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QUANTUM FILTERING IN COHERENT STATES

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Abstract. We derive the form of the Belavkin-Kushner-Stratonovich equation describing the filtering of a continuous observed quantum system via non-demolition measurements when the statistics of the input field used for the indirect measurement are in a general coherent state.

1. Introduction

One of the most remarkable consequences of the Hudson-Parthasarathy quantum stochastic calculus [21] is V. P. Belavkin's formulation of a quantum theory of filtering based on non-demolition measurements of an output field that has interacted with a given system [4, 6, 7, 8]. Specifically, we must measure a particular feature of the field, for instance a field quadrature, or the count of the field quanta, and this determines a self-commuting, therefore essentially classical, stochastic process. The resulting equations have structural similarities with the classical analogues appearing in the work of Kallianpur, Striebel, Kusnher, Stratonovich, Zakai, Duncan and Mortensen on nonlinear filtering, see [16, 23, 24, 28]. This showed that the earlier models of repeated quantum photon counting measurements developed by Davies [14, 15] could be realized using a concrete theory: this was first shown by taking the pure-jump process limit of diffusive quantum filtering problems [3].

There has been recent interest amongst the physics community in quantum filtering as an applied technique in quantum feedback and control [1, 2, 10, 11, 12, 17, 19, 22, 26, 27]. An additional driver is the desire to go beyond the situation of a vacuum field and derive the filter for other physically important states such as thermal, squeezed, single photon states, etc. In this note we wish to present the filter for non-demolition quadrature and photon-counting measurements when the choice of state for the input field is a coherent state with intensity function β. The resulting filters are a deformation of the vacuum filters and reduce to the latter when we take $\beta \equiv 0$, this is perhaps to be expected given that the coherent states have a continuous-in-time tensor product factorization property. We derive the filters using the reference probability approach, as well as the characteristic function approach.

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1.1. Classical Non-linear Filtering. We consider a state based model where the state X_t evolves according to a stochastic dynamics and we make noisy observations Y_t on the state. The dynamics-observations equations are the SDEs

$$dX_t = v(X_t)\,dt + \sigma_X(X_t)\,dW_t^{\mathrm{proc}}, \tag{1.1}$$

$$dY_t = h(X_t)\,dt + \sigma_Y dW_t^{\mathrm{obs}} \tag{1.2}$$

and we assume that the process noise W^{proc} and the observation noise W_{obs} are uncorrelated multi-dimensional Wiener processes. The generator of the state diffusion is then

$$\mathcal{L} = v^i\partial_i + \frac{1}{2}\Sigma_{XX}^{ij}\partial_{ij}^2,$$

where $\Sigma_{XX} = \sigma_X\sigma_X^\top$. The aim of filtering theory to obtain a least squares estimate for the state dynamics. More specifically, for any suitable function f of the state, we would like to evaluate the conditional expectation

$$\pi_t(f) := \mathbb{E}[f(X_t) \mid \mathcal{F}_{t]}^Y],$$

with $\mathcal{F}_{t]}^Y$ being the σ-algebra generated by the observations up to time t.

1.1.1. *Kallianpur-Striebel Formula.* By introducing the Kallianpur-Striebel likelihood function

$$L_t(\mathbf{x}|\mathbf{y}) = \exp\int_0^t \left\{h(x_s)^\top dy_s - \frac{1}{2}h(x_s)^\top h(x_s)\,ds\right\}$$

for sample state path $\mathbf{x} = \{x_s : 0 \le s \le t\}$ conditional on a given sample observation $\mathbf{y} = \{y_s : 0 \le s \le t\}$, we may represent the conditional expectation as

$$\pi_t(f) = \left.\frac{\int_{C_0^x[0,t]} f(x_t)\,L_t(\mathbf{x}|\mathbf{y})\,\mathbb{P}[d\mathbf{x}]}{\int_{C_0^x[0,t]} L_t(\mathbf{x}|\mathbf{y})\,\mathbb{P}[d\mathbf{x}]}\right|_{\mathbf{y}=Y(\omega)} = \frac{\sigma_t(f)}{\sigma_t(1)}$$

where $\mathbb{P}$ is canonical Wiener measure and

$$\sigma_t(f)(\omega) = \int_{C_0^x[0,t]} f(x_t)\,L_t(\mathbf{x}|Y(\omega))\,\mathbb{P}[d\mathbf{x}].$$

1.1.2. *Duncan-Mortensen-Zakai and Kushner-Stratonovich Equations.* Using the Itō calculus, we may obtain the Duncan-Mortensen-Zakai equation for the unnormalized filter $\sigma_t(f)$, and the Kushner-Stratonovich equation for the normalized version $\pi_t(f)$. These are

$$d\sigma_t(f) = \sigma_t(\mathcal{L}f)\,dt + \sigma_t\left(fh^\top\right)dY_t,$$

$$d\pi_t(f) = \pi_t(\mathcal{L}f)\,dt + \left[\pi_t\left(fh^\top\right) - \pi_t(f)\,\pi_t\left(h^\top\right)\right]dI_t,$$

where (I_t) are the innovations:

$$dI_t := dY_t - \pi_t(h)\,dt,\ I(0) = 0.$$

We note that there exist variants of these equations for more general processes than diffusions (in particular for point processes which will be of relevance for photon counting), and for the case where the process and observation noises are correlated.

1.1.3. *Pure versus Hybrid Filtering Problems.* We remark that we follow the traditional approach of adding direct Wiener noise W_{obs} to the observations. We could of course consider a more general relation of the form $dY_t = h\left(X_t\right) dt + \sigma_Y dW_t^{\text{obs}}$ but for constant coefficients σ_Y a simple rearrangement returns us to the above setup.

The situation where we envisage $dY_t = h\left(X_t\right) dt + \sigma_Y\left(X_t\right) dW_t^{\text{obs}}$, with σ_Y a known function of the unobserved state, must be considered as being *too good to be true* since we can then obtain information about the unobserved state by just examining the quadratic variation of the observations process, since we then have $dY\left(t\right) dY\left(t\right)^{\top} = \sigma_Y(X_t)\sigma_Y\left(X_t\right)^{\top} dt$. For instance, in the case of scalar processes, if we have $\sigma_Y\left(X\right) = \gamma|X|$ then knowledge of the quadratic variation yields the magnitude $|X_t|$ of the signal without any need for filtering. Such situations are rarely if ever arise in practice, and one naturally restricts to pure filtering problems.

2. Quantum Filtering

We wish to describe the quantum mechanical analogue of the classical filtering problem. To begin with, we note that in quantum theory the physical degrees of freedom are modeled as observables, that is self-adjoint operators on a fixed Hilbert space $\mathfrak{h}$. The observables will generally not commute with each other. In place of the classical notion of a state, we will have a normalized vector $\psi \in \mathfrak{h}$ and the averaged of an observable X will be give by the real number $\langle\psi|X\psi\rangle$. (Here we following the physicist convention of taking the inner product $\langle\psi|\phi\rangle$ to be linear in the second argument ϕ and conjugate linear in the first ψ.) More generally we define a quantum state to be a positive, normalized linear functional $\mathbb{E}$ on the set of operators. Every such expectation may be written as

$$\mathbb{E}\left[X\right] = \mathrm{tr}_{\mathfrak{h}}\left\{\varrho X\right\}$$

where ϱ is a positive trace-class operator normalized so that $\mathrm{tr}_{\mathfrak{h}}\varrho = 1$. The operator ϱ is referred to as a *density matrix*. The set of all states is a convex set whose extreme points correspond to the density matrices that are rank-one projectors onto the subspace spanned by a unit vectors $\psi \in \mathfrak{h}$.

To make a full analogy with classical theory, we should exploit the mathematical framework of quantum probability which gives the appropriate generalization of probability theory and stochastic processes to the quantum setting. The standard setting is in terms of a von Neumann algebra of observables over a fixed Hilbert space, which will generalize the notion of an algebra of bounded random variables, and take the state to be an expectation functional which is continuous in the normal topology. The latter condition is equivalent to the σ-finiteness assumption in probability theory and results in all the states of interest being equivalent to a density matrix.

In any given experiment, we may only measure commuting observables. Quantum estimation theory requires that the only observables that we may estimate based on a particular experiment are those which commute with the measured observables. In practice, we do not measure a quantum system directly, but apply an input field and measure a component of the output field. The input field results in a open dynamics for the system while measurement of the output ensures that we met so called non-demolition conditions which guarantee that quantum measurement process itself does not destroy the statistical features which we would like to infer. We will now describe these elements in more detail below.

2.1. Quantum Estimation. We shall now describe the reference probability approach to quantum filtering. Most of our conventions following the presentation of Bouten and van Handel [12]. For an alternative account, including historical references, see [9].

Let $\mathfrak{A}$ be a von Neumann algebra and $\mathbb{E}$ be a normal state. In a given experiment one may only measure a set of *commuting* observables $\{Y_\alpha : \alpha \in A\}$. Define the measurement algebra to be the commutative von Neumann algebra generated by the chosen observables

$$\mathfrak{M} = \mathrm{vN}\{Y_\alpha : \alpha \in A\} \subset \mathfrak{A}.$$

We may estimate an observable $X \in \mathfrak{A}$ from an experiment with measurement algebra $\mathfrak{M}$ if and only if

$$X \in \mathfrak{M}' := \{A \in \mathfrak{A} : [A, Y] = 0, \forall\, Y \in \mathfrak{M}\},$$

That is, if it is physically possible to measure X in addition to all the Y_α. Therefore the algebra $\mathrm{vN}\{X, Y_\alpha : \alpha \in A\}$ must again be commutative. We may then set about defining the conditional expectation of estimable observables onto the measurement algebra.

Definition 2.1. For commutative von Neumann algebra $\mathfrak{M}$, the *conditional expectation* onto $\mathfrak{M}$ is the map

$$\mathbb{E}[\cdot \mid \mathfrak{M}] : \mathfrak{M}' \mapsto \mathfrak{M}$$

defined by

$$\mathbb{E}[\,\mathbb{E}[X \mid \mathfrak{M}]\, Y] \equiv \mathbb{E}[XY], \forall Y \in \mathfrak{M}. \tag{2.1}$$

In contrast to the general situation regarding conditional expectations in the non-commutative setting of von Neumann algebras [25], this particular definition is always nontrivial insofar as existence is guaranteed. Introducing the norm $\|A\|^2 := \mathbb{E}[A^\dagger A]$, we see that the conditional expectation always exists and is unique up to norm-zero terms. It moreover satisfies the least squares property

$$\|X - \mathbb{E}[X \mid \mathfrak{M}]\| \leq \|X - Y\|, \quad \forall Y \in \mathfrak{M}.$$

As the set $\mathrm{vN}\{X, Y_\alpha : \alpha \in A\}$ is a commutative von Neumann algebra for each $X \in \mathfrak{M}'$, it will be isomorphic to the space of bounded functions on a measurable space by Gelfand's theorem. The state induces a probability measure on this space and we may obtain the standard conditional expectation of the random variable corresponding to X onto the σ-algebra generated by the functions corresponding to the Y_α. This classical conditional expectation then corresponds to a unique element $\mathbb{E}[X \mid \mathfrak{M}] \in \mathfrak{M}$ and this gives the construction of the quantum conditional expectation. We sketch the conditional expectation in figure 1. Note that while this may seem trivial at first sight, it should be stressed that the commutant $\mathfrak{M}'$ itself will typically be a non-commutative algebra, so that while our measured observables commute, and what we wish to estimate must commute with our measured observables, the object we can estimate need not commute amongst themselves.

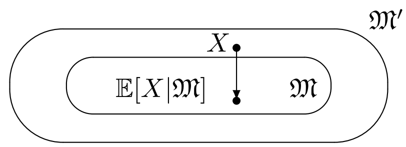

FIGURE 1. Quantum Conditional Expectation $\mathbb{E}[\cdot|\mathfrak{M}]$ as a least squares projection onto $\mathfrak{M}$ from its commutant.

The following two lemmas will be used extensively, see [12] and [13].

Lemma 2.2 (Unitary rotations)**.** *Let U be unitary and define $\tilde{\mathbb{E}}\left[X\right] := \mathbb{E}\left[U^\dagger X U\right]$ and let $\tilde{\mathfrak{M}} = U^\dagger \mathfrak{M} U$. Then*

$$\mathbb{E}[U^\dagger X U \mid \tilde{\mathfrak{M}}] = U^\dagger \tilde{\mathbb{E}}\left[X \mid \mathfrak{M}\right] U.$$

Here we think of going from the Schrödinger picture where the state $\mathbb{E}$ is fixed and observables evolve to $U^* X U$, to the Heisenberg picture where the state evolves to $\tilde{\mathbb{E}}$ and the observables are fixed. Lemma 2.2 tells us how we may transform the conditional expectation between these two picture.

Lemma 2.3 (Quantum Bayes' formula)**.** *Let $F \in \mathfrak{M}'$ with $\mathbb{E}\left[F^\dagger F\right] = 1$ and set $\mathbb{E}_F\left[X\right] := \mathbb{E}\left[F^\dagger X F\right]$. Then*

$$\mathbb{E}_F\left[X \mid \mathfrak{M}\right] = \frac{\mathbb{E}\left[F^\dagger X F \mid \mathfrak{M}\right]}{\mathbb{E}\left[F^\dagger F \mid \mathfrak{M}\right]}.$$

Proof. For all $Y \in \mathfrak{M}$,

$$\begin{aligned}
\mathbb{E}\left[\mathbb{E}\left[F^\dagger X F \mid \mathfrak{M}\right] Y\right] &= \mathbb{E}\left[F^\dagger X F Y\right] \\
&= \mathbb{E}_F[XY], \qquad \text{since } [F, Y] = 0, \\
&= \mathbb{E}_F\left[\mathbb{E}_F\left[X \mid \mathfrak{M}\right] Y\right] \\
&= \mathbb{E}\left[F^\dagger F \mathbb{E}_F\left[X \mid \mathfrak{M}\right] Y\right], \text{ since } F \in \mathfrak{M}' \\
&= \mathbb{E}\left[\mathbb{E}\left[F^\dagger F \mid \mathfrak{M}\right] \mathbb{E}_F\left[X \mid \mathfrak{M}\right] Y\right].
\end{aligned}$$

□

Note that the proof only works if $F \in \mathfrak{M}'$!

2.2. Quantum Stochastic Processes. We begin by reviewing the theory of quantum stochastic calculus developed by Hudson and Parthasarathy [21] which gives the mathematical framework with which to generalize the notions of the classical Itō integration theory.

We take $\mathbb{R}_+ = [0, \infty)$. We shall denote by $L^2_{\text{symm}}(\mathbb{R}^n_+)$ the space of all square-integrable functions of n positive variables that are completely symmetric: that is, invariant under interchange of any pair of its arguments. The Bose Fock space over $L^2(\mathbb{R}_+)$ is then the infinite direct sum Hilbert space

$$\mathfrak{F} := \bigoplus_{n=0}^{\infty} L^2_{\text{symm}}(\mathbb{R}^n_+)$$

with the $n = 0$ space identified with $\mathbb{C}$. An element of $\mathfrak{F}$ is then a sequence $\Psi = (\psi_n)_{n=0}^{\infty}$ with $\psi_n \in L^2_{\text{symm}}(\mathbb{R}^n_+)$ and $\|\Psi\|^2 = \sum_{n=0}^{\infty} \int_{[0,\infty)^n} |\psi_n (t_1, \cdots, t_n)|^2 dt_1 \cdots dt_n < \infty$. Moreover, the Fock space has inner product

$$\langle \Psi \mid \Phi \rangle = \sum_{n=0}^{\infty} \int_{[0,\infty)^n} \psi_n (t_1, \cdots, t_n)^* \phi (t_1, \cdots, t_n).$$

Th physical interpretation is that $\Psi = (\psi_n)_{n=0}^{\infty} \in \mathfrak{F}$ describes the state of a quantum field consisting of an indefinite number of indistinguishable (Boson) particles on the half-line $\mathbb{R}_+$. A simple example is the *vacuum vector* defined by

$$\Omega := (1, 0, 0, \cdots)$$

clearly corresponding to no particles. (Note that the no-particle state is a genuine physical state of the field and is not just the zero vector of $\mathfrak{F}$!) An important class of vectors are the *coherent states* $\Psi(\beta)$ defined by

$$[\Psi(\beta)]_n (t_1, \cdots, t_n) := e^{-\|\beta\|^2} \frac{1}{\sqrt{n!}} \beta(t_1) \cdots \beta(t_n),$$

for $\beta \in L^2(\mathbb{R}_+)$. (The $n = 0$ component understood as $e^{-\|\beta\|^2}$.) The vacuum then corresponds to $\Psi(0)$.

For each $t > 0$ we define the operators of annihilation $B(t)$, creation $B^*(t)$ and gauge $\Lambda(t)$ by

$$\begin{aligned}
[B(t)\Psi]_n (t_1, \cdots, t_n) &:= \sqrt{n+1} \int_0^t \psi_{n+1} (s, t_1, \cdots, t_n)\, ds, \\
[B^*(t)\Psi]_n (t_1, \cdots, t_n) &:= \frac{1}{\sqrt{n}} \sum_{j=1}^{n} 1_{[0,t]}(t_j) \psi_{n-1} \left(t_1, \cdots, \widehat{t_j}, \cdots, t_n\right), \\
[\Lambda(t)\Psi]_n (t_1, \cdots, t_n) &:= \sum_{j=1}^{n} 1_{[0,t]}(t_j) \psi_n (t_1, \cdots, t_n).
\end{aligned}$$

The creation and annihilation process are adjoint to each other and the gauge is self-adjoint. We may define a field quadrature by

$$Q(t) = B(t) + B^*(t)$$

and this yields a quantum stochastic process which is essentially classical in the sense that it is self-adjoint and self-commuting, that is $[Q(t), Q(s)] = 0$ for all $t, s \geq 0$. We remark that for each $\theta \in [0, 2\pi)$ we may define quadratures $Q_\theta(t) = e^{-i\theta} B(t) + e^{i\theta} B^*(t)$ which again yield essentially classical processes, however different quadratures will not commute! For the choice of the vacuum state, $\{Q(t) : t \geq 0\}$ then yields a representation of the Wiener process: for real $k(\cdot)$

$$\langle \Omega \mid e^{i \int_0^\infty k(t) dQ(t)} \Omega \rangle = e^{-\frac{1}{2} \int_0^\infty k(t)^2 dt}.$$

TABLE 1. Quantum Itō Table

$\times$	dB	$d\Lambda$	dB^*	dt
dB	0	dB	dt	0
$d\Lambda$	0	$d\Lambda$	dB^*	0
dB^*	0	0	0	0
dt	0	0	0	0

We also note that $\{\Lambda(t) : t \geq 0\}$ is also an essentially classical process and for the choice of a coherent state yields a non-homogeneous Poisson process: for real $k(\cdot)$

$$\langle \Psi(\beta) \mid e^{i\int_0^\infty k(t)d\Lambda(t)} \Psi(\beta) \rangle = \exp \int_0^\infty |\beta(t)|^2 \left(e^{ik(t)} - 1 \right) dt.$$

We consider a quantum mechanical system with Hilbert space $\mathfrak{h}$ being driven by an external quantum field input. the quantum field will be modeled as an idealized Bose field with Hilbert space $\Gamma\left(L^2(\mathbb{R}_+, dt)\right)$ which is the Fock space over the one-particle space $L^2(\mathbb{R}_+, dt)$. Elements of the Fock space may be thought of as vectors $\Psi = \oplus_{n=0}^\infty \psi_n$ where $\psi_n = \psi_n(t_1, \cdots, t_n)$ is a completely symmetric functions with

$$\sum_{n=0}^\infty \int_{[0,\infty)^n} |\psi_n(t_1, \cdots, t_n)|^2 dt_1 \cdots dt_n < \infty.$$

The Hudson-Parthasarathy theory of quantum stochastic calculus gives a generalization of the Itō theory of integration to construct integral processes with respect to the processes of annihilation, creation, gauge and, of course, time. This leads to the quantum Itō table 1.

We remark that the Fock space carries a natural filtration in time obtained from the decomposition $\mathfrak{F} \cong \mathfrak{F}_{t]} \otimes \mathfrak{F}_{(t}$ into past and future subspaces: these are the Fock spaces over $L^2[0,t]$ and $L^2(t,\infty)$ respectively.

2.3. Continuous-Time Quantum Stochastic Evolutions. On the joint space $\mathfrak{h} \otimes \mathfrak{F}$, we consider the quantum stochastic process $V(\cdot)$ satisfying the QSDE

$$dV(t) = \left\{ \begin{array}{c} (S - I) \otimes d\Lambda(t) + L \otimes dB^*(t) \\ -L^* S \otimes dB(t) - (\frac{1}{2} L^* L + iH) \otimes dt \end{array} \right\} V(t),$$

with $V(0) = 1$, and where S is unitary, L is bounded and H self-adjoint. This specific form of QSDE may be termed the *Hudson-Parthasarathy equation* as the algebraic conditions on the coefficients are necessary and sufficient to ensure unitarity (though the restriction for L to be bounded can be lifted). The process is also adapted in the sense that for each $t > 0$, $V(t)$ acts non-trivially on the component $\mathfrak{h} \otimes \mathfrak{F}_{t]}$ and trivially on $\mathfrak{F}_{(t}$.

2.3.1. *The Heisenberg-Langevin Equations.* For a fixed system operator X we set

$$j_t(X) := V^\dagger(t) [X \otimes I] V(t). \tag{2.2}$$

Then from the quantum Itō calculus we get

$$\begin{aligned} dj_t(X) &= j_t(\mathcal{L}_{11} X) \otimes d\Lambda(t) + j_t(\mathcal{L}_{10} X) \otimes dB^*(t)) \\ &\quad + j_t(\mathcal{L}_{01} X) \otimes dB(t) + j_t(\mathcal{L}_{00} X) \otimes dt \end{aligned} \tag{2.3}$$

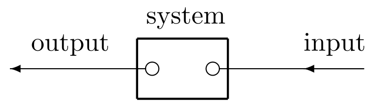

FIGURE 2. Input - Output component

where the Evans-Hudson maps $\mathcal{L}_{\mu\nu}$ are explicitly given by

$$\begin{aligned}
\mathcal{L}_{11}X &= S^*XS - X, \\
\mathcal{L}_{10}X &= S^*[X, L], \\
\mathcal{L}_{01}X &= [L^*, X]S \\
\mathcal{L}_{00}X &= \mathcal{L}_{(L,H)}
\end{aligned}$$

and in particular $\mathcal{L}_{00}$ takes the generic form of a Lindblad generator:

$$\mathcal{L}_{(L,H)} = \frac{1}{2}L^*[X, L] + \frac{1}{2}[L^*, X]L - i\,[X, H]\,. \tag{2.4}$$

2.3.2. *Output Processes.* We introduce the processes

$$\begin{aligned}
B^{\mathrm{out}}(t) &:= V^\dagger(t)\,[I \otimes B(t)]\,V(t)\,, \\
\Lambda^{\mathrm{out}}(t) &:= V^\dagger(t)\,[I \otimes \Lambda(t)]\,V(t)\,.
\end{aligned} \tag{2.5}$$

We note that we equivalently have $B^{\mathrm{out}}(t) \equiv V^\dagger(T)\,[1 \otimes B(t)]\,V(T)$, for $t \leq T$. Again using the quantum Itō rules, we see that

$$\begin{aligned}
dB_{\mathrm{out}} &= j_t(S)dB(t) + j_t(L)dt, \\
d\Lambda_{\mathrm{out}} &= d\Lambda(t) + j_t(L^*S)dB^*(t) + j_t(S^*L)dB(t) + j_t(L^*L)dt.
\end{aligned} \tag{2.6}$$

2.3.3. *The Measurement Algebra.* We wish to consider the problem of continuously measuring a quantum stochastic process associated with the output field. we shall chose to measure an observable process of the form

$$Y^{\mathrm{out}}(t) := V(t)^\dagger[I \otimes Y^{\mathrm{in}}(t)]V(t) \tag{2.7}$$

which corresponds to a quadrature of the field when

$$Y^{\mathrm{in}}(t) = Q(t) = B^*(t) + B(t),$$

or counting the number of output photons when

$$Y^{\mathrm{in}}(t) = \Lambda(t).$$

We introduce von Neumann algebra

$$\mathfrak{Y}^{\mathrm{in}}_{t]} = \mathrm{vN}\left\{Y^{\mathrm{in}}(s) : 0 \leq s \leq t\right\},$$

and define the measurement algebra up to time t to be

$$\mathfrak{Y}^{\mathrm{out}}_{t]} = \mathrm{vN}\left\{Y^{\mathrm{out}}(s) : 0 \leq s \leq t\right\} \equiv V(t)^\dagger\,\mathfrak{Y}^{\mathrm{in}}_{t]}V(t)\,. \tag{2.8}$$

Note that both algebras are commutative:

$$\left[Y^{\mathrm{out}}(t)\,, Y^{\mathrm{out}}(s)\right] = V(T)^\dagger\left(I \otimes \left[Y^{\mathrm{in}}(t)\,, Y^{\mathrm{in}}(s)\right]\right)V(T) = 0$$

for $T = t \vee s$. The family $\left\{ \mathfrak{Y}_{t]}^{\text{out}} : t \geq 0 \right\}$ then forms an increasing family (*filtration*) of von Neumann algebras.

2.3.4. *The Non-Demolition Property.* The system observables may be estimated from the current measurement algebra

$$j_t(X) \in \left(\mathfrak{Y}_{t]}^{\text{out}} \right)'. \tag{2.9}$$

The proof follows from the observation that for $t \geq s$

$$\left[j_t(X), Y^{\text{out}}(s) \right] = V(t)^\dagger \left[X \otimes I, I \otimes Y^{\text{in}}(s) \right] V(t) = 0.$$

2.4. Constructing The Quantum Filter. The filtered estimate for $j_t(X)$ given the measurements of the output field is then

$$\pi_t(X) := \mathbb{E}\left[j_t(X) \mid \mathfrak{Y}_{t]}^{\text{out}} \right].$$

Let $\tilde{\mathbb{E}}_t[X] = \mathbb{E}[j_t(X)]$, then by lemma 2.2

$$\pi_t(X) = \mathbb{E}\left[j_t(X) \mid \mathfrak{Y}_{t]}^{\text{out}} \right] = V(t)^\dagger \tilde{\mathbb{E}}_t \left[X | \mathfrak{Y}_{t]}^{\text{in}} \right] V(t).$$

2.4.1. *Reference Probability Approach.* Suppose that there is an adapted process $F(\cdot)$ such that $F(t) \in \left(\mathfrak{Y}_{t]}^{\text{in}} \right)'$ and $\tilde{\mathbb{E}}_t[X|] = \mathbb{E}\left[F(t)^\dagger (X \otimes 1) F(t) \right]$ for all system operators X, then by lemma 2.3

$$\begin{aligned}
\pi_t(X) &= \mathbb{E}\left[j_t(X) \mid \mathfrak{Y}_{t]}^{\text{out}} \right] \\
&= V(t)^\dagger \tilde{\mathbb{E}}_t \left[X \mid \mathfrak{Y}_{t]}^{\text{in}} \right] V(t) \\
&= V(t)^\dagger \frac{\mathbb{E}\left[F(t)^\dagger (X \otimes 1) F(t) \mid \mathfrak{Y}_{t]}^{\text{in}} \right]}{\mathbb{E}\left[F(t)^\dagger F(t) \mid \mathfrak{Y}_{t]}^{\text{in}} \right]} V(t)
\end{aligned}$$

This is essentially a non-commutative version of the Girsanov transformation from stochastic analysis. The essential feature is that the transformation operators $F(t)$ giving the change of representation for the expectation lie in the commutant of the measurement algebra up to time t.

We therefore obtain an operator-valued Kallianpur-Striebel relation

$$\pi_t(X) = \frac{\sigma_t(X)}{\sigma_t(1)}, \tag{2.10}$$

which may be called the *quantum Kallianpur-Striebel*, where

$$\sigma_t(X) := V(t)^\dagger \mathbb{E}\left[F(t)^\dagger (X \otimes 1) F(t) \mid \mathfrak{Y}_{t]}^{\text{in}} \right] V(t). \tag{2.11}$$

3. Coherent State Filters

We shall consider the class of states

$$\mathbb{E}^{\beta}\left[\,\cdot\,\right] = \langle\psi^{\beta}|\cdot\psi^{\beta}\rangle \tag{3.1}$$

of the form

$$\psi^{\beta} = \phi\otimes\Psi\left(\beta\right) \tag{3.2}$$

where ϕ is a normalized vector in the system Hilbert space and $\Psi\left(\beta\right)$ is the coherent state with test function $\beta\in L^{2}[0,\infty)$. We note that

$$\begin{aligned} dB\left(t\right)\ \Psi\left(\beta\right) &= \beta\left(t\right)dt\ \Psi\left(\beta\right), \\ d\Lambda\left(t\right)\ \Psi\left(\beta\right) &= \beta\left(t\right)dB^{*}(t)\ \Psi\left(\beta\right). \end{aligned}$$

We see that

$$\mathbb{E}^{\beta}[dj_{t}\left(X\right)] = \mathbb{E}^{\beta}[j_{t}\left(\mathcal{L}^{\beta(t)}X\right)]dt \tag{3.3}$$

where

$$\mathcal{L}^{\beta(t)}X = \mathcal{L}_{00}X + \beta\left(t\right)^{*}\mathcal{L}_{10}X + \beta\left(t\right)\mathcal{L}_{01}X + |\beta\left(t\right)|^{2}\mathcal{L}_{11}X \tag{3.4}$$

The generator is again of Lindblad form and in particular we have

$$\mathcal{L}^{\beta(t)} \equiv \mathcal{L}_{\left(L^{\beta(t)},H\right)}$$

with

$$L^{\beta(t)} = S\beta\left(t\right) + L. \tag{3.5}$$

e may define a parameterized family of density matrices on the system by setting $\mathrm{tr}_{\mathfrak{h}}\left\{\varrho_{t}X\right\} = \mathbb{E}^{\beta}[j_{t}\left(X\right)]$, in which case we deduce the master equation

$$\dot{\varrho}_{t} = \mathcal{L}^{\beta(t)\prime}\left(\varrho\right),$$

where the adjoint is defined through the duality $\mathrm{tr}_{\mathfrak{h}}\left\{\varrho\,\mathcal{L}X\right\} = \mathrm{tr}_{\mathfrak{h}}\left\{\mathcal{L}'\varrho\,X\right\}$.

From the input-output relation for the field

$$dB_{\mathrm{out}} = j_{t}\left(S\right)dB\left(t\right) + j_{t}\left(L\right)dt$$

we obtain the average

$$\begin{aligned} \mathbb{E}^{\beta}\left[dB_{\mathrm{out}}\right] &= \left\{j_{t}\left(S\right)\beta\left(t\right) + j_{t}\left(L\right)\right\}dt \\ &= j_{t}\left(L^{\beta(t)}\right)dt. \end{aligned}$$

3.1. Quadrature Measurement. We take $Y^{\mathrm{in}}\left(t\right) = B\left(t\right) + B^{*}\left(t\right)$ which is a quadrature of the input field. Setting $\psi\left(t\right) = V\left(t\right)\psi$ we have that

$$d\psi\left(t\right) = \left(\left(S-1\right)\beta\left(t\right) + L\right)dB^{*}\left(t\right)\psi\left(t\right) - \left(L^{*}S\beta\left(t\right) + \frac{1}{2}L^{*}L + iH\right)dt\,\psi\left(t\right) \tag{3.6}$$

At this stage we apply a trick which is essentially a *quantum Girsanov transformation*. This trick is due to Belavkin [5] and Holevo [20]. We now add a term proportional to $dB\left(t\right)\psi\left(t\right)$ to get

$$\begin{aligned} d\psi(t) &= ((S-1)\beta(t)+L)\left[dB^*(t)+dB(t)\right]\psi(t) \\ &\quad -\left(L^*S\beta(t)+\frac{1}{2}L^*L+iH+((S-1)\beta(t)+L)\beta(t)\right)dt\,\psi(t) \\ &\equiv \tilde{L}_t dY^{\text{in}}(t)\,\psi(t)+\tilde{K}_t dt\,\psi(t), \end{aligned}$$

where

$$\begin{aligned} \tilde{L}_t &= L+(S-I)\beta(t)=L^{\beta(t)}-\beta(t), \\ \tilde{K}_t &= -L^*S\beta(t)-\frac{1}{2}L^*L-iH-L^{\beta(t)}\beta(t)+\beta(t)^2. \end{aligned}$$

It follows that $\psi(t)\equiv F(t)\psi$ where $F(t)$ is the adapted process satisfying the QSDE

$$dF(t)=\tilde{L}_t dY^{\text{in}}(t)\,F(t)+\tilde{K}_t dt\,F(t),\ F(0)=I.$$

Moreover $F(t)$ is in the commutant of $\mathfrak{Y}^{\text{in}}_{t]}$ and therefore allows us to perform the non-commutative Girsanov trick.

From the quantum Itō product rule we then see that

$$\begin{aligned} d\left[F^*(t)XF(t)\right] &= F^*(t)\left(X\tilde{L}_t+\tilde{L}_t^*X\right)F(t)\,dY^{\text{in}}(t) \\ &\quad +F^*(t)\left(\tilde{L}_t^*X\tilde{L}_t+X\tilde{K}_t+\tilde{K}_tX\right)F(t)\,dt \end{aligned}$$

and this leads to the SDE for the un-normalized filter

$$\begin{aligned} d\sigma_t(X) &= \sigma_t\left(X\tilde{L}_t+\tilde{L}_t^*X\right)dY^{\text{out}}(t) \\ &\quad +\sigma_t\left(\tilde{L}_t^*X\tilde{L}_t+X\tilde{K}_t+\tilde{K}_tX\right)dt. \end{aligned}$$

After a small bit of algebra, this may be written in the form

$$d\sigma_t(X)=\sigma_t\left(X\tilde{L}_t+\tilde{L}_t^*X\right)\left[dY^{\text{out}}(t)-\left(\beta(t)+\beta(t)^*\right)dt\right]+\sigma_t\left(\mathcal{L}^{\beta(t)}X\right)dt.$$

This is the quantum Zakai equation for the filter based on continuous measurement of the output field quadrature.

To obtain the quantum Kushner-Stratonovich equation we first observe that the normalization satisfies the SDE

$$d\sigma_t(I)=\sigma_t\left(\tilde{L}_t+\tilde{L}_t^*\right)\left[dY^{\text{out}}(t)-\left(\beta(t)+\beta(t)^*\right)dt\right]$$

and by Itō's formula

$$d\frac{1}{\sigma_t(I)}=-\frac{\sigma_t\left(\tilde{L}_t+\tilde{L}_t^*\right)}{\sigma_t(I)^2}\left[dY^{\text{out}}(t)-\left(\beta(t)+\beta(t)^*\right)dt\right]+\frac{\sigma_t\left(\tilde{L}_t+\tilde{L}_t^*\right)^2}{\sigma_t(I)^3}dt.$$

The product rule then allows us to determine the SDE for $\pi_t(X)=\dfrac{\sigma_t(X)}{\sigma_t(I)}$:

$$d\pi_t(X)=\pi_t\left(\mathcal{L}^{\beta(t)}X\right)+dt\left\{\pi_t\left(X\tilde{L}_t+\tilde{L}_t^*X\right)-\pi_t(X)\,\pi_t\left(\tilde{L}_t+\tilde{L}_t^*\right)\right\}dI(t),$$

where the innovations process satisfies

$$dI\left(t\right)=dY^{\mathrm{out}}\left(t\right)-\left[\pi_{t}\left(\tilde{L}_{t}+\tilde{L}_{t}^{*}\right)+\beta\left(t\right)+\beta\left(t\right)^{*}\right]dt.$$

We note that the innovations is martingale with respect to filtration generated by the output process for the choice of probability measure determined by the coherent state.

3.1.1. *The Quadrature Measurement Filter for a Coherent state.* In it convenient to write the filtering equations in terms of the operators $L^{\beta(t)}$. The result is the *Belavkin-Kushner-Stratonvich equation* for the filtered estimate based on optimal estimation of continuous non-demolition field-quadrature measurements in a coherent state $\beta(\cdot)$.

$$\begin{aligned} d\pi_{t}\left(X\right) &= \pi_{t}\left(\mathcal{L}^{\beta(t)}X\right)dt \\ &\quad+\left\{\pi_{t}\left(XL^{\beta(t)}+L^{\beta(t)*}X\right)-\pi_{t}\left(X\right)\pi_{t}\left(L^{\beta(t)}+L^{\beta(t)*}\right)\right\}dI^{\mathrm{quad}}\left(t\right), \end{aligned} \tag{3.7}$$

with the innovations

$$dI^{\mathrm{quad}}\left(t\right)=dY^{\mathrm{out}}\left(t\right)-\pi_{t}\left(L^{\beta(t)}+L^{\beta(t)*}\right)dt. \tag{3.8}$$

3.2. Photon Counting Measurement. For convenience we shall derive the filter based on measuring the number of output photons under the assumption that the function β is bounded away from zero. We discuss how this restriction can be removed later. We now set $Y^{\mathrm{in}}\left(t\right)=\Lambda\left(t\right)$. We again seek to construct an adapted process $F\left(t\right)$ in the commutant of $\mathfrak{Y}_{t]}^{\mathrm{in}}$ such that $\psi\left(t\right)=F\left(t\right)\psi$. We start with 3.6 again but now note that

$$dB^{*}\left(t\right)\psi\left(t\right)=\frac{1}{\beta\left(t\right)}d\Lambda(t)\psi\left(t\right)$$

and making this substitution gives

$$d\psi\left(t\right)=\frac{1}{\beta\left(t\right)}\tilde{L}_{t}dY^{\mathrm{in}}(t)\psi\left(t\right)-\left(\frac{1}{2}L^{*}L+iH+L^{*}S\beta\left(t\right)\right)dt\,\psi\left(t\right).$$

We are then lead to the Zakai equation

$$\begin{aligned} d\sigma_{t}\left(X\right) &= \frac{1}{|\beta\left(t\right)|^{2}}\sigma_{t}\left(\tilde{L}_{t}^{*}X\tilde{L}_{t}+\beta(t)^{*}X\tilde{L}_{t}+\tilde{L}_{t}^{*}X\beta\left(t\right)\right)dY^{\mathrm{out}}\left(t\right) \\ &\quad-\sigma_{t}\left(\frac{1}{2}XL^{*}L+\frac{1}{2}L^{*}LX+i\left[X,H\right]-XL^{*}S\beta\left(t\right)-\beta\left(t\right)^{*}S^{*}LX\right)dt \end{aligned}$$

which may be rearranged as

$$\begin{aligned} d\sigma_{t}\left(X\right) &= \sigma_{t}\left(\mathcal{L}^{\beta}X\right)dt \\ &\quad+\frac{1}{|\beta\left(t\right)|^{2}}\sigma_{t}\left(\tilde{L}_{t}^{*}X\tilde{L}_{t}+\beta(t)^{*}X\tilde{L}_{t}+\tilde{L}_{t}^{*}X\beta\left(t\right)\right)\left[dY^{\mathrm{out}}\left(t\right)-|\beta\left(t\right)|^{2}dt\right]. \end{aligned}$$

To determine the normalized filter, we note that

$$d\sigma_t(I) = \frac{1}{|\beta(t)|^2}\sigma_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right)\left[dY^{\mathrm{out}}(t) - |\beta(t)|^2 dt\right]$$

and that

$$\begin{aligned} d\frac{1}{\sigma_t(I)} &= \frac{\sigma_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right)}{\sigma_t(I)^2}dt \\ &\quad - \frac{\sigma_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right)}{\sigma_t(I)\left[\sigma_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right) + |\beta(t)|^2\right]}dY^{\mathrm{out}}(t). \end{aligned}$$

Applying the Itō product formula yields the quantum analogue of the Kushner-Stratonovich equation for the normalized filter;

$$\begin{aligned} d\pi_t(X) &= \pi_t\left(\mathcal{L}^\beta X\right)dt + \frac{1}{\pi_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right) + |\beta(t)|^2}dI(t) \\ &\times\left\{\pi_t(\tilde{L}_t^* X\tilde{L}_t + \beta(t)^* X\tilde{L}_t + \tilde{L}_t^* X\beta(t)) - \pi_t(X)\,\pi_t(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t))\right\} \end{aligned}$$

and the innovations process is now

$$dI(t) = dY^{\mathrm{out}}(t) - \left[\pi_t\left(\tilde{L}_t^*\tilde{L}_t + \beta(t)^*\tilde{L}_t + \tilde{L}_t^*\beta(t)\right) + |\beta(t)|^2\right]dt.$$

We note that the innovations is again a martingale with respect to filtration generated by the output process for the choice of probability measure determined by the coherent state.

The derivation above relied on the assumption that $\beta(t) \neq 0$, however this is not actually essential. In the case of a vacuum input, it is possible to apply an additional rotation $W(t)$ satisfying $dW(t) = [z^* dB(t) - z dB^*(t) - \frac{1}{2}|z|^2 dt]W(t)$, with $W(0) = I$, and apply the reference probability technique to the von Neumann algebra generated by $N(t) = W(t)\Lambda(t)W(t)^*$: this leads to a Zakai equation that explicitly depends on the choice of $z \in C$ however the Kushner-Stratonovich equation for the normalized filter will be z-independent. Similarly for the general coherent state considered here, we could take z to be a function of t in which case we must chose z so that $\beta(t) + z(t) \neq 0$. The Kushner-Stratonovich equation obtained will then be identical to what we have just derived.

3.2.1. *Photon Counting Measurement in a Coherent State.* Again it is convenient to re-express the filter in term of $L^{\beta(t)}$. We now obtain the Belavkin-Kushner-Stratonvich equation for the filtered estimate based on optimal estimation of continuous non-demolition field-quanta number measurements in a coherent state $\beta(\cdot)$.

$$d\pi_t(X) = \pi_t\left(\mathcal{L}^{\beta(t)}X\right)dt + \left\{\frac{\pi_t\left(L^{\beta(t)*}XL^{\beta(t)}\right)}{\pi_t\left(L^{\beta(t)*}L^{\beta(t)}\right)} - \pi_t(X)\right\}dI^{\mathrm{num}}(t), \tag{3.9}$$

with the innovations

$$dI^{\mathrm{num}}(t) = dY^{\mathrm{out}}(t) - \pi_t\left(L^{\beta(t)*}L^{\beta(t)}\right)dt. \tag{3.10}$$

4. Characteristic Function Approach

As an alternative to the reference probability approach, we apply a method based on introducing a process $C(t)$ satisfying the QSDE

$$dC(t) = f(t)\,C(t)\,dY(t)\,, \tag{4.1}$$

with initial condition $C(0) = I$. Here we assume that f is integrable, but otherwise arbitrary. This approach is a straightforward extension of a classical procedure and as far as we are aware was first used in the quantum domain by Belavkin [4]. The technique is to make an ansatz of the form

$$d\pi_t(X) = \mathcal{F}_t(X)\,dt + \mathcal{H}_t(X)\,dY(t) \tag{4.2}$$

where we assume that the processes $\mathcal{F}_t(X)$ and $\mathcal{H}_t(X)$ are adapted and lie in $\mathfrak{Y}_{t]}$. These coefficients may be deduced from the identity

$$\mathbb{E}\left[(\pi_t(X) - j_t(X))\,C(t)\right] = 0$$

which is valid since $C(t) \in \mathfrak{Y}_{t]}$. We note that the Itō product rule implies $I + II + III = 0$ where

$$\begin{aligned} I &= \mathbb{E}\left[(d\pi_t(X) - dj_t(X))\,C(t)\right], \\ II &= \mathbb{E}\left[(\pi_t(X) - j_t(X))\,dC(t)\right], \\ III &= \mathbb{E}\left[(d\pi_t(X) - dj_t(X))\,dC(t)\right]. \end{aligned}$$

We illustrate how this works in the case of quadrature and photon counting in a coherent state. For convenience of notation we shall write S_t for $j_t(S)$, etc.

4.1. Quadrature Measurement.

Here we have

$$dY(t) = S_t dB(t) + S_t^* dB(t)^* + (L_t + L_t^*)\,dt$$

so that

$$\begin{aligned} I &= \mathbb{E}_\beta\left[\mathcal{F}_t(X)\,C(t) + \mathcal{H}_t(X)\,(S_t\beta_t + S_t^*\beta_t^* + L_t + L_t^*)\,C(t)\right]dt \\ &\quad -\mathbb{E}_\beta\left[\left\{(\mathcal{L}_{00}X)_t + (\mathcal{L}_{01}X)_t\,\beta_t + (\mathcal{L}_{10}X)_t\,\beta_t^* + (\mathcal{L}_{11}X)_t\,|\beta_t|^2\right\}C(t)\right]dt, \\ II &= \mathbb{E}_\beta\left[(\pi_t(X) - X_t)\,f(t)\,C(t)\,(S_t\beta_t + S_t^*\beta_t^* + L_t + L_t^*)\right]dt, \\ III &= \mathbb{E}_\beta\left[\left\{\mathcal{H}_t(X) - (\mathcal{L}_{01}X)_t\,S_t^*\beta_t^* - (\mathcal{L}_{11}X)_t\,S_t^*\beta_t^*\right\}f(t)\,C(t)\right]dt. \end{aligned}$$

Now from the identity $I + II + III = 0$ we may extract separately the coefficients of $f(t)\,C(t)$ and $C(t)$ as $f(t)$ was arbitrary to deduce

$$\begin{aligned} \pi_t\left((\pi_t(X) - X_t)\,(S_t\beta_t + S_t^*\beta_t^* + L_t + L_t^*)\right) & \\ +\pi_t\left(\mathcal{H}_t(X) - (\mathcal{L}_{01}X)_t\,S_t^*\beta_t^* - (\mathcal{L}_{11}X)_t\,S_t^*\beta_t^*\right) &= 0, \\ \pi_t\left(\mathcal{F}_t(X) + \mathcal{H}_t(X)\,(S_t\beta_t + S_t^*\beta_t^* + L_t + L_t^*) - \left(\mathcal{L}^{\beta(t)}X\right)_t\right) &= 0. \end{aligned}$$

Using the projective property of the conditional expectation ($\pi_t \circ \pi_t = \pi_t$) and the assumption that $\mathcal{F}_t(X)$ and $\mathcal{H}_t(X)$ lie in $\mathfrak{Y}_{t]}$, we find after a little algebra that

$$\begin{aligned}\mathcal{H}_t(X) &= \pi_t\left(XL^{\beta(t)} + L^{\beta(t)*}X\right) - \pi_t(X)\,\pi_t\left(L^{\beta(t)} + L^{\beta(t)*}\right),\\ \mathcal{F}_t(X) &= \pi_t\left(\mathcal{L}^{\beta(t)}X\right) - \mathcal{H}_t(X)\,\pi_t\left(L^{\beta(t)} + L^{\beta(t)*}\right),\end{aligned}$$

so that the equation (4.2) reads as

$$d\pi_t(X) = \pi_t\left(\mathcal{L}^{\beta(t)}X\right)dt + \mathcal{H}_t(X)\left[dY(t) - \pi_t\left(L^{\beta(t)} + L^{\beta(t)*}\right)dt\right].$$

4.2. Photon Counting Measurement. We now have

$$dY(t) = d\Lambda(t) + L_t^* S_t dB(t) + S_t^* L_t dB(t)^* + L_t^* L_t dt$$

so that

$$\begin{aligned} I &= \mathbb{E}_\beta\left[\left\{\mathcal{F}_t(X) + \mathcal{H}_t(X)\left(|\beta_t|^2 + L_t^* S_t \beta_t + S_t^* L_t \beta_t^* + L_t^* L_t\right)\right\} C(t)\right] dt \\ & \quad \mathbb{E}_\beta\left[\left\{-\left(\mathcal{L}^{\beta(t)}X\right)_t\right\} C(t)\right] dt \\ II &= \mathbb{E}_\beta\left[(\pi_t(X) - X_t)\, f(t)\, C(t)\left(|\beta_t|^2 + L_t^* S_t \beta_t + S_t^* L_t \beta_t^* + L_t^* L_t\right)\right] dt, \\ III &= \mathbb{E}_\beta\left[\mathcal{H}_t(X)\left(|\beta_t|^2 + L_t^* S_t \beta_t + S_t^* L_t \beta_t^* + L_t^* L_t\right) f(t)\, C(t)\right] dt \\ & \quad -\mathbb{E}_\beta\left[\left\{(\mathcal{L}_{01}X)_t\, S_t^* L_t + (\mathcal{L}_{01}X)_t\, \beta_t\right\} f(t)\, C(t)\right] dt \\ & \quad -\mathbb{E}_\beta\left[\left\{(\mathcal{L}_{11}X)_t\, |\beta_t|^2 + (\mathcal{L}_{11}X)_t\, S_t^* L_t \beta_t^*\right\} f(t)\, C(t)\right] dt. \end{aligned}$$

This time, the identity $I + II + III = 0$ implies

$$\begin{aligned} &\pi_t\left((\pi_t(X) - X_t)\, L_t^{\beta(t)*} L_t^{\beta(t)} + \mathcal{H}_t(X)\, L_t^{\beta(t)*} L_t^{\beta(t)}\right) \\ &\quad -\pi_t\left((\mathcal{L}_{01}X)_t\, S_t^* L_t^{\beta(t)} + (\mathcal{L}_{11}X)_t\, S_t^* L_t^{\beta(t)} \beta_t^*\right) = 0, \\ &\pi_t\left(\mathcal{F}_t(X) + \mathcal{H}_t(X)\, L_t^{\beta(t)*} L_t^{\beta(t)} - \left(\mathcal{L}^{\beta(t)}X\right)_t\right) = 0. \end{aligned}$$

Again, after a little algebra, we find that

$$\begin{aligned}\mathcal{H}_t(X) &= \frac{\pi_t\left(L^{\beta(t)*} X L^{\beta(t)}\right)}{\pi_t\left(L^{\beta(t)*} L^{\beta(t)}\right)} - \pi_t(X),\\ \mathcal{F}_t(X) &= \pi_t\left(\mathcal{L}^{\beta(t)}X\right) - \mathcal{H}_t(X)\,\pi_t\left(L^{\beta(t)*} L^{\beta(t)}\right),\end{aligned}$$

so that the equation (4.2) reads as

$$d\pi_t(X) = \pi_t\left(\mathcal{L}^{\beta(t)}X\right)dt + \mathcal{H}_t(X)\left[dY(t) - \pi_t\left(L^{\beta(t)*} L^{\beta(t)}\right)dt\right].$$

In both cases, the form of the filter is identical to what we found using the reference probability approach.

5. Conclusion

Both the quadrature filter (3.7) and the photon counting filter (3.9) take on the same form as in to the vacuum case and of course reduce to these filters when we set $\beta \equiv 0$. In both cases it is clear that averaging over the output gives

$$\mathbb{E}^{\beta}\left[d\pi_t\left(X\right)\right] = \mathbb{E}^{\beta}\left[j_t\left(\mathcal{L}^{\beta(t)}X\right)\right]dt$$

which is clearly the correct unconditioned dynamics in agreement with (3.3), and we obtain the correct master equation.

It is worth commenting on the fact that the pair of equations now replacing the dynamical and observation relations (1.1,1.2) are the Heisenberg-Langevin equation (2.3) and the appropriate component of the input-output relation (2.6). The process and observation noise have the same origin however the nature of the quantum filtering based on a non-demolition measurement scheme results in a set of equations that resemble the uncorrelated classical Kushner-Stratonovich equations.

5.1. Is Quantum Filtering still a Pure Filtering Problem? The form of the input-output relations (2.6) might suggest that it is possible to learn something about the system dynamics by examining the quadratic variation of the output process, however, this is not the case! We in fact have an enforced "too good to be true" situation here as the output fields satisfy the same canonical commutation relations as the inputs with the result that the quantum Itō table for the output processes B_{out}, B^*_{out} and Λ_{out} has precisely the same structure as table 1. Therefore we always deal with a pure filtering theory in the quantum models considered here.

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