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SURFACE MEASURES ON THE DUAL SPACE OF THE SCHWARTZ SPACE

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Abstract. We adopt the Airault–Malliavin method to construct surface measures within the framework of white noise analysis for surfaces in the dual space of the Schwartz space. The main ingredient of our construction is the integral representation theorem of positive generalized functions. We also give some examples to illustrate the precise description of surface measures in terms of the Laplace transform.

1. Introduction

In 1972 Goodman [4] gave the first construction of surface measures on infinite dimensional spaces and proved an infinite dimensional divergence theorem (see also [6]). This construction is rather complicated and difficult to compute.

In 1974 Skorohod [13] gave another approach to construct surface measures in infinite dimensional Hilbert space, which was further developed by Uglanov [15, 16]. This approach is based on the construction of a local surface measure on sufficiently small neighborhoods of the points on the underlying surface. Later Pugachev [11] generalized Uglanov's method.

In 1988 Airault and Malliavin [1] introduced a new method of constructing surface measures for the Gaussian case. This method was improved by Bogachev [2] and later generalized by Pugachev [12] to construct surface measures for the non-Gaussian case.

One disadvantage of Skorohod's approach is the fact that the existence of local surface measure may not yield the existence of a global surface measure. On the contrary, the Airault–Malliavin method produces a measure on the entire surface. We point out that an important ingredient in this method is the integral representation of positive distributions by measures [14, 17]).

In this paper we will construct surface measures within the framework of white noise analysis for nuclear spaces. Similar to the Airault–Malliavin method, our main tool is the integral representation of positive white noise distributions by measures as obtained in [10]. In Section 2 we will recall basic spaces for our framework. In Section 3 we construct surface measures on the dual space of the Schwartz space, i.e., the space of tempered distributions. The main results are given by Theorems 3.2 and 3.4. In sections 4 and 5 we will study surface measures

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on hyperplanes and quadric surfaces, respectively. Finally we will study in Section 6 surface measures on infinite dimensional spheres.

2. Basic Spaces of the Framework

The infinite dimensional space on which we will construct surface measures is the dual space $S'(\mathbb{R})$ of the Schwartz space $S(\mathbb{R})$. The basic tool of our construction is the representation in Theorem 2.1 for positive generalized functions defined on the space $S'(\mathbb{R})$. We will briefly review these basic spaces below. For detail, see the paper [3] and the book [7].

2.1. The Schwartz space and its dual space. A function ξ on $\mathbb{R}$ is called *rapidly decreasing* if it is a smooth function such that $|t^n\xi^{(k)}(t)| \to 0$ as $|t| \to \infty$ for any positive integers n and k. The space $S(\mathbb{R})$ consisting of all rapidly decreasing functions is called the *Schwartz space* on $\mathbb{R}$. It is a Fréchet space with the following family of norms

$$|\xi|_{n,k} = \left(\int_{\mathbb{R}} |t^n\xi^{(k)}(t)|^2 dt\right)^{\frac{1}{2}},$$

where $n, k \in \mathbb{N}$. On the other hand, let $H_n(x) = (-1)^n e^{x^2}(d/dx)^n e^{-x^2}$ be the Hermite polynomial of degree $n \geq 0$ and let

$$e_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\, 2^n n!}} H_n(x) e^{-x^2/2}, \quad n \geq 0, \tag{2.1}$$

be the Hermite functions. It is a well-known fact that the collection $\{e_n; n \geq 0\}$ is an orthonormal basis for the Hilbert space $\mathcal{L}^2(\mathbb{R}, dt)$.

Now, let $f \in \mathcal{L}^2(\mathbb{R}, dt)$. For each $p \geq 0$, define

$$|f|_p = \left(\sum_{n=0}^{\infty} (2n+2)^{2p} (f, e_n)^2\right)^{1/2}, \tag{2.2}$$

where $(\cdot\,,\cdot)$ is the inner product of $\mathcal{L}^2(\mathbb{R}, dt)$. Define

$$S_p(\mathbb{R}) = \big\{f \in \mathcal{L}^2(\mathbb{R}, dt);\ |f|_p < \infty\big\}.$$

Then we have an increasing sequence $\{S_p(\mathbb{R}); p \in \mathbb{N}\}$ of Hilbert spaces such that the inclusion mapping $S_{p+1}(\mathbb{R}) \hookrightarrow S_p(\mathbb{R})$ is Hilbert Schmidt operator. It follows that the projective limit

$$\operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}) := \bigcap_{p\geq 0} S_p(\mathbb{R})$$

being equipped with the projective limit topology is a nuclear space.

It is well known that the families $\{|\cdot|_{n,k}\ ;\ n, k \geq 0\}$ and $\{|\cdot|_p\ ;\ p \geq 0\}$ are equivalent, i.e., they generate the same topology on $S(\mathbb{R})$. This implies that

$$S(\mathbb{R}) = \operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}).$$

Moreover, we have the following equality

$$S'(\mathbb{R}) = \operatorname*{ind-lim}_{p\to\infty} S_{-p}(\mathbb{R}),$$

where the inductive limit space is equipped with the inductive limit topology and $S_{-p}(\mathbb{R})$ is the completion of $\mathcal{L}^2(\mathbb{R}, dt)$ with respect to the norm

$$|f|_{-p} = \left(\sum_{n=0}^{\infty} (2n+2)^{-2p} (f, e_n)^2 \right)^{1/2}, \quad p \geq 0. \tag{2.3}$$

Use the Riesz representation theorem to identify the space $\mathcal{L}^2(\mathbb{R}, dt)$ with its dual space. Then we get a Gel'fand triple:

$$S(\mathbb{R}) = \operatorname*{proj-lim}_{p\to\infty} S_p(\mathbb{R}) \hookrightarrow \mathcal{L}^2(\mathbb{R}, dt) \hookrightarrow \operatorname*{ind-lim}_{p\to\infty} S_{-p}(\mathbb{R}) = S'(\mathbb{R}).$$

By the Minlos theorem (see, e.g., [5] or [9]), there exists a unique probability measure γ on $S'(\mathbb{R})$ such that

$$\int_{S'(\mathbb{R})} e^{i\langle y, \xi\rangle}\, d\gamma(y) = e^{-\frac{1}{2}|\xi|_0^2}, \quad \xi \in S(\mathbb{R}),$$

where $\langle \cdot, \cdot \rangle$ denotes the dual paring between $S'(\mathbb{R})$ and $S(\mathbb{R})$ and $|\cdot|_0$ is the norm on $\mathcal{L}^2(\mathbb{R}, dt)$.

The space $S'(\mathbb{R})$ is our infinite dimensional analogue of $\mathbb{R}^n$. Since there is no infinite dimensional Lebesgue measure, we will take the Gaussian measure γ on $S'(\mathbb{R})$ as the infinite dimensional replacement of the finite dimensional Lebesgue measure. Hence surface measures on subsets of $S'(\mathbb{R})$ will be respective to γ.

2.2. Spaces of test and generalized functions. We first recall that a function $\theta : \mathbb{R}_+ \to \mathbb{R}$ is called a *Young function* if it is continuous, convex, strictly increasing, $\theta(0) = 0$, and

$$\lim_{t\to\infty} \frac{\theta(t)}{t} = \infty.$$

Let B be a complex Banach space with norm $|\cdot|$. A holomorphic function f on B is said to have *exponential growth of order θ with finite type $m > 0$* if it satisfies the following condition,

$$\|f\|_{\theta,m} := \sup_{z\in B} |f(z)|\, e^{-\theta(m|z|)} < \infty.$$

Let $\mathrm{Exp}(B, \theta, m)$ denote the collection of such functions, namely,

$$\mathrm{Exp}(B, \theta, m) = \Big\{ f \, : \, \|f\|_{\theta,m} < \infty \Big\}. \tag{2.4}$$

Then $\mathrm{Exp}(B, \theta, m)$ is a complex Banach space with norm $\|\cdot\|_{\theta,m}$.

Apply Equation (2.4) to the case when B is the complexification $S_{-p,\mathbb{C}}(\mathbb{R})$ of the space $S_{-p}(\mathbb{R})$, $p \geq 0$. Then we have the spaces $\mathrm{Exp}(S_{-p,\mathbb{C}}(\mathbb{R}), \theta, m)$ for $m > 0$ and $p \geq 0$. The space of *test functions* on the infinite dimensional space $S'(\mathbb{R})$ is defined to be the space

$$\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})) := \bigcap_{m,p} \mathrm{Exp}(S_{-p,\mathbb{C}}(\mathbb{R}), \theta, m),$$

where $S'_{\mathbb{C}}(\mathbb{R})$ is the complexification of $S'(\mathbb{R})$. The space of *generalized functions* on $S'(\mathbb{R})$ is defined to be the strong topological dual space $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ of $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$.

We will also assume that the Young function θ satisfies the condition

$$\lim_{t\to\infty} \frac{\theta(t)}{t^2} < \infty. \tag{2.5}$$

Under the condition in Equation (2.5) we have a Gel'fand triple by [3]:

$$\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})) \subset L^2(S'(\mathbb{R}), \gamma) \subset \mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R})).$$

A test function $f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ is called *positive* if it satisfies the condition

$$f(y + i0) \geq 0, \quad \forall\, y \in S'(\mathbb{R}),$$

where we write $y + i0$ instead of y in order to emphasize the situation that f is restricted to the real space $S'(\mathbb{R})$.

We call a generalized function $\Psi \in \mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ *positive* if

$$\Psi(f) \geq 0, \quad \forall\, \text{positive } f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})).$$

The main tool of this paper is the following integral representation theorem from [10].

Theorem 2.1. *Let Ψ be a positive generalized function. Then there exists a unique finite positive Borel measure μ on $S'(\mathbb{R})$ such that*

$$\Psi(f) = \int_{S'(\mathbb{R})} f(y + i0)\, d\mu(y), \quad \forall\, f \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})). \tag{2.6}$$

We will also need the next theorem from [10]. For convenience, we will say that μ *represents a generalized function* Ψ in $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ when Equation (2.6) holds.

Theorem 2.2. *Let μ be a finite Borel measure on $S'(\mathbb{R})$. Then μ represents a generalized function in $\mathcal{F}^*_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ if and only if there exist $p \in \mathbb{N}$ and $m > 0$ such that μ is supported by $S_{-p}(\mathbb{R})$ and*

$$\int_{S_{-p}(\mathbb{R})} e^{\theta(m|y|_{-p})}\, d\mu(y) < \infty. \tag{2.7}$$

3. Construction of Surface Measures

3.1. Absolute continuity of induced signed measures on $\mathbb{R}$. We first state a well-known lemma from [8] (page 246, Lemma 3.1.1), which we will need for the proof of Theorem 3.2.

Lemma 3.1. *Let κ be a finite positive Borel measure on $\mathbb{R}$. Suppose there exists a positive constant c such that*

$$\Big| \int_{\mathbb{R}} \varphi'(x)\, d\kappa(x) \Big| \leq c \sup_{x\in\mathbb{R}} |\varphi(x)|$$

for every bounded C^1-function φ on $\mathbb{R}$. Then κ is absolutely continuous with respect to the Lebesgue measure dx on $\mathbb{R}$. Moreover, the Radon-Nikodym derivative $K(x) = \frac{d\kappa}{dx}(x)$ belongs to $\mathcal{L}^2(\mathbb{R}, dx)$ and

$$\int_{\mathbb{R}} K(x)^2\, dx \leq c\, \kappa(\mathbb{R}).$$

Next we recall the divergence operator which we need for the statement of Theorem 3.2. For a read-valued function f on $S'(\mathbb{R})$, its *directional derivative* at x in the direction $h \in \mathcal{L}^2(\mathbb{R}, dt)$ is defined by

$$D_h f(x) = \lim_{\epsilon \to 0} \frac{1}{\epsilon}\big(f(x+\epsilon h) - f(x)\big),$$

where the limit is taken in $S'(\mathbb{R})$ almost everywhere with respect to the Gaussian measure γ on $S'(\mathbb{R})$.

The *gradient* of f is a vector field $\nabla f : S'(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}, dt)$ defined by

$$\langle \nabla f(x), h\rangle = D_h f(x), \quad \forall\, h \in \mathcal{L}^2(\mathbb{R}, dt),$$

where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathcal{L}^2(\mathbb{R}, dt)$.

The *divergence* of a vector field $U : S'(\mathbb{R}) \to \mathcal{L}^2(\mathbb{R}, dt)$ is defined to be the function δU on $S'(\mathbb{R})$ satisfying the equality

$$\int_{S'(\mathbb{R})} f(x)(\delta U)(x)\, d\gamma(x) = \int_{S'(\mathbb{R})} \langle \nabla f(x), U(x)\rangle\, d\gamma(x) \tag{3.1}$$

for all smooth real-valued functions f on $S'(\mathbb{R})$.

For the rest of the paper we will use $\gamma \circ \Phi^{-1}$ to denote the distribution of Φ with respect to γ, i.e., $(\gamma \circ \Phi^{-1})(A) = \gamma\big(\Phi^{-1}(A)\big)$, $A \in \mathcal{B}(\mathbb{R})$.

Now we can state the first main result of this paper.

Theorem 3.2. *Suppose Φ is a real-valued function on $S'(\mathbb{R})$ such that*

(a) $\frac{1}{|\nabla\Phi(\cdot)|_0} \in \mathcal{L}^2(S'(\mathbb{R}), \gamma)$,

(b) $\delta \frac{\nabla\Phi(\cdot)}{|\nabla\Phi(\cdot)|_0^2} \in \mathcal{L}^1(S'(\mathbb{R}), \gamma)$.

Then the probability measure $\gamma \circ \Phi^{-1}$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure.

Proof. For any bounded C^1-function φ on $\mathbb{R}$, we use the chain rule to get

$$\nabla(\varphi \circ \Phi)(y) = \varphi'(\Phi(y))\nabla\Phi(y),$$

which implies that

$$\Big\langle \nabla(\varphi \circ \Phi)(y), \frac{\nabla\Phi(y)}{|\nabla\Phi(y)|_0^2}\Big\rangle = \varphi'(\Phi(y)).$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}} \varphi'(y)\, d(\gamma \circ \Phi^{-1})(x) &= \int_{S'(\mathbb{R})} \varphi'(\Phi(y))\, d\gamma(y)\\ &= \int_{S'(\mathbb{R})} \Big\langle \nabla(\varphi \circ \Phi)(y), \frac{\nabla\Phi(y)}{|\nabla\Phi(y)|_0^2}\Big\rangle\, d\gamma(y).\end{aligned}$$

Then apply Equation (3.1) to obtain the equality

$$\int_{\mathbb{R}} \varphi'(y)\, d(\gamma \circ \Phi^{-1})(x) = \int_{S'(\mathbb{R})} \varphi(\Phi(y))\, \Big(\delta \frac{\nabla\Phi(\cdot)}{|\nabla\Phi(\cdot)|_0^2}\Big)(y)\, d\gamma(y).$$

Hence by the assumption on Φ we have

$$\Big|\int_{\mathbb{R}} \varphi'(y)\, d(\gamma\circ\Phi^{-1})(x)\Big| \le \sup_{x\in\mathbb{R}} |\varphi(x)| \int_{S'(\mathbb{R})} \Big|\Big(\delta \frac{\nabla\Phi(\cdot)}{|\nabla\Phi(\cdot)|_0^2}\Big)(y)\Big|\, d\gamma(y)$$
$$< \infty.$$

Thus by Lemma 3.1 the probability measure $\gamma\circ\Phi^{-1}$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure. □

Let g be a γ-integrable function on $S'(\mathbb{R})$. For convenience, we will use γ_g to denote the signed measure $d\gamma_g = g\, d\gamma$. Suppose Φ a read-valued measurable function on $S'(\mathbb{R})$. Then it is obvious that the induced signed measure $\gamma_g\circ\Phi^{-1}$ is absolutely continuous with respect to the induced probability measure $\gamma\circ\Phi^{-1}$. This fact and Theorem 3.2 yield immediately the next corollary.

Corollary 3.3. *Let Φ be a real-valued function on $S'(\mathbb{R})$ satisfying conditions* (a) *and* (b) *in Theorem 3.2. Then for any $g \in \mathcal{L}^2(S'(\mathbb{R}), \gamma)$, the signed measure $\gamma_g\circ\Phi^{-1}$ on $\mathbb{R}$ is absolutely continuous with respect to the Lebesgue measure.*

3.2. Surface measure on $S'(\mathbb{R})$. Recall that γ denotes the standard Gaussian measure on $S'(\mathbb{R})$. In this subsection we will assume that Φ is a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma\circ\Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, for instance, when Φ satisfies conditions (a) and (b) in Theorem 3.2.

The type of surfaces in $S'(\mathbb{R})$ on which we will construct surface measures is specified by

$$V^a = \big\{y \in S'_{\mathbb{C}}(\mathbb{R})\,;\, \Phi(y) = a\big\},$$

where a is a real number.

Since $\gamma\circ\Phi^{-1}$ is assumed to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, the signed measure $\gamma_g\circ\Phi^{-1}$ for $g \in \mathcal{L}^2(S'(\mathbb{R}), \gamma)$ is also absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$.

For convenience, we introduce the notation

$$K(a) = \frac{d(\gamma\circ\Phi^{-1})}{dx}(a), \quad K_g(a) = \frac{d(\gamma_g\circ\Phi^{-1})}{dx}(a), \quad a \in \mathbb{R}. \tag{3.2}$$

and let $\mathcal{O}$ denote the set

$$\mathcal{O} = \{a \in \mathbb{R}\,;\, K(a) \neq 0\}.$$

Theorem 3.4. *Assume that Φ is a real-valued measurable function on $S'(\mathbb{R})$ such that $\gamma\circ\Phi^{-1}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$. Then for each $a \in \mathcal{O}$, there exists a unique probability measure ν^a on $S'(\mathbb{R})$ satisfying the following equality:*

$$\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y) = \frac{K_g(a)}{K(a)}, \quad \forall\, g \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})). \tag{3.3}$$

Moreover, there exist $p \in \mathbb{N}$ and $m > 0$ such that ν^a is supported by $S_{-p}(\mathbb{R})$ and

$$\int_{S_{-p}(\mathbb{R})} e^{\theta(m|y|_{-p})}\, d\nu^a(y) < \infty. \tag{3.4}$$

Proof. The uniqueness of a probability measure ν^a satisfying Equation (3.3) is obvious. To prove the existence of ν^a, let $a \in \mathcal{O}$ and consider the function

$$\Psi_a : g \longmapsto \frac{K_g(a)}{K(a)}, \quad g \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})).$$

It can be easily checked that Ψ_a is a continuous linear functional on the space $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R}))$ of test functions. Thus Ψ_a is a generalized function on $S'(\mathbb{R})$. Moreover, it is obvious that Ψ_a is positive. Hence by Theorem 2.1 there exists a unique finite positive Borel measure ν^a such that

$$\Psi_a(g) = \int_{S'(\mathbb{R})} g(y)\, d\nu^a(y), \quad \forall\, g \in \mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})),$$

which implies that Equation (3.3) holds. Note that

$$\nu^a\big(S'(\mathbb{R})\big) = \Psi_a(1) = \frac{K(a)}{K(a)} = 1.$$

Hence ν^a is a probability measure. Finally, we observe that the assertion regarding to Equation (3.4) follows from Theorem 2.2. □

Theorem 3.5. *The probability measure ν^a given by Theorem 3.4 is supported by the surface V^a and the following equality*

$$\int_{S'(\mathbb{R})} v(\Phi(x))g(x)\, d\gamma(x) = \int_{\mathbb{R}} v(a)K(a)\bigg(\int_{V^a} g(y)\, d\nu^a(y)\bigg) da \tag{3.5}$$

holds for any bounded smooth function v on $\mathbb{R}$ and test function g in $\mathcal{F}_\theta(S'_{\mathbb{C}}(\mathbb{R})$.

Remark 3.6. By Theorems 3.4 and 3.5, the probability measure ν^a is supported by the set $V^a \cap S_{-p}(\mathbb{R})$ for some $p \in \mathbb{N}$ (which may depend on $a \in \mathbb{R}$). Note that Equation (3.5) gives a decomposition of the standard Gaussian measure γ in term of the surface measure ν^a on V^a and the Lebesgue measure on $\mathbb{R}$. Thus ν^a can be regarded as the conditional law of γ given that $\Phi = a$.

Proof. By Equation (3.3) we have

$$K_g(a) = K(a) \int_{S'(\mathbb{R})} g(y)\, d\nu^a(y).$$

It follows from Equations (3.2) and (3.3) that

$$\frac{d(\gamma_g \circ \Phi^{-1})}{dx}(a) = K(a) \int_{S'(\mathbb{R})} g(y)\, d\nu^a(y),$$

which is equivalent to

$$\int_{S'(\mathbb{R})} v(\Phi(x))g(x)\, d\gamma(x) = \int_{\mathbb{R}} v(a)K(a)\bigg(\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y)\bigg) da$$

for any bounded smooth function v on $\mathbb{R}$. Therefore, to finish the proof of this theorem, it only remains to show that ν^a is supported by the surface V^a.

Suppose g is a test function vanishing on V^a. Take a sequence $\{u_n^a\}$ of smooth functions converging to the Dirac function δ_a in the distribution sense, for instance, $u_n^a(x) = (\pi n)^{-1/2} e^{-n(x-a)^2}$. Then by Equation (3.3) we have

$$\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y) = \frac{K_g(a)}{K(a)} = \lim_{n\to\infty} \int_{\mathbb{R}} u_n^a(y) \frac{K_g(y)}{K(y)}\, dy.$$

Make a change of variables $y = \Phi(x)$ to rewrite this equation as

$$\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y) = \lim_{n\to\infty} \int_{S'(\mathbb{R})} u_n^a(\Phi(x)) \frac{g(x)}{K(\Phi(x))}\, d\gamma(x). \tag{3.6}$$

Since g vanishes on V^a, we see that

$$\begin{aligned} \int_{S'(\mathbb{R})} u_n^a(\Phi(x)) \frac{g(x)}{K(\Phi(x))}\, d\gamma(x) &= \int_{S'(\mathbb{R})\setminus V^a} u_n^a(\Phi(x)) \frac{g(x)}{K(\Phi(x))}\, d\gamma(x) \\ &= \int_{\mathbb{R}\setminus\{a\}} u_n^a(y) \frac{K_g(y)}{K(y)}\, dy \\ &\to \int_{\mathbb{R}\setminus\{a\}} \frac{K_g(y)}{K(y)}\, d\delta_a(y) \\ &= 0. \end{aligned} \tag{3.7}$$

It follows from Equations (3.6) and (3.7) that

$$\int_{S'(\mathbb{R})} g(y)\, d\nu^a(y) = 0$$

for all test functions g vanishing on V^a. This implies that ν^a is supported by V^a and the proof is complete. □

4. Surface Measures on Hyperplanes

In this section we will describe surface measures on hyperplanes V^a specified by the function $\Phi(x) = \langle x, \xi\rangle$ with ξ being a fixed nonzero function in $S(\mathbb{R})$.

Lemma 4.1. *Let $\Phi(x) = \langle x, \xi\rangle$ with $\xi \in S(\mathbb{R})$, $\xi \neq 0$. Then the corresponding function $K(y)$ in Equation (3.2) is given by*

$$K(y) = \frac{1}{\sqrt{2\pi}\, |\xi|_0}\, e^{-\frac{y^2}{2|\xi|_0^2}}, \quad y \in \mathbb{R}. \tag{4.1}$$

Moreover, for the test function $g(x) = e^{\langle x, \eta\rangle}$ with $\eta \in S(\mathbb{R})$, the corresponding function $K_g(y)$ in Equation (3.2) is given by

$$K_g(y) = \frac{e^{\frac{|\eta|_0^2}{2}}}{\sqrt{2\pi}\, |\xi|_0}\, e^{-\frac{(y - \langle \xi, \eta\rangle)^2}{2|\xi|_0^2}}, \quad y \in \mathbb{R}. \tag{4.2}$$

Proof. The first assertion is obvious since the random variable Φ is Gaussian with mean 0 and variance $|\xi|_0^2$. To prove the second assertion, observe that for any bounded continuous function $F(x)$ on $\mathbb{R}$, we have

$$\int_{\mathbb{R}} F(y)\, d(\gamma_g \circ \Phi^{-1})(y) = \int_{S'(\mathbb{R})} F(\Phi(x)) g(x)\, d\gamma(x). \tag{4.3}$$

We can evaluate the right-hand side of Equation (4.3) with $F(y) = e^{\lambda y}$, $\lambda \in \mathbb{R}$, and the given functions Φ and g to get

$$\int_{S'(\mathbb{R})} F(\Phi(x))g(x)\,d\gamma(x) = \int_{S'(\mathbb{R})} e^{\langle x, \lambda\xi+\eta\rangle}\,d\gamma(x)$$
$$= e^{\frac{1}{2}\lambda^2|\xi|_0^2+\lambda\langle\xi,\eta\rangle+\frac{1}{2}|\eta|_0^2}. \tag{4.4}$$

It follows from Equations (4.3) and (4.4) that

$$\int_{\mathbb{R}} e^{\lambda y}\,d(\gamma_g\circ\Phi^{-1})(y) = e^{\frac{1}{2}\lambda^2|\xi|_0^2+\lambda\langle\xi,\eta\rangle+\frac{1}{2}|\eta|_0^2},$$

which gives the Laplace transform of the measure $\gamma_g\circ\Phi^{-1}$. Take the inverse Laplace transform to obtain

$$\frac{d(\gamma_g\circ\Phi^{-1})}{dx}(y) = \frac{e^{\frac{|\eta|_0^2}{2}}}{\sqrt{2\pi}\,|\xi|_0}\,e^{-\frac{(y-\langle\xi,\eta\rangle)^2}{2|\xi|_0^2}}, \quad y\in\mathbb{R},$$

that is, Equation (4.2) holds. □

In the next theorem we use the Laplace transform to describe surface measures arising from the function $\Phi(x) = \langle x, \xi\rangle$ with $\xi \in S(\mathbb{R})$, $\xi \neq 0$.

Theorem 4.2. *Let $\xi \in S(\mathbb{R})$, $\xi \neq 0$ and $a \in \mathbb{R}$. Then the Laplace transform of the surface measure ν^a on $V^a = \{x \in S'(\mathbb{R})\,;\,\langle x,\xi\rangle = a\}$ is given by*

$$(\mathcal{L}\nu^a)(\eta) = \exp\left(|\eta|_0^2 - \frac{\langle\xi,\eta\rangle^2}{2|\xi|_0^2} + a\frac{\langle\xi,\eta\rangle}{|\xi|_0^2}\right), \quad \eta\in S(\mathbb{R}). \tag{4.5}$$

Proof. Apply Equation (3.3) to the function $g(y) = e^{\langle y,\eta\rangle}$ to get

$$\int_{S'(\mathbb{R})} e^{\langle y,\eta\rangle}\,d\nu^a(y) = \frac{K_g(a)}{K(a)}. \tag{4.6}$$

Equation (4.5) follows immediately from Equations (4.1), (4.2), and (4.6). □

5. Surface Measures on Quadric Surfaces

In this section we will describe surface measures on quadric surfaces V^a specified by the function $\Phi(x) = \langle x,\xi\rangle^2$ with ξ being a fixed nonzero function in $S(\mathbb{R})$.

Lemma 5.1. *Let $\Phi(x) = \langle x,\xi\rangle^2$ with $\xi\in S(\mathbb{R})$, $\xi\neq 0$. Then the corresponding function $K(y)$ in Equation (3.2) is given by*

$$K(y) = \frac{1}{\sqrt{2\pi y}\,|\xi|_0}\,e^{-\frac{y}{2|\xi|_0^2}}, \quad y > 0. \tag{5.1}$$

Moreover, for the test function $g(x) = e^{\langle x,\eta\rangle}$ with $\eta\in S(\mathbb{R})$, the corresponding function $K_g(y)$ in Equation (3.2) is given by

$$K_g(y) = \frac{e^{\frac{1}{2}\left(|\eta|_0^2-\frac{\langle\xi,\eta\rangle^2}{|\xi|_0^2}\right)}}{\sqrt{2\pi y}\,|\xi|_0}\,e^{-\frac{y}{2|\xi|_0^2}}\cosh\left(\frac{\langle\xi,\eta\rangle}{|\xi|_0^2}\sqrt{y}\right), \quad y>0. \tag{5.2}$$

Remark 5.2. Recall that the density function of a gamma distribution $\Gamma(\alpha,\lambda)$ with parameters $\alpha>0$ and $\lambda>0$ is given by

$$\frac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1}e^{-\lambda x}, \quad x>0.$$

Thus in view of Equation (5.1) the measure $\gamma\circ\Phi^{-1}$ is the gamma distribution $\Gamma(\frac{1}{2}, \frac{1}{2|\xi|_0^2})$.

Proof. First note that $\gamma\circ\Phi^{-1}\big((-\infty,y]\big)=0$ when $y\le 0$. If $y>0$, then

$$\begin{aligned}\gamma\circ\Phi^{-1}\big((-\infty,y]\big) &= \gamma\{x; \langle x,\xi\rangle^2\le y\}\\ &= \gamma\Big\{x; \Big|\Big\langle x, \frac{\xi}{|\xi|_0}\Big\rangle\Big| \le \frac{\sqrt{y}}{|\xi|_0}\Big\}\\ &= 2\int_0^{\frac{\sqrt{y}}{|\xi|_0}} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\, dx.\end{aligned}$$

Upon differentiating both sides, we see that for $y>0$,

$$K(y)=\frac{d(\gamma\circ\Phi^{-1})}{dx}(y)=\frac{d}{dy}\big[\gamma\circ\Phi^{-1}\big((-\infty,y]\big)\big]=\frac{1}{\sqrt{2\pi y}\,|\xi|_0}\, e^{-\frac{y}{2|\xi|_0^2}},$$

which proves Equation (5.1).

To prove the second assertion, we apply Equation (4.3) with $\Phi(x)=\langle x,\xi\rangle^2$, $g(x)=e^{\langle x,\eta\rangle}$, and $F(y)=e^{\lambda y}$ to get

$$\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g\circ\Phi^{-1})(y)=\int_{S'(\mathbb{R})} e^{\lambda\langle x,\,\xi\rangle^2+\langle x,\,\eta\rangle}\, d\gamma(x).$$

The integral in the right-hand side can be computed by writing $\eta=c\xi+\tilde{\xi}$ with $\tilde{\xi}\perp\xi$ and $c=\langle\xi,\,\eta\rangle/|\xi|_0^2$ and carrying out straightforward calculations. Then

$$\begin{aligned}&\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g\circ\Phi^{-1})(y)\\ &= \frac{1}{\sqrt{1-2\lambda|\xi|_0^2}}\, \exp\bigg(\frac{1}{2}|\eta|_0^2+\frac{1}{2\big(1-2\lambda|\xi|_0^2\big)}\,\frac{\langle\xi,\eta\rangle^2}{|\xi|_0^4}-\frac{1}{2}\,\frac{\langle\xi,\eta\rangle^2}{|\xi|_0^2}\bigg),\end{aligned} \tag{5.3}$$

where $\lambda<\frac{1}{2|\xi|_0^2}$.

To derive the inverse Laplace transform of Equation (5.3), we use the following equality which can be checked by direct calculation:

$$\begin{aligned}&\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\Big(-\frac{1}{2}\big(1-2\lambda|\xi|_0^2\big)y^2+\frac{\langle\xi,\eta\rangle}{|\xi|_0}y\Big)\, dy\\ &= \frac{1}{\sqrt{1-2\lambda|\xi|_0^2}}\, \exp\bigg(\frac{1}{2\big(1-2\lambda|\xi|_0^2\big)}\,\frac{\langle\xi,\eta\rangle^2}{|\xi|_0^4}\bigg),\end{aligned} \tag{5.4}$$

By Equations (5.3) and (5.4), we have

$$\begin{aligned}&\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g\circ\Phi^{-1})(y)\\ &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \exp\Big(\frac{1}{2}|\eta|_0^2-\frac{1}{2}\,\frac{\langle\xi,\eta\rangle^2}{|\xi|_0^2}-\frac{1}{2}\big(1-2\lambda|\xi|_0^2\big)y^2+\frac{\langle\xi,\eta\rangle}{|\xi|_0}y\Big)\, dy,\end{aligned}$$

which can be rewritten as

$$\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g \circ \Phi^{-1})(y)$$
$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty \Big(e^{\frac{\langle \xi,\eta\rangle}{|\xi|_0} y} + e^{-\frac{\langle \xi,\eta\rangle}{|\xi|_0} y}\Big) \exp\Big(\frac{1}{2}|\eta|_0^2 - \frac{1}{2}\,\frac{\langle \xi,\eta\rangle^2}{|\xi|_0^2} - \frac{1}{2}\big(1-2\lambda|\xi|_0^2\big)y^2\Big)\, dy,$$

Finally, make a change of variables $y = \sqrt{x}/|\xi|_0$ to show that

$$\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g \circ \Phi^{-1})(y) = \int_{\mathbb{R}} e^{\lambda x} H(x)\, dx,$$

where the function $H(x)$ is given by

$$H(x) = \frac{e^{\frac{1}{2}\left(|\eta|_0^2 - \frac{\langle \xi,\eta\rangle^2}{|\xi|_0^2}\right)}}{2\sqrt{2\pi x}\, |\xi|_0}\, e^{-\frac{x}{2|\xi|_0^2}} \Big(e^{\frac{\langle \xi,\eta\rangle}{|\xi|_0^2}\sqrt{x}} + e^{-\frac{\langle \xi,\eta\rangle}{|\xi|_0^2}\sqrt{x}}\Big)$$
$$= \frac{e^{\frac{1}{2}\left(|\eta|_0^2 - \frac{\langle \xi,\eta\rangle^2}{|\xi|_0^2}\right)}}{\sqrt{2\pi x}\, |\xi|_0}\, e^{-\frac{x}{2|\xi|_0^2}} \cosh\Big(\frac{\langle \xi,\eta\rangle}{|\xi|_0^2}\sqrt{x}\Big), \quad x > 0.$$

Hence the function $K_g(y)$ in Equation (3.2) for our present case is given by the above function $H(y)$ and Equation (5.2) is proved. □

We can describe surface measures arising from the function $\Phi(x) = \langle x, \xi\rangle^2$ with $\xi \in S(\mathbb{R})$, $\xi \neq 0$ in terms of the Laplace transform as stated in the next theorem.

Theorem 5.3. *Let $\xi \in S(\mathbb{R})$, $\xi \neq 0$ and $a > 0$. Then the Laplace transform of the surface measure ν^a on $V^a = \{x \in S'(\mathbb{R})\,;\, \langle x, \xi\rangle^2 = a\}$ is given by*

$$\mathcal{L}(\nu^a)(\eta) = e^{\frac{1}{2}\left(|\eta|_0^2 - \frac{\langle \xi,\eta\rangle^2}{|\xi|_0^2}\right)} \cosh\Big(\frac{\langle \xi,\eta\rangle}{|\xi|_0^2}\sqrt{a}\Big), \quad \eta \in S(\mathbb{R}). \tag{5.5}$$

Proof. Just apply Equation (3.3) to the present case with $K(a)$ and $K_g(a)$ as given by Equations (5.1) and (5.2), respectively. □

6. Surface Measures on Infinite Dimensional Spheres

Recall the Hilbert space $S_{-p}(\mathbb{R})$ with norm $|\cdot|_{-p}$ defined by Equation (2.3). It is well known that the triple $S_p(\mathbb{R}) \subset \mathcal{L}^2(\mathbb{R}, dt) \subset S_{-p}(\mathbb{R})$ is an abstract Wiener space for any $p > 1/2$. Hence the standard Gaussian measure γ on $S'(\mathbb{R})$ is actually supported by $S_{-p}(\mathbb{R})$ for any $p > 1/2$.

In this section we will study the surface measures arising from the function $\Phi(x) = |x|_{-p}^2$ for $p > 1/2$. The associated surface $V^a = \{x \in S_{-p}(\mathbb{R}); \Phi(x) = a\}$ for $a > 0$ is an infinite dimensional sphere with radius $\sqrt{a}$.

By using the Hermite functions defined by Equation (2.1) and the norm given by Equation (2.3), we can rewrite $\Phi(x)$ as

$$\Phi(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+2)^{2p}}\, \langle x, e_n\rangle^2 \tag{6.1}$$

We introduce a notation for the infinite convolution product of functions:

$$*\prod_{n=1}^{\infty} F_n = F_1 * F_2 * \cdots * F_n * \cdots$$

For convenience, let $G_{a,b}$ with $a > 0$ and $b \in \mathbb{R}$ denote the following function:

$$G_{a,b}(y) = \frac{a}{\sqrt{2\pi y}}\, e^{-\frac{1}{2}a^2 y} \cosh\big(ab\sqrt{y}\,\big), \quad y > 0, \tag{6.2}$$

and $G_{a,b}(y) = 0$ if $y \le 0$. Note that $G_{a,0}$ is the $\Gamma(\frac{1}{2}, \frac{a^2}{2})$-distribution. Moreover, we have the Laplace transform of $G_{a,b}$ given by

$$\int_{\mathbb{R}} e^{\lambda y} G_{a,b}(y)\, dy = \frac{1}{\sqrt{1-\frac{2\lambda}{a^2}}} \exp\left(\frac{b^2}{2\big(1-\frac{2\lambda}{a^2}\big)}\right), \quad \lambda < \frac{a^2}{2}. \tag{6.3}$$

Lemma 6.1. *Let $\Phi(x) = |x|_{-p}^2$ with $p > 1/2$. Then the corresponding function $K(y)$ in Equation* (3.2) *is given by*

$$K(y) = \Big(*\prod_{n=0}^{\infty} G_{(2n+2)^p,\,0}\Big)(y), \quad y > 0, \tag{6.4}$$

where $G_{a,0}$ is the function defined by Equation (6.2) *with $b = 0$ and $*\prod_{n=0}^{\infty}$ denotes the infinite convolution product. Moreover, for the test function $g(x) = e^{\langle x,\eta\rangle}$ with $\eta \in S(\mathbb{R})$, the corresponding function $K_g(y)$ in Equation* (3.2) *is given by*

$$K_g(y) = \Big(*\prod_{n=0}^{\infty} G_{(2n+2)^p,\,\langle \eta,\, e_n\rangle}\Big)(y), \quad y > 0, \tag{6.5}$$

where $G_{a,b}$ is the function defined by Equation (6.2).

Remark 6.2. As a consequence of the proof below, the infinite convolution products in Equations (2.6) and (3.1) exist.

Proof. For $\lambda < 2^{2p-1}$, we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma\circ\Phi^{-1})(y) &= \int_{S'(\mathbb{R})} e^{\lambda \sum_{n=0}^{\infty} \frac{1}{(2n+2)^{2p}} \langle x, e_n\rangle^2}\, d\gamma(x) \\
&= \prod_{n=0}^{\infty} \int_{\mathbb{R}} e^{\frac{\lambda}{(2n+2)^{2p}} y^2} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy \qquad (6.6)\\
&= \prod_{n=0}^{\infty} \frac{1}{\sqrt{1-\frac{2\lambda}{(2n+2)^{2p}}}}. \qquad (6.7)
\end{aligned}$$

On the other hand, by Equation (6.3) with $b = 0$, we have

$$\int_{\mathbb{R}} e^{\lambda y} \Big(*\prod_{n=0}^{\infty} G_{(2n+2)^p,\,0}\Big)(y)\, dy = \prod_{n=0}^{\infty} \frac{1}{\sqrt{1-\frac{2\lambda}{(2n+2)^{2p}}}}. \tag{6.8}$$

Obviously, Equations (6.7) and (6.8) yield Equation (6.4).

The proof of Equation (6.5) is similar, but with some modifications. Instead of Equation (6.6), we now have

$$\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g \circ \Phi^{-1})(y) = \prod_{n=0}^{\infty} \int_{\mathbb{R}} e^{\frac{\lambda}{(2n+2)^{2p}} y^2 + \langle \eta, e_n\rangle y} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy. \tag{6.9}$$

By direct computation we can easily derive the following equality:

$$\int_{\mathbb{R}} e^{\frac{\lambda}{a^2} y^2 + by} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy = \frac{1}{\sqrt{1-\frac{2\lambda}{a^2}}} \exp\left(\frac{b^2}{2\left(1-\frac{2\lambda}{a^2}\right)}\right), \quad \lambda < \frac{a^2}{2}. \tag{6.10}$$

Then from Equations (6.3) and (6.10) we have the equality

$$\int_{\mathbb{R}} e^{\frac{\lambda}{a^2} y^2 + by} \frac{1}{\sqrt{2\pi}}\, e^{-\frac{y^2}{2}}\, dy = \int_{\mathbb{R}} e^{\lambda y} G_{a,b}(y)\, dy.$$

Now, apply this equality with $a = (2n+2)^p$ and $b = \langle \eta, e_n\rangle$ to the right-hand side of Equation (6.9) to get

$$\begin{aligned} &\int_{\mathbb{R}} e^{\lambda y}\, d(\gamma_g \circ \Phi^{-1})(y) \\ &= \prod_{n=0}^{\infty} \int_{\mathbb{R}} e^{\lambda y} G_{(2n+2)^p, \langle \eta,\, e_n\rangle}(y)\, dy \\ &= \int_{\mathbb{R}} e^{\lambda y} \Big(* \prod_{n=0}^{\infty} G_{(2n+2)^p,\, \langle \eta,\, e_n\rangle} \Big)(y)\, dy, \end{aligned}$$

which implies Equation (6.5). □

Theorem 6.3. *Let $a > 0$. Then the Laplace transform of the surface measure ν^a on $V^a = \{x \in S'(\mathbb{R})\,;\, |x|_{-p}^2 = a\}$ is given by*

$$\mathcal{L}(\nu^a)(\eta) = \frac{\Big(* \prod_{n=0}^{\infty} G_{(2n+2)^p,\, 0} \Big)(a)}{\Big(* \prod_{n=0}^{\infty} G_{(2n+2)^p,\, \langle \eta,\, e_n\rangle} \Big)(a)}, \quad \eta \in S(\mathbb{R}),$$

where the function $G_{a,b}$ is defined by Equation (6.2).

Proof. Just use Equation (3.3) for the present case with $K(a)$ and $K_g(a)$ given by Equations (6.4) and (6.5), respectively. □

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