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SUFFICIENT CONDITIONS OF OPTIMALITY FOR BACKWARD STOCHASTIC EVOLUTION EQUATIONS

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Abstract. In this paper we derive for a controlled stochastic evolution system on Hilbert space H a sufficient condition for optimality. The result is derived by using its adjoint backward stochastic evolution equation.

1. Introduction

In this work we consider the following controlled stochastic system on a separable Hilbert space H:

$$\begin{cases} dX(t) = (\, A\,X(t) + G(X(t)) + B\,\nu(t)\,)\; dt + \; \mathcal{Q}^{1/2}\; dW(t), \quad t \in [0,T], \\ X(0) = x \in H, \end{cases}$$

where A is an unbounded linear operator on H, W is a cylindrical Wiener process on H and $\nu(\cdot)$ represents a control taking its values in a separable Hilbert space U and B is a bounded operator from U into H. We are concerned with minimizing the cost functional, which is defined by the equation (3.2) below, over the set of admissible controls. Our aim is to apply the theory of backward stochastic evolution equations (see [3]) to obtain a sufficient condition of optimality which gives us the minimization.

Such a problem was studied by many authors; among those are [6], [12] and [4] and all related references therein. One can see also [10] for the deterministic case. There are variety of methods in the literature for studying such stochastic infinite dimensional systems. They include studying Hamilton-Jacobi-Bellman equation by using semi-group techniques or the techniques of viscosity solutions. We refer the reader also to [8], [13], [11], [12] and [14] for the applications of backward stochastic differential equations in optimal control.

In the present paper we focus our attention to apply our earlier results in [3] of existence and uniqueness of solutions of the adjoint equation to derive a minimum principle for the above system. We shall provide a sufficient condition for optimality of the control $\nu(\cdot)$ and the corresponding solution $X(\cdot, \nu(\cdot))$.

The paper is organized as follows. Section 2 contains the required information on backward stochastic evolution equations. We establish our control problem and prove the main results in Section 3. Some examples are given in Section 4.

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2. Backward Stochastic Evolution Equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and denote by $\mathcal{N}$ the collection of $\mathbb{P}$- null sets of $\mathcal{F}$. Let $\{W(t),\ 0 \leq t \leq T\}$ be a cylindrical Wiener process on H with its completed natural filtration $\mathcal{F}_t = \sigma\{\ell \circ W(s)\,,\ 0 \leq s \leq t\,, \ell \in H^*\} \vee \mathcal{N},\ t \geq 0$; cf. [1].

For a separable Hilbert space $\tilde{H}$ let $L^2_{\mathcal{F}}(0,T;\tilde{H})$ denote the space of all $\{\mathcal{F}_t, 0 \leq t \leq T\}$- progressively measurable processes $\tilde{f}$ with values in $\tilde{H}$ such that

$$\mathbb{E}\,[\int_0^T |\tilde{f}(s)|^2_{\tilde{H}}\ ds] < \infty.$$

Thus $L^2_{\mathcal{F}}(0,T;\tilde{H})$ is a Hilbert space with the norm

$$||\tilde{f}||\ = \big(\mathbb{E}\,[\int_0^T |\tilde{f}(s)|^2_{\tilde{H}}\ ds]\big)^{1/2}.$$

It is known that if f evolves in the space $L_2(H)$ of all Hilbert-Schmidt operators on H, and lies in $L^2_{\mathcal{F}}(0,T;L_2(H))$ we can define the stochastic integrals $\int f(s)dW(s)$; for example see [5].

Let us now consider the following equation:

$$\begin{cases} -\,dY(t) = (A\,Y(t) + f(t,Y(t),Z(t)))\ dt - Z(t)\ dW(t), \\ \quad Y(T) = \xi. \end{cases} \tag{2.1}$$

This equation is called *a backward stochastic evolution equation* and is denoted shortly by BSEE. The operator A will be assumed to be the infinitesimal generator of a C_0- semigroup $\{S(t),\ t \geq 0\}$.

Definition 2.1. A *mild* solution (or simply a solution) of the BSEE (2.1) is a pair (Y,Z) in $L^2_{\mathcal{F}}(0,T;H) \times L^2_{\mathcal{F}}(0,T;L_2(H))$ such that the following equality holds $\mathbb{P}$- a.s.

$$\begin{aligned} Y(t) \quad &= \quad S(T)\,\xi + \int_t^T S(s-t)\ f(s,Y(s),Z(s))\ ds \\ &\quad - \int_t^T S(s-t)\ Z(s)\ dW(s),\ 0 \leq t \leq T. \end{aligned} \tag{2.2}$$

For existence and uniqueness of the solutions of these BSEEs we impose the following assumptions.

- (H1) f is a mapping from $[0,T] \times \Omega \times H \times L_2(H)$ to H that is $\mathcal{P} \otimes \mathcal{B}(H) \otimes \mathcal{B}(L_2(H))/\mathcal{B}(H)$- measurable and satisfies

$$f(\cdot,0,0) \in L^2_{\mathcal{F}}(0,T;H),$$

where $\mathcal{P}$ is the σ- algebra of all $\mathcal{F}_*$- progressively measurable subsets of $[0,T] \times \Omega$.

- (H2) $\exists\, k > 0$ such that $\forall\ y, y' \in H$ and $\forall\ z, z' \in L_2(H)$

$$|f(t,y,z) - f(t,y',z')|^2_H \leq c(|y-y'|^2_H) + k\ |z-z'|^2_{L_2(H)}\,,$$

for a.e. $(\omega, t) \in \Omega \times [0,T]$, where c is a concave nondecreasing continuous function from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that $c(0) = 0, c(x) > 0$ for all $x > 0$ and

$$\int_{0^+}^{a} \frac{dx}{c(x)} = \infty,$$

for any sufficiently small (and so for all) $a > 0$.

Theorem 2.2. *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ be given. Assume that f satisfies (H1) and (H2). Then there exists a unique solution (Y, Z) of (2.1) in $L^2_{\mathcal{F}}(0,T;H) \times L^2_{\mathcal{F}}(0,T;L_2(H))$.*

The proof of this theorem can be found in [3]. One can also see [7] for the case when the coefficients satisfy a Lipchitz condition. In [2] we studied the regularity of such mild solutions. In particular it was shown under certain conditions that a weak solution and a strong solution exist for the equation (2.1). In the next section we shall study the case when we install a control in that equation (2.1).

3. Main Results

Let $G : H \to H$ be a continuous mapping and $\mathcal{Q}$ a symmetric nonnegative nuclear operator on H. Consider the following controlled system:

$$\begin{cases} dX(t) = (\, A\, X(t) + G(X(t)) + B\, \nu(t)\,)\, dt + \, \mathcal{Q}^{1/2}\, dW(t), \quad t \in [0,T], \\ X(0) = x \in H, \end{cases} \tag{3.1}$$

where $\nu(\cdot) \in \mathcal{U}_{ad}$ represents the control. This set $\mathcal{U}_{ad}$ of *admissible controls* consists of all progressively measurable square-integrable processes $\nu : [0,T] \times \Omega \to U$, i.e. $\nu \in L^2_{\mathcal{F}}(0,T;U)$. A solution of (3.1) will be denoted by X^{ν} to indicate the presence of the control.

Denote by $C^1_b(K_1\,; K_2)$, where K_i, $i = 1, 2$, are separable Hilbert spaces, to the space of all continuous and bounded real-valued functions defined on K_1 and take values in K_2, which are Fréchet differentiable on K_1 with a continuous and bounded derivative. We shall use $C^1_b(K_1)$ for $C^1_b(K_1\,; \mathbb{R})$.

Let g and ϕ be two elements of the space $C^1_b(H)$. Assume now that we are given a convex C^1_b function $h : U \to [0,\infty)$ and wish to minimize the *cost functional*

$$J(\nu(\cdot)) := \mathbb{E}\, [\, \int_0^T (\, g(X^{\nu}(t)) + h(\nu(t)\,)\,)\, dt + \phi(X^{\nu}(T))\,] \tag{3.2}$$

over all admissible controls. In particular the optimal control problem of the system (3.1) is to find the *value function* J^* and *an optimal control* $\nu^*(\cdot) \in \mathcal{U}_{ad}$ such that

$$J^* := \inf\{ J(\nu(\cdot)) :\ \nu(\cdot) \in \mathcal{U}_{ad} \} = J(\nu^*(\cdot)), \tag{3.3}$$

in which case the corresponding solution X^{ν^*} is called *an optimal solution* of the stochastic control problem (3.1), (3.3) and $(X^{\nu^*}, \nu^*(\cdot))$ is called *an optimal pair.*

Let us now recall the following result.

Theorem 3.1. *Assume that A is an unbounded linear operator on H that generates a C_0-semigroup $\{S(t),\ t \geq 0\}$ on H and $G \in C^1_b(H; H)$. Let $\nu(\cdot) \in \mathcal{U}_{ad}$.*

Then (3.1) *has a unique mild solution* X^ν *on* $[0,T]$. *That is* X^ν *is a progressively measurable stochastic process such that* $X(0)=x$ *and for all* $t\in[0,T]$,

$$\begin{aligned} X^\nu(t) &= S(t)\,x + \int_0^t S(t-s)\,B\,\nu(s)\,ds \\ &\quad + \int_0^t S(t-s)\,G(X^\nu(s))\,ds + \int_0^t S(t-s)\,\mathcal{Q}^{1/2}\,dW(s). \end{aligned} \tag{3.4}$$

The proof of this theorem can be derived in a similar way to those in [6, Chapter 7] or [9].

As it is known that backward stochastic differential equations play an important role in deriving the maximum (or the minimum) principle of SDEs, it is natural to search for such a role for the equation (3.1) by dealing with its adjoint equation. For this we define the *Hamiltonian*: $\mathcal{H}: H\times U\times H\times L_2(H)\to\mathbb{R}$ by

$$\mathcal{H}(x,\nu,y,z) := g(x) + \big\langle G(x),y\big\rangle_H + h(\nu) + \big\langle B\,\nu,y\big\rangle_H + \big\langle \mathcal{Q}^{1/2},z\big\rangle_{L_2(H)}. \tag{3.5}$$

And consider the following *adjoint BSEE* on H:

$$\begin{cases} -\,dY^\nu(t) = & [\,A^*\,Y(t) + \nabla_x\mathcal{H}(X^\nu(t),\nu(t),Y^\nu(t),Z^\nu(t))\,]\,dt \\ & \qquad -Z(t)\,dW(t), \quad 0\le t\le T, \\ \;\; Y^\nu(T) = & \nabla\phi(X^\nu(T))\in H, \end{cases} \tag{3.6}$$

where $\nabla\phi$ denotes the gradient of ϕ, which is defined, by using the directional derivative $D\phi(x)(h)$ of ϕ at a point $x\in H$ in the direction of $h\in H$, as $\big\langle\nabla\phi(x),h\big\rangle_H = D\phi(x)(h)$. Here $A^*:\mathcal{D}(A^*)\subset H\to H$ denotes the adjoint operator of A, which is the infinitesimal generator of the adjoint semigroup $\{S^*(t),\ t\ge 0\}$ of $\{S(t)\,,t\ge 0\}$. Thus in particular a mild solution of (3.6) is a pair $(Y,Z)\in L^2_{\mathcal{F}}(0,T;H)\times L^2_{\mathcal{F}}(0,T;L_2(H))$ such that we have $\mathbb{P}$-a.s. for all $t\in[0,T]$

$$\begin{aligned} Y^\nu(t) &= S^*(T-t)\,\nabla\phi(X^\nu(T)) \\ &\quad + \int_t^T S^*(s-t)\,[\,\nabla_x\mathcal{H}(X^\nu(t),\nu(t),Y^\nu(t),Z^\nu(t))\,]\,ds \\ &\quad - \int_t^T S^*(s-t)\,Z(s)\,dW(s). \end{aligned} \tag{3.7}$$

We now state our main result.

Theorem 3.2. *Assume that* $\nu^*(\cdot)$ *is an admissible control such that the solutions* X^{ν^*} *and* (Y^{ν^*},Z^{ν^*}) *of the corresponding equations (3.1) and (3.6) respectively exist. Assume that*

(i) ϕ *is convex,*
(ii) $\phi,g\in C_b^1(H),\ h\in C_b^1(U),\ G\in C_b^1(H;H)$,
(iii) $\mathcal{H}(\cdot,\cdot,Y^{\nu^*}(t),Z^{\nu^*}(t))$ *is convex for all* $t\in[0,T]$, $\mathbb{P}$ *- a.s.,*
(iv) $\mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t)) = \inf_{\nu\in U}\mathcal{H}(X^{\nu^*}(t),\nu,Y^{\nu^*}(t),Z^{\nu^*}(t))$ *for a.e.* $t\in[0,T]$, $\mathbb{P}$ *- a.s.*

Then $(X^{\nu^*},\nu^*(\cdot))$ *is an optimal pair for the problem (3.1), (3.3) .*

Proof. Let $\nu(\cdot)$ be an arbitrary admissible control. From the definitions in (3.3) and (3.2) we obtain

$$\begin{aligned} J(\nu^*(\cdot)) - J(\nu(\cdot)) &= \inf_{\nu(\cdot)\in\mathcal{U}_{ad}} J(\nu(\cdot)) - J(\nu(\cdot)) \\ &= \mathbb{E}\,[\int_0^T (g(X^{\nu^*}(t)) + h(\nu^*(t)))\,dt + \phi(X^{\nu^*}(T))\,] \\ &\quad - \mathbb{E}\,[\int_0^T (g(X^{\nu}(t)) + h(\nu(t)))dt + \phi(X^{\nu}(T))\,] \\ &= \mathbb{E}\,[\int_0^T (g(X^{\nu^*}(t)) - g(X^{\nu}(t)))dt\,] + \mathbb{E}\,[\int_0^T (h(\nu^*(t)) - h(\nu(t)))dt\,] \\ &\quad + \mathbb{E}\,[\,\phi(X^{\nu^*}(T)) - \phi(X^{\nu}(T))\,]. \end{aligned} \tag{3.8}$$

But

$$\begin{aligned} g(X^{\nu^*}(t)) - g(X^{\nu}(t)) &= \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) \\ &\quad - \mathcal{H}(X^{\nu}(t), \nu(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) \\ &\quad - \big\langle G(X^{\nu^*}(t)) - G(X^{\nu}(t))\,, Y^{\nu^*}(t) \big\rangle \\ &\quad - \big\langle B\,\nu^*(t) - B\,\nu(t)\,, Y^{\nu^*}(t) \big\rangle \\ &\quad - [\,h(\nu^*(t)) - h(\nu(t))\,]\;\; a.s. \end{aligned}$$

Therefore (3.8) becomes

$$\begin{aligned} J(\nu^*(\cdot)) - J(\nu(\cdot)) &= \mathbb{E}\,[\int_0^T (\,\mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t)) \\ &\quad - \mathcal{H}(X^{\nu}(t), \nu(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\,) \\ &\quad - \big\langle G(X^{\nu^*}(t)) - G(X^{\nu}(t))\,, Y^{\nu^*}(t) \big\rangle \\ &\quad - \big\langle B\,\nu^*(t) - B\,\nu(t)\,, Y^{\nu^*}(t) \big\rangle\,]\,dt \\ &\quad + \mathbb{E}\,[\,\phi(X^{\nu^*}(T)) - \phi(X^{\nu}(T))\,]. \end{aligned} \tag{3.9}$$

By the convexity assumption on ϕ in (i) we get

$$\phi(X^{\nu^*}(T)) - \phi(X^{\nu}(T)) \le \big\langle\, \nabla\phi(X^{\nu^*}(T))\,, X^{\nu^*}(T) - X^{\nu}(T)\,\big\rangle\; a.s. \tag{3.10}$$

Then, since $\nabla\phi(X^{\nu^*}(T)) = Y^{\nu^*}(T)$, we deduce that

$$\mathbb{E}\,[\,\phi(X^{\nu^*}(T)) - \phi(X^{\nu}(T))\,] \le \mathbb{E}\,[\,\big\langle\, Y^{\nu^*}(T)\,, X^{\nu^*}(T) - X^{\nu}(T)\,\big\rangle\,]. \tag{3.11}$$

Next let $\psi_1(t) := G(X^{\nu^*}(t)) - G(X^{\nu}(t)) + B\,(\nu^*(t) - \nu(t))$, where $t \in [0,T]$. Then thanks to (ii) and (3.1) we see that $\psi_1 \in L^2_{\mathcal{F}}(0,T;H)$.

Now multiply the adjoint equation(3.7) by $\psi_1(t)$, integrate with respect to $t \in [0,T]$, take the expectation and use stochastic Fubini's theorem to get the following result.

$$
\begin{aligned}
&\mathbb{E}\,[\int_0^T \langle\, Y^{\nu^*}(t)\,,\psi_1(t)\,\rangle\,dt\,]\\
&\quad=\mathbb{E}\,[\int_0^T \langle\, S^*(T-t)\,Y^{\nu^*}(T)\,,\psi_1(t)\,\rangle\,dt\,]\\
&\qquad+\mathbb{E}\,[\int_0^T \langle\, \int_t^T S^*(s-t)\,\nabla_x\mathcal{H}(X^{\nu^*}(s),\nu^*(s),Y^{\nu^*}(s),Z^{\nu^*}(s))\,,\psi_1(t)\,\rangle\,dt\,]\\
&\quad=\mathbb{E}\,[\,\langle\, Y^{\nu^*}(T)\,,\int_0^T S(T-t)\,\psi_1(t)\,dt\,\rangle\,]\\
&\qquad+\mathbb{E}\,[\int_0^T \langle\, \nabla_x\mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t))\,,\int_0^t S^*(t-s)\,\psi_1(s)\,ds\,\rangle\,dt\,]\\
&\quad=\mathbb{E}\,[\,\langle\, Y^{\nu^*}(T)\,,X^{\nu^*}(T)-X^{\nu}(T)\,\rangle\,]\\
&\qquad+\mathbb{E}\,[\int_0^T \langle\, \nabla_x\mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t))\,,X^{\nu^*}(t)-X^{\nu}(t)\,\rangle\,dt\,].
\end{aligned}
$$

We have used here

$$
\begin{aligned}
X^{\nu^*}(t)-X^{\nu}(t) \quad = \quad & \int_0^t S(t-s)\,[\,(\,G(X^{\nu^*}(s))-G(X^{\nu}(s))\,)\\
&+(\,\nu^*(s)-\nu(s)\,)\,]\,ds, \qquad 0\le t\le T.
\end{aligned}
$$

Thus in particular

$$
\begin{aligned}
&\mathbb{E}\,[\,\langle\, Y^{\nu^*}(T)\,,X^{\nu^*}(T)-X^{\nu}(T)\,\rangle\,]=\mathbb{E}\,[\int_0^T \langle\, Y^{\nu^*}(t)\,,\psi_1(t)\,\rangle\,dt\,]\\
&\quad-\mathbb{E}\,[\int_0^T \langle\, \nabla_x\mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t))\,,X^{\nu^*}(t)-X^{\nu}(t)\,\rangle\,dt\,]. \qquad (3.12)
\end{aligned}
$$

Now by applying (3.8), (3.11) and this fact (3.12) we see actually that

$$
\begin{aligned}
J(\nu^*(\cdot))-J(\nu(\cdot)) &\le \mathbb{E}\,[\int_0^T \langle\, Y^{\nu^*}(t)\,,\psi_1(t)\,\rangle\,dt\,]\\
&\quad-\mathbb{E}\,[\int_0^T \langle\, \nabla_x\mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t))\,,X^{\nu^*}(t)-X^{\nu}(t)\,\rangle\,dt\,]\\
&\qquad\qquad+\mathbb{E}\,[\int_0^T \psi_2(t)dt\,], \qquad (3.13)
\end{aligned}
$$

where, for $t\in[0,T]$,

$$
\psi_2(t)=\delta\mathcal{H}(t)+\langle\,-\psi_1(t)\,,Y^{\nu^*}(t)\,\rangle
$$

and

$$
\begin{aligned}
\delta\mathcal{H}(t) \quad = \quad & \mathcal{H}(X^{\nu^*}(t),\nu^*(t),Y^{\nu^*}(t),Z^{\nu^*}(t))\\
&-\mathcal{H}(X^{\nu}(t),\nu(t),Y^{\nu^*}(t),Z^{\nu^*}(t)).
\end{aligned}
$$

This implies that

$$J(\nu^*(\cdot)) - J(\nu(\cdot)) \leq \mathbb{E}\,[\, \int_0^T \delta\mathcal{H}(t)\; dt \,]$$
$$- \mathbb{E}\,[\, \int_0^T \left\langle\, \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\; , X^{\nu^*}(t) - X^{\nu}(t) \,\right\rangle dt \,]. \quad (3.14)$$

From the convexity of $\mathcal{H}$ in the condition (iii) we see that the following inequality holds for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s.

$$\begin{aligned}\delta\mathcal{H}(t) \quad \leq \quad & \left\langle\, \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\;,\; X^{\nu^*}(t) - X^{\nu}(t) \,\right\rangle \\ & + \left\langle\, \nabla_\nu \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\;,\; \nu^*(t) - \nu(t) \,\right\rangle_U .\end{aligned}$$

But the minimum condition (iv) implies

$$\left\langle\, \nabla_\nu \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\;,\; \nu^*(t) - \nu(t) \,\right\rangle_U \leq 0$$

for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s.; see e.g. [10]. Consequently, for a.e. $t \in [0,T]$, $\mathbb{P}$-a.s.,

$$\delta\mathcal{H}(t) \;-\left\langle\, \nabla_x \mathcal{H}(X^{\nu^*}(t), \nu^*(t), Y^{\nu^*}(t), Z^{\nu^*}(t))\;,\; X^{\nu^*}(t) - X^{\nu}(t) \,\right\rangle \;\leq 0.$$

By applying this result in (3.14) we deduce finally that $J(\nu^*(\cdot)) \leq J(\nu(\cdot))$. This completes the proof. □

4. Examples

We shall introduce here some examples to illustrate the result of the previous section.

Example 4.1. Keeping the notations used earlier we shall study a special case when the mapping $G = 0$ and in particular we shall consider the following controlled SEE:

$$\begin{cases} dX^{\nu}(t) = (\, A\, X^{\nu}(t) + B\, \nu(t)\,)\; dt + \;\mathcal{Q}^{1/2}\; dW(t), \;\; t \in [0,T], \\ \;X^{\nu}(0) = x \in H. \end{cases} \quad (4.1)$$

The solution of this equation is given through Theorem 3.1 by the formula (3.4) but obviously without the second integral.

Given $\phi \in C_b^1(H)$ as in Theorem 3.2 let us consider the cost functional as follows:

$$J(\nu(\cdot)) = \mathbb{E}\,[\, \int_0^T |\nu(t)|_U^2\; dt \,] + \mathbb{E}\,[\, \phi(X^{\nu}(T))\,]. \quad (4.2)$$

This is similar to (3.2) when we set the functions $g(x) = 0$ and $h(\nu) = |v|_U^2$ for $x \in H$ and $\nu \in U$. We define the value function as

$$J^* = \inf\{J(\nu(\cdot)) :\; \nu(\cdot) \in \mathcal{U}_{ad}\}. \quad (4.3)$$

Then the Hamiltonian is defined by

$$\mathcal{H}(x, \nu, y, z) = |\nu|_U^2 + \left\langle B\,\nu\,, y\right\rangle_H + \left\langle \mathcal{Q}^{1/2}\,, z\right\rangle_{L_2(H)},$$

where $(x, \nu, y, z) \in H \times U \times H \times L_2(H)$.

Now the adjoint BSEE becomes

$$\begin{cases} -\,dY^{\nu}(t) = & A^*\, Y^{\nu}(t) - Z^{\nu}(t)\; dW(t), \quad t \in [0,T], \\ \quad Y^{\nu}(T) = & \nabla\phi(X^{\nu}(T)). \end{cases}$$

This equation attains an explicit solution given by the pair

$$Y^{\nu}(t) = \mathbb{E}\,[\; S^{*}(T-t)\,\nabla\phi(X^{\nu}(T)) \mid \mathcal{F}_t \;]$$

and

$$Z^{\nu}(t) = S^{*}(T-t)\,R^{\nu}(t).$$

Here $S^{*}(\cdot)$ is the C_0-semigroup generated by A^{*} and R^{ν} is the unique process in $L^2_{\mathcal{F}}(0,T;L_2(H))$ such that

$$\nabla\phi(X^{\nu}(T)) = \mathbb{E}\,[\; \nabla\phi(X^{\nu}(T)) \;] + \int_0^T R^{\nu}(t)\,dW(t).$$

We refer the reader to [3, Lemma 3.1] for more details.

Note that for fixed (x,y,z), the function $\nu \mapsto H(x,\nu,y,z)$ attains its minimum at $\nu = \frac{1}{2}\,B^{*}\,y\ (\in U\,)$. So we take

$$\nu^{*}(t,\omega) = \frac{1}{2}\,B^{*}\,Y^{\nu^{*}}(t,\omega) = \frac{1}{2}\,B^{*}\,\mathbb{E}\,[\; S^{*}(T-t)\,\nabla\phi(X^{\nu^{*}}(T)) \mid \mathcal{F}_t \;](\omega)$$

as a candidate optimal control.

It is easy to see that with these choices all the requirements of Theorem 3.2 are verified. Hence this candidate $\nu^{*}(\cdot)$ is indeed an optimal control for the problem (4.1)-(4.3) and it corresponding optimal solution $X^{\nu^{*}}$ is the solution of the following SEE:

$$\begin{cases} dX^{\nu^{*}}(t) = (A\,X^{\nu^{*}}(t) + \frac{1}{2}\,B\,B^{*}\,Y^{\nu^{*}}(t))\,dt + \mathcal{Q}^{1/2}\,dW(t), \quad t \in [0,T], \\ X^{\nu^{*}}(0) = x. \end{cases} \tag{4.4}$$

The value function can then be computed by using the formula

$$J^{*} \;\; = \;\; \frac{1}{4}\,\mathbb{E}\,[\int_0^T |\,B^{*}\,Y^{\nu^{*}}(t)|^2_U\;dt\,] + \mathbb{E}\,[\,\nabla\phi(X^{\nu^{*}}(T))\,].$$

Remark 4.2. If in the above example X is deterministic (e.g. when $\mathcal{Q}=0$), then using the above formula of R^{ν}, Y^{ν} and Z^{ν} we deduce immediately that $Z^{\nu}(t)=0$ for each t and Y^{ν} (or just written as Y) is also deterministic. The following example treats a case where Y is deterministic although X is not.

The above example or rather our main result (Theorem 3.2) are in the direction of generalizing some of the deterministic control problems, e.g. as in [10].

Example 4.3. Let $H = U = L^2(\mathbb{R}^d),\ d \geq 1$. Consider the SEE (4.1) with $A = \frac{1}{2}\Delta$, half-Laplacian, and $B = id_H$. Suppose that for $x \in H$, $\phi(x) = \big\langle \ell\,, x \big\rangle_H$, where ℓ is a given fixed element of H. We shall let the following cost functional be

$$J(x,\nu(\cdot)) = \mathbb{E}\,[\int_0^T |\nu(t)|^2_U\;dt\,] + \mathbb{E}\,[\;\big\langle \ell\,, X^{\nu}(T) \big\rangle_H\;]. \tag{4.5}$$

By doing the same business as in Example 4.1 we find that the adjoint BSEE is the following.

$$\begin{cases} -\,dY^{\nu}(t) = & \frac{1}{2}\Delta\,Y^{\nu}(t) - Z^{\nu}(t)\,dW(t), \quad t \in [0,T], \\ Y^{\nu}(T) = & \ell. \end{cases}$$

Since $Y^\nu(T)$ is not random we can choose $Z^\nu(t) = 0$ for each $t \in [0, T]$. That is we are in charge of the deterministic heat equation:

$$\begin{cases} \frac{\partial}{\partial t} Y(t) = -\frac{1}{2} \Delta Y(t) \\ Y(T) = \ell. \end{cases} \tag{4.6}$$

For this reason we dropped off here the dependence of Y on ν and wore merely Y instead Y^ν.

The solution of (4.6) takes actually the formula:

$$Y(t) = S(T-t)\,\ell, \tag{4.7}$$

where

$$(S(r)\,\ell)(\rho) = (2\pi r)^{-d/2} \int_{\mathbb{R}^d} \ell(w)\, e^{(\frac{-|\rho - w|^2}{2r})}\, dw\,, \quad \rho \in \mathbb{R}^d, r > 0.$$

Continuing as in Example 4.1 we get an optimal control given by the identity

$$\nu^*(t) = \frac{1}{2} S(T-t)\,\ell \ \ (= \frac{1}{2} Y(t)\,)$$

and an optimal solution being the solution of the equation (4.4) with $A = \frac{1}{2}\Delta$ and Y as in (4.7).

Finally we mention the value function for convenience:

$$J^* \ = \ \frac{1}{4} \int_0^T |\, S(T-t)\,\ell\,|_H^2\; dt + \mathbb{E}\,[\,\left\langle \ell\,, \phi(X^{\nu^*}(T))\right\rangle_H\,].$$

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