

THE ITÔ INTEGRAL FOR A CERTAIN CLASS OF LÉVY PROCESSES AND ITS APPLICATION TO STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. Fix $p \in (0, 1]$. Let E and Z be two Banach spaces, let η be a time homogeneous Poisson random measure and $\xi : [0, \infty) \rightarrow L^p(Z, \nu; E)$ be a progressively measurable process such that the composition of the intensity measure of η and ξ is symmetric. We will show that the Itô integral

$$\int_0^t \int_Z \xi(s, z) \eta(dz, ds), \quad t \geq 0,$$

is well defined. In addition we state some inequalities which the Itô Integral will satisfy. In the second part of this paper, we apply our result to SPDEs of parabolic type and show existence and uniqueness of certain SPDEs.

1. Introduction

Stochastic integration with respect to a Wiener process in Banach spaces have been considered by several authors (Brzeźniak [5], Dettweiler [13], Neidhart [22] and Van Neerven, Veraar and Weiss [30]). Similarly, stochastic integration with respect to Lévy processes in Banach spaces is of increasing interest. So, these articles [2, 3, 7, 15, 24, 25] are devoted to this topic. Nevertheless, in the articles above the focus was on Lévy processes of finite p -variation, where $p \in (1, 2]$. In this paper, our focus will be on the Itô integral driven by Lévy processes of finite p -variation, $p \in (0, 1]$ - an issue in which Laurent Schwartz [27] was interested. We will show under which conditions the Itô integral is well defined even for $0 < p \leq 1$ and will present some inequalities satisfied by the Itô integral.

Additionally, we apply our result to SPDEs. To illustrate the consequences of our results, let us state the following example. Let \mathcal{O} be a bounded domain in \mathbb{R}^d with smooth boundary and A be an infinitesimal generator of an analytic semigroup on $L^q(\mathcal{O})$, $1 \leq q < \infty$. Let (Z, \mathcal{Z}) be a measurable space and η be a time homogeneous Poisson random measure defined on Z having as intensity measure a finite Lévy measure ν on Z . Then it is a sufficient condition for the equation

$$\begin{cases} du(t) &= Au(t) dt + \int_Z g(u(t); z) \eta(dz; dt), \\ u(0) &= u_0. \end{cases} \quad (1.1)$$

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to have a unique mild solution being integrable on $L^q(\mathcal{O})$ that $g(x, \cdot) \circ \nu$ induces a symmetric measure on $L^q(\mathcal{O})$ for all $x \in L^q(\mathcal{O})$ and that there exists a natural number $n > 0$ such that $(-A)^{-n}g : L^q(\mathcal{O}) \rightarrow L^1(Z, \nu; L^q(\mathcal{O}))$ is Lipschitz continuous and bounded.

Furthermore, our result can be applied to an SPDE driven by space time Lévy white noise (see for a Definition the book [23] or [8, Chapter 7]). To be more precise, let \mathcal{O} be a bounded domain with Lipschitz boundary and $L = \{L(t, \xi) : \xi \in \mathcal{O}, 0 \leq t < \infty\}$ be a space time Lévy white noise with symmetric jump size measure such that $\int_Z |z| \nu(dz) < \infty$. Moreover, let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow L(\mathbb{R}, \mathbb{R})$ be Lipschitz continuous and bounded. We consider the following SPDE which we write heuristically as follows

$$\begin{cases} \frac{d}{dt}u(t, \xi) &= \Delta u(t, \xi) + g(u(t, \xi)) \dot{L}(t, \xi), & \xi \in \mathcal{O}, t \geq 0, \\ u(0, \xi) &= u_0(\xi) \\ u(t, \xi) &= 0, & \xi \in \partial\mathcal{O}. \end{cases} \tag{1.2}$$

Here $\dot{L}(t, \xi)$ denotes, roughly stated, the Radon Nikodym derivative of L with respect to time. We say a (measured valued) process is a solution of Equation (1.2), iff the process u is weakly càdlàg (that is for all $\phi \in C_0^\infty(\mathcal{O})$, $\mathbb{R}_+ \ni t \mapsto \langle \phi, u(t) \rangle \in \mathbb{R}$ is càdlàg) and satisfies for all $\phi \in C_0^\infty([0, \infty) \times \mathcal{O})$ the following integral equation \mathbb{P} -a.s.

$$\begin{aligned} & \int_{\mathcal{O}} u(T, \xi) \phi(T, \xi) d\xi - \int_{\mathcal{O}} u(0, \xi) \phi(0, \xi) d\xi + \int_0^T \int_{\mathcal{O}} u(t, \xi) \phi_t(t, \xi) d\xi \\ &= \int_0^T \int_{\mathcal{O}} u(t, \xi) \Delta \phi(\xi, t) d\xi + \int_0^T \int_{\mathcal{O}} \phi(\xi, t) g(u(t, \xi)) [L(dt, d\xi)], \quad T > 0. \end{aligned}$$

Our main result says that, if $u_0 \in L^1(\mathcal{O})$, then such a solution u exists. Observe, that in spite the fact that for all $T > 0$

$$\int_0^T \|u(t)\|_{L^1(\mathcal{O})} dt < \infty,$$

u is not necessarily càdlàg in $L^1(\mathcal{O})$.

For related works see also Mueller [21]. He considered a parabolic SPDE driven by a Lévy noise with only positive jumps and investigated under which conditions a solution exists. Nevertheless, the method he applies differs from the method we apply.

Notation. By \mathbb{N} we denote the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \dots\}$ and by $\bar{\mathbb{N}}$ we denote the set $\mathbb{N} \cup \{+\infty\}$. Whenever we speak about \mathbb{N} (or $\bar{\mathbb{N}}$)-valued measurable functions we implicitly assume that that set is equipped with the trivial σ -field $2^{\mathbb{N}}$ (or $2^{\bar{\mathbb{N}}}$).

By \mathbb{R}_+ we will denote the interval $[0, \infty)$ and by \mathbb{R}^* the set $\mathbb{R} \setminus \{0\}$. If X is a topological space, then by $\mathcal{B}(X)$ we will denote the Borel σ -field on X . By λ we will denote the Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, by λ_d the Lebesgue measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

For a measurable space (S, \mathcal{S}) we will denote by $M_{\bar{\mathbb{N}}}(S)$ the set of all $\bar{\mathbb{N}}$ -valued measures on S and by $\mathcal{M}_{\bar{\mathbb{N}}}(S)$ will denote the σ -field on $M_{\bar{\mathbb{N}}}(S)$ generated by functions $i_B : M_{\bar{\mathbb{N}}}(S) \ni \mu \mapsto \mu(B) \in \bar{\mathbb{N}}$. By $M^+(S)$ we will denote the set of all

σ -finite positive measures on S and by $M_F^+(S)$ we will denote the set of all finite positive measures on S .

2. Stochastic Preliminaries

The classical Itô integral has been generalised in several directions, for example it has been generalised to Banach spaces of martingale type p for $1 < p \leq 2$. A short account on martingale type p Banach spaces and the maximal inequalities are given in the Appendix of [7]. Moreover, a short summary of stochastic integration in Banach spaces of martingale type p with respect to Lévy processes is given in Albeverio and Rüdiger [2], Applebaum [3], and Brzeźniak and Hausenblas [7]. However, here we are interested in $0 < p \leq 1$.

Since on one hand we are working with Poisson random measures and on the other hand we claim to be able to handle SPDEs driven by Lévy processes, we start this section with pointing out the connection between Lévy processes and Poisson random measures. Therefore, let us recall firstly the definition of a Lévy process.

Definition 2.1. Let E be a Banach space and let $L = \{L(t) : t \geq 0\}$ be an E -valued stochastic process over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. L is a *Lévy process* if the following conditions are satisfied.

- (i) for any choice $n \in \mathbb{N}$ and $0 \leq t_0 < t_1 < \dots < t_n < \infty$, the random variables $L(t_0), L(t_1) - L(t_0), \dots, L(t_n) - L(t_{n-1})$ are independent;
- (ii) the increments are stationary, in particular, for any $t \geq 0$ and $h \geq 0$ the law of $L(t+h) - L(t)$ does not depend on t ;
- (iii) $L_0 = 0$ a.s.;
- (iv) L is stochastically continuous, in particular, for all $A \in \mathcal{B}(E)$ the function $(0, \infty) \ni t \mapsto \mathbb{E}1_A(L(t))$ is continuous;
- (v) L is $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted.

The characteristic function of a Lévy process is uniquely defined and is given by the Lévy-Khinchin formula. For completeness, before stating the Lévy Khinchin-formula, let us recall the definition of a Lévy measure.

Definition 2.2. (see Linde [19, Chapter 5.4]) Let E be a separable Banach space and let E' be its dual. A symmetric¹ σ -finite Borel measure ν on E is called a *Lévy measure* if and only if

- (i) $\nu(\{0\}) = 0$, and
- (ii) the function²

$$E' \ni a \mapsto \exp \left(\int_E (\cos \langle x, a \rangle - 1) \nu(dx) \right)$$

is a characteristic function of a Radon measure on E .

¹i.e. $\nu(A) = \nu(-A)$ for all $A \in \mathcal{B}(E)$,

²As remarked in Linde [19, Chapter 5.4] we do not need to suppose that the integral $\int_E (\cos \langle x, a \rangle - 1) \nu(dx)$ is finite. However, see Corollary 5.4.2 in ibidem, if ν is a symmetric Lévy measure, then, for each $a \in E'$, the integral in question is finite.

An arbitrary σ -finite Borel measure ν on E is called a Lévy measure provided its symmetric part $\frac{1}{2}(\nu + \nu^-)$, where $\nu^-(A) := \nu(-A)$, $A \in \mathcal{B}(E)$, is a Lévy measure. The class of all Lévy measures on $(E, \mathcal{B}(E))$ will be denoted by $\mathcal{L}(E)$.

Now, if E is a Banach space, then for any E -valued Lévy process $L = \{L(t) : t \geq 0\}$ there exists a positive operator $Q : E' \rightarrow E$, a non negative measure $\nu \in \mathcal{L}(E)$ and an element $m \in E'$ such that

$$E' \ni x \mapsto \mathbb{E}e^{i\langle L(1), x \rangle} = \exp\left(i\langle m, x \rangle - \frac{1}{2}\langle Qx, x \rangle + \int_E \left(1 - e^{i\langle x, y \rangle} + 1_{(-1,1)}(|y|)i\langle x, y \rangle\right) \nu(dy)\right)$$

is the characteristic function of a Radon measure (see [19]).

We call the measure ν *characteristic measure* of the Lévy process L . Moreover, the triplet (Q, m, ν) uniquely determines the law of the Lévy process. Now, starting with an E -valued Lévy process over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, one can construct an integer valued random measure as follows. For each $B \times I \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}_+)$ let

$$\mu_L(B \times I) := \#\{s \in I \mid \Delta_s L \in B\} \in \bar{\mathbb{N}}_0. \quad (2.1)$$

Here, the *jump process* $\Delta X = \{\Delta_t X : 0 \leq t < \infty\}$ of a process X is given by $\Delta_t X(t) := X(t) - X(t-) = X(t) - \lim_{\varepsilon \rightarrow 0} X(t - \varepsilon)$, $t > 0$, $\Delta_0 X = 0$. Such a random measure μ_L is a time homogeneous Poisson random measure, whose definition is introduced in the next paragraph. Note that the notion of Poisson random measures is slightly more general as in the setting described by the counting measure defined in (2.1). In particular, the Poisson random measure can be defined on any measurable space, whereas the Lévy process which induces the counting measure in (2.1) can be defined only on a topological vector space.

Definition 2.3. (see [17], Def. I.8.1) Let (Z, \mathcal{Z}) be a measurable space, $\nu \in M^+(Z)$, and let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space. A *time homogeneous Poisson random measure* η on Z with intensity ν over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$, is a measurable function $\eta : (\Omega, \mathcal{F}) \rightarrow (M_{\bar{\mathbb{N}}}(Z), \mathcal{M}_{\bar{\mathbb{N}}}(Z))$, such that

- (i) for each $B \times I \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+)$, $\eta(B \times I) := i_{B \times I} \circ \eta : \Omega \rightarrow \bar{\mathbb{N}}$ is a Poisson random variable with parameter³ $\nu(B)\lambda(I)$. Here, $i_{B \times I} : M_{\bar{\mathbb{N}}}(Z) \ni \mu \mapsto \mu(B \times I) \in \bar{\mathbb{N}}$.
- (ii) η is independently scattered, i.e. if the sets $B_j \times I_j \in \mathcal{Z} \otimes \mathcal{B}(\mathbb{R}_+)$, $j = 1, \dots, n$ are disjoint, then the random variables $\eta(B_j \times I_j)$, $j = 1, \dots, n$ are mutually independent;
- (iii) for each $B \in \mathcal{Z}$, the $\bar{\mathbb{N}}$ -valued process $(N(t, B))_{t \geq 0}$ defined by

$$N(t, B) := \eta(B \times (0, t]), \quad t \geq 0$$

is $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted and its increments are independent of the past, i.e. if $t > s \geq 0$, then $N(t, B) - N(s, B) = \eta(B \times (s, t])$ is independent of \mathcal{F}_s .

³If $\nu(B) = \infty$ or $\lambda(I) = \infty$, then obviously $\eta(B \times I) = \infty$ a.s..

Throughout this paper let η be a time homogeneous Poisson random measure on a measurable space (Z, \mathcal{Z}) over a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with intensity measure $\nu \in M^+(Z)$. Let E be a separable Banach, $p \in (0, 1]$, and ξ be an $L^p(Z, \nu, E)$ -valued and adapted process. Now we are interested under which conditions on E , ν and p the Itô integral $\int_0^t \int_Z \xi(s, z) \eta(dz, ds)$ does exist. We will show that, if the integrand ξ composed with the intensity measure induces a symmetric Lévy measure, if there exists a deterministic function $g : Z \rightarrow E$ such that $|\xi(s, z)| \leq |g(z)|$ for all $(s, z) \in [0, \infty) \times Z$, and $\int_Z |g(z)|^p \nu(dz) < \infty$, then the stochastic integral in E can be defined in the Itô sense. In order to define a Itô integral one uses the fact that the underlying space is of martingale type p . Usually in the definition of martingale type p Banach spaces p is supposed to be between 1 and 2. However, one can transfer the definition of martingale type p Banach spaces to $0 < p \leq 1$.

Definition 2.4. Assume that $p \in (0, 2]$ is fixed. A Banach space E is of *martingale type p* iff there exists a constant $L_p(E) > 0$ such that for all E -valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\sup_{0 \leq n \leq N} \mathbb{E}|M_n|_E^p \leq L_p(E) \mathbb{E} \sum_{n=0}^N |M_n - M_{n-1}|_E^p, \tag{2.2}$$

where as usually, we put $M_{-1} = 0$.

Here, also the natural question appears, which spaces are of martingale type p , $p \in (0, 1]$.

Lemma 2.5. For $0 < p \leq 1$ any Banach space is of martingale type p with constants $L_p(E) = 1$.

Proof. Let $0 < p \leq 1$ and E be a Banach space. Let $\{M_n\}_{n=0}^N$ be a finite E -valued martingale. Then we have by the Minkowski inequality

$$\sup_{1 \leq n \leq N} \mathbb{E}|M_n|^p \leq \mathbb{E} \left(\sum_{n=1}^N |M_n - M_{n-1}| \right)^p.$$

Since for the function $f : [0, \infty) \ni x \mapsto x^p$ we have $f(x + y) \leq f(x) + f(y)$, $x, y \in [0, \infty)$, we obtain by induction

$$\mathbb{E}|M_n|^p \leq \mathbb{E} \sum_{n=1}^N |M_n - M_{n-1}|^p,$$

which is the assertion. □

Even a stronger inequality as inequality (2.2) holds.

Lemma 2.6. For $0 < p \leq 1$ any Banach space, then for any $p \leq q < \infty$ the following inequality holds

$$\sup_{1 \leq n \leq N} \mathbb{E}|M_n|^q \leq \mathbb{E} \left(\sum_{n=1}^N |M_n - M_{n-1}|^p \right)^{\frac{q}{p}}.$$

Proof. The proof is essentially the same as the proof of 2.5. □

Throughout this Section, let E be a separable Banach space. Moreover, before proceeding, we will introduce certain spaces. Let $p \in (0, 1]$ and $q \in [1, \infty)$ and X be a Banach space. Let $\mathcal{N}_{\text{step}}(a, b; X)$ be the space of all $\xi \in \mathcal{N}(a, b; X)$ for which there exists a partition $a = t_0 < t_1 < \dots < t_n < b$ such that for $k \in \{1, \dots, n\}$, for $t \in (t_{k-1}, t_k]$, $\xi(t) = \xi(t_k)$ is $\mathcal{F}_{t_{k-1}}$ -measurable and $\xi(t) = 0$ for $t \in (t_n, b)$. Let $\mathcal{N}([0, \infty); X)$ be the space of (equivalence classes of) progressively-measurable processes $\xi : [a, b) \times \Omega \rightarrow X$. Let For $r \in (0, \infty)$ let

$$L_{\text{sym}}^r(Z, \nu; E) := \left\{ \xi \in L^r(Z, \nu; E), \nu(\xi^{-1}(C)) = \nu(\xi^{-1}(-C)), \right. \\ \left. \forall C \in \mathcal{B}(E), \text{ and } \int_Z 1_{\{0\}}(\xi(z))\nu(dz) = 0 \right\},$$

equipped by the metric d_{L^r} defined for $\xi_1, \xi_2 \in L_{\text{sym}}^r(Z, \nu; E)$ by

$$d_{L^r}(\xi_1, \xi_2) := \begin{cases} \left(\int_Z |\xi_1(z) - \xi_2(z)|_E^r \nu(dz) \right)^{\frac{1}{r}}, & \text{if } r \geq 1, \\ \int_Z |\xi_1(z) - \xi_2(z)|_E^r \nu(dz), & \text{if } 0 < r < 1. \end{cases}$$

Here, for $C \in \mathcal{B}(E)$, $-C = \{x \in E, -x \in C\}$. Furthermore, we set

$$\mathcal{M}^q([0, \infty); L_{\text{sym}}^p(Z, \nu; E)) = \\ \left\{ \xi \in \mathcal{N}([0, \infty); L_{\text{sym}}^p(Z, \nu; E)) : \mathbb{E} \int_0^\infty d(0, \xi(t))_{L^p}^q dt < \infty \right\}. \quad (2.3)$$

We equip the space $\mathcal{M}^q([0, \infty); L_{\text{sym}}^p(Z, \nu; E))$ by the metric $d_{\mathcal{M}^q(L^p)}$ defined for $\xi_1, \xi_2 \in \mathcal{M}^q([0, \infty); L_{\text{sym}}^p(Z, \nu; E))$ by

$$d_{\mathcal{M}^q(L^p)}(\xi_1, \xi_2) := \left(\mathbb{E} \int_0^\infty d_{L^p}(\xi_1(s), \xi_2(s))^q ds \right)^{\frac{1}{q}}.$$

Note that $\mathcal{M}^1([0, \infty); L_{\text{sym}}^p(Z, \nu; E))$ is a complete metrizable topological vector space, nevertheless, not locally convex. Finally, we put $\mathcal{M}_{\text{step}}^q = \mathcal{M}^q \cap \mathcal{N}_{\text{step}}$.

Fix $p \in (0, 1]$. For any $\xi \in \mathcal{M}_{\text{step}}^1([0, \infty); L_{\text{sym}}^p(Z, \nu; E))$ having representation

$$\xi(r) = \sum_{j=1}^n 1_{(t_{j-1}, t_j]}(r) \xi_j, \quad r \geq 0,$$

where $\{t_0 = 0 < t_1 < \dots < t_n < \infty\}$ is a finite partition of $[0, \infty)$ and for all j , $\xi_j : \Omega \rightarrow L_{\text{sym}}^p(Z, \nu; E)$ is $\mathcal{F}_{t_{j-1}}$ measurable, we put

$$I(\xi) := \sum_{j=1}^n \int_Z \xi_j(x) \eta(dx, (t_{j-1}, t_j]). \quad (2.4)$$

Now, similarly to Lemma C.1 in [7] following Lemma can be shown.

Lemma 2.7. *There exists a unique bounded linear operator*

$$\tilde{I} : \mathcal{M}^1([0, \infty), L_{\text{sym}}^p(Z, \nu; E)) \rightarrow L^p(\Omega, \mathcal{F}, E)$$

such that for $\xi \in \mathcal{M}_{\text{step}}^1((0, \infty], L_{\text{sym}}^p(Z, \nu; E))$ we have $I(\xi) = \tilde{I}(\xi)$. In particular, for every $\xi \in \mathcal{M}^1((0, \infty], L_{\text{sym}}^p(Z, \nu; E))$,

$$\mathbb{E}|I(\xi)|_E^p \leq \mathbb{E} \int_0^\infty \int_Z |\xi(t, x)|_E^p \nu(dx) dt. \quad (2.5)$$

Proof. If ξ is simple, the assertions of Lemma 2.7 follow by Proposition 4.4. Therefore, it remains to show that the set of finite valued simple function are dense in $\mathcal{M}^p([0, \infty); L_{\text{sym}}^p(\nu, Z; E))$, but this holds if E is separable. \square

For short, let us define $\int_0^t \int_Z \xi(r, z) \eta(dz, dr) := I(1_{[0, t]} \xi)$, $t \geq 0$. Here, the process $1_{[0, t]} \xi$ is defined by $[1_{[0, t]} \xi](r, x; \omega) := 1_{[0, t]}(r) \xi(r, x, \omega)$, $t \geq 0$, $r \in \mathbb{R}_+$, $x \in Z$ and $\omega \in \Omega$.

If $0 < p < 1$, then the classical maximal inequality does not hold. However, by following Lemma one gets bounds on higher order moments.

Lemma 2.8. *Assume that $p \in (0, 1]$, $1 \leq q < \infty$. Then there exists a constant $C > 0$, only depending on p and q , such that for any time homogeneous Poisson random measures η on (Z, \mathcal{Z}) over a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with intensity measure ν and any $\xi \in \mathcal{M}^{\frac{q}{p}}([0, \infty), L_{\text{sym}}^p(Z, \nu; E)) \cap \mathcal{M}^1([0, \infty), L_{\text{sym}}^p(Z, \nu; E))$ the following inequality holds*

$$\mathbb{E} \left| \int_0^\infty \int_Z \xi(r, z) \eta(dz, dr) \right|_E^q \leq C \mathbb{E} \left(\int_0^\infty \int_Z |\xi(t, z)|_E^p \eta(dz, dt) \right)^{\frac{q}{p}}. \quad (2.6)$$

Proof. Lemma 2.8 is an application of Lemma 4.5, the fact that the step function are dense in $\mathcal{M}^{\frac{q}{p}}([0, \infty), L_{\text{sym}}^p(Z, \nu; E)) \cap \mathcal{M}^1([0, \infty), L_{\text{sym}}^p(Z, \nu; E))$ and the definition of the stochastic integral. \square

Using the same idea as in [10, Corollary 2.10-(ii)] one can show the following.

Corollary 2.9. *Under the assumption of Lemma 2.8 there exists a constant $C > 0$, depending on p and q , such that for any time homogeneous Poisson random measures η on (Z, \mathcal{Z}) over a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with intensity measure $\nu \in M^+(Z)$ and any $\xi \in \mathcal{M}^1([0, \infty); L_{\text{sym}}^p(Z, \nu; E)) \cap \mathcal{M}^q([0, \infty); L_{\text{sym}}^q(Z, \nu; E))$ the following inequality holds*

$$\begin{aligned} & \mathbb{E} \left| \int_0^\infty \int_Z \xi(r, z) \eta(dz, dr) \right|_E^q \quad (2.7) \\ & \leq C \left\{ \left(\int_0^\infty \int_Z \mathbb{E} |\xi(t, z)|_E^p \nu(dz) dt \right)^{\frac{q}{p}} + \left(\int_0^\infty \int_Z \mathbb{E} |\xi(t, z)|_E^q \nu(dz) dt \right) \right\}. \end{aligned}$$

Since the proof is similar to the proof of [10, Corollary 2.10-(ii)] we omit the proof.

If $p = 1$, then the proof of the Burkholder Davis Gundy inequality given in [11] can be transferred.

Lemma 2.10. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be a non decreasing, convex and continuous function with $\Phi(0) = 0$ such that there exists a strictly increasing and nonnegative*

function $\phi : [0, \infty) \rightarrow \mathbb{R}$ with $\Phi(t) := \int_0^t \phi(s) ds$. Additionally, there exists a constants $c > 0$ with

$$\Phi(2\lambda) \leq c\Phi(\lambda), \quad \lambda \in [0, \infty). \quad (2.8)$$

Let us denote the smallest constant satisfying (2.8) by c_ϕ .

Then, there exists a constant $C > 0$, only depending on Φ , such that for any time homogeneous Poisson random measure η on (Z, \mathcal{Z}) with intensity measure ν and every $\xi \in \mathcal{M}^1((0, \infty]; L_{\text{sym}}^1(Z, \nu; E)) \cap \mathcal{M}^{c_\phi}((0, \infty]; L_{\text{sym}}^{c_\phi}(Z, \nu; E))$ the following inequality holds

$$\mathbb{E} \Phi \left(\sup_{0 \leq t \leq T} \left| \int_0^t \int_Z \xi(r, x) \eta(dx, dr) \right|_E \right) \leq C \mathbb{E} \Phi \left(\int_0^T \int_Z |\xi(r, x)| \eta(dx, dr) \right).$$

Proof. The proof of Lemma 2.10 follows by Lemma 4.6, the fact that the step function are dense in $\mathcal{M}^1((0, \infty]; L_{\text{sym}}^1(Z, \nu; E)) \cap \mathcal{M}^{c_\phi}((0, \infty]; L_{\text{sym}}^{c_\phi}(Z, \nu; E))$ and by the definition of the stochastic integral. \square

However, the linear function $x \mapsto x$ does not satisfy the assumption of Lemma of [11]. Nevertheless, by means of [12] the case of a linear function, i.e. Lemma 2.11, is also covered.

Lemma 2.11. *There exists a constant $C > 0$ such that for any time homogeneous Poisson random measure η on (Z, \mathcal{Z}) with intensity measure $\nu \in M^+(Z)$ and every $\xi \in \mathcal{M}^1([0, \infty); L_{\text{sym}}^1(Z, \nu; E))$ the following inequality holds*

$$\mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_Z \xi(r, x) \eta(dx, dr) \right|_E \leq C \mathbb{E} \int_0^T \int_Z |\xi(r, x)| \eta(dx, dr).$$

Proof. The proof of Lemma 2.11 follows by Lemma A.2, the fact that the step function are dense in $\mathcal{M}^1((0, \infty); L_{\text{sym}}^1(Z, \nu; E))$ and by the definition of the stochastic integral. \square

Corollary 2.12. *If*

$$\int_0^\infty \int_Z |\xi(x, r)| \nu(dz) dr < \infty,$$

then there exists a càdlàg modification of the stochastic integral.

Proof. The proof of Corollary follows by Theorem 1 [14, p. 181] and Lemma 2.11. \square

3. Semilinear SPDEs With Respect to Lévy Processes

In Section 2 we have seen that the Itô Integral can also be defined for $p \in (0, 1]$. In this Section we will investigate the consequences of applying the results of Section 2 to parabolic SPDEs driven by Poisson random measures or Lévy processes with finite p -variation, where p is less than or equal to one.

Therefore, throughout the whole Section let $0 < p \leq 1$. Additionally, let us fix the following Hypothesis.

(H1) B is an Banach space of martingale type p .

(H2-a) A is a positive operator in B , i.e. a densely defined and closed operator for which there exists $M > 0$ such that for $\lambda \geq 0$

$$\|(A + \lambda)^{-1}\| \leq \frac{M}{1 + \lambda}, \lambda > 0.$$

Moreover, $(A + \lambda)^{-1} : B \rightarrow B$ is assumed to be a compact operator.

(H2-b) $-A$ is a generator of an analytic semigroup $\{e^{-tA}\}_{t \geq 0}$ on B .

(H3) There exist positive constants K and ϑ satisfying $\vartheta < \frac{\pi}{2}$ such that $\|A^{is}\| \leq Ke^{\vartheta|s|}$, $s \in \mathbb{R}$.

Let (Z, \mathcal{Z}) be a measurable space and $\nu \in M^+(Z)$. Let η be a time homogeneous Poisson random measure over a given probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ defined on (Z, \mathcal{Z}) with ν as intensity measure. Let $G : B \times Z \rightarrow E$ be a densely defined mapping specified later. We consider the following SPDE written in the Itô-form

$$\begin{cases} du(t) &= Au(t) dt + \int_Z G(t, u(t); z) \eta(dz; dt), \\ u(0) &= u_0. \end{cases} \tag{3.1}$$

Since there exist different types of solution we give here a definition of the solution we are working with.

Definition 3.1. Suppose that G is a densely defined function from $[0, \infty) \times B$ into $L^p(Z, \nu, B)$ and $u_0 \in B$. Suppose that $u = \{u(t) : 0 \leq t < \infty\}$ is a B -valued, càdlàg and adapted process such that $\{G(t, u(t))\}_{t \geq 0}$ is a well defined $L^p(Z, \nu, B)$ -valued progressively measurable process. Then u is called a *mild solution* on B to the Problem (3.1) iff for any $t > 0$,

$$\tag{3.2}$$

$$\int_0^t \int_Z \left| e^{-(t-r)A} G(r, u(r); z) \right|_B^p \nu(dz) dr < \infty$$

and, \mathbb{P} -a.s.,

$$u(t) = e^{-tA}u_0 + \int_0^t \int_Z e^{-(t-r)A} G(r, u(r); z) \eta(dz, dr).$$

In [15] we have shown that iff $p \in (1, 2]$ and G is Lipschitz continuous in a certain sense, then a mild solution to Problem (3.1) exists. Using Lemma 2.8 we will show that a similar assertion holds for $p \in (0, 1]$ provided G composed with ν is symmetric. To formulate the assumption of G let us fix an auxiliary Banach space E .

Remark 3.2. Later on we take E to be a real interpolation space⁴ of the form $D_A^B(\delta, p)$, where $\delta \geq 0$. Since $D_A^B(\delta, p)$ is also a Banach space and since $\{e^{-tA}\}_{t \geq 0}$ restricted to $D_A^B(\delta, p)$ will also a C_0 -semigroup, the hypothesis **(H1)**, **(H2)** and **(H3)** are satisfied on E under the usual conditions.

⁴Recall, for $0 < \delta < 1$, $1 \leq p \leq \infty$, $D_A^B(\delta, p) := \{x \in B : t^{1-\delta}|Ae^{-tA}x| \in L_*^p(0, 1)\}$ (see e.g. Bergh and Löfström [4, Chapter 6.7]). If $\delta \in (1, 2]$, then $D_A^B(\delta, p) = D_{A^2}^B(\delta/2, p)$ (see Lunardi [20, Proposition 3.1.8, p. 63]).

Assumption 3.3. There exists a $\delta_G \in [0, \frac{1}{p})$ such that the function

$$A^{-\delta_G} G : \mathbb{R}^+ \times E \rightarrow L^p_{\text{sym}}(Z, \nu; E)$$

is measurable, Lipschitz continuous with respect to the second variable and uniformly continuous with respect to the first one. In particular, there exists a constant $L_G > 0$ such that

$$\int_Z |A^{-\delta_G} [G(t, x; z) - G(t, y; z)]|_E^p \nu(dz) \leq L_G |x - y|_E^p, \quad x, y \in E, t \in \mathbb{R}_+.$$

Assumption 3.4. There exists numbers $\delta_B \geq \delta_G$, $q \in [1, \infty)$ and a constant $R_G > 0$ such that

$$\left(\int_Z |A^{-\delta_B} G(t, x; z)|_E^q \nu(dz) \right)^{\frac{1}{q}} \leq R_G \quad x \in E, t \in \mathbb{R}_+, \quad (3.3)$$

and

$$\int_Z |A^{-\delta_B} G(t, x; z)|_E \nu(dz) \leq R_G \quad x \in E, t \in \mathbb{R}_+. \quad (3.4)$$

Finally, we impose the following assumption on the initial condition.

Assumption 3.5. There exists $\delta_I < 1$ such that $A^{-\delta_I} u_0 \in E$.

Theorem 3.6. *Let $\delta_G > 0$, $\delta_B > 0$ and $\delta_I > 0$ be fixed and let δ be a constant such that $\delta > \max(\delta_B + \frac{1}{q}, \delta_I)$. Suppose, that for a function G the Assumption 3.3 and Assumption 3.4 with $E = D_A^B(\delta, p)$ is satisfied. Then, for any u_0 satisfying Assumption 3.5, there exists a mild solution $\{u(t) : 0 \leq t < \infty\}$ on B of Problem (3.1). Moreover, there exists a $\lambda \in \mathbb{R}$ with*

$$\int_0^\infty e^{-\lambda t} \mathbb{E} |u(t)|_E^p dt < \infty.$$

For clarity we postpone the proof to page 416 and continue with presenting a Corollary.

Corollary 3.7. *Let Z be a topological vector space and ν be a finite and symmetric measure on $(Z, \mathcal{B}(Z))$ such that there exists a neighbourhood \mathcal{U} of $\{0\}$ with $\nu(\mathcal{U}) = 0$. Let η be a time homogeneous Poisson random measure on $(Z, \mathcal{B}(Z))$ with intensity measure ν .*

Let B be a Banach space and G be a mapping for which a real positive number $n \in \mathbb{N}$ exists such that $G : [0, \infty) \times D_A^B(n+1, p) \rightarrow L(Z; B)$ is Lipschitz continuous in the second variable and bounded in the first variable. Then there exists a mild solution $\{u(t) : 0 \leq t < \infty\}$ on B to the following Problem (see (1.1))

$$\begin{cases} du(t) &= Au(t) dt + \int_Z G(t, u(t); z) \eta(dz; dt), \\ u(0) &= u_0. \end{cases}$$

In particular, for any $0 < p < \frac{1}{n}$

$$\mathbb{E} \int_0^\infty e^{-\lambda t} |(I + A)^{n-1} u(t)|_B^p dt < \infty.$$

Proof. We will show that Theorem 3.6 is applicable. First, note that, since there exists a neighbourhood \mathcal{U} of $\{0\}$ with $\nu(\mathcal{U}) = 0$, and, since G is linear, for any $0 < p \leq 1$, $\int_Z |G(t, z)|^p \nu(dz) < \infty$ for all $t \geq 0$. Fix $p \leq \frac{1}{n}$ and put $E = D_A^B(n + 1, p)$. Then Assumption 3.3 is satisfied, and, hence, Theorem 3.6 is applicable. \square

Theorem 3.6 was formulated in terms of an abstract mapping G . However, we can apply our main results to an SPDE driven by space time Lévy white noise. Here, for a exact reference we refer to [23] or [8]. Therefore, let \mathcal{O} be a bounded open domain with Lipschitz boundary and $L = \{L(t, \xi) : \xi \in \mathcal{O}, 0 \leq t < \infty\}$ be a space time Lévy white noise with symmetric jump size measure with $\int_Z |z| \nu(dz) < \infty$. Let us consider the following SPDE

$$\begin{cases} \frac{d}{dt}u(t, \xi) &= \Delta u(t, \xi) + f(u(t, \xi)) + g(u(t, \xi)) \dot{L}(t, \xi), & \xi \in \mathcal{O}, t \geq 0, \\ u(0, \xi) &= u_0(\xi) \\ u(t, \xi) &= 0, & \xi \in \partial\mathcal{O}, \end{cases} \tag{3.5}$$

where $\dot{L}(t, \xi)$ denotes the Radon Nikodym derivative of L with respect to time, $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous and $g : \mathbb{R} \rightarrow L(\mathbb{R}, \mathbb{R})$ Lipschitz continuous. Since the solution process will be in fact measured valued, we introduce the notion of weak solution.

A (measured valued) process is a weak solution of Equation (3.5), iff for any $\phi \in C_0^\infty(\mathcal{O})$ the real valued process

$$t \mapsto \langle u(t), \phi \rangle = \int_{\mathcal{O}} \phi(\xi) u(t, \xi) d\xi$$

is càdlàg and for any $\phi \in C_b^\infty([0, \infty) \times \mathcal{O})$ and any $T > 0$ the following integral equation holds \mathbb{P} -a.s.

$$\begin{aligned} & \int_{\mathcal{O}} u(T, \xi) \phi(T, \xi) d\xi - \int_{\mathcal{O}} u(0, \xi) \phi(0, \xi) d\xi + \int_0^T \int_{\mathcal{O}} u(t, \xi) \phi_t(t, \xi) d\xi \\ &= \int_0^T \int_{\mathcal{O}} f(u(t, \xi)) \varphi(\xi, t) d\xi dt + \int_0^T \int_{\mathcal{O}} u(t, \xi) \Delta \varphi(\xi, t) d\xi \\ & \quad + \int_0^T \int_{\mathcal{O}} g(u(t, \xi)) \varphi(\xi, t) L(dt, d\xi). \end{aligned}$$

Theorem 3.8. *For any $\gamma > \frac{1}{q}$ and $u_0 \in L^1(\mathcal{O})$ the equation given in (3.5) has a weak solution u on $W^{-\gamma, 1}(\mathcal{O})$. Moreover, there exists a $\lambda > 0$ such that*

$$\int_0^\infty e^{-\lambda t} \mathbb{E}|u(t)|_{L^1(\mathcal{O})} dt < \infty.$$

In particular, for any $\phi \in W_0^{\gamma, 1}(\mathcal{O})$ the real valued process $t \mapsto \langle u(t), \phi \rangle$ has a càdlàg version.

For clarity we postpone the proof of Theorem 3.8 to page 416. Moreover, we will introduce in the following pages some notation and will show some technical Lemmata.

We have seen, for $p \in (0, 1]$ the space $L^p(Z, \nu; E)$ is a Fréchet space, i.e. a locally bounded F -space, i.e. a locally bounded topological vector space, where

the topology is induced by a complete invariant metric (see e.g. [18, 26]). For $\lambda \geq 0$ the space

$$L_\lambda^p([0, \infty); E) := \left\{ u : [0, \infty) \rightarrow E, \int_0^\infty e^{-\lambda t} |u(t)|_E^p dt < \infty \right\}$$

is also a Fréchet space, i.e. a locally bounded F -space, i.e. a locally bounded topological vector space, where the topology is induced by a complete invariant metric (see e.g. [18, 26]). Therefore, $L_\lambda^p([0, \infty); E)$ is a complete metric space with metric d_λ given by

$$d_\lambda(f, g) := \mathbb{E} \int_0^\infty e^{-\lambda t} |f(t) - g(t)|_E^p dt, \quad f, g \in L_\lambda^p([0, \infty); E).$$

Similarly, the spaces

$$\mathcal{M}_\lambda^p([0, \infty); E) := \left\{ u : \Omega \times [0, \infty) \rightarrow E, \right. \\ \left. u \text{ is progressively measurable and } \int_0^\infty e^{-\lambda t} |u(t)|_E^p dt < \infty \right\}$$

and

$$\mathcal{M}_\lambda^1([0, \infty); L^p(Z, \nu; E)) := \left\{ \xi : \Omega \times [0, \infty) \rightarrow E, \right. \\ \left. u \text{ is progressively measurable and } \int_0^\infty e^{-\lambda t} |\xi(t)|_{L^p(Z, \nu; E)} dt < \infty \right\}$$

are Fréchet spaces and, as an important fact, a version of the Banach Fixed Point theorem can be used to get the existence of a mild solution to (3.1).

Theorem 3.9. (see Agrawal [1, Theorem 1.1]) *Let (X, d) be a complete metric space and let $F : X \rightarrow X$ be a contraction with Lipschitz constant K . Then F has a unique fixed point $x^* \in X$. Furthermore, for any $x \in X$ we have $\lim_{n \rightarrow \infty} F^n(x) = x^*$, with $d(F^n(x), x^*) \leq K^n / (1 - K) d(F(x), x^*)$.*

Let \mathfrak{S} denote the stochastic convolution operator defined for an appropriate process $\xi \in \mathcal{M}_\lambda([0, \infty); L^p(Z, \nu; E))$ by the formula,

$$\mathfrak{S}(\xi) = \left\{ \mathbb{R}^+ \ni t \mapsto \int_0^t e^{-(t-s)A} \xi(s; z) \eta(dz, ds) \right\}. \quad (3.6)$$

Before starting with the proof let us state two auxiliary Propositions.

Proposition 3.10. *Assume that the conditions **(H1)** and **(H2)** are satisfied. Assume also that for a $\nu \geq 0$, $A + \nu I$ satisfies the condition **(H3)**. Assume that $0 \leq \rho < \frac{1}{p} - \delta_G$. Then there exists a constant $C > 0$ such that for any $\lambda \in \mathbb{R}$ and any process ξ with $A^{-\delta_G} \xi \in \mathcal{M}_\lambda^p([0, \infty); L^p(Z, \nu; E))$ we have*

$$\begin{aligned} \mathbb{E} \int_0^\infty e^{-\lambda t} |A^\rho \mathfrak{S}(\xi)(t)|_E dt &= |A^\rho \mathfrak{S} \xi|_{\mathcal{M}_\lambda^p(\mathbb{R}_+; E)} \\ &\leq C \lambda^{(\delta_G + \rho)p - 1} |A^{-\delta_G} \xi|_{\mathcal{M}_\lambda^1(\mathbb{R}_+; L^p(Z, \nu; E))}. \end{aligned} \quad (3.7)$$

Proof. The proof is similarly to the proof of Theorem 2.1 in [7]. By Lemma 2.7 and the analyticity of the semigroup generated by the operator $-A$ we infer the following sequence of inequalities

$$\begin{aligned}
& \mathbb{E} \int_0^\infty e^{-\lambda t} |A^\rho \mathfrak{S}(\xi)(t)|_E^p dt \\
&= \int_0^T e^{-\lambda t} \mathbb{E} \left| \int_0^t \int_Z A^\rho e^{-(t-s)A} \xi(s; z) \eta(ds, dz) \right|_E^p dt \\
&\leq C \int_0^\infty e^{-\lambda t} \int_0^t \int_Z \mathbb{E} \left| A^\rho e^{-(t-s)A} \xi(s; z) \right|^p \nu(dz) ds dt \\
&\leq C \int_0^\infty e^{-\lambda t} \int_0^t \int_Z \mathbb{E} \left| A^{\rho+\delta_G} e^{-(t-s)A} A^{-\delta_G} \xi(s; z) \right|^p \nu(dz) ds dt \\
&\leq C \int_0^\infty \int_0^t \int_Z e^{-\lambda(t-s)} \left| A^{\rho+\delta_G} e^{-(t-s)A} \right|_{L(E,E)}^p \\
&\quad \times e^{-\lambda s} \mathbb{E} \left| A^{-\delta_G} \xi(s; z) \right|^p \nu(dz) ds dt \\
&\leq C \int_0^\infty \int_0^t \int_Z e^{-\lambda(t-s)} (t-s)^{-(\rho+\delta_G)p} \\
&\quad \times e^{-\lambda s} \mathbb{E} \left| A^{-\delta_G} \xi(s; z) \right|^p \nu(dz) ds dt.
\end{aligned}$$

The Young inequality for convolutions implies

$$\begin{aligned}
\dots &\leq C \int_0^\infty \mathbb{E} e^{-\lambda t} \int_Z \left| A^{-\delta_G} \xi(s; z) \right|^p \nu(dz) ds \int_0^\infty e^{-\lambda t} t^{-(\delta_G+\rho)p} dt \\
&\leq C \lambda^{(\delta_G+\rho)p-1} \int_0^\infty e^{-\lambda s} \mathbb{E} \int_Z \left| A^{-\delta_G} \xi(s; z) \right|^p \nu(dz) ds.
\end{aligned}$$

This concludes the proof of inequality (3.8). \square

Proposition 3.11. *Let E be a Banach space. Assume that the conditions **(H1)** and **(H2)** are satisfied. Assume also that for a $\nu \geq 0$, $A + \nu I$ satisfies the condition **(H3)**.*

Let $\delta > \max(\delta_B + \frac{1}{q}, \delta_G)$. Then, there exists a constant $C > 0$ and a constant $\gamma > 0$ such that for any process ξ with $A^{-\delta_B} \xi \in \mathcal{M}^1([0, \infty); L^1(Z, \nu; E)) \cap \mathcal{M}^q([0, \infty); L^q(Z, \nu; E))$

$$\begin{aligned}
& \int_0^T \int_Z \mathbb{E} \left| A^{-\delta_B} e^{-\lambda s} \xi(s; z) \right|_E \nu(dz) ds \\
&+ \left(\int_0^T \int_Z \mathbb{E} \left| A^{-\delta_B} e^{-\lambda s} \xi(s; z) \right|_E^q \nu(dz) ds \right)^{\frac{1}{q}} < \infty
\end{aligned} \tag{3.8}$$

we have

$$\mathbb{E} \sup_{t_1 \leq t \leq t_2} e^{-\lambda t} \left| A^{-\delta} \left[\int_{t_1}^t \int_Z e^{-\lambda s} \xi(s; z) \eta(ds, dz) \right] \right|_E \leq C (t_2 - t_1)^\gamma C_\xi(t_1, t_2),$$

where

$$\begin{aligned} C_\xi(t_1, t_2) &:= \int_{t_1}^{t_2} \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E \nu(dz) ds \\ &+ \left(\int_{t_1}^{t_2} \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E^q \nu(dz) ds \right)^{\frac{1}{q}} < \infty \end{aligned} \quad (3.9)$$

Proof. Due to the time homogeneity of the Poisson random measure there is no loss of generality setting $t_1 = 0$ and $t_2 - t_1 = T$. The stochastic Fubini Theorem and the equality $\int_r^t (-A)e^{-(t-s)A} ds = I - e^{-(t-r)A}$, $0 \leq r < t$, give

$$\begin{aligned} \int_0^t \int_Z e^{-(t-s)A} \xi(s; z) \eta(dz, ds) &= \int_0^t \int_Z \xi(s; z) \eta(dz; ds) \\ &+ \int_0^t e^{-(t-s)A} A \int_0^s \int_Z \xi(r; z) \eta(dz; dr) ds. \end{aligned}$$

Therefore, we can infer

$$\begin{aligned} \mathbb{E} \sup_{0 \leq t \leq T} \left| A^{-\delta} \left(\int_0^t \int_Z e^{-(t-s)A} \xi(s; z) \eta(dz, ds) \right) \right|_E &\leq \\ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_Z A^{-\delta} \xi(s; z) \eta(dz; ds) \right|_E &+ \\ \mathbb{E} \sup_{0 \leq t \leq T} \left| \int_0^t e^{-(t-s)(A+\lambda)} A^{-\delta} A \int_0^s \int_Z \xi(r; z) \eta(dz; dr) ds \right|_E & \\ =: I + II. & \end{aligned} \quad (3.10)$$

Applying Lemma 2.11 gives the following estimate for the first term in (3.10)

$$\begin{aligned} I &\leq C \left(\int_0^T \int_Z \mathbb{E} |A^{-(\delta-\delta_B)} A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E \nu(dz) ds \right) \\ &\leq C \left(\int_0^T \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E \nu(dz) ds \right). \end{aligned}$$

In order to deal with the second term we put $\varepsilon := \delta - \delta_B < 1/q$ and recall that $\|e^{-(t-s)A} A^{(1-\varepsilon)}\| \leq C(t-s)^{\varepsilon-1}$ for $t > s \geq 0$. Then the Young inequality for convolutions gives

$$II \leq CT^{\varepsilon-\frac{1}{q}} \mathbb{E} \sup_{0 \leq t \leq T} \left(\int_0^t \left| \int_0^s \int_Z (A+\lambda)^\varepsilon A^{-\delta} \xi(r; z) \eta(dz; dr) \right|_E^q ds \right)^{\frac{1}{q}}.$$

Applying Lemma 2.7 we continue

$$\begin{aligned}
 II &\leq CT^{\varepsilon-\frac{1}{q}} \mathbb{E} \left(\int_0^T \left| \int_0^s \int_Z A^{-\delta_B} e^{-\lambda r} \xi(r; z) \eta(dz; dr) \right|_E^q ds \right)^{\frac{1}{q}} \\
 &\leq C \left(\int_0^T \left[\int_0^s \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E \nu(dz) ds \right]^q \right. \\
 &\quad \left. + \int_0^s \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E^q \nu(dz) ds \right)^{\frac{1}{q}} \\
 &\leq C T^\varepsilon \left[\int_0^T \int_Z \mathbb{E} |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E ds \right. \\
 &\quad \left. + \left(\int_0^T \int_Z |A^{-\delta_B} e^{-\lambda s} \xi(s; z)|_E^q \nu(dz) ds \right)^{\frac{1}{q}} \right].
 \end{aligned}$$

By Assumption 3.8 we know that II is bounded. Combining all together gives the assertion. \square

To prove Theorem 3.8 we have also to handle the deterministic convolution. Therefore, let us define the deterministic convolution process for a process $u \in L^1([0, \infty); E)$ by

$$\Lambda(u) = \left\{ \mathbb{R}^+ \ni t \mapsto \int_0^t e^{-(t-s)A} u(s) ds \right\}. \tag{3.11}$$

Proposition 3.12. *Assume that the conditions (H1) and (H2) are satisfied. Assume also that for a $\nu \geq 0$, $A + \nu I$ satisfies the condition (H3). Assume that $0 \leq \rho < \delta_F < 1$. Then there exists a constant $C > 0$ such that for any $\lambda \in \mathbb{R}$ and any process u with $A^{-\delta_F} u \in L^1_\lambda([0, \infty); E)$ we have*

$$\begin{aligned}
 \mathbb{E} \int_0^\infty e^{-\lambda t} |A^\rho \Lambda(u)(t)|_E dt &= |A^\rho \Lambda(u)|_{L_\lambda(\mathbb{R}_+; E)} \\
 &\leq C \lambda^{(\delta_F + \rho) - 1} |A^{-\delta_F} u|_{L^1_\lambda(\mathbb{R}_+; E)}. \tag{3.12}
 \end{aligned}$$

Proof. The proof follows by direct calculations. To be more precise, by the Fubini Theorem, the Minkowski inequality, the analyticity of the semigroup generated by the operator $-A$ and the Young inequality we infer the following sequence of inequalities, which gives the assertion.

$$\begin{aligned}
 \mathbb{E} \int_0^\infty e^{-\lambda t} |A^\rho \Lambda(u)(t)|_E dt &= \int_0^T e^{-\lambda t} \mathbb{E} \left| \int_0^t A^\rho e^{-(t-s)A} u(s) ds \right|_E dt \\
 &\leq C \int_0^\infty \int_0^t e^{-\lambda(t-s)} |A^{\rho+\delta_F} e^{-(t-s)A}|_{L(E, E)} e^{-\lambda s} \mathbb{E} |A^{-\delta_F} u(s)| ds dt
 \end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^\infty \int_0^t e^{-\lambda(t-s)} (t-s)^{-(\rho+\delta_F)} e^{-\lambda s} \mathbb{E} |A^{-\delta_F} u(s)| \, ds \, dt \\
&\leq C \int_0^\infty \mathbb{E} e^{-\lambda t} |A^{-\delta_F} u(s)| \, ds \int_0^\infty e^{-\lambda t} t^{-(\delta_F+\rho)} \, dt \\
&\leq C \lambda^{(\delta_F+\rho)-1} \int_0^\infty e^{-\lambda s} \mathbb{E} |A^{-\delta_F} u(s)| \, ds.
\end{aligned}$$

This concludes the proof of inequality (3.12). \square

Proof of Theorem 3.6. Let \mathfrak{G} denote the stochastic convolution operator defined by

$$\mathfrak{G}(u) = \left\{ \mathbb{R}^+ \ni t \mapsto \int_0^t e^{-(t-s)A} G(s; u(s); z) \eta(dz, ds) \right\}. \quad (3.13)$$

Note that $\mathfrak{G}(u) = \mathfrak{G}(\xi)$, where $\xi(s; z) = G(s; u(s); z)$, $s \in [0, \infty)$ and \mathfrak{G} is defined in (3.13). Thus, in some non-rigorous way, the map \mathfrak{G} is a composition of the map G with the stochastic convolution operator \mathfrak{G} .

By Proposition 3.10 and Assumption 3.3 there exists a $\lambda > 0$ such that the mapping \mathfrak{G} is a contraction from $\mathcal{M}_\lambda^p([0, \infty); E)$ into itself. Moreover, $\mathcal{M}_\lambda^p([0, \infty); E)$ is a complete metrizable space. Thus, the assumption of Theorem 3.9 are satisfied and we can apply it. It follows that there exists a unique fixed point $u^* \in \mathcal{M}_\lambda^p([0, \infty); E)$. It remains to show that u^* is càdlàg in B . Here, we will use Corollary 2.10 [9]. First, let us observe that by Proposition 3.11 and Assumption 3.4 following holds

$$\mathbb{E} \left[\sup_{s \leq r \leq t} |A^{-\delta} u^*(r)|_E \mid \mathcal{F}_s \right] \leq C(t-s).$$

Moreover, setting in Proposition 3.11 $\delta > \bar{\delta} > \delta_B + 1/q$, $s = 0$ and $t = T$ and using the fact that $A^{-\delta-\bar{\delta}} : B \rightarrow B$ is compact, it follows that the assumptions (i) and (ii) of Corollary 2.10 [9] are satisfied. Therefore, Corollary 2.10 [9] implies that the trajectories of $\{A^{-\delta} u^*(t) : 0 \leq t < \infty\}$ belong a.s. to $\mathbb{D}([0, \infty); E)$. \square

Proof of Theorem 3.8. In [8, Chapter 7] we have seen that the space time Lévy white noise can be embedded into the Besov space $B_{1,\infty}^{1,0}(\mathcal{O})$. Moreover, for any $\delta > 0$, $B_{1,\infty}^{1,0}(\mathcal{O}) \hookrightarrow W^{1,-\delta}(\mathcal{O})$ continuously. Therefore, fixing $\delta > 0$, by Proposition D.1 in [8] Assumption 3.3 holds with $E = L^1(\mathcal{O})$, $\delta_G = \delta$ and $B = W^{1,-\delta}(\mathcal{O})$.

Again, as before we introduce the operator \mathfrak{G} which is defined by (3.13). To handle the deterministic term we introduce first a mapping $F : [0, \infty) \times L^1(\mathcal{O}) \rightarrow L^1(\mathcal{O})$ by putting $F(s, u)(\xi) := f(s, u(\xi))$, $(s, \xi) \in [0, \infty) \times \mathcal{O}$. Next, we introduce the operator Γ by the composition of Λ , defined in (3.11) and F by putting $\Gamma(u) = \Lambda(v)$, where $v(s) := F(s; u(s))$, $s \in [0, \infty)$. Now, by Proposition 3.10 we know that there exists a $\lambda_1 > 0$ such that

$$|\mathfrak{G}u|_{\mathcal{M}_{\lambda_1}^1([0, \infty); L^1(\mathcal{O}))} \leq \frac{1}{2} |u|_{\mathcal{M}_{\lambda_1}^1([0, \infty); L^1(\mathcal{O}))}.$$

To handle the deterministic perturbation first notice that the Nemytskii operator F is Lipschitz continuous on $L^1(\mathcal{O})$. Similarly, Proposition (3.12) (see.g. also

Proposition 2.2 of [6]) gives that there exists a $\lambda_2 > 0$ such that

$$|\Gamma u|_{\mathcal{M}_{\lambda_2}^1([0, \infty); L^1(\mathcal{O}))} \leq \frac{1}{2} |u|_{\mathcal{M}_{\lambda_2}^1([0, \infty); L^1(\mathcal{O}))}.$$

Taking $\lambda = \max(\lambda_1, \lambda_2)$ we know by Theorem 3.9 that the operator \mathcal{K} defined by

$$\mathcal{K} : \mathcal{M}_{\lambda}^1([0, \infty); L^1(\mathcal{O})) \ni u \mapsto e^{-\cdot A} u_0 + \mathfrak{G}u + \Gamma u \in \mathcal{M}_{\lambda}^1([0, \infty); L^1(\mathcal{O}))$$

is well defined and has a fixed point. Let us denote the fixed point by u^* .

It remains to show that the trajectories of u^* are càdlàg in $W^{-\gamma, 1}(\mathcal{O})$. Note, that, again by Proposition D.1 in [8], we know that Assumption 3.4 is valid for $p = 1$ and any $\delta_B > 0$. Since $\gamma > 1$, Proposition 3.11 implies that there exist two constants $C > 0$ and $\varepsilon > 0$ such that $\mathbb{E} [\sup_{s \leq r \leq t} |u^*(r)|_{W^{-\gamma, 1}} | \mathcal{F}_s] \leq C(t - s)^\varepsilon$. Again, since $\gamma > 1$, Proposition 3.11 implies that there exist a number $\rho < \gamma$ and a constant $C = C(T) > 0$ such that $\mathbb{E} \sup_{0 \leq t \leq T} |u^*(t)|_{W^{-\rho, 1}} \leq C$. Since $W^{-\rho, 1}(\mathcal{O}) \hookrightarrow W^{-\gamma, 1}(\mathcal{O})$ compactly, Assumptions (i) and (ii) of Corollary 2.10 in [9] are satisfied and we infer that the trajectories of the process $\mathfrak{G}u^*$ belong \mathbb{P} -a.s. to $\mathbb{D}([0, \infty); W^{-\gamma, 1}(\mathcal{O}))$. Similarly, Lemma 2.4 of [6], respective Proposition 3.12, gives that the trajectories of the process Γu^* belong \mathbb{P} -a.s. to $C([0, \infty); W^{-\gamma, 1}(\mathcal{O}))$. Therefore, and since the sum of a continuous function and a càdlàg function is càdlàg, we know that the trajectories of the process $u^* = e^{\cdot A} u_0 + \mathfrak{G}u^* + \Gamma u^*$ belong to $\mathbb{D}([0, \infty); W^{-\gamma, 1}(\mathcal{O}))$. Now, Corollary 3.8 is proved by the fact that a mild solution is also a weak solution. \square

4. Some Technical Propositions and Lemmata

The definition of martingale type p Banach spaces is based on martingale differences. Therefore, to show inequality (2.5) we approximate the integrand by simple functions and write the integral as a sum of martingale differences. Then we apply Definition 2.4 to get inequality (2.5) for simple integrands. The main difference between the proof of inequality (2.5) and the proof of the inequality (2.3) in paper [7] is, that, in order to get martingale differences, we consider here a Poisson random measure where the intensity measure composed with the integrand is symmetric, and in [7], a compensated Poisson random measure. This difference is reflected in the fact that we used here Lemma 4.1 instead of Lemma C.3, which we have used in [7]. Beside this point, the remaining part of the proof here follows along the same lines as the corresponding part of the proof of inequality (2.3) in [7, Appendix C].

Throughout this section we will assume that (Z, \mathcal{Z}) is a measurable space, $\nu \in M_+(Z)$ is a non negative and σ -finite measure and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is a filtered probability space with right continuous filtration. Moreover, we assume that η is a time homogeneous Poisson random measure on Z over $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with intensity measure ν . Let us begin with the following auxiliary results.

Lemma 4.1. *Suppose that $X \sim \text{Poiss}(\lambda)$, where $\lambda > 0$. Then, for all $p \in (0, 1]$,*

$$\mathbb{E}|X|^p \leq \lambda.$$

Proof. The following calculations

$$\mathbb{E}|X|^p = \sum_{k=1}^{\infty} \mathbb{P}(X = k) k^p \leq \sum_{k=1}^{\infty} \mathbb{P}(X = k) k = \lambda$$

prove the lemma. \square

Lemma 4.2. *Assume that $p \in (0, 1]$ and that E is a Banach space. Then for any $0 \leq a < b < \infty$ and any finite valued $\xi \in L^p(\Omega; L^p_{\text{sym}}(Z, \nu, E) \cap L^p_{\text{step}}(Z, \nu, E))$ being \mathcal{F}_a -measurable we have*

$$\mathbb{E} \left[\int_Z \xi(z) \eta(dz, (a, b)) \right] = 0,$$

and

$$\mathbb{E} \left| \int_Z \xi(z) \eta(dz, (a, b)) \right|_E^p \leq (b - a) \mathbb{E} \int_Z |\xi(z)|^p \nu(dz),$$

where the constant $C > 0$ only depends on p and E .

Proof. Since ξ is a finite valued step function belonging to $L^p_{\text{step}}(Z, \nu, E)$, there exist two numbers $K > 0$ and $J > 0$, there exist a family of sets $\{Z_k^+, Z_k^- : k = 1 \dots K\} \subset Z$, a family of sets $\{\Omega_{k,j} : k = 1, \dots, K, j = 1, \dots, J\} \subset \mathcal{F}_a$ and a family of elements $\{b_j^+, b_j^- : j = 1 \dots J\} \subset E$ such that $b_j^+ = -b_j^-$, $j = 1, \dots, J$, $\nu(Z_k^+) = \nu(Z_k^-)$, $k = 1, \dots, K$, and ξ can be written as

$$\xi : \Omega \times Z \ni (\omega, z) \mapsto \sum_{k=1}^K \sum_{j=1}^J 1_{\Omega_{k,j}}(\omega) \left(1_{Z_k^+} b_j^+ + 1_{Z_k^-} b_j^- \right).$$

Now,

$$\begin{aligned} \mathbb{E} \int_Z \xi(z) \eta(dz, (a, b)) &= \mathbb{E} \sum_{k=1}^K \sum_{i=1}^J 1_{\Omega_{k,i}} (\eta(Z_k^+ \times (a, b)) b_i^+ + \eta(Z_k^- \times (a, b)) b_i^-) \\ &= \mathbb{E} \sum_{k=1}^K \sum_{j=1}^J 1_{\Omega_{k,j}} \mathbb{E} [(\eta(Z_j^+ \times (a, b)) b_j^+ + \mathbb{E} \eta(Z_j^- \times (a, b)) b_j^-) \mid \mathcal{F}_a] \quad (4.1) \\ &= (b - a) \mathbb{E} \sum_{k=1}^K \sum_{j=1}^J 1_{\Omega_{k,j}} (\nu(Z_j^+) b_j^+ + \nu(Z_j^-) b_j^-) = 0. \end{aligned}$$

Again, since $b_k^+ = -b_k^-$, $k = 1, \dots, K$, the inner part of the sum in (4.1) are martingale differences. By Lemma 2.5, E is of martingale type p . Therefore, by the martingale type p property, the tower property, and Lemma 4.1,

$$\begin{aligned} &\mathbb{E} \left| \int_Z \xi(z) \eta(dz, I) \right|_E^p \\ &\leq \mathbb{E} \sum_{k=1}^K \sum_{i=1}^J 1_{\Omega_{k,i}} \mathbb{E} \left[|\eta(Z_j^+ \times (a, b)) b_j^+ + \eta(Z_j^- \times (a, b)) b_j^-|_E^p \mid \mathcal{F}_a \right] \\ &\leq (b - a) \mathbb{E} \sum_{k=1}^K \sum_{i=1}^J \left[\nu(Z_j^+) |b_j^+|_E^p + \nu(Z_j^-) |b_j^-|_E^p \right]. \quad (4.2) \end{aligned}$$

The definition of the integral on the RHS of inequality (4.2) gives the assertion. \square

Corollary 4.3. *Assume that $p \in (0, 1]$ and that E is a separable Banach. Let $0 \leq a < b < \infty$. Then there exists a unique, linear and bounded operator*

$$\tilde{I}_{(a,b)} : L^p((\Omega, \mathcal{F}_a); L^p_{\text{sym}}(Z, \nu, E)) \rightarrow L^p(\Omega, \mathcal{F}_a; E)$$

such that for any $\xi \in L^p_{\text{sym}}(Z, \nu, E) \cap L^p_{\text{step}}(Z, \nu, E)$, we have $\mathbb{E}\tilde{I}_{(a,b)}\xi = 0$ and $\tilde{I}_{(a,b)}\xi = \int_Z \xi(z)\eta(dz, (a, b))$. In particular,

$$\mathbb{E} \left| \int_Z \xi(z)\eta(dz, (a, b)) \right|_E^p \leq (b - a) \mathbb{E} \int_Z |\xi(z)|^p \nu(dz).$$

Proof. It remains to show that the set of finite valued symmetric step function is dense in $L^p_{\text{sym}}(Z, \nu, E)$. Indeed, let us fix $\varepsilon > 0$. Since E is separable, $L^p(Z, \nu, E)$ is separable. Thus, for any $\varepsilon > 0$ there exists an ε -covering $\{X_i : i \in \mathbb{N}\}$ of $L^p(Z, \nu, E)$. From $\{X_i : i \in \mathbb{N}\}$ we construct an 2ε -covering of $L^p_{\text{sym}}(Z, \nu, E)$ in the following way. Let

$$X'_i := \begin{cases} X_i \cap L^p_{\text{sym}}(Z, \nu, E) & \text{if } X_i \cap L^p_{\text{sym}}(Z, \nu, E) \neq \emptyset, \\ \emptyset & \text{if } X_i \cap L^p_{\text{sym}}(Z, \nu, E) = \emptyset. \end{cases}$$

Remunerating gives a new family of sets called for simplicity again $\{X'_i, i \in \mathbb{N}\}$. For any $i \in \mathbb{N}$ let $\zeta_i \in X'_i \cap L^p_{\text{sym}}(Z, \nu, E)$. By the triangle inequality and by the fact that $L^p_{\text{sym}}(Z, \nu, E) \subset L^p(Z, \nu, E)$ it follows that $\{X'_i : i \in \mathbb{N}\}$ is an 2ε -covering of $L^p_{\text{sym}}(Z, \nu, E)$. Hence for any $\xi \in L^p_{\text{sym}}(Z, \nu, E)$ there exists a sequence $\{\xi_n : n \in \mathbb{N}\} \subset L^p_{\text{sym}}(Z, \nu, E)$ converging to ξ in $L^p(Z, \nu, E)$. By Lemma 4.2 it follows the assertion. \square

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space and let X be a complete metric space with metric d_X . Later on we will take X to be one of the spaces E or $L^p_{\text{sym}}(Z, \nu, E)$. For $a < b \in [0, \infty]$ let $\mathcal{N}(a, b; X)$ be the space of (equivalence classes of) progressively measurable processes $\xi : (a, b] \times \Omega \rightarrow X$.

For $q \in (0, \infty)$ we set

$$\mathcal{M}^q((a, b]; X) = \left\{ \xi \in \mathcal{N}_{\text{sym}}(a, b; X) : \mathbb{E} \int_a^b d_X(0, \xi(t))^q dt < \infty \right\}.$$

Let $\mathcal{N}_{\text{step}}(a, b; X)$ be the space of all $\xi \in \mathcal{N}(a, b; X)$ for which there exists a partition $a = t_0 < t_1 < \dots < t_n < b$ such that for $k \in \{1, \dots, n\}$, for $t \in (t_{k-1}, t_k]$, $\xi(t) = \xi(t_k)$ is $\mathcal{F}_{t_{k-1}}$ -measurable and $\xi(t) = 0$ for $t \in (t_n, b)$. We put $\mathcal{M}^q_{\text{step}} = \mathcal{M}^q \cap \mathcal{N}_{\text{step}}$. Note that $\mathcal{M}^q((a, b]; X)$ is a closed subspace of $L^q([a, b] \times \Omega; X) \cong L^q([a, b]; L^q(\Omega; X))$.

In what follows for any $\xi \in \mathcal{M}^p_{\text{step}}([0, \infty); L^p_{\text{sym}}(Z, \nu; E))$ having representation

$$\xi(s) = \sum_{n=1}^N 1_{(s_{n-1}, s_n]}(s) \xi_n \tag{4.3}$$

where $\pi = \{0 = s_0 < s_1 < \dots < s_n < \infty\}$ is a partition of $[0, \infty)$, $\xi_n : \Omega \rightarrow L^p_{\text{sym}}(Z, \nu; E)$ is $\mathcal{F}_{s_{n-1}}$ -measurable, we put

$$\tilde{I}(\xi) := \sum_{n=1}^N \int_Z \xi_n(z) \eta(dz, (s_{n-1}, s_n]). \tag{4.4}$$

Obviously, $\tilde{I}(\xi)$ is an \mathcal{F} -measurable map from Ω into E and $\mathbb{E}\tilde{I}(\xi) = 0$. Now, we can formulate the following auxiliary result.

Proposition 4.4. *Assume that $p \in (0, 1]$ and that E is a separable Banach space. Then, for any $\xi \in \mathcal{M}^1_{\text{step}}([0, \infty); L^p_{\text{sym}}(Z, \nu; E))$ having representation given in (4.3) we have*

$$\mathbb{E}|\tilde{I}(\xi)|^p_E \leq \int_0^\infty \int_Z \mathbb{E}|\xi(s, z)|^p_E \nu(dz) ds. \tag{4.5}$$

Proof. Let $\xi \in \mathcal{M}^1_{\text{step}}([0, \infty); L^p_{\text{sym}}(Z, \nu; E))$ having representation as given in (4.3). Then, $\tilde{I}(\xi)$ is defined in (4.4). Since $\mathbb{E} \int_Z \xi_n(z) \eta(dz, (s_{n-1}, s_n]) = 0$ for any $n = 1, \dots, N$, the inner part of (4.4) are martingale differences and the martingale type p property of E gives

$$\mathbb{E}|\tilde{I}(\xi)|^p_E = \mathbb{E} \left| \sum_{n=1}^N \int_Z \xi_n(z) \eta(dz, (s_{n-1}, s_n]) \right|^p_E \tag{4.6}$$

$$\leq \sum_{n=1}^N \mathbb{E} \left| \int_Z \xi_n(z) \eta(dz, (s_{n-1}, s_n]) \right|^p_E \tag{4.7}$$

$$\leq \sum_{n=1}^N (s_n - s_{n-1}) \int_Z \mathbb{E}|\xi_n(z)|^p_E \nu(dz) = \int_0^\infty \int_Z \mathbb{E}|\xi(z, s)|^p_E \nu(dz) ds. \tag{4.8}$$

□

A similar, but slightly different lemma is proved below.

Lemma 4.5. *Let $p \in (0, 1]$, $p \leq q < \infty$, and let E be a separable Banach space. Then, for any $\xi \in \mathcal{M}^1_{\text{step}}([0, \infty); L^p_{\text{sym}}(Z, \nu; E) \cap L^p_{\text{step}}(Z, \nu; E))$, we have*

$$\mathbb{E}|I(\xi)|^q_E \leq \mathbb{E} \left(\int_0^\infty \int_Z |\xi(t, x)|^p_E \eta(dx, dt) \right)^{\frac{q}{p}}.$$

Proof. Let $\{M_m\}_{m=0}^{NJ}$ be a sequence defined by $M_0 = 0$ and for $m = nJ + j$

$$M_m = M_{m-1} + \sum_{i=1}^I 1_{\Omega_{i,j,n}} (\eta(S_j^+ \times (t_n, t_{n+1}]) - \eta(S_j^- \times (t_n, t_{n+1}])) x_i. \tag{4.9}$$

Then, the sequence $\{M_m\}_{m=0}^{NJ}$ is an E -valued martingale (with respect to the filtration $(\tilde{\mathcal{F}}_m)_{m=0}^{NJ}$, where $\tilde{\mathcal{F}}_m = \mathcal{F}_{t_n}$, $m = nJ + j$). The martingale type p

property of the space E verifies the following sequence of inequalities

$$\begin{aligned} \mathbb{E}|I(\xi)|_E^q &= \mathbb{E}|M_{NJ}|_E^q \leq \mathbb{E} \left(\sum_{m=1}^{NJ} |M_m - M_{m-1}|_E^p \right)^{\frac{q}{p}} \\ &\leq \mathbb{E} \left(\sum_{n=1}^N \sum_{j=1}^J \left| \sum_{i=1}^I 1_{\Omega_{i,j,n}} (\eta(S_j^+ \times (t_n, t_{n+1})) - \eta(S_j^- \times (t_n, t_{n+1}))) x_i \right|^p \right)^{\frac{q}{p}}. \end{aligned}$$

Now we will use the fact that for fixed n and j the sets $\Omega_{i,j,n}$, $i = 1, \dots, I$ are disjoint. Let $e_1, \dots, e_{I^{JN}} \in \{1, 2, \dots, I\}^{NJ}$ vectors such that $\cup_{k=1}^{I^{JN}} e_k = \{1, 2, \dots, I\}^{NJ}$. Set $\tilde{\Omega}_k := \cap_{k=1}^{I^{JN}} \Omega_{e(k), k-[k/J], [k/n]}$. Then $\Omega = \cup_{k=1}^{I^{JN}} \tilde{\Omega}_k$ and

$$\begin{aligned} \mathbb{E}|I(\xi)|_E^q &\leq \sum_{k=1}^{I^{JN}} \\ &\mathbb{E} 1_{\tilde{\Omega}_k} \left(\sum_{n=1}^N \sum_{j=1}^J \left| \sum_{i=1}^I 1_{\Omega_{i,j,n}} (\eta(S_j^+ \times (t_n, t_{n+1})) - \eta(S_j^- \times (t_n, t_{n+1}))) x_i \right|^p \right)^{\frac{q}{p}}. \end{aligned}$$

Now, for any k only the inner sum differs from zero only for exactly one index i . Thus, we get

$$\begin{aligned} \mathbb{E}|\tilde{I}(\xi)|_E^q &\leq \sum_{k=1}^{I^{JN}} \mathbb{E} 1_{\tilde{\Omega}_k} \\ &\left(\sum_{n=1}^N \sum_{j=1}^J \sum_{i=1}^I |1_{\Omega_{i,j,n}} (\eta(S_j^+ \times (t_n, t_{n+1})) - \eta(S_j^- \times (t_n, t_{n+1}))) x_i|^p \right)^{\frac{q}{p}} \\ &= \mathbb{E} \left(\sum_{n=1}^N \sum_{j=1}^J \sum_{i=1}^I |1_{\Omega_{i,j,n}} (\eta(S_j^+ \times (t_n, t_{n+1})) - \eta(S_j^- \times (t_n, t_{n+1}))) x_i|^p \right)^{\frac{q}{p}}. \end{aligned}$$

Using the fact that $0 < p \leq 1$ and

$$\begin{aligned} &|\eta(S_j^+ \times (t_n, t_{n+1})) - \eta(S_j^- \times (t_n, t_{n+1}))|^p \\ &\leq ||\eta(S_j^+ \times (t_n, t_{n+1}))| + |\eta(S_j^- \times (t_n, t_{n+1}))||^p \\ &\leq |\eta(S_j^+ \times (t_n, t_{n+1}))| + |\eta(S_j^- \times (t_n, t_{n+1}))|, \end{aligned}$$

we arrive at

$$\begin{aligned} \mathbb{E}|I(\xi)|_E^q &\leq \\ &\mathbb{E} \left(\sum_{n=1}^N \sum_{j=1}^J \sum_{i=1}^I |1_{\Omega_{i,j,n}} x_i|^p (|\eta(S_j^+ \times (t_n, t_{n+1}))| + |\eta(S_j^- \times (t_n, t_{n+1}))|) \right)^{\frac{q}{p}}. \end{aligned}$$

Substituting the definition of $I(\xi)$ we get the assertion. \square

Finally, if $p = 1$, using Theorem A.1 one gets the following Lemma.

Lemma 4.6. *Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be a non decreasing, convex and continuous function with $\Phi(0) = 0$ such that there exists a strictly increasing and nonnegative function $\phi : [0, \infty) \rightarrow \mathbb{R}$ with $\Phi(t) := \int_0^t \phi(s) ds$. Additionally, there exists a constant $c > 0$ with*

$$\Phi(2\lambda) \leq c\Phi(\lambda), \quad \lambda \in [0, \infty). \quad (4.10)$$

We denote the smallest constant satisfying (4.10) by c_Φ . Let E be a separable Banach space. Then, there exists a constant C , only depending on Φ , such that for all $\xi \in \mathcal{M}_{\text{step}}^1([0, \infty); L_{\text{step}}^p(Z, \nu; E) \cap L_{\text{sym}}^p(Z, \nu; E)) \cap \mathcal{M}^{c_\Phi}([0, \infty); L_{\text{sym}}^p(Z, \nu; E))$ the following inequality holds

$$\mathbb{E} \Phi \left(\sup_{0 \leq t \leq T} |\tilde{I}(\xi)|_E \right) \leq C \mathbb{E} \Phi \left(\int_0^t \int_Z |\xi(s, z)| \eta(dz, ds) \right).$$

Proof. In order to show Lemma 4.6 we start as in Lemma 4.5. The difference is that in Lemma 4.6 we use Theorem A.1 instead of inequality 2.2. \square

Note that Lemma 4.6 does not imply

$$\mathbb{E} \sup_{0 \leq t \leq T} |\tilde{I}(\xi)|_E \leq C \mathbb{E} \int_0^t \int_Z |\xi(s, z)| \eta(dz, ds).$$

To give an estimate of $\sup_{0 \leq t \leq T} |\tilde{I}(\xi)|_E$ we use Theorem 1 of [12].

Corollary 4.7. *Assume that E is a Banach space E of martingale type 1. Then, there exists a constant $C > 0$ such that the following inequality holds*

$$\mathbb{E} \sup_{0 \leq t \leq T} |\tilde{I}(\xi)|_E \leq C \mathbb{E} \int_0^t \int_Z |\xi(s, z)| \eta(dz, ds).$$

Proof of Lemma 4.7. Again the proof of Corollary 4.7 we start as in Lemma 4.5. The difference is that in 4.6 we use inequality A.1 instead of inequality 2.2. \square

Appendix A. Doob's Maximal Inequality

In this section we will verify the question if a maximal inequality holds. First, note that Doob's maximal inequality in the classical sense holds. To be precise, for any non-negative real valued submartingale $\{M_n\}_{n=0}^N$ and any $\lambda > 0$ we have

$$\lambda \mathbb{P}(M_n^* \geq \lambda) \leq \mathbb{E}(M_n 1_{\{M_n^* \geq \lambda\}}).$$

where $M_n^* = \max_{0 \leq k \leq n} M_k$, $n = 0, \dots, N$. Analysing the proof of Doob's maximal L^2 inequality one sees that the proof cannot be transferred to $L^1(\Omega)$. Nevertheless, for $1 \leq r < \infty$ we get following version by setting $\tilde{M}_n := |M_n|^r$

$$\lambda^r \mathbb{P}((M_n^*)^r \geq \lambda) \leq \mathbb{E}(|M_n|^r),$$

an inequality which is used by Davis, Burkholder and Gundy to derive the Davis Burkholder Gundy inequality.

We are interested if a version of the Burkholder Davis Gundy inequality holds also for $p \in (0, 1]$ similarly to Theorem A.5 of [16]. Analysing the proof (e.g. in [16] one sees that in the Proof of Theorem A.5 it is essential that $p \geq 1$ (to be precise see the third line after equation (A.16))). On the other side, verifying step by step one sees that the case $p = 1$ is covered. That is, the following holds.

Theorem A.1. Let $\Phi : [0, \infty) \rightarrow \mathbb{R}$ be a non decreasing, convex and continuous function with $\Phi(0) = 0$ such that there exists a strictly increasing and nonnegative function $\phi : [0, \infty) \rightarrow \mathbb{R}$ with $\Phi(t) := \int_0^t \phi(s) ds$. Additionally, there exists a constant $c > 0$ with

$$\Phi(2\lambda) \leq c\Phi(\lambda), \quad \lambda \in [0, \infty).$$

Assume that E is a Banach space. Then, there exists a constant $C = C(E, \phi) > 0$ such that for all E -valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\mathbb{E}\Phi\left(\sup_{0 \leq n \leq N} |M_n|_E\right) \leq C \mathbb{E}\Phi\left(\sum_{n=0}^N |M_n - M_{n-1}|_E\right),$$

where as usually, we put $M_{-1} = 0$.

Proof. The proof can be found by tracing the proof of Theorem A.5 in [16] and verifying at each step if p can be chosen to be one. \square

The linear function $x \mapsto x$ does not satisfy the assumption of Theorem A.1. However, analysing the proof of Theorem 1 [12], one sees that the Theorem can be transferred to Banach spaces of martingale type 1. In particular one gets

Theorem A.2. Assume that E is a Banach space E . Then, there exists a constant $C = C(E, \phi) > 0$ such that for all E -valued finite martingale $\{M_n\}_{n=0}^N$ the following inequality holds

$$\mathbb{E} \sup_{0 \leq n \leq N} |M_n|_E \leq C \mathbb{E} \sum_{n=0}^N |M_n - M_{n-1}|_E, \quad (\text{A.1})$$

where as usually, we put $M_{-1} = 0$.

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