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SAMPLE PROPERTIES OF RANDOM FIELDS
III: DIFFERENTIABILITY

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Abstract. Differentiability properties of random fields indexed by an open connected subset O of $\mathbb{R}^m$ are studied. It is shown that a real valued random field indexed by O has a sample differentiable modification if is differentiable in quadratic mean, with partial derivatives in quadratic mean, which are continuous in quadratic mean and admit a sample continuous modification.

1. Introduction

This is the third in a series of three papers on sample properties of random fields (cf. also [6, 7]), and in the present paper the question of the existence of a sample differentiable modification of a given random field indexed by an open connected subset of $\mathbb{R}^m$, $m \in \mathbb{N}$, is discussed.

In contrast to the question under which conditions a given stochastic process has a continuous modification, the problem of sample differentiable modifications has not received much attention in the standard literature. In view of the path properties of Brownian motion and the stochastic processes derived thereof this seems also very natural. On the other hand, for instance for random fields which are defined as solutions of space-time stochastic partial differential equations this question seems more meaningful, because such random fields often have samples which behave in the time direction similarly as the paths of a Brownian motion, but in the space direction the samples may be smooth if their space correlation is appropriate.

For stochastic processes differentiability of the sample paths has been discussed in [5] and in [1], by two quite different methods. (In [4] Loève has announced without proof a result concerning the sample differentiability of random fields. In his book [5], which appeared later, Loève unfortunately only treats the case of a one-dimensional parameter domain, i.e., of stochastic processes.) The construction of sample differentiable modification of a stochastic processes in [1] uses polygonal approximation of the paths and a regularization argument. It is rather difficult to generalize this method to the higher dimensional case. The method of Loève in [5] is quite ingenious: Loève develops first a differential and integral calculus of stochastic processes in quadratic mean (QM) sense, including a version of the

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fundamental theorem of calculus. Then he uses the observation that for stochastic processes which have a continuous modification the Riemann integral in the sense of QM — which appears in the fundamental theorem of calculus in QM — is almost surely equal to the sample integral. Thus one can construct the modification as the sample integral of the continuous modification of the derivative in QM. This way one obtains also very natural criteria for the existence of the sample differentiable modification in terms of statistical quantities.

In the present paper, we shall follow the above sketched method of Loève. A first attempt in this direction can be found in [2]. The generalization of Loève's method to the case of a higher dimensional parameter domain necessitates special care of two aspects: Firstly one has to prove the "right" version of the fundamental theorem of calculus in QM, and secondly the control of the exceptional sets becomes much more involved. (Basically, the reason for this is the trivial fact that on the line there is only one continuous path from point to another which has no double points, while in the higher dimensional situation there are uncountably many.)

Consequently, a large portion, namely section 2, of this paper deals with calculus in QM, the ultimate aim being the fundamental theorem of calculus in QM in the form of theorem 2.14. As in the case of usual calculus, the higher dimensional case needs a good account of the calculus of one variable, and the calculus of stochastic processes in quadratic mean has been worked out in an appendix. (Many of the results proved in this appendix are — explicitly or implicitly — contained in [5], but the form in which they were given there was not always appropriate for the present paper. Another appendix is devoted to continuity in QM.) In section 3 the main result of the paper is proved (theorem 3.2): If a random field defined on an open, pathwise connected subset of $\mathbb{R}^m$ is such that it is differentiable in QM, the partial derivatives in QM are continuous in QM, and they have a sample continuous modification, then the random field has a sample differentiable modification. Observe, that all these conditions can be guaranteed by appropriate conditions on statistical quantities, viz., moment or tail estimates. As an example, in section 4 the case of a Gaussian random field is worked out explicitly.

2. Differential Calculus in Quadratic Mean

We begin by setting up some notation. Let $(\Omega, \mathcal{A}, P)$ be a standard probability space. The space of real valued random variables on $(\Omega, \mathcal{A}, P)$ with finite second moment is denoted by $\mathcal{L}^2(P)$. For X, $Y \in \mathcal{L}^2(P)$, we set

$$\langle X, Y\rangle := \mathbb{E}(X\,Y).$$

Then $\langle\cdot,\cdot\rangle$ is a positive semidefinite symmetric bilinear form on $\mathcal{L}^2(P)$, which defines the seminorm

$$\|X\|_2 := \sqrt{\langle X, X\rangle}, \qquad X \in \mathcal{L}^2(P),$$

and the pseudo-metric

$$d_2(X, Y) := \|X - Y\|_2, \qquad X, Y \in \mathcal{L}^2(P),$$

on $\mathcal{L}^2(P)$. We consider $\mathcal{L}^2(P)$ as equipped with the (non-Hausdorff) topology defined by d_2. It is clear that if $(X_n, n \in \mathbb{N})$ is a sequence in $\mathcal{L}^2(P)$ which converges to $X \in \mathcal{L}^2(P)$, and if X^P denotes the P-class of X (i.e., the set of all random variables X' so that $P(X = X') = 1$), then $(X_n, n \in \mathbb{N})$ converges to $X' \in \mathcal{L}^2(P)$ if and only if $X' \in X^P$. We recall that the Riesz-Fischer-theorem states that $\mathcal{L}^2(P)$ is complete with respect to d_2.

We consider differentiability properties and the differential calculus of real valued random fields ϕ indexed by an open subset $O \subset \mathbb{R}^m$,

$$\phi : O \times \Omega \to \mathbb{R},$$

such that for every $x \in O$, $\phi(x) \in \mathcal{L}^2(P)$. Henceforth such random fields will be called *second order random fields.*

Throughout this section we shall view ϕ as a mapping from O into $\mathcal{L}^2(P)$, and hence — following [5] — the differential calculus to be developed is called *differential calculus in quadratic mean (QM).*

2.1. Derivatives in QM. A mapping η from a neighborhood of zero in $\mathbb{R}^m$ into $\mathcal{L}^2(P)$ is called *tangential at zero*, if $\|\eta(h)\|_2 = o(|h|)$ as h tends to zero in $\mathbb{R}^m$, where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^m$.

Definition 2.1. Let $x \in O$. ϕ is called *differentiable in quadratic mean in* x or *qm-differentiable in* x, if there exists a linear mapping $D_{qm}\phi(x)$ from $\mathbb{R}^m$ into $\mathcal{L}^2(P)$ so that the mapping

$$h \mapsto \phi(x+h) - \phi(h) - \big(D_{qm}\phi(x)\big)(h)$$

is tangential at zero. Then $D_{qm}\phi(x)$ is called a *derivative in quadratic mean of* ϕ *in* x or a *qm-derivative of* ϕ *in* x. If for all $x \in O$ ϕ is qm-differentiable in x, then ϕ is called *differentiable in quadratic mean in* O or qm-*differentiable in* O.

Lemma 2.2. *Let $x \in O$, and assume that ϕ is qm-differentiable in x. Then ϕ is* qm-continuous *in x, i.e., as a mapping from O into $\mathcal{L}^2(P)$, ϕ is continuous in x.*

Proof. Let $h \in \mathbb{R}^m$ be such that $x + h \in O$. Then

$$\begin{aligned}\|\phi(x+h) - \phi(x)\|_2 &\le \big\|\big(D_{qm}\phi(x)\big)(h)\big\|_2 + o(|h|)\\ &= O(|h|),\end{aligned}$$

which concludes the proof. □

Remark 2.3. Continuity in quadratic mean is studied in more detail in appendix A.

Lemma 2.4. *Let $x \in O$, and suppose that ϕ is qm-differentiable in x with qm-derivative $D_{qm}\phi(x)$. Then $D_{qm}\phi(x)$ is P-a.s. unique in the following sense: If L is another linear mapping from $\mathbb{R}^m$ into $\mathcal{L}^2(P)$ so that*

$$h \mapsto \phi(x+h) - \phi(x) - L(h)$$

is tangential at zero, then there exists a P-null set $N \in \mathcal{A}$ so that on its complement we have for all $h \in \mathbb{R}^m$, $(D_{qm}\phi(x))(h) = L(h).$,

Proof. By assumption $D_{qm}\phi(x) - L$ is tangential at zero. Therefore, we have by linearity for all $h \in \mathbb{R}^m$

$$\big\|\big(D_{qm}\phi(x)\big)(h) - L(h)\big\|_2 = 0.$$

Hence for every $h \in \mathbb{R}^m$, there exists a P-null set $N_h \in \mathcal{A}$ so that on its complement $(D_{qm}\phi(x))(h)$ and $L(h)$ coincide. Let $\{b_1, \dots, b_m\}$ be a basis of $\mathbb{R}^m$, and set

$$N := \bigcup_{i=1}^{m} N_{b_i}.$$

Then on N^c, $(D_{qm}\phi(x))(b_i) = L(b_i)$ for all $i = 1, \dots, d$. On the other hand, for every $\omega \in \Omega$, $D_{qm}\phi(x, \omega)$ and $L(\omega)$ are linear mappings from $\mathbb{R}^m$ into $\mathbb{R}$, and therefore on N^c for all $h \in \mathbb{R}^m$, $(D_{qm}\phi(x))(h) = L(h)$. □

Remark 2.5. In the sense of lemma 2.4 we shall speak henceforth of *the* qm-derivative $D_{qm}\phi(x)$ of ϕ in x.

2.2. Partial derivatives in QM. Let $\{e_1, e_2, \dots, e_m\}$ denote the standard cartesian basis of $\mathbb{R}^m$.

Definition 2.6. Let $x \in O$, $i \in \{1, \dots, m\}$. If the limit of

$$\lambda^{-1}\big(\phi(x + \lambda e_i) - \phi(x)\big), \qquad \lambda \neq 0 \tag{2.1}$$

exists in $\mathcal{L}^2(P)$ as $\lambda \to 0$, it is said that ϕ has a *partial derivative in quadratic mean in x in direction i.*

It is clear that if ϕ has a partial derivative in quadratic mean in x in direction i, then the limit of (2.1) as $\lambda \to 0$ is P-a.s. unique. We select an arbitrary element of the P-class of limits and denote it by $D^i_{qm}\phi(x)$. As in remark 2.5, we shall speak of *the* partial derivative of ϕ in quadratic mean in x in direction i or *the* partial qm-derivative of ϕ in x in direction i.

The following lemma is trivial:

Lemma 2.7. *Let $x \in O$, and assume that ϕ is qm-differentiable in x. Then for every $i \in \{1, \dots, m\}$ ϕ has a partial qm-derivative in x in direction i, and P-a.s. for every $i \in \{1, \dots, m\}$,*

$$D^i_{qm}\phi(x) = \big(D_{qm}\phi(x)\big)(e_i).$$

Moreover, P-a.s. for every $h \in \mathbb{R}^m$,

$$\big(D_{qm}\phi(x)\big)(h) = \sum_{i=1}^{m} h_i D^i_{qm}\phi(x). \tag{2.2}$$

As before, we assume that ϕ is a real valued second order random field with index set $O \subset \mathbb{R}^m$, O open. For x, $y \in O$ we denote

$$\begin{aligned}
\mu_\phi(x) &:= \mathbb{E}\big(\phi(x)\big),\\
\Gamma_\phi(x, y) &:= \mathbb{E}\big(\phi(x)\phi(y)\big),\\
C_\phi(x, y) &:= \mathrm{Cov}\big(\phi(x), \phi(y)\big)\\
&= \Gamma_\phi(x, y) - \mu_\phi(x)\,\mu_\phi(y).
\end{aligned}$$

If there is no danger of confusion, we occasionally drop the subscript ϕ.

Let $i \in \{1, \dots, m\}$, $x \in O$ and assume that the limit

$$D^{ii}\Gamma_\phi(x,x) := \lim_{h,h'} \frac{1}{hh'}\Big(\Gamma_\phi(x+he_i, x+h'e_i) - \\ - \Gamma_\phi(x+he_i, x) - \Gamma_\phi(x, x+h'e_i) + \Gamma_\phi(x,x)\Big)$$

exists, where h and h' tend to 0 in such a way that $h \neq 0$ and $h' \neq 0$. Then this limit is called the *generalized partial second derivative of* Γ *in* $(x,x) \in O \times O$ in direction i. Obviously, we have

Lemma 2.8. *Let $x \in O$. ϕ has an partial derivative in quadratic mean in x in direction i, if and only if Γ_ϕ has a generalized partial second derivative in (x,x) in direction i.*

We use the following notation for the partial derivatives of real valued functions f in $O \times O$. For $i \in \{1, \dots, m\}$, and $(x,y) \in O \times O$,

$$D_1^i f(x,y) = \lim_{h\to 0,\, h\neq 0} \frac{1}{h}\big(f(x+he_i, y) - f(x,y)\big),$$
$$D_2^i f(x,y) = \lim_{h\to 0,\, h\neq 0} \frac{1}{h}\big(f(x, y+he_i) - f(x,y)\big),$$

whenever the limits exist.

The following lemma is trivial:

Lemma 2.9. *Let $x \in O$, and suppose that the partial derivative of ϕ in quadratic mean in x in direction i exists, or equivalently that the generalized partial second derivative of Γ_ϕ exists in (x,x) in direction i. Then the following statements hold:*

(a) $D^i\mu_\phi(x)$ *exists, and* $D^i\mu_\phi(x) = \mathbb{E}(D^i_{qm}\phi(x))$.

(b) $D_1^i\Gamma_\phi(x,x)$ *and* $D_2^i\Gamma_\phi(x,x)$ *exist, and*

$$D_1^i\Gamma_\phi(x,x) = D_2^i\Gamma_\phi(x,x) = \mathbb{E}\big((D^i_{qm}\phi(x))\,\phi(x)\big).$$

(c) $D_1^iC_\phi(x,x)$ *and* $D_2^iC_\phi(x,x)$ *exist, and*

$$D_1^iC_\phi(x,x) = D_2^iC_\phi(x,x) = \operatorname{Cov}\big(D^i_{qm}\phi(x), \phi(x)\big).$$

(d) $D_1^iD_2^i\Gamma_\phi(x,x)$ *and* $D_2^iD_1^i\Gamma(x,x)$ *exist, and*

$$D_1^iD_2^i\Gamma_\phi(x,x) = D_2^iD_1^i\Gamma_\phi(x,x) = \mathbb{E}\big((D^i_{qm}\phi(x))^2\big).$$

(e) $D_1^iD_2^iC_\phi(x,x)$ *and* $D_2^iD_1^iC_\phi(x,x)$ *exist, and*

$$D_1^iD_2^iC_\phi(x,x) = D_2^iD_1^iC_\phi(x,x) = \operatorname{Var}\big(D^i_{qm}\phi(x)\big).$$

Lemma 2.10. *Suppose that all partial derivatives in quadratic mean of ϕ exist in O and are qm-continuous. Then ϕ is qm-differentiable in O.*

Proof. We only give the proof for $d = 2$, the general case is treated with obvious modifications. Let $x \in O$ and let $\epsilon > 0$ be such that $\overline{B_\epsilon(x)} \subset O$. Let $h = (h_1, h_2) \in \mathbb{R}^2$ be such that $|h| < \epsilon$. We have to show that

$$h \mapsto \psi(h) := \phi(x+h) - \phi(x) - h_1 D^1_{qm}\phi(x) - h_2 D^2_{qm}\phi(x)$$

is tangential at zero. To this end write

$$\begin{aligned}\phi(x+h) &- \phi(x) \\ &= \phi(x+h_1e_1+h_2e_2) - \phi(x+h_1e_1) + \phi(x+h_1e_1) - \phi(x).\end{aligned}$$

Suppose that h_1, $h_2 > 0$. Since the line segments from $x+h_1e_1$ to $x+h_1e_1+h_2e_2$ and from x to $x+h_1e_1$ belong to O, and since $y \mapsto D^i_{qm}\phi(y)$, $i = 1, 2$, are qm-continuous in O, we can apply corollary B.11, and obtain P-a.s.

$$\phi(x+h) - \phi(x) = {}_{qm}\!\int_0^{h_2} D^2_{qm}\phi(x+h_1e_1+\lambda e_2)\,d\lambda + {}_{qm}\!\int_0^{h_1} D^1_{qm}\phi(x+\lambda e_1)\,d\lambda,$$

where the integrals are Riemann integrals in quadratic mean as defined in appendix B (cf. also [5]). Thus we have P-a.s.

$$\begin{aligned}\psi(h) = {}_{qm}\!\int_0^{h_2} &\big(D^2_{qm}\phi(x+h_1e_1+\lambda e_2) - D^2_{qm}\phi(x+h_1e_1)\big)\,d\lambda \\ &+ {}_{qm}\!\int_0^{h_1} \big(D^1_{qm}\phi(x+\lambda e_1) - D^1_{qm}\phi(x)\big)\,d\lambda.\end{aligned}$$

Now we can use lemma B.7 for the following estimation

$$\begin{aligned}\|\psi(h)\|_2 &\le \int_0^{h_2} \|D^2_{qm}\phi(x+h_1e_1+\lambda e_2) - D^2_{qm}\phi(x+h_1e_1)\|_2\,d\lambda \\ &\qquad + \int_0^{h_1} \|D^1_{qm}\phi(x+\lambda e_1) - D^1_{qm}\phi(x)\|_2\,d\lambda \\ &= h_2\big\|D^2_{qm}\phi(x+h_1e_1+\lambda_2e_2) - D^2_{qm}\phi(x+h_1e_1)\big\|_2 \\ &\qquad + h_1\big\|D^1_{qm}\phi(x+\lambda_1e_1) - D^1_{qm}\phi(x)\big\|_2,\end{aligned}$$

with $0 \le \lambda_1 \le h_1$, $0 \le \lambda_2 \le h_2$, by the mean value theorem for Riemann integrals. The cases $h_1 < 0$ or $h_2 < 0$ can be handled similarly. Thus ψ is tangential at zero. □

Corollary 2.11. *Assume that*

(a) for every $i \in \{1,\dots,m\}$ and all $x \in O$ the generalized partial second derivative $D^{ii}\Gamma_\phi(x,x)k$ exists in (x,x) in direction i, and that
(b) for every $i \in \{1,\dots,m\}$, the mapping $x \mapsto D^{ii}\Gamma_\phi(x,x)$ from O into $\mathbb{R}$ is continuous.

Then ϕ is qm-differentiable in O.

2.3. Chain rule in QM. It is straightforward to prove the chain rule in the setting of derivatives in quadratic mean, cf., e.g., [8]. Here we shall only give it in a form in which we shall need it in section 3.

Theorem 2.12. *Let ϕ be a second order random field indexed by $O \subset \mathbb{R}^m$, O open. Suppose that f is a mapping from $U \subset \mathbb{R}^k$, U open, into O which is differentiable in a point $t \in U$. Suppose furthermore that ϕ is qm-differentiable in $f(t)$. Then $\phi \circ f$ is qm-differentiable in t, and*

$$D_{qm}(\phi\circ f)(t) = D_{qm}\phi\big(f(t)\big)\circ Df(t), \qquad P\text{-a.s.}, \tag{2.3}$$

where Df is the usual Jacobian of f.

Similarly, there is a chain rule for partial derivatives in quadratic mean:

Theorem 2.13. *Let ϕ be a second order random field indexed by $O \subset \mathbb{R}^m$, O open. Assume that f is a mapping from $U \subset \mathbb{R}^k$, U open, with existing partial derivative in $t \in U$ in direction $i \in \{1, 2, \dots, k\}$. Suppose furthermore that ϕ is qm-differentiable in $f(t)$. Then $\phi \circ f$ has a partial derivative in quadratic mean in t in direction i, and*

$$D^i_{qm} \varphi \circ f(t) = \big(D_{qm} \varphi(f(t))\big)\big(D^i f(t)\big), \qquad P\text{-a.s.} \tag{2.4}$$

2.4. Fundamental theorem of calculus in QM. In this subsection we shall assume that $O \subset \mathbb{R}^m$ is open and pathwise connected.

By a *smooth directed curve* γ in O we mean the image of $[0, 1]$ under a continuous and piecewise continuously differentiable map $\hat{\gamma}$ from $[0, 1]$ into O. γ is called *closed* if $\hat{\gamma}(0) = \hat{\gamma}(1)$.

Let ψ be a *second order random vector field indexed by* O, i.e.,

$$\psi : O \times \Omega \to \mathbb{R}^m$$

so that $\psi(x, w) = (\psi_1(x, w), \dots, \psi_m(x, w))$, and for every $i \in \{1, \dots, m\}$, ψ_i is a second order random field. Let γ be a smooth directed curve in O with parameterization $\hat{\gamma}$, and let $I_1, \dots, I_N$ be a decomposition of $[0, 1]$ so that $\hat{\gamma}$ is continuously differentiable on I_k for every $k \in \{1, \dots, N\}$. Then we can define the *line integral in quadratic mean of ψ along γ* by

$$_{qm}\!\int_\gamma \psi \, d\gamma := \sum_{k=1}^{N} \sum_{i=1}^{m} {}_{qm}\!\int_{I_k} \psi_i \circ \hat{\gamma}(t) \, \hat{\gamma}_i'(t) \, dt, \tag{2.5}$$

provided the Riemann integrals in quadratic mean on the right hand side of (2.5) exist (cf. appendix B). As sufficient condition for this to be the case is that ψ is qm-continuous in O (corollary B.3).

Consider a second order random vector field ψ as above. A second order random field ϕ indexed by O is called a *qm-antiderivative of* ψ, if all partial derivatives in quadratic mean of ϕ exist in O, and if for all $i \in \{1, \dots, m\}$, $x \in O$, $D^i_{qm}\phi(x) = \psi_i(x)$ holds P-a.s.

Theorem 2.14. *Let O be an open, pathwise connected subset of $\mathbb{R}^m$. Assume that ψ is a second order random vector field indexed by O which is qm-continuous in O. Then ψ has a qm-antiderivative if and only if for every smooth directed closed curve γ in O*

$$_{qm}\!\int_\gamma \psi \, d\gamma = 0, \qquad P\text{-a.s.}, \tag{2.6}$$

holds.

Proof. Suppose that ψ has a qm-antiderivative ϕ: $D^i_{qm}\phi(x) = \psi_i(x)$ for all $i \in \{1, \dots, m\}$, $x \in O$, P-a.s. Then all partial derivatives in quadratic mean of ϕ exist in every $x \in O$, and for every $i \in \{1, \dots, m\}$, $x \mapsto D^i_{qm}\phi(x)$ is qm-continuous in O. Lemma 2.10 implies that ϕ is qm-differentiable in O.

Let γ be a smooth directed closed curve in O with parameterization $\hat\gamma$. There will be no loss of generality if we assume that $\hat\gamma$ is C^1 on $[0,1]$. Since ϕ is qm-differentiable in O, we get by the chain rule, theorem 2.12, and by equation (2.2) that for all $t \in [0,1]$, P-a.s.

$$\begin{aligned} D_{qm}\phi \circ \hat\gamma(t) &= \sum_{i=1}^m D^i_{qm}\phi\bigl(\hat\gamma(t)\bigr)\,\hat\gamma_i'(t) \\ &= \sum_{i=1}^m \psi_i\bigl(\hat\gamma(t)\bigr)\,\hat\gamma_i'(t), \end{aligned}$$

that is $D_{qm}\phi \circ \hat\gamma$ and $\sum_{i=1}^m (\psi_i \circ \hat\gamma)\,\hat\gamma_i'$ are modifications of each other. Lemma B.6 implies that P-a.s.

$$\begin{aligned} {}_{qm}\!\int_\gamma \psi\, d\gamma &= {}_{qm}\!\int_0^1 D_{qm}\phi \circ \hat\gamma(t)\, dt \\ &= \phi \circ \hat\gamma(1) - \phi \circ \hat\gamma(0) \\ &= 0, \end{aligned}$$

where we used corollary B.11 in the second step.

Conversely, suppose that for every smooth directed closed curve γ equality 2.6 holds. Fix a point $x_0 \in O$, let $x \in O$, and choose a smooth directed curve γ_x from x_0 to x. Define

$$\phi(x) := {}_{qm}\!\int_{\gamma_x} \psi\, d\gamma_x.$$

By hypothesis, for every $x \in O$, $\phi(x)$ is P-a.s. well-defined, in the sense that if η_x is a different smooth directed curve from x_0 to x, and we set

$$\chi(x) := {}_{qm}\!\int_{\eta_x} \psi\, d\eta_x,$$

then $P\bigl(\phi(x) = \chi(x)\bigr) = 1$.

Let $x \in O$, $i \in \{1, \dots, m\}$, $h \neq 0$, such that $x + he_i \in O$, and consider

$$\phi(x + he_i) - \phi(x).$$

In the sequel we shall only treat the case $h > 0$, the case $h < 0$ can handled in an analogous way. If γ is any smooth directed curve from x to $x + he_i$, then by construction, our hypothesis we have P-a.s.

$$\phi(x + he_i) - \phi(x) = {}_{qm}\!\int_\gamma \psi\, d\gamma.$$

In particular, if $h > 0$ is small enough, we may choose γ as parameterized by $\hat\gamma(t) = x + the_i$, $t \in [0,1]$. Then

$$\phi(x + he_i) - \phi(x) = h\ {}_{qm}\!\int_0^1 \psi_i(x + t\, he_i)\, dt, \qquad P\text{-a.s.}$$

and therefore

$$\frac{1}{h}\bigl(\phi(x + he_i) - \phi(x)\bigr) = {}_{qm}\!\int_0^1 \psi_i(x + t\, he_i)\, dt, \qquad P\text{-a.s..}$$

Lemmas B.1 and B.5 yield then

$$\begin{aligned}\Big\|\frac{1}{h}\big(\phi(x+he_i)-\phi(x)\big)-\psi_i(x)\Big\|_2^2 &= \Big\|\frac{1}{h}\ {}_{qm}\!\int_0^1 \psi_i(x+t\,he_i)dt-\psi_i(x)\Big\|_2^2\\ &= \frac{1}{h^2}\int_0^1\int_0^1 \Gamma_{\psi_i}(x+she_i, x+the_i)\,ds\,dt\\ &\quad -\frac{2}{h}\int_0^1 \Gamma_{\psi_i}(x+the_i, x)\,dt+\Gamma_{\psi_i}(x,x).\end{aligned}$$

By corollary A.5, the assumption that ψ is qm-continuous implies that for every $i\in\{1,\dots,m\}$, Γ_{ψ_i} is continuous on $O\times O$. Therefore the mean value theorem shows that

$$\lim_{h\to 0,\,h>0}\Big\|\frac{1}{h}\big(\phi(x+he_i)-\phi(x)\big)-\psi_i(x)\Big\|_2=0.$$

Hence the partial derivatives in quadratic mean of ϕ exist in O, and for all $i\in\{1,\dots,m\}$, $x\in O$, $D^i_{qm}\phi(x)=\psi_i(x)$ P-a.s. □

3. Sample Differentiability

Throughout this section we assume that O is an open, pathwise connected subset of $\mathbb{R}^m$. The derivative of a function $f: O\to\mathbb{R}$ at $x\in O$ will be denoted by $Df(x)$.

Consider a real valued random field ϕ indexed by an open subset O of $\mathbb{R}^m$.

Definition 3.1. ϕ is called *P-a.s. sample differentiable in O*, if there exists a P-null set $N\in\mathcal{A}$ so that for all $\omega\in N^c$ the mapping $x\mapsto\phi(x,\omega)$ from O into $\mathbb{R}$ is differentiable. ϕ is called *sample differentiable in O* if for all $\omega\in\Omega$ the mapping $x\mapsto\phi(x,\omega)$ from O into $\mathbb{R}$ is differentiable.

Theorem 3.2. *Let O be an open, pathwise connected subset of $\mathbb{R}^m$, and assume that ϕ is a real valued second order random field indexed by O. Assume furthermore that ϕ is qm-differentiable in O, that the partial derivatives in quadratic mean of ϕ are qm-continuous and that they have a P-a.s. sample continuous modification. Then ϕ has a sample differentiable modification ψ with $D\psi=D_{qm}\phi$ P-a.s.*

Proof. Without loss of generality we may suppose that for every $i\in\{1,\dots,m\}$, $D^i_{qm}\phi$ is qm-continuous and sample continuous in O. In view of lemma 2.7 we shall identify $D_{qm}\phi$ with the random vector field given by $(D^1_{qm}\phi,\dots,D^m_{qm}\phi)$. Furthermore we denote $O_{\mathbb{Q}}:=O\cap\mathbb{Q}^m$.

Fix some point $x_0\in O$, and choose a family $(\gamma_x, x\in O)$ of smooth directed curves so that γ_x begins in x_0 and ends in x. For $x\in O_{\mathbb{Q}}$ define

$$\phi_1(x):=\phi(x_0)+\int_{\gamma_x} D_{qm}\phi\,d\gamma_x \tag{3.1}$$

where the integral on the right hand side of equation (3.1) is the line integral (in the usual Riemannian sense) of the sample continuous random vector field $D_{qm}\phi$ along γ_x. By theorem 2.14, ϕ_1 is P-a.s. well-defined on $O_{\mathbb{Q}}$, in the sense that a different choice of the family $(\gamma_x,\, x \in O_{\mathbb{Q}})$ leads to a modification of ϕ_1. Note that by lemma B.8 this integral is P-a.s. equal to the qm-integral of $\nabla_{qm}\phi$ along γ_x. Hence

$$\begin{aligned}\int_{\gamma_x} D_{qm}\phi\, d\gamma_x &= {}_{qm}\!\int_{\gamma_x} D_{qm}\phi\, d\gamma_x\\ &= \phi(x) - \phi(x_0),\end{aligned}$$

where the last equality follows from corollary B.11 as in the proof of theorem 2.14. Thus ϕ_1 is a modification of ϕ on $O_{\mathbb{Q}}$.

Let $\Lambda_{\mathbb{Q}}$ denote the set of all pairs $(x, y) \in O_{\mathbb{Q}} \times O_{\mathbb{Q}}$ so that the directed line segment $l_{x,y}$ from x to y belongs to O. Obviously $\Lambda_{\mathbb{Q}}$ is countable. We parameterize $l_{x,y}$ by $\hat{l}_{x,y}(t) = x + t(y - x)$, $t \in [0, 1]$. Let $(x, y) \in \Lambda_{\mathbb{Q}}$. By theorem 2.14 there is a P-null set $N_{x,y} \in \mathcal{A}$ so that on its complement

$$\phi_1(y) - \phi_1(x) = \int_{l_{x,y}} D_{qm}\phi\, dl_{x,y} \tag{3.2}$$

holds, because the composition of γ_x, $l_{x,y}$ and the smooth directed curve given by γ_y with reverse direction is a smooth directed closed curve in O. Define the P-null set

$$N := \bigcup_{(x,y)\in\Lambda_{\mathbb{Q}}} N_{x,y}.$$

Then on N^c equation (3.2) holds for all $(x, y) \in \Lambda_{\mathbb{Q}}$.

Let $\omega \in \Omega$, and suppose that $C \subset O$ is a compact convex set. Consider the mapping from $C \times C$ into $\mathbb{R}$, which maps $(x, y) \in C \times C$ to the right hand side of (3.2) evaluated in $\omega \in \Omega$

$$\int_{l_{x,y}} D_{qm}\phi(\cdot,\omega)\, dl_{xy} = \sum_{i=1}^{m} (y - x)_i \int_0^1 D^i_{qm}\phi\big(x + t(y - x), \omega\big)\, dt.$$

By our assumptions on ϕ, for every $i \in \{1, \dots, m\}$ the mapping

$$(t, x, y) \mapsto D^i_{qm}\phi\big(x + t(y - x), \omega\big)$$

is continuous on $[0, 1] \times C \times C$, and hence it is uniformly continuous. It follows that for all $i \in \{1, \dots, m\}$ the mapping

$$(x, y) \mapsto \int_0^1 D^i_{qm}\phi\big(x + t(y - x), \omega\big)\, dt$$

is continuous from $C \times C$ into $\mathbb{R}$, and therefore it is bounded. Thus

$$(x, y) \mapsto \int_{l_{x,y}} D_{qm}\phi(\cdot,\omega)\, dl_{x,y}$$

is uniformly continuous from $C \times C$ into $\mathbb{R}$, and for every $x \in C$,

$$\lim_{y\to x} \int_{l_{x,y}} D_{qm}\phi(\cdot,\omega)\, dl_{x,y} = 0.$$

In particular, on N^c the random field ϕ_1 is continuous on O_Q by equation (3.2).

Define a random field ψ indexed by O as follows: On N set $\psi \equiv 0$. On N^c let ψ be the unique continuous extension of $x \mapsto \phi_1(x)$ from $O_{\mathbb{Q}}$ to O.

We show that ψ is a modification of ϕ. Because ψ is equal to ϕ_1 on $O_{\mathbb{Q}}$, and ϕ_1 is a modification of ϕ on $O_{\mathbb{Q}}$, we get for all $x \in O_{\mathbb{Q}}$, $P\big(\psi(x) = \phi(x)\big) = 1$. Lemma A.4 implies that the restriction of ψ to $O_{\mathbb{Q}}$ is qm-continuous. Let $x \in O$, and let $(x_n, n \in \mathbb{N})$ be a sequence in $O_{\mathbb{Q}}$ which converges to x. Then $(\psi(x_n), n \in \mathbb{N})$ is Cauchy in $\mathcal{L}^2(P)$. By the Riesz-Fischer-theorem, this sequence has a limit, say $\chi(x) \in \mathcal{L}^2(P)$. A subsequence converges P-a.s. to $\chi(x)$, and therefore we have $P\big(\chi(x) = \psi(x)\big) = 1$. On the other hand, the sequence $(\psi(x_n), n \in \mathbb{N})$ converges in $\mathcal{L}^2(P)$ to $\phi(x)$, because for all $x \in O_{\mathbb{Q}}$, $P\big(\psi(x) = \phi(x)\big) = 1$ and ϕ is qm-continuous, by lemma 2.2. Thus ψ is a modification of ϕ.

We have already shown that the right hand side of equation (3.2) is continuous in x and in y. Therefore, by continuity we have on N^c

$$\psi(y) - \psi(x) = \int_{l_{x,y}} D_{qm}\phi \, dl_{x,y}. \tag{3.3}$$

Let $x \in O$, choose $i \in \{1, \dots, m\}$, and let $h > 0$ such that $x + he_i \in O$. Then on N^c equation (3.3) entails

$$\begin{aligned} \frac{1}{h}\big(\psi(x+he_i) - \psi(x)\big) &= \frac{1}{h}\int_0^1 \big(D_{qm}\phi(x+the_i)\big)(he_i)\, dt \\ &= \frac{1}{h}\int_0^h D^i_{qm}\phi(x+te_i)\, dt. \end{aligned}$$

By the mean value theorem the last expression converges with $h \to 0$ to $D^i_{qm}\phi(x)$. The case $h < 0$ is treated similarly. Thus we have proved that on N^c ψ is in every $x \in O$ partially differentiable in every direction, and for every $i \in \{1, 2, \dots, m\}$, $D^i\psi(x) = D^i_{qm}\phi(x)$. Finally, since $D^i_{qm}\phi$ is continuous, it follows that ψ is sample differentiable. □

4. An Application

In this section theorem 3.2 is illustrated with a simple application to Gaussian random fields. Let O be an open, pathwise connected subset of $\mathbb{R}^m$, and assume that ϕ is a Gaussian random field indexed by O. Without loss of generality we may suppose that ϕ is centered, so that with the notation introduced above its covariance is equal to Γ_ϕ.

By lemma 2.8 ϕ is differentiable in quadratic mean in O, if and only if the second generalized derivative $D^{ii}\Gamma_\phi(x,x)$ exists for all $i \in \{1, 2, \dots, m\}$ and all $x \in O$. It is easy to check that a sufficient condition for this to be the case is that $\Gamma_\phi \in C^{1,1}(O \times O)$. Here $C^{1,1}(O \times O)$ denotes the space of functions f in $O \times O$, so that for all i, $j \in \{1, 2, \dots, m\}$ and all $(x,y) \in O \times O$ the partial derivatives $D^i_1 f(x,y)$, $D^j_2 f(x,y)$, and $D^i_1 D^j_2 f(x,y)$ exist and are continuous in $O \times O$. Under this hypothesis it also follows that $x \mapsto D^{ii}\Gamma_\phi(x,x)$ is continuous in O. On the other hand it is clear that for every $i \in \{1, 2, \dots, m\}$, $D^i\phi$ is a centered Gaussian random field with covariance $D^{ii}\Gamma_\phi$. Hence from corollary A.5 and lemma 2.10 we get the following result:

Lemma 4.1. *Suppose that ϕ is a centered Gaussian random field on $O \subset \mathbb{R}^m$, O open, with covariance Γ_ϕ in $C^{1,1}(O \times O)$. Then ϕ is differentiable in quadratic mean, and $D_{qm}\phi$ is continuous in quadratic mean.*

In order to establish that ϕ has a sample differentiable modification with the help of theorem 3.2, we need a condition which guarantees that $D_{qm}\phi$ has a continuous modification. To this end we use the following sufficient condition which has been proved in [6, corollary 4.6, cf. also remark 4.4].

For $\rho > 0$, small enough, and $h \in (0, \rho]$, $\alpha \geq 0$, $\beta \geq 0$ set

$$t(h) := \Big(2 \ln(2) \Big(m \log_2(h)^{2\beta+1} + \alpha \log_2(h^{-1})^{2\beta} \log_2\big(\log_2(h^{-1})\big)\Big)\Big)^{-1}, \quad (4.1)$$

and $t(h) := 0$, if $h = 0$.

Lemma 4.2. *Suppose that χ is a centered Gaussian random field indexed by $O \subset \mathbb{R}^m$, O open, such that for all x, $y \in O$, $\sigma(x,y)^2 := \mathrm{Var}\big(\chi(x) - \chi(y)\big) > 0$. Assume furthermore that there exist $\alpha > 1$, $\beta > 1$, $\rho > 0$ so that*

$$\sigma(x,y)^2 \leq t\big(d(x,y)\big)$$

for all x, $y \in M$ with $d(x,y) \leq \rho$. Then χ has a modification which is sample continuous in O.

We apply this with $\chi = D^i_{qm}\phi$, $i \in \{1, 2, \ldots, m\}$, and obtain the following result:

Theorem 4.3. *Let ϕ be a centered Gaussian random field indexed by $O \subset \mathbb{R}^m$, O open, with covariance $\Gamma_\phi \in C^{1,1}(O \times O)$. For x, $y \in O$, $i \in \{1, 2, \ldots, m\}$ set*

$$\sigma_i(x,y)^2 := D^{ii}\Gamma_\phi(x,x) - 2D^{ii}\Gamma_\phi(x,y) + D^{ii}\Gamma_\phi(y,y). \quad (4.2)$$

If for all $i \in \{1, 2, \ldots, m\}$, x, $y \in O$, $\sigma_i(x,y)^2 > 0$, and there exist $\alpha > 1$, $\beta > 1$, $\rho > 0$ such that

$$\sigma_i(x,y)^2 \leq t\big(d(x,y)\big)$$

whenever $d(x,y) \leq \rho$, where t is defined in (4.1), then ϕ has a sample differentiable modification, which is a centered Gaussian random field with covariance $D^i_1 D^j_2 \Gamma_\phi$, i, $j \in \{1, 2, \ldots, m\}$.

The following result follows from theorem 4.3 with an easy calculation:

Corollary 4.4. *Assume that ϕ is a Gaussian random field indexed by $O \subset \mathbb{R}^m$, O open, with covariance $\Gamma_\phi \in C^{1,1}(O \times O)$. If there exist K, ρ, $\gamma > 0$ such that for all $i \in \{1, 2, \ldots, m\}$, and all x, $y \in O$ with $d(x,y) < \rho$,*

$$\sigma_i(x,y)^2 \leq K\, d(x,y)^\gamma$$

holds, where $\sigma_i(x,y)$ is given by equation (4.2), then ϕ has a sample differentiable modification, which is a centered Gaussian random field with covariance $D^i_1 D^j_2 \Gamma_\phi$, i, $j \in \{1, 2, \ldots, m\}$.

Appendix A. Continuity in Quadratic Mean

The following lemma, which is a direct consequence of the completeness of $\mathcal{L}^2(P)$, i.e., of the Riesz-Fischer-theorem, will be very useful (cf. also [5, p. 135]).

Lemma A.1. *A sequence $(X_n, n \in \mathbb{N})$ in $\mathcal{L}^2(P)$ converges in $\mathcal{L}^2(P)$, if and only if the doubly indexed sequence $\big(\langle X_n, X_m\rangle,\, n, m \in \mathbb{N}\big)$ converges.*

Let (M, d) be a metric space, and consider a *second order random field* $\phi = (\phi(x), x \in M)$ indexed by M, i.e., for every $x \in M$, $\phi(x) \in \mathcal{L}^2(P)$.

Definition A.2. Let $x_0 \in M$. ϕ is called *continuous in quadratic mean in* x_0 or *qm-continuous in* x_0, if the mapping $\phi : M \to \mathcal{L}^2(P)$ is continuous in x_0. ϕ is called *continuous in quadratic mean* or *qm-continuous* if ϕ is qm-continuous in x for all $x \in M$.

We denote by μ_ϕ the mean of ϕ,

$$\mu_\phi(x) := \mathbb{E}\big(\phi(x)\big), \qquad x \in M,$$

and by C_ϕ its covariance,

$$C_\phi(x, y) := \mathbb{E}\Big(\big(\phi(x) - \mu_\phi(x)\big)\big(\phi(y) - \mu_\phi(y)\big)\Big), \qquad x, y \in M.$$

Moreover, we let Γ_ϕ denote the (mixed) second moment of ϕ:

$$\Gamma_\phi(x, y) := \mathbb{E}\big(\phi(x)\phi(y)\big), \qquad x, y \in M.$$

Thus

$$C_\phi(x, y) = \Gamma_\phi(x, y) - \mu_\phi(x)\,\mu_\phi(y), \qquad x, y \in M.$$

If there is no danger of confusion, we also drop the subscript ϕ in μ_ϕ, C_ϕ, Γ_ϕ.

The following is trivial:

Lemma A.3. *Assume that ϕ' is a modification of ϕ. Then $\mu_\phi = \mu_{\phi'}$, $C_\phi = C_{\phi'}$, and $\Gamma_\phi = \Gamma_{\phi'}$.*

We consider $M \times M$ as equipped with the product topology.

Lemma A.4. *Suppose that ϕ is a second order random field indexed by M, and let $x_0 \in M$. The following are equivalent:*

(a) ϕ is qm-continuous in x_0.
(b) Every modification of ϕ is qm-continuous in x_0.
(c) Γ_ϕ is continuous in $(x_0, x_0) \in M \times M$.
(d) μ_ϕ is continuous in x_0 and C_ϕ is continuous in $(x_0, x_0) \in M \times M$.

Proof. Assume that we have proved the equivalence of statements (a) and (c). Then the equivalence of statements (a) and (b) follows from lemma A.3. Suppose that (a) holds, and let $(x_n\,, n \in \mathbb{N})$, $(y_n, n \in \mathbb{N})$ be two sequences in M converging to x_0 with respect to d. Then we have

$$\begin{aligned}\lim_{n,m\to\infty} \Gamma_\phi(x_m, y_n) &= \lim_{n,m\to\infty} \langle \phi(x_m), \phi(y_n)\rangle \\ &= \langle \phi(x_0), \phi(x_0)\rangle \\ &= \Gamma_\phi(x_0, x_0),\end{aligned}$$

because $\langle \cdot, \cdot \rangle$ is continuous. Conversely, suppose that (c) holds, then (a) follows from

$$\mathbb{E}\Big(\big(\phi(x_0) - \phi(x)\big)^2\Big) = \Gamma_\phi(x_0, x_0) - 2\Gamma_\phi(x_0, x) + \Gamma_\phi(x, x),$$

for any $x \in M$. Finally we show the equivalence of (d) and (a), (b), (c): Assume that (a) holds. Then because of the continuity of $\langle \cdot, \cdot \rangle$

$$x \mapsto \mu_\phi(x) = \langle \phi(x), 1 \rangle$$

is continuous in x_0. Since Γ_ϕ is continuous in (x_0, x_0), so is C_ϕ, and (d) is proved. Conversely, if (d) holds, then (c) is true. □

Corollary A.5. *The following are equivalent:*

(a) ϕ *is qm-continuous;*
(b) Γ_ϕ *is continuous in every diagonal point* $(x, x) \in M \times M$
(c) μ_ϕ *is continuous on* M*, and* C_ϕ *is continuous in every diagonal point* $(x, x) \in M \times M$*;*
(d) Γ_ϕ *is continuous on* $M \times M$*;*
(e) μ_ϕ *is continuous on* M*, and* C_ϕ *is continuous on* $M \times M$*.*

Proof. In view of lemma A.4 we only need to prove that (b) implies (d). But (b) entails (a), and therefore for any $x, y \in M$ and all sequences $(x_m, m \in \mathbb{N})$, $(y_n, n \in \mathbb{N})$ converging to x, y respectively, we have

$$\begin{aligned}\lim_{n,m\to\infty} \Gamma_\phi(x_m, y_n) &= \langle \phi(x), \phi(y) \rangle \\ &= \Gamma_\phi(x, y),\end{aligned}$$

which finishes the proof. □

Appendix B. Calculus in QM for Stochastic Processes

B.1. Riemann integration in QM. Consider a finite interval $I = [a, b]$ and let X be a second order stochastic process indexed by I.

We denote by $\mathcal{R}_I$ the set of all pairs R of the form

$$R = \big((t_0, t_1, \ldots, t_N), (\tau_0, \tau_1, \ldots, \tau_{N-1})\big), \tag{B.1}$$

with $N \in \mathbb{N}$, $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$, and $\tau_0 \in [t_0, t_1)$, $\tau_1 \in [t_1, t_2)$, $\ldots$, $\tau_{N-1} \in [t_{N-1}, t_N]$. Set

$$|R| := \max\big\{|t_k - t_{h-1}|, k = 1, \ldots, N\big\}.$$

For $R \in \mathcal{R}_I$ of the form (B.1) we define the Riemann sum

$$S_R(X) := \sum_{h=1}^{N} X(\tau_{k-1})(t_k - t_{k-1}). \tag{B.2}$$

Suppose that for every sequence $(R_n, n \in \mathbb{N})$ in $\mathcal{R}_I$, with $|R_n| \to 0$ as $n \to +\infty$, the sequence $(S_{R_n}(x), n \in \mathbb{N})$ converges in $\mathcal{L}^2(P)$, and that P-a.s. the limit does not depend on the choice of the sequence $(R_n, n \in \mathbb{N})$. (I.e., if $(R_n^1, n \in \mathbb{N})$, $(R_n^2, n \in \mathbb{N})$ are sequences in $\mathcal{R}_I$ with $|R_n^i| \to 0$, $n \to +\infty$, $i = 1, 2$, then their limits in $\mathcal{L}^2(P)$ coincide P-a.s.) Then X *is called Riemann integrable in quadratic mean in* I or *qm-integrable in* I. Assume that X is qm-integrable in I and let

$S(X)$ denote the P-class of limits of the sequences $(S_{R_n}(X),\, n \in \mathbb{N})$ for sequences $(R_n,\, n \in \mathbb{N})$ in $\mathcal{R}_I$ with $|R_n| \to 0$ as $n \to +\infty$. An element in $S(X)$ is called a *Riemann integral in quadratic mean of X in I* or a *qm-integral of X in I*. We select an arbitrary element in $S(X)$, and denote it by

$$_{qm}\!\int_a^b X(s)\, ds.$$

Because this element is P-a.s. unique as the limit of the sequence $(S_{R_n}(X),\, n \in N)$, we shall also call it *the* qm-integral of X.

Lemma B.1. *X is qm-integrable in $I = [a,b]$, if and only if Γ_X is Riemann integrable in $I \times I$. In that case*

$$\mathbb{E}\Big(\Big(_{qm}\!\int_a^b X(s)\, ds\Big)^2\Big) = \int_a^b\!\int_a^b \Gamma_X(s,t)\, ds\, dt. \tag{B.3}$$

Proof. Let $(R_n,\, n \in \mathbb{N})$ be a sequence in $\mathcal{R}_I$ with $|R_n| \to 0$, $u \to +\infty$, of the form

$$R_n = \big((t_{n,0}, t_{n,1}, \dots, t_{n,N_n}), (\tau_{n,0}, \tau_{n,1}, \dots, \tau_{n,N_n-1})\big). \tag{B.4}$$

For m, $n \in \mathbb{N}$ let $S_{R_m}(X)$, $S_{R_n}(X)$ be defined as in (B.2). Then

$$\begin{aligned}
&\mathbb{E}\big(S_{R_m}(X)\, S_{R_n}(X)\big)\\
&\qquad = \sum_{k=1}^{N_m}\sum_{l=1}^{N_n} \mathbb{E}\big(X(\tau_{m,k-1})X(\tau_{n,l-1})\big)(t_{m,k} - t_{m,k-1})(t_{n,l} - t_{n,l-1})\\
&\qquad = \sum_{k=1}^{N_m}\sum_{l=1}^{N_n} \Gamma(\tau_{m,k-1}, \tau_{n,l-1})(t_{m,k} - t_{m,k-1})(t_{n,l} - t_{n,l-1}).
\end{aligned} \tag{B.5}$$

Therefore, by lemma A.1 $(S_{R_n}(X),\, n \in \mathbb{N})$ converges in $\mathcal{L}^2(P)$, if and only if the limit of the right hand side of (B.5) exists as n, $m \to +\infty$. In that case, the limit of $(S_{R_n}(X),\, n \in \mathbb{N})$ is P-a.s. unique with respect to the choice of the sequence $(R_n,\, n \in \mathbb{N})$ in $\mathcal{R}_I$, if and only if the limit of the right hand side is independent of the choice of the sequence $(R_n,\, n \in \mathbb{N})$. This proves the first statement of the lemma. The second statement follows by taking the limit $m, n \to +\infty$ of equation (B.5). □

Corollary B.2. *Assume that X is a second order stochastic process which is qm-integrable in the interval I. Then it is qm-integrable in every subinterval of I.*

Proof. By lemma B.1, Γ_X is Riemann integrable on $I \times I$. Hence for every subinterval J of I, Γ_X is Riemann integrable in $J \times J$. Therefore X is qm-integrable in J by lemma B.1 □

From lemma B.1 and lemma A.1 we get directly

Corollary B.3. *If X is qm-continuous in $[a,b]$, then it is qm-integrable in $[a,b]$.*

Lemma B.4. *Suppose that X and Y are second order stochastic processes indexed by $[a,b]$ which are qm-integrable in $[a,b]$.*

(a) Let $\alpha, \beta \in \mathbb{R}$. Then $\alpha X + \beta Y$ is qm-integrable in $[a, b]$ and

$$\begin{aligned}{}_{qm}\int_a^b \big(\alpha\, X(s) + \beta\, Y(s)\big)\, ds \\ &= \alpha \;{}_{qm}\int_a^b X(s)\, ds + \beta \;{}_{qm}\int_a^b Y(s)\, ds, \qquad P\text{-a.s.}\end{aligned}$$

(b) Let $c \in [a, b]$, then

$${}_{qm}\int_a^b X(s)\, ds = {}_{qm}\int_a^c X(s)\, ds + {}_{qm}\int_c^b X(s)\, ds, \qquad P\text{-a.s.}$$

(c) Assume that P-a.s. $X \geq 0$, then

$${}_{qm}\int_a^b X(s)\, ds \geq 0, \quad P\text{-a.s.}$$

Proof. As the proofs of the first two statements follow directly from the definition of the integral and from corollary B.2 with the usual refinement argument, they can be left to the interested reader. For the proof of (c), let $(R_n,\, n \in \mathbb{N})$ be a sequence in $\mathcal{R}_I$, $I = [a, b]$ with $|R_n| \to 0$ as $n \to +\infty$. Then $(S_{R_n}(X),\, n \in \mathbb{N})$ converges in $\mathcal{L}^2(P)$ to

$${}_{qm}\int_a^b X(s)\, ds,$$

and hence a subsequence $(S_{R_{n'}}(X))$ converges to this random variable P-a.s. Let $N_0 \in \mathcal{A}$ denote a P-null set so that on its complement this subsequence converges pointwise to the qm-integral of X. Note that for every n', $S_{R_{n'}}(X) \geq 0$ P-a.s. Thus for every n' there is a P-null set $N_{n'}$ so that on its complement $S_{R_{n'}}(X) \geq 0$. Set

$$N := \Big(\bigcup_{n'} N_{n'}\Big) \cup N_0.$$

Then N is a P-null set, and on its complement the qm-integral of X is the pointwise limit of a sequence of non-negative random variables, and therefore it is non-negative. □

Lemma B.5. *Let X be qm-integrable in $[a, b]$, and let $Z \in \mathcal{L}^2(P)$. Then*

$$\mathbb{E}\Big(\Big({}_{qm}\int_a^b X(s)\, ds\Big) Z\Big) = \int_a^b \mathbb{E}\big(X(s) Z\big)\, ds, \tag{B.6}$$

where the right hand side is a Riemann integral, and in particular

$$\mathbb{E}\Big({}_{qm}\int_a^b X(s)\, ds\Big) = \int_a^b \mu_X(s)\, ds$$

Proof. For every sequence $(R_n,\, n \in \mathbb{N})$ in $\mathcal{R}_I$, $I = [a, b]$, as in (B.4) with $|R_n| \to 0$ as $n \to +\infty$, we have by construction

$$\mathbb{E}\Big(\Big({}_{qm}\int_a^b X(s) ds\Big) Z\Big) = \lim_{n\to\infty} \sum_{k=1}^{N_n} \mathbb{E}\big(X(\tau_{n,k-1}) Z\big)(t_{n,k} - t_{n,k-1}),$$

and the limit on the right hand side exists, and is independent of the choice of $(R_n,\, n \in \mathbb{N})$. Thus $s \mapsto \mathbb{E}(X(s)Z)$ is Riemann integrable in $[a,b]$, and equation (B.6) holds. □

Lemma B.6. *Assume that X is qm-integrable in $[a,b]$, and that Y is a second order stochastic process indexed by $[a,b]$ which is a modification of X. Then Y is qm-integrable in $[a,b]$, and the qm-integrals of X and of Y in $[a,b]$ coincide P-a.s.*

Proof. Clearly, for all $s,t \in [a,b]$, we have

$$\mathbb{E}\big(X(s)X(t)\big) = \mathbb{E}\big(X(s)Y(t)\big) = \mathbb{E}\big(Y(s)X(t)\big) = \mathbb{E}\big(Y(s)Y(t)\big) = \Gamma_X(s,t).$$

Therefore the first statement follows from lemma B.1. With lemma B.5 we get

$$\begin{aligned}
&\mathbb{E}\Big(\Big({}_{qm}\!\int_a^b X(s)\,ds - {}_{qm}\!\int_a^b Y(s)\,ds\Big)^2\Big)\\
&= \int_a^b\int_a^b \big|\mathbb{E}\big(X(s)X(t)\big) - \mathbb{E}\big(X(s)Y(t)\big) - \mathbb{E}\big(Y(s)X(t)\big) + \mathbb{E}\big(Y(s)Y(t)\big)\big|\, dsdt\\
&= 0.
\end{aligned}$$

Thus

$${}_{qm}\!\int_a^b X(s)\,ds = {}_{qm}\!\int_a^b Y(s)\,ds, \quad P\text{-a.s.},$$

and the proof is done. □

Lemma B.7. *Let X be qm-integrable in $[a,b]$. Then*

$$\Big\| {}_{qm}\!\int_a^b X(s)\,ds\Big\|_2 \le \int_a^b \big\|X(s)\big\|_2\, ds.$$

Proof. The statement follows from lemma B.1 and Schwarz' inequality. □

Let X be second order stochastic process which is qm-continuous and P-a.s. sample continuous in $I = [a,b]$. Let $N \in \mathcal{A}$ be a P-null set so that for all ω in its complement N^c, $t \mapsto X(t,\omega)$ is continuous. The *sample integral*

$$\int_a^b X(t)\,dt$$

of X is defined as follows. For $\omega \in N$, set

$$\int_a^b X(t,\omega)\,dt := 0,$$

and for $\omega \in N^c$

$$\int_a^b X(t,\omega)\,dt$$

is the usual Riemann integral. Fubini's theorem and Schwarz' inequality give

$$\mathbb{E}\Big(\Big(\int_a^b X(t)\,dt\Big)^2\Big) \le \Big(\int_a^b \big\|X(t)\big\|_2\,dt\Big)^2,$$

which is finite because $t \mapsto \big\|X(t)\big\|_2$ is continuous in $[a,b]$. Thus the sample integral belongs to $\mathcal{L}^2(P)$.

Lemma B.8. *Suppose that X is a second order stochastic process which is qm-continuous and P-a.s. sample continuous in $[a,b]$. Then P-a.s. the qm-integral and the sample integral of X in $[a,b]$ coincide.*

Proof. Choose a sequence $(R_n,\, n \in \mathbb{N})$ in $\mathcal{R}_{[a,b]}$ with $|R_n| \to 0$ as $n \to \infty$. By assumption, the sequence $(S_{R_n}(X),\, n \in \mathbb{N})$ of Riemann sums converges in $\mathcal{L}^2(P)$ to the qm-integral of X. Therefore, a subsequence $(S_{R_{n'}}(X),\, n \in \mathbb{N})$ converges P-a.s. to the qm-integral of X. On the other hand the sequence of Riemann sums $(S_{R_{n'}}(X),\, n \in \mathbb{N})$ converges P-a.s. to the sample integral of X in $[a,b]$. Thus qm- and sample integral are P-a.s. equal. □

B.2. Fundamental theorem of calculus in QM. Suppose that X is a second order stochastic process which is qm-continuous in $[a,b]$. Define a second order stochastic process Y indexed by $[a,b]$ by

$$Y(t) := {}_{qm}\!\int_a^t X(s)ds, \qquad t \in [a,b]. \tag{B.7}$$

Lemma B.9. *Assume that X is a second order stochastic process which is qm-continuous in $[a,b]$. Then the stochastic process $Y = (Y(t), t \in [a,b])$ defined by equation* (B.7) *is qm-differentiable in $[a,b]$, and for every $t \in [a,b]$, $D_{qm}Y(t) = X(t)$ P-a.s. If Z is another second order process which is qm-differentiable in $[a,b]$ so that for all $t \in [a,b]$, $D_{qm}Z(t) = X(t)$ P-a.s., then for all $t \in [a,b]$,*

$$Z(t) = Z(0) + Y(t), \qquad P\text{-a.s.}$$

Remark B.10. Of course, at the endpoints of the interval $[a,b]$ a right derivative, resp. left derivative, in quadratic mean is understood.

Proof. Let $t \in [a,b)$, $h > 0$ such that $t+h \in [a,b]$. By lemmas B.1 and B.5 we get

$$\mathbb{E}\Big(\Big(\frac{1}{h}\,{}_{qm}\!\int_t^{t+h} X(s)\,ds - X(t)\Big)^2\Big) = \frac{1}{h^2}\int_t^{t+h}\int_t^{t+h} \Gamma_X(u,s)\,du\,ds - \frac{2}{h}\int_t^{t+h} \Gamma_X(s,t)\,ds + \Gamma_X(t,t).$$

X is qm-continuous, and hence Γ_X is continuous on $[a,b]\times[a,b]$ by corollary A.5. Therefore, by the mean value theorem the last expression tends to zero with $h \to 0$. The cases $h < 0$ and $t = b$ are treated similarly, and the first statement is proved.

Let Z be as in the second statement of the lemma. Set

$$W(t) = Z(t) - Z(0) - Y(t).$$

Then $W(0) = 0$, and for $t \in (a,b]$

$$\begin{aligned}\frac{d}{dt}\,\mathbb{E}\big(W(t)^2\big) &= \lim_{h\to 0,\, h\neq 0} \mathbb{E}\Big(\frac{1}{h}(W(t+h)^2 - W(t)^2)\Big)\\ &= \lim_{h\to 0,\, h\neq 0}\Big(\mathbb{E}\Big(W(t+h)\,\frac{1}{h}\big(W(t+h) - W(t)\big)\Big)\\ &\qquad + \mathbb{E}\Big(\frac{1}{h}\big(W(t+h) - W(t)\big)W(t)\Big)\Big)\\ &= 0,\end{aligned}$$

because by construction W is qm-differentiable with $D_{qm}W(t) = 0$ for all $t \in [a,b]$, and W is qm-continuous in $[a,b]$ by lemma 2.2. Thus for all $t \in [a,b]$, $\mathbb{E}(W(t)^2) = 0$ and hence $W(t) = 0$ P-a.s. □

Corollary B.11. *Assume that X is a second order stochastic process which is qm-differentiable in $[a,b]$, and such that $D_{qm}X$ is qm-continuous in $[a,b]$. Then for all $s,t \in [a,b]$, $s \leq t$,*

$$X(t) - X(s) = {}_{qm}\!\int_s^t D_{qm}X(u)\,du, \qquad P\text{-}a.s. \tag{B.8}$$

Proof. First consider the case $s = 0$. X is a qm-antiderivative of $D_{qm}X$, and hence equation(B.8) follows directly from lemma B.9. The general case can now be obtained from the P-a.s. additivity of the qm-integral (lemma B.4.b). □

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