_2 \in L^4([0, T] \times G) \cap V$, we have

$$\|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)\|_{L^4} \leq C\|\mathbf{w}_1 - \mathbf{w}_2\|_{L^4} (\|\mathbf{w}_1 + \mathbf{z}\|_{L^4} + \|\mathbf{w}_2 + \mathbf{z}\|_{L^4})$$

for a suitable constant C .

Indeed, by the basic estimates on \mathbf{B} , and the Jensen inequality, one gets

$$\|\mathbf{B}(\mathbf{y}_1) - \mathbf{B}(\mathbf{y}_2)\|_{L^2(0, T; V')} \leq C\|\mathbf{y}_1 - \mathbf{y}_2\|_{L^4} (\|\mathbf{y}_1\|_{L^4} + \|\mathbf{y}_2\|_{L^4})$$

and thus

$$\begin{aligned} & \|\mathbf{B}(\Lambda(\mathbf{w}_1) + \mathbf{z}) - \mathbf{B}(\Lambda(\mathbf{w}_2) + \mathbf{z})\|_{L^2(0, T; V')} \\ & \leq C\|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)\|_{L^4} (\|\Lambda(\mathbf{w}_1) + \mathbf{z}\|_{L^4} + \|\Lambda(\mathbf{w}_2) + \mathbf{z}\|_{L^4}). \end{aligned}$$

Let $\mathbf{y}_j = \Lambda(\mathbf{w}_j) + \mathbf{z}$ for $j = 1, 2$. Using again the Sobolev embedding of $L^4(G)$ in $W^{1/2, 2}$, we get

$$\begin{aligned} \|\Lambda(\mathbf{w}_1) - \Lambda(\mathbf{w}_2)\|_{L^4} & \leq C\|\mathbf{y}_1 - \mathbf{y}_2\|_{L^4} (\|\mathbf{y}_1\|_{L^4} + \|\mathbf{y}_2\|_{L^4}) \\ & = C\|\mathbf{w}_1 - \mathbf{w}_2\|_{L^4} (\|\mathbf{w}_1 + \mathbf{z}\|_{L^4} + \|\mathbf{w}_2 + \mathbf{z}\|_{L^4}). \end{aligned}$$

STEP 3: From the previous step, we get that the operator

$$\Lambda : \begin{cases} L^4 \mapsto L^4 \\ \mathbf{w} \mapsto \Lambda(\mathbf{w}) := S(\cdot)\mathbf{y}_0 - \int_0^\cdot S(\cdot - s)\mathbf{B}(\mathbf{w}(s) + \mathbf{z}(s))ds \end{cases}$$

is well-defined as mapping L^4 to itself. By the hypotheses of the theorem, \mathbf{z} is in $L^4(\Omega \times [0, T] \times G)$, which implies that $\mathbf{z} \in L^4$ almost surely. Fix any ω in this almost sure set. Note that

$$\Lambda(0) = S(\cdot)\mathbf{y}_0 - \int_0^\cdot S(\cdot - s)\mathbf{B}(\mathbf{z}(s))ds$$

is then a fixed function on $[0, T] \times G$, and a member of L^4 . The fixed point theorem can be applied to $\{\Lambda^n(0) : n \geq 1\}$ in a small interval $[0, T_1]$. The unique fixed point of the map Λ is the unique solution in L of equation (6.7) restricted to $[0, T_1]$. The solution can be continued to the entire interval $[0, T]$ by a standard argument for mild solutions. Therefore, for almost every ω , the unique solution exists on the entire time interval $[0, T]$, and belongs to L^4 . \square

Remark 6.5. In Equation (6.1), an additive force $\sigma(t)dW_t$ where W is an infinite-dimensional Wiener process can be introduced. If σ satisfies the condition that the process $\int_0^t S(t-s)\sigma(s)dW_s$ belongs to $L^4(\Omega \times [0, T] \times G)$, the existence and uniqueness of a mild solution follows from the above proof. To keep the discussion simple, we have avoided the introduction of such extra forces.

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