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QUANTUM QUASI-MARKOV PROCESSES, L-DYNAMICS, AND NONCOMMUTATIVE GIRSANOV TRANSFORMATION

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Abstract. We review the basic concepts of quantum stochastics using the universal Itô *-algebra approach. The main notions and results of classical and quantum stochastics are reformulated in this unifying approach. The thermal quantum Lévy process with commuting increments is defined in terms of the modular *-Itô algebra. The quasi-Markov quantum stochastic processes over increasing W*-algebras generated by a quantum Lévy process is characterized in terms of their quantum stochastic germs. The corresponding quantum stochastic master equation on the increasing predual L-spaces is derived as a noncommutative and non-Markov generalization of the Zakai equation driven by the dual quantum Lévy process. This is done by a noncommutative analog of the Girsanov transformation which we introduce here in full generality.

1. Introduction

Noncommutative dynamics is the unified dynamics of quantum and classical systems on a C*-algebra represented in a Hilbert space Γ with a generating state vector $\eta \in \Gamma$ of norm one. Every 'classical' probability space $(\Omega, \mathfrak{F}, \mathsf{P})$ in the sense of [24] can be canonically represented by a 'quantum' one $(\Gamma, \mathbb{M}, \mathsf{E})$ on the Hilbert space $\Gamma = L^2(\mathsf{P})$ such that $\langle 1|X1\rangle = \int x \mathrm{d}\mathsf{P}$, where X is a 'diagonal' operator $(Xf)(\omega) = x(\omega)f(\omega)$ from the C*-algebra of diagonal operators as pointwise multiplications by $\mathfrak{F}$-measurable bounded complex random variables $x : \Omega \to \mathbb{C}$, and the functional of expectation $\mathsf{E}(X) = \langle 1|X1\rangle$ defined as the linear positive normalized functional on $\mathbb{M}$ by the probability vector $1(\omega) = 1$ for all $\omega \in \Omega$. The converse is true only for commutative $*$-subalgebras $\mathbb{M} \subseteq \mathfrak{L}(\Gamma)$ when all operators have a joint spectrum Ω, i.e. for any Abelian C*-algebra $\mathbb{M}$ [19]. Then one can take the Gel'fand transform $x(\omega) = \omega(\chi) \equiv \check{\chi}(\omega)$ on the vectors $\chi = \mathrm{X}\eta$. This proves considerably greater generality of the non-commutative probability theory, also covering the purely quantum case which corresponds to a simple or irreducible algebra $\mathbb{M} = \mathfrak{L}(\Gamma)$ of all linear continuous operators in a Hilbert space Γ.

The reader should notice that the terms classical and quantum are used here not for continuity or discontinuity of underlying stochastic processes but as an indication of the smallest and largest categories of commutative (Kolmogorov) and

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noncommutative (von Neumann) probability theory of stochastic processes in the unifying representation of Fock space. The word 'classical' applies only to the commutative case when the algebra generating the Hilbert space is minimal, while the word 'quantum' usually applies to the opposite case when the generating C*-algebra is the maximal coinciding with the algebra of all bounded operators in the Hilbert case. In both cases the dynamics can be deterministic, usually smooth, and stochastic, usually diffusive nonsmooth or even discontinuous. While the deterministic smooth quantum dynamics described by a Schrödinger or Heisenberg equation is well-studied in the traditional quantum mechanics and quantum field theory of closed Hamiltonian systems as main objects of the theoretical physics of last century, the nonsmooth quantum jump and diffusive noncommutative dynamics, which has become the major object of study in the experimental physics of individual quantum objects, can be properly described only in the framework of quantum stochastics as a noncommutative generalization of the classical stochastic dynamics.

Classical stochastic calculus, developed by Itô in the mid last Century as a generalization of Newton-Leibniz calculus in [23], was extended in the 80's into the noncommutative domain by Hudson and Parthasarathy who built a quantum stochastic calculus in Fock space [22]. Soon both calculi were unified in a universal Itô $\star$-algebra approach by Belavkin [3], and the unified stochastic analysis in Fock scales was developed in [8][11]. It was applied to quantum statistics of continuous observation by building a noncommutative extension of the Girsanov transformation and quantum nonlinear filtering in [4][6][9].

1.1. Hudson-Parthasarathy Itô calculus. Non-commutative stochastic analysis and calculus appeared in the eighties as a result of the mathematical justification of the notions of quantum white noise and the corresponding 'Langevin equations' discussed by physicists from the sixties onwards in connection with stochastic models of quantum optics and radio-physics [20][21][25]. The first rigorous results in quantum stochastic calculus are due to Hudson and Parthasarathy [22], who gave the quantum Itô formula for multiplication of operator-valued integrals with respect to non-commutating *vacuum martingales* of Bosonic *annihilation* $\mathrm{A}_-(t)$, *gauge* (or vacuum quanta number) $\mathrm{N}(t)$ and *creation* $\mathrm{A}^+(t)$ commuting with their independent increments which satisfy a noncommutative multiplication table

$$\mathrm{dA}_-\mathrm{dA}^+ = \mathrm{I}dt, \quad \mathrm{dA}_-\mathrm{dN} = \mathrm{dA}_- \quad \mathrm{dNdN} = \mathrm{dN}, \quad \mathrm{dNdA}^+ = \mathrm{dA}^+, \tag{1.1}$$

with all other combinations equal to zero. Represented in the symmetric Fock space $\mathcal{F} = \Gamma(\mathcal{K})$ over $\mathcal{K} = L^2(\mathbb{R}_+)$ by noncommuting operators but mutually commuting with the increments at each t with self-adjoint $\mathrm{N} = \mathrm{N}^*$ but $\mathrm{A}^+ = \mathrm{A}_-^*$, they determine three linear-independent self-adjoint combinations

$$\mathrm{B}_0 = \mathrm{N}, \quad \mathrm{B}_1 = \frac{1}{2}\left(\mathrm{A}_- + \mathrm{A}^+\right), \quad \mathrm{B}_2 = \frac{\mathrm{i}}{2}\left(\mathrm{A}_- - \mathrm{A}^+\right) \tag{1.2}$$

with a noncommutative multiplication table induced from (1.1). Each operator-valued function $\mathrm{B}_i(t) \equiv \mathrm{B}_i^t$ represents on the vacuum state vector $\delta_\emptyset \in \mathcal{F}$ a real 'classical' Lévy process $B_i^t = \check{\chi}_i^t$ with zero expectation given by a Gel'fand transformation $\check{\chi}_i^t(\omega) = \omega(\chi_i^t)$ of Fock vector $\chi_i^t = \mathrm{B}_i^t \delta_\emptyset$. The probability law P_i induced on the spectrum Ω_i of each B_i by the characteristic functional

$$\check{\mathsf{P}}_i(g) = \langle 1_\emptyset \mid \mathrm{V}_i(g) 1_\emptyset \rangle \equiv \mathsf{E}_\emptyset[\mathrm{V}_i(g)] \quad i = 0, 1, 2$$

on $\mathrm{V}_i(g) = \exp\left[\mathrm{i} \int g(t) \mathrm{dB}_i^t\right]$ is such that B_1^t and B_2^t are classical Brownian motions identical in distribution, $\mathsf{P}_1 = \mathsf{P}_2$, but B_0^t is even nonstochastic martingale identical in distribution $\mathsf{P}_0 = \delta_0$ to zero. However, due to mutual noncommutativity $[\mathrm{B}_i, \mathrm{B}_k] \neq 0$ they do not have joint probability law $\check{\mathsf{P}}(g_\bullet)$ coinciding with $\mathsf{E}_\emptyset[\mathrm{V}(g_\bullet)]$ for

$$\mathrm{V}(g_\bullet) = \exp\left[\mathrm{i} \sum_i \int g_i(t) \mathrm{dB}_i^t\right].$$

Therefore, these classical martingales cannot be *simultaneously represented* as a vector-valued stochastic process

$$t \mapsto B_\bullet^t(\omega) = \left(B_0^t, B_1^t, B_2^t\right)(\omega) = \check{\chi}_\bullet^t(\omega)$$

in any classical Kolmogorovian probability space $(\Omega, \mathfrak{F}, \mathsf{P})$. If they could, they would have a commutative multiplication table, but due to the noncommutativity they do not have even joint spectrum Ω which is required for the Kolmogorovian functional representation. They are *quantum Lévy noises* as the *quantum martingales with independent increments* with respect to the conditional expectations $\mathsf{E}^t : \mathbb{A} \to \mathbb{A}^t$ on an operator algebra $\mathbb{A} = \mathfrak{L}(\Gamma)$ of multiple *quantum stochastic integrals* generated by $\mathrm{V}(g_\bullet)$ in a bigger, noncommutative probability category. It is defined by the filtration $\mathbb{A}^r \subseteq \mathbb{A}^t$ of the operator algebras $\mathbb{A}^t = \mathfrak{L}(\Gamma_t)$ corresponding to a filtration $\Gamma_r \subseteq \Gamma_t$ for all $r \leq t$, the natural filtration $\{\Gamma_t = \Gamma(\mathcal{K}_t) : t \in \mathbb{R}_+\}$ of the Fock subspaces $\Gamma_t = \mathcal{F}_t$, defined by the subspaces $\mathcal{K}_t \subset \mathcal{K}$ of the integrable functions $f : \mathbb{R}_+ \to \mathbb{C}$ with the support in $[0, t]$ and the unit vector $1_\emptyset \in \cap_{t>0} \mathcal{F}_t$ of the vacuum state $\mathsf{E}[\mathrm{V}] = \langle 1_\emptyset \mid \mathrm{V} 1_\emptyset \rangle$. The triple $(\Gamma, \mathbb{A}, \mathsf{E})$ is said to be a 'quantum probability space', and in general it consists of a Hilbert space Γ, a unital algebra $\mathbb{A}$ of operators in Γ with involution, Hermitian conjugation $\mathrm{V} \longmapsto \mathrm{V}^* \in \mathbb{A}$ and the functional of mathematical expectation $\mathsf{E} : \mathbb{A} \to \mathbb{C}$ defined as the scalar product $\langle 1_\emptyset \mid \varphi \rangle$ of the unit vector $1 \in \Gamma$ and the vector $\varphi = \mathrm{V} 1_\emptyset$.

1.2. HP Itô algebra and matrix notations. Now we introduce the convenient tensor notation $\mathrm{A}_\gamma^\iota = \mathrm{B}^i$ indexing by pairs $i = (\iota, \gamma)$ of two-valued indices $\iota \in \{\circ, +\} \equiv J_+$ and $\gamma \in \{-, \circ\} \equiv J_-$ the canonical quadruple of basic quantum stochastic integrators including $\mathrm{A}_-^+ = \mathrm{B}^\emptyset$ and $\mathrm{A}_\circ^\circ = \mathrm{B}^0$ such that

$$\mathrm{A}_-^+(t) = t\mathrm{I}, \;\; \mathrm{A}_-^\circ = \mathrm{A}_-, \;\; \mathrm{A}_\circ^\circ = \mathrm{N}, \;\; \mathrm{A}_\circ^+ = \mathrm{A}^+. \tag{1.3}$$

Identifying $+ \equiv (+, \circ)$, $0 \equiv (\circ, \circ)$, $- \equiv (\circ, -)$ and $\emptyset \equiv (+, -)$, the Einstein summation convention holds for $\mathrm{B}^i \equiv \mathrm{B}_\gamma^\iota$ as the trace convolution convention

$$\mathrm{K}_i \mathrm{B}^i := \sum_{i \in J_+ \times J_-} \mathrm{K}_i \mathrm{B}^i = \sum_{\iota \in J_+, , \kappa \in J_-} \mathrm{K}_\iota^\gamma \mathrm{B}_\gamma^\iota \equiv \mathrm{K}_\iota^\gamma \mathrm{B}_\gamma^\iota.$$

Note that the rule of identification $\mathrm{K}_{\iota,\gamma} \equiv \mathrm{K}_\iota^\gamma$ is contravariant to $\mathrm{B}^{\iota,\gamma} \equiv \mathrm{B}_\gamma^\iota$. This allows to introduce also the covariant notation $\mathrm{B}_{\sim i} = \mathrm{B}_\gamma^\iota$ in terms of the transposition $\sim (\iota, \gamma) = (\gamma, \iota)$ mapping $\mathbb{J}^\circ = J_+ \times J_-$ onto $\mathbb{J}_\circ = J_- \times J_+$, with the inverse mapping $j \mapsto i$ of $j = (\gamma, \iota)$ also denoted by $\sim j = i$.

In these notation the pseudo-Hermiticity

$$(\mathrm{B}_{\emptyset}, \mathrm{B}_-, \mathrm{B}_0, \mathrm{B}_+)^{\star} = (\mathrm{B}_{\emptyset}, \mathrm{B}_-, \mathrm{B}_0, \mathrm{B}_+) = \left(\mathrm{B}^*_{\emptyset}, \mathrm{B}^*_+, \mathrm{B}^*_0, \mathrm{B}^*_-\right)$$

of the canonical integrators $\mathrm{B}_j = \mathrm{A}^{\iota}_{\gamma}$ is simply written as $\mathrm{B}^{\star}_j = \mathrm{B}_j$ in terms of $\mathrm{B}^{\star}_j := \mathrm{B}^*_{j^{\star}}$ with respect to the involution $j^{\star} = (-\iota, -\gamma)$ given by the transposition of $j = (\gamma, \iota)$ and the reflection $-+ = -$, $-\circ = \circ$, $-- = +$ on the joint index set $J = \{-, \circ, +\} = J_- \cup J_+$ of ι and γ such that $\mathrm{B}^{\iota *}_{\gamma} = \mathrm{B}^{-\gamma}_{-\iota}$.

The convenience of this matrix notation allows to express the HP multiplication table (1.1) for the *canonical basis* $\mathrm{B}^{\nu}_{\mu} = \mathrm{A}^{\nu}_{\mu}$ in the simple form $\mathrm{dA}^{\iota}_{\gamma}\mathrm{dA}^{\lambda}_{\kappa} = \delta^{\iota}_{\kappa}\mathrm{dA}^{\lambda}_{\gamma}$ as suggested in [3][14]. However, it is a particular case of the universal noncommutative Itô multiplication table $\mathrm{dB}_j\mathrm{dB}_k = \varepsilon^m_{j,k}\mathrm{dB}_m$ defining in a pseudo-Hermitian basis $\mathrm{B}^i = \mathrm{B}^*_{-i} = \mathrm{B}_{\sim i}$ the associative QS Itô covariations

$$\mathrm{dX}_t\mathrm{dY}_t = \mathrm{K}_i\varepsilon^{i,l}_{\emptyset}\mathrm{L}_l\mathrm{d}t + \sum_{n\neq\emptyset}\mathrm{K}_i\varepsilon^{i,,l}_n\mathrm{L}_l\mathrm{dB}^n_t = \mathrm{K}_i\varepsilon^{i,l}_n\mathrm{L}_l\mathrm{dB}^n_t \tag{1.4}$$

in terms of the structural constants $\varepsilon^{i,l}_n = \varepsilon^{\sim n}_{\sim l,\sim i}$ for the product of quantum-stochastic differentials $\mathrm{dX}_t = \mathrm{K}_i\mathrm{dB}^i_t$ and $\mathrm{dY}_t = \mathrm{L}_l\mathrm{dB}^l_t$. Here $\varepsilon^{j^{\star},l}_{k^{\star}} = (\varepsilon^{l^{\star},j}_k)^*$ are $\star$-Hermitian structural coefficients which are not necessarily symmetric, $\varepsilon^{i,l}_n \neq \varepsilon^{l,i}_n$ if $\mathrm{dB}^i\mathrm{dB}^l \neq \mathrm{dB}^l\mathrm{dB}^i$, but satisfying the associativity condition corresponding to

$$(\mathrm{dX}_t\mathrm{dY}_t)\,\mathrm{dZ}_t = \mathrm{dX}_t\,(\mathrm{dY}_t\mathrm{dZ}_t)\,.$$

Moreover, these complex coefficients should satisfy a positivity condition corresponding to the semi-positivity $\mathrm{dXdX}^* \geq 0$, where $\mathrm{dX}^*_t = \mathrm{K}^{\star}_i\mathrm{dB}^i_t$ is defined by $\mathrm{K}^{\star}_i = \mathrm{K}^*_{i^{\star}}$, such that the covariance matrix $\boldsymbol{\kappa} = [\varepsilon^{\emptyset}_{j^{\star},k}]$ with $\varepsilon^{\emptyset}_{j^{\star},k}\mathrm{d}t = \mathsf{E}\left[\mathrm{dB}^*_j\mathrm{dB}_k\right]$ should be Hermitian-positive but could be complex due to the noncommutativity even for the Hermitian $\mathrm{B}^*_j = \mathrm{B}_j$.

The HP table (1.1) in the canonical basis (1.3) corresponds to the case

$$\varepsilon^{i,l}_n := \delta^{\mu}_{\gamma}\delta^{\iota}_{\kappa}\delta^{\lambda}_{\nu} \equiv \varepsilon^m_{j,k} \quad \forall j = (\gamma, \iota)\,,\ k = (\kappa, \lambda)\,,\ m = (\mu, \nu)$$

of matrix units multiplication table $\mathbf{e}_j\mathbf{e}_k = \delta^{\iota}_{\kappa}\mathbf{e}^{\lambda}_{\gamma}$ in terms of 3×3-matrices

$$\mathbf{e}^{\lambda}_{\gamma} = \left[e^{\lambda,\mu}_{\gamma,\nu}\right]^{\mu=-,\circ,+}_{\nu=-,\circ,+}, \quad e^{\lambda,\mu}_{\gamma,\nu} = \delta^{\mu}_{\gamma}\delta^{\lambda}_{\nu}$$

with matrix elements of $\mathbf{e}_j \equiv \mathbf{e}^{\iota}_{\gamma}$ and $\mathbf{e}_k \equiv \mathbf{e}^{\lambda}_{\kappa}$ defining the structural constants $\varepsilon^m_{j,k}$ as $e^{\iota,\mu}_{\gamma,\kappa}\delta^{\lambda}_{\nu} = \delta^{\mu}_{\gamma}e^{\lambda,\iota}_{\kappa,\nu}$. The set of these units indexed by $\mathbb{J}_{\circ} \simeq \{\emptyset, -, 0, +\}$ forms the basis for the HP Itô $\star$-algebra canonically represented by the triangular matrices $\mathbf{K} = [\mathrm{K}^{\mu}_{\nu}] = \mathrm{K}^{\gamma}_{\iota}\mathbf{e}^{\iota}_{\gamma}$ with the usual matrix product

$$(\mathbf{KL})^{\mu}_{\nu} = \mathrm{K}_i\varepsilon^{i,l}_n\mathrm{L}_l = \mathrm{K}^{\mu}_{\iota}\mathrm{L}^{\iota}_{\nu}$$

in terms of the elements $\mathrm{K}^{\gamma}_{\iota} \equiv \mathrm{K}_i$ and $\mathrm{L}^{\kappa}_{\lambda} \equiv \mathrm{L}_l$. Thus, the quantum Itô product (1.4) is represented in terms of the matrix product $\mathbf{L}^{\ddagger}\mathbf{L}$ but with unusual Hermitian adjoint $\mathbf{L}^{\ddagger} = \bar{\mathbf{L}}^{\intercal}$ representing the pseudo-adjoint $\mathrm{L}^{\star}_{\cdot} = (\mathrm{L}^*_{\emptyset}, \mathrm{L}^*_+, \mathrm{L}^*_0, \mathrm{L}^*_-)$ of $\mathrm{L}_{\cdot} = (\mathrm{L}_{\emptyset}, \mathrm{L}_-, \mathrm{L}_0, \mathrm{L}_+)$ by matrix elements $\left(\mathbf{L}^{\ddagger}\right)^{\gamma}_{\iota} = \mathrm{L}^*_{i^{\star}} = \bar{\mathrm{L}}^{\intercal}_i$ for $i = (\iota, \gamma) \in J_+ \times J_-$ and zeroth otherwise, given by the usual matrix transposition $\bar{\mathrm{L}}^{\intercal}_i = \bar{\mathrm{L}}^{\iota}_{\gamma}$ of $\bar{\mathrm{L}}^{\iota}_{\gamma} := \mathrm{L}^*_{-j} \equiv \bar{\mathrm{L}}_j$.

1.3. Matrix representation of classical and Itô algebras. The deterministic Newton-Leibniz differential calculus based on the nilpotent multiplication rule $(\mathrm{d}t)^2 = 0$ can be easily extended from smooth to continuous trajectories $x(t)$ having right, say $\mathbb{C}$-valued, derivatives $\kappa(t)$. The differentials $\mathrm{d}x = x(t+\mathrm{d}t) - x(t) = \kappa \mathrm{d}t$ form obviously one-dimensional $\star$-algebra with the trivial products $\kappa \cdot \kappa^* = 0$ representing $|\mathrm{d}x|^2 = 0$. This formal algebra of differential calculus was generalized by Itô [23] to non-smooth continuous diffusions having conditionally independent increments $\mathrm{d}x \propto \sqrt{\mathrm{d}t}$ with no derivative at any t or having the right derivative for almost all t with forward conditionally independent jumps $\mathrm{d}x \propto \{0,1\}$ of the trajectories at some random t with right limits defined as $x(t^+) = x(t) + \mathrm{d}x(t)$ at every t. The first can be achieved by adding the increment of a real Wiener process $w(t) = w(t)^*$ in $\mathrm{d}x = \kappa\mathrm{d}t + \zeta\mathrm{d}w$ with the expectation $\mathsf{E}(\mathrm{d}w) = 0$ and the standard multiplication table

$$(\mathrm{d}w)^2 = \mathrm{d}t,\ \mathrm{d}w\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}w,\ (\mathrm{d}t)^2 = 0.$$

The second can be achieved by adding the increment of a compensated Poisson process $m(t) = \varepsilon n(t) - t/\varepsilon$ in $\mathrm{d}x = \kappa\mathrm{d}t + \zeta\mathrm{d}m$ with the multiplication table

$$(\mathrm{d}m)^2 = \mathrm{d}t + \varepsilon\mathrm{d}m, \quad \mathrm{d}m\mathrm{d}t = 0 = \mathrm{d}t\mathrm{d}m$$

following from $(\mathrm{d}n)^2 = \mathrm{d}n$ for a counting Poisson process $n(t) = n(t)^*$ of the intensity $\lambda(t) > 0$ defining its expectation by $\mathsf{E}(\mathrm{d}n) = \lambda\mathrm{d}t$ and $\varepsilon = \lambda^{1/2}$. These rules are known respectively as the differential multiplications for the standard Wiener process $w(t)$ and for the forward differentials of the standard Poisson process $n(t)$ compensated by its mean value t.

The linear complex spans $b_0 = \kappa d + \zeta e_w$ of formal increments $\{d, e_w\}$, having the only nonzero product $e_w^2 := e_w \cdot e_w = d$ given by a nilpotent in second order element $e_w = e_w^\star$, is two-dimensional commutative $\star$-algebra called the Wiener-Itô algebra $\mathfrak{a}_0$ with the state $l(b) = \kappa$ defining the drift $\mathsf{E}(\mathrm{d}x_0) = \kappa\mathrm{d}t$. Another basic Itô $\star$-algebra is the direct sum $\mathfrak{b}_1 = \mathfrak{d} \oplus \mathfrak{a}$ of the nilpotent $\mathfrak{d}$ and the unital $\star$-algebra $\mathfrak{a} = \mathbb{C}e_n$ with usual multiplication of complex numbers rescalling the unit jumps $e_n = e_n \cdot e_n \equiv e_n^2 \propto \mathrm{d}n$ of the standard Poisson process $n(t) = (m(t) + t/\varepsilon)/\varepsilon$ corresponding to $\varepsilon = 1$. It can be also written in the form of spans $b_1 = \kappa d + \zeta e_m$, where κ is defined by the drift $\mathsf{E}(\mathrm{d}x_1) = \kappa\mathrm{d}t$ of $\mathrm{d}x_1 = \kappa\mathrm{d}t + \zeta\mathrm{d}m$, with the formal increments $e_m = \varepsilon e_n - d/\varepsilon \propto \mathrm{d}m$ of the compensated Poisson process $m(t)$ having the products

$$e_m^2 = d + \varepsilon e_m = \varepsilon^2 p,\ e_m \cdot d = 0 = d \cdot e_m,$$

where $p = p \cdot p^\star$ is the Poisson idempotent $e_n = p = e_n^*$. It is also associative and commutative algebra called the Itô-Poisson algebra with the state $l(b) = \kappa$. As in the case of Newton algebra $\mathfrak{d} = \mathbb{C}d$, the Itô $\star$-algebras $\mathfrak{b}_0$ and $\mathfrak{b}_1$ have no Euclidean operator realization, but they can be represented by nonunital $\star$-algebras of triangular 3×3-matrices $\mathbf{b}_0 = \kappa\mathbf{d} + \zeta\mathbf{e}_w$, $\mathbf{b}_1 = \kappa\mathbf{d} + \zeta\mathbf{e}_m$ in the matrix pseudo-Hermitian basis

$$\mathbf{d} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{e}_w = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \mathbf{e}_m = \begin{bmatrix} 0 & 1 & 0 \\ 0 & \varepsilon & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{d}^{\ddagger}:=\boldsymbol{I}\mathbf{d}^{\dagger}\boldsymbol{I}=\mathbf{d},\quad \mathbf{e}_{w}^{\ddagger}:=\boldsymbol{I}\mathbf{e}_{w}^{\dagger}\boldsymbol{I}=\mathbf{e}_{w},\quad \mathbf{e}_{m}^{\ddagger}:=\boldsymbol{I}\mathbf{e}_{m}^{\dagger}\boldsymbol{I}=\mathbf{e}_{m}\ .$$

with respect to the 'falling identity' (Itô) metric $\boldsymbol{I}=\left[\delta_{-\nu}^{\mu}\right]_{\nu=-,\circ,+}^{\mu=-,\circ,+}$, not to be mixed with the identity $\mathbf{I}=[\delta_{\nu}^{\mu}]$. These commutative nilpotent and idempotent $\star$-algebras $\mathfrak{b}=\mathfrak{b}_0,\mathfrak{b}_1$ are the only possible two-dimensional extensions of the Newton $\star$-algebra $\mathfrak{d}$ such that $\mathfrak{d}\mathfrak{b}=0=\mathfrak{b}\mathfrak{d}$.

The algebraic Lévy-Khinchin theorem [16] states that every classical Itô $\star$-algebra $\mathfrak{b}$ as a commutative extension of $\mathfrak{d}$ can be decomposed into the orthogonal sum $\mathfrak{b}_0+\mathfrak{b}_1$, $\mathfrak{b}_0\mathfrak{b}_1=0=\mathfrak{b}_1\mathfrak{b}_0$ uniquely up to the $\star$-ideal $\mathfrak{d}=\mathfrak{b}_0\cap\mathfrak{b}_1$. In particular, every nilpotent $\mathbf{e}_w$ must be orthogonal to every idempotent $\mathbf{e}_n$ in any classical Itô $\star$-algebra $\mathfrak{b}=\mathfrak{b}_0+\mathfrak{b}_1$, $\mathbf{e}_w\mathbf{e}_n=0=\mathbf{e}_n\mathbf{e}_w$, given simply as $\mathbf{e}_n=\mathbf{e}_m+\mathbf{d}$ with $\varepsilon=1$. This suggests that the minimal triangular matrix implementation of any commutative Itô $d+1$-dimensional algebra $\mathfrak{b}$ in the pseudo-Euclidean space $\mathbb{k}=\mathbb{C}\oplus\mathfrak{k}\oplus\mathbb{C}$, where $\mathfrak{k}=\mathbb{C}^d$ with $d=d_w+d_n$ given by $\dim\mathfrak{b}_0=d_w+1$ and $\dim\mathfrak{b}_1=d_n+1$.

1.4. The $\star$-symmetric basis and duality of modular Itô algebras. Let us now consider a general noncommutative Itô B*-algebra [17], i.e. a Banach subalgebra $\mathfrak{b}\subseteq\mathfrak{b}(\mathfrak{k})$ of block-matrices $\mathbf{b}=[b_{\nu}^{\mu}]$ indexed by $\mu,\nu=-,\circ+$ with $b_{\nu}^{\mu}=0$ if $\mu=+$ or $\nu=-$ and involution $b\mapsto b^{\star}$ represented by $\mathbf{b}^{\ddagger}=\left[\bar{b}_{\mu}^{\nu}\right]=\bar{\mathbf{b}}^{\intercal}$, where $\bar{b}_{\mu}^{-\nu}=b_{-\mu}^{\nu *}$. The linear functional $l:b\mapsto b_{+}^{-}$ describes the 'vacuum' state on the maximal Itô algebra $\mathfrak{b}(\mathfrak{k})$ of quadruples $(b_{\nu}^{\mu})_{\nu=+,\circ}^{\mu=-,\circ}$ with independent values $b_{+}^{-}\in\mathbb{C}$, $b_{+}^{\circ}\in\mathfrak{k}$, $b_{\circ}^{-}\in\mathfrak{k}^{*}$ and $b_{\circ}^{\circ}\in\mathfrak{L}(\mathfrak{k})$ for a Hilbert space $\mathfrak{k}$. The Itô algebra $\mathfrak{b}$ is called modular (non-vacuum, or thermal) if the state

$$l(b)=(\mathbf{v}_{\emptyset}|\mathbf{b}\mathbf{v}_{\emptyset}):=\mathbf{v}_{\emptyset}^{\ddagger}\mathbf{b}\mathbf{v}_{\emptyset}=b_{+}^{-},$$

induced in the canonical representation by the 'vacuum' vector $\mathbf{v}_{\emptyset}=\left(\delta_{+}^{\mu}\right)$ with the covector $\mathbf{v}_{\emptyset}^{\ddagger}=(1,0,0)=\mathbf{v}_{\emptyset}^{\dagger}\boldsymbol{I}$, has the faithfulness property $\left(\mathbf{b}^{\ddagger}\mathbf{b}\right)_{+}^{-}=0\Rightarrow b=0$ on $\mathfrak{b}_{\circ}=\{b-l(b)\,d:b\in\mathfrak{b}\}$, as it is always assumed by default in the commutative case. This makes from the quotient $\mathfrak{b}/\mathfrak{d}\simeq\mathfrak{b}_{\circ}$ a left Hilbert $\star$-algebra dense in the Hilbert subspace $\mathfrak{k}_{\circ}\subseteq\mathfrak{k}$ generated by $\mathfrak{b}\mathbf{v}_{\emptyset}$ with respect to the scalar product

$$(b|b):=(\mathbf{b}\mathbf{v}_{\emptyset}|\mathbf{b}\mathbf{v}_{\emptyset})=\left(\mathbf{b}^{\ddagger}\mathbf{b}\right)_{+}^{-}\equiv b_{+}^{\circ *}b_{+}^{\circ}\geq 0\quad\forall b\in\mathfrak{b}_{\circ}\ .$$

Let $\mathfrak{k}$ be separable and $J_{\circ}=J_{\circ}^{\star}$ be an index set with involution $j=i^{\star}\Leftrightarrow i=j^{\star}$ for a $\star$-symmetric basis

$$e_{J_{\circ}}:=\left(e_{j}\right)_{j=J_{\circ}},\quad e_{i}^{\star}=e_{i^{\star}}\in\mathfrak{b}_{\circ}\ \ \forall i,j\in J_{\circ}\ .$$

A basis $e_{J_{\circ}}=\left(e_{j}\right)_{j\neq\emptyset}$ in $\mathfrak{b}_{\circ}$ is called *standard* if the Hermitian semipositive Gram matrix $\boldsymbol{\kappa}=[\kappa_{i,j}]$ of

$$\kappa_{i,j}:=e_{i}^{\circ *}e_{j}^{\circ}=l\left(e_{i}^{\star}\cdot e_{j}\right)\equiv\sigma_{i^{\star},j}\tag{1.5}$$

for the basic vectors $e_{i}^{\circ}=\left(\mathbf{e}_{i}\right)_{+}^{\circ}\in\mathfrak{k}_{\circ}$ has quasiinverse $\tilde{\boldsymbol{\kappa}}=\boldsymbol{\kappa}_{\star}^{*}$ as its $\star$-conjugation $\tilde{\kappa}^{i,j}=\kappa_{i^{\star},j^{\star}}^{*}$, the complex conjugation of $\boldsymbol{\kappa}_{\star}=[\sigma_{i,j^{\star}}]$ corresponding to the Hermitian adjoint covariance matrix $\boldsymbol{\sigma}_{\star}=[\sigma_{i^{\star},j^{\star}}]=\boldsymbol{\sigma}^{\dagger}$.

The Gram matrix $\boldsymbol{\kappa}$ of a modular nonstandard basis has quasiinverse $\tilde{\boldsymbol{\kappa}}$ with a common $\star$-real support $\tilde{\boldsymbol{\kappa}}\boldsymbol{\kappa} = \boldsymbol{\pi} = \boldsymbol{\kappa}\tilde{\boldsymbol{\kappa}}$ defined as the minimal $\star$-real orthoprojection matrix $\boldsymbol{\pi} = [\pi^i_j] = \boldsymbol{\pi}^*_\star$ such that

$$\boldsymbol{\kappa}\boldsymbol{\pi} = \boldsymbol{\kappa}, \quad \boldsymbol{\pi}\tilde{\boldsymbol{\kappa}} = \tilde{\boldsymbol{\kappa}}, \quad \boldsymbol{\pi}^2 = \boldsymbol{\pi} = \boldsymbol{\pi}^\dagger, \quad \boldsymbol{\pi}^\dagger = \boldsymbol{\pi}^{*\intercal}. \tag{1.6}$$

If the basis is not overcomplete, then $\boldsymbol{\pi} = \mathbf{1}$ such that for the standard basis $\boldsymbol{\kappa}^* = \boldsymbol{\kappa}_\star^{-1}$, otherwise $\boldsymbol{\kappa}^*_\star \tilde{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\tilde{\boldsymbol{\kappa}}$, where $\tilde{\boldsymbol{\gamma}} = \tilde{\boldsymbol{\gamma}}^*_\star$ is quasiinverse to a $\star$-real Hermitian matrix $\boldsymbol{\gamma} = [\gamma_{i,j}] = \boldsymbol{\gamma}^\dagger$ with $\tilde{\boldsymbol{\gamma}}\boldsymbol{\gamma} = \boldsymbol{\pi} = \boldsymbol{\gamma}\tilde{\boldsymbol{\gamma}}$. One can take semipositve such $\boldsymbol{\gamma}$, uniquely defined as the geometric mean

$$\boldsymbol{\gamma} = \boldsymbol{\kappa}^{\frac{1}{2}} \left(\tilde{\boldsymbol{\kappa}}^{\frac{1}{2}} \boldsymbol{\kappa}^*_\star \tilde{\boldsymbol{\kappa}}^{\frac{1}{2}} \right)^{\frac{1}{2}} \boldsymbol{\kappa}^{\frac{1}{2}}$$

of $\boldsymbol{\kappa}^*_\star$ and $\boldsymbol{\kappa}$. The semipositive matrix $\tilde{\boldsymbol{\kappa}}$ is dual Gram matrix

$$\tilde{\kappa}^{i,j} := \bar{e}^i_\circ \bar{e}^{j*}_\circ = l\left(\bar{e}^i \cdot \bar{e}^{j\star}\right) \equiv \tilde{\sigma}^{i^\star,j}$$

for adjoint covectors $\bar{e}^i_\circ = \left(\bar{\mathbf{e}}^i\right)^-_\circ \in \mathfrak{k}^*_\circ$ such that $\overline{e_i} = \gamma_{i,j}\bar{e}^j$ is the modular $\star$-conjugation $\overline{e_{i^\star}} = \overline{e_i}^\star$ represented by $\overline{\mathbf{e}_i} = \mathrm{J}^*\mathbf{e}_i\mathrm{J}$ in terms of an antilinear isometry $\mathrm{J}^* = s\mathrm{J} = \mathrm{J}^{-1}$, where $s = s^\ddagger = \overline{s}$ is any signature $s^2 = 1$ commuting with $\mathfrak{b}$. The dual $\star$-Hermitian matrix $\tilde{\boldsymbol{\sigma}} = \left[\tilde{\sigma}^{i,j}\right] = \tilde{\boldsymbol{\sigma}}^\dagger_\star$ is defined by the covariances $\tilde{\sigma}^{i,j} = l\left(\tilde{e}^i \cdot \tilde{e}^j\right)$ of $\tilde{e}^i = \tilde{\gamma}^{i,j}\widetilde{e_j}$ like $\boldsymbol{\sigma} = [\sigma_{i,j}] = \boldsymbol{\sigma}^\dagger_\star$ by (1.5), where $\widetilde{e_i} = \overline{e_i^\star}$ represented by $\widetilde{\mathbf{e}_i} = \mathrm{J}^*\mathbf{e}_i^\ddagger\mathrm{J}$.

In the classical case we have always $\boldsymbol{\sigma}^*_\star = \boldsymbol{\sigma}$ due to the usual symmetricity $\boldsymbol{\sigma} = \boldsymbol{\sigma}^\intercal$ following from the commutativity $e_i \cdot e_j = e_j \cdot e_i$. Therefore in the standard basis $\tilde{\boldsymbol{\kappa}} = \boldsymbol{\kappa}$ corresponding to orthonormality $\boldsymbol{\kappa} = \mathbf{1}$ of the basis $e^\circ_{J_\circ}$ in $\mathfrak{k}_\circ$, or, if the classical standard basis is overcomplete, $\boldsymbol{\kappa}$ satisfies the projectivity condition $\boldsymbol{\kappa} = \boldsymbol{\pi} = \overline{\boldsymbol{\kappa}}$ corresponding to the normality of overcomplete basis in $\mathfrak{k}_\circ$. Thus, the classical standard basis is always selfdual in the sense that the completeness condition

$$e^\circ_j \bar{e}^j_\circ = \mathrm{I}^\circ_\circ = \bar{e}^\circ_j e^j_\circ \tag{1.7}$$

in the Hilbert space $\mathfrak{k}_\circ$ is satisfied for $\bar{e}^i = e^{i^\star}$ i.e. $\tilde{e}^i = e^i$ implying symmetricity $\widetilde{\mathbf{e}_i} = \mathbf{e}_i$ of the matrix representation $\mathfrak{b} \subset \mathfrak{b}\left(\mathfrak{k}_\circ\right)$ for the abelian $\mathfrak{b}$ in any $\star$-Hermitian basis.

For the general modular algebra the standard basis in $\mathfrak{k}_\circ$ cannot be selfdual, unless the Itô state l is tracial,

$$l\left(e_i^\star \cdot e_k\right) = l\left(e_k \cdot e_i^\star\right) \quad \forall i.k \in J_\circ \ .$$

which does not imply the commutativity $e_i^\star \cdot e_k = e_k \cdot e_i^\star$. However, one can prove that the *dual Itô algebra* $\overline{\mathfrak{b}} = \widetilde{\mathfrak{b}}$ generated in $\mathfrak{b}\left(\mathfrak{k}_\circ\right)$ by the basis $\mathbf{e}^J$ of the transposed matrices $\widetilde{\mathbf{e}}_i = \overline{\mathbf{e}_{i^\star}}$, commutes in the standard representation with the modular Itô algebra $\mathfrak{b}$ since $\bar{\mathbf{e}}^i\mathbf{e}_k = \mathbf{e}_k\bar{\mathbf{e}}^i$ due to $\pi^{i^\star}_{k^\star} = \pi^{i*}_k$ for $\pi^i_k = \left(\bar{\mathbf{e}}^i\mathbf{e}_k\right)^-_+ = \bar{e}^i_\circ e^\circ_k$ and $\mathbf{m}^{i^\star}_{k^\star} = \mathbf{m}^{i\ddagger}_k$ for the 'martingale' part $\mathbf{m}^i_k = \tilde{\mathbf{e}}^i\mathbf{e}_k - \pi^i_k\mathbf{d}$ of $\bar{\mathbf{e}}^i\mathbf{e}_k$ in the following multiplication table

$$\mathbf{e}_i\mathbf{e}_k = \sigma_{i,k}\mathbf{d} + \varepsilon^j_{i,k}\mathbf{e}_j, \ \bar{\mathbf{e}}^i\mathbf{e}_k = \pi^i_k\mathbf{d} + \mathbf{m}^i_k, \ \bar{\mathbf{e}}^i\bar{\mathbf{e}}^k = \bar{\sigma}^{i,k}\mathbf{d} + \bar{\varepsilon}^{i,k}_j\bar{\mathbf{e}}^j. \tag{1.8}$$

Here $\sigma_{i,k} = \sigma^*_{k^\star,i^\star}$ such that $\bar{\boldsymbol{\sigma}} = \bar{\boldsymbol{\sigma}}^\dagger_\star \equiv \tilde{\boldsymbol{\sigma}}_\star$ in the standard basis is $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}^\intercal_\star$, $\boldsymbol{\pi} = \left[\pi^i_k\right] = \widetilde{\boldsymbol{\pi}}$ is $\star$-symmetric orthoprojector, $\varepsilon^j_{i,k} = \varepsilon^{j^\star *}_{k^\star,i^\star}$ are associative structural coefficients, $\bar{\boldsymbol{\varepsilon}} = \bar{\boldsymbol{\varepsilon}}^\dagger_\star \equiv \tilde{\boldsymbol{\varepsilon}}_\star$ coincide for the standard basis with $\boldsymbol{\varepsilon}^* = \boldsymbol{\varepsilon}^\intercal_\star$. Note that $\boldsymbol{\gamma}_\star \bar{\boldsymbol{\varepsilon}}_i \boldsymbol{\gamma} = \gamma_{i,j}\boldsymbol{\varepsilon}^{j*}$, including $\boldsymbol{\gamma}_\star \bar{\boldsymbol{\sigma}} \boldsymbol{\gamma} = \boldsymbol{\sigma}^*$ as the case $\bar{\boldsymbol{\varepsilon}}_\emptyset = \bar{\boldsymbol{\sigma}}$ for $i = \emptyset$, where $\boldsymbol{\varepsilon}^*$ coincides in the commutative case $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^\intercal$ with $\boldsymbol{\varepsilon}^\dagger = \left[\varepsilon^{j*}_{k,i}\right] = \boldsymbol{\varepsilon}_\star$.

The components of the quantized vector-processes $\hat{e}_j(t) = \Lambda(t, \mathbf{e}_j)$ and $\check{e}^j(t) = \Lambda\left(t, \bar{\mathbf{e}}^j\right)$, defined on the Fock-Guichardet space $F = \Gamma(K)$ over $K = \mathfrak{k} \otimes L^2(\mathbb{R}_+) \equiv L^2_{\mathfrak{k}}$ as in [11][14], in general do not commute with their adjoints but commute with their tensor-independent increments as in the classical Lévy case. However, the dual quantum standard Lévy processes $\hat{e}_{J_\circ}(t)$, $\check{e}^{J_\circ}(t)$ do not coincide as in the classical case but are *maximally entangled* having mutually commuting maximally correlated $\star$-Hermitian components $\hat{e}_{i^\star} = \hat{e}^*_i$, $\check{e}^{i^\star} = \check{e}^{i*}$. They generate the dual W*-filtration of maximal mutually commuting W*-algebras $\mathbb{B}^t_0$ and $\widetilde{\mathbb{B}}^t_0 = \mathrm{J}^* \mathbb{B}^t_0 \mathrm{J}$ on the Fock filtration over $K^t_0 = L^2[0,t]$ which are antiisomorphic by the transposition $\widetilde{\mathrm{B}} = \overline{\mathrm{B}}^*$, and both have vacuum vector as a separating cyclic vector common in all $F^t_0 = \Gamma(K^t_0)$ inducing a faithful complex-conjugated white noise thermostate consistent on all $\mathbb{B}^t_0$ and $\widetilde{\mathbb{B}}^t_0$.

2. Quantum Itô Semimartingales and Stochastic Covariations

2.1. Quantum filtration and quantum martingales. Here we use some definitions and facts from noncommutative quantum probability summarized in the Appendix.

Let $\mathbb{B}^r \subseteq \mathbb{B}^t$ for $r < t \in \mathbb{T}$ denote the increasing unital operator subalgebras which are assumed to be modular, or contain weakly dense modular subalgebras $\mathbb{L}^r_\star \subseteq \mathbb{L}^t_\star$ as preduals of their L^1-completions $\mathbb{L}^r \subseteq \mathbb{L}^t$ with respect to the symmetric pairings $\langle \mathrm{q} \cdot \mathrm{p} \rangle^r_\mu = \mu^r(\mathrm{q} \cdot \mathrm{p})$ of $\mathbb{B}^r =$ and $\mathbb{L}^r$ induced by the reference weights $\mu^r = \mu^t | \mathbb{L}^r$. An adapted $\mathbb{B}$-process with respect to the *filtration* $(\mathbb{B}^t)$ is described by a function $\mathrm{Q}(t) = \mathrm{q}^t \in \mathbb{B}^t$ mapping a totally ordered set $\mathbb{T}$ into the corresponding $*$-subalgebras of $\mathbb{B} := \cup \mathbb{B}^t$. The embeddings $\iota^t(r) : \mathbb{L}^r_\star \hookrightarrow \mathbb{L}^t_\star$ are assumed to be regular such that they are adjointable, $\iota^t(r) := \mu^r(t)_\star \equiv \mu^t_\star(r)$ in the sense

$$(\mathrm{q}|\mu^r(t,\mathrm{p}))^r_\mu := \langle \mathrm{q}^* \cdot \mathrm{p} \rangle^t_\mu \equiv \left(\iota^t(r,\mathrm{q})|\mathrm{p}\right)^t_\mu \quad \forall \mathrm{q} \in \mathbb{B}^r, \mathrm{p} \in \mathbb{L}^t$$

with respect to the symmetric pairing given by μ. Then the adjoint comorphisms $\mu^r(t) := \iota^t(r)^\star_\mu$ form a *forward hemigroup* of surjections $\mu^r(t) : \mathbb{L}^t \to \mathbb{L}^r$ in the sense

$$\mu^r(s) \circ \mu^s(t) = \mu^r(t) \quad \forall r < s < t,$$

such that $\mu^r = \mu^r(t) \circ \mu^t$ for all *localizations* $\mu^t : \mathbb{L} \to \mathbb{L}^t$ at $t \geq r$ defined on $\mathbb{L} = \mathbb{B}^\star$ as the comorphisms onto $\mathbb{L}^t$ adjoint to the embeddings $\iota(t) : \mathbb{B}^t \hookrightarrow \mathbb{B}$. If each $\mathbb{B}^t$ is invariant with respect to the involution $\dagger = \dagger_\mu$, then, obviously $\dagger^t = \dagger | \mathbb{B}^t$, and the localizations are simply restrictions $\mu^r(t) = \mathsf{E}^r | \mathbb{L}^t$ of the conditional expectations $\mathsf{E}^r : \mathbb{L} \to \mathbb{L}^r$ as the positive projections onto $\mathbb{L}^r$ embedded into the L^1-completion $\mathbb{L}$ of $\mathbb{B} \supseteq \mathbb{B}^r$.

An $\mathbb{L}$-adapted process $\mathrm{Z}(t) = \mathrm{z}^t \in \mathbb{L}^t$ is called L^1-martingale if

$$\mu_r\left(t, \mathrm{z}^t\right) = \mathrm{z}^r \;\; (\text{or } \mathsf{E}^r\left[\mathrm{z}^t\right] = \mathrm{z}^r) \; \forall r < t,$$

and is called global martingale if $\mathrm{z}^t = \mu^t(\mathrm{z})$ (or $\mathrm{z}^t = \mathsf{E}^t[\mathrm{z}]$). Every positive normalized L^1-martingale $t \mapsto \mathrm{p}^t$ defines a state $\pi^t(\mathrm{q}) = \langle \mathrm{q} \cdot \mathrm{p}^t \rangle$ on $\mathbb{B}$ which is called global if p^t is the global martingale given by a positive normalized $\mathrm{p} \in \mathbb{L}$.

2.2. Quantum adapted processes and martingales. From now on we assume the invariance of all $\mathbb{B}^t \subseteq \mathbb{B}$ with respect to left (and right) modular involutions with respect to the symmetric pairing. Then the weight $\mu(\mathrm{p}) = \langle 1 \cdot \mathrm{p} \rangle$ on $\mathbb{L}$ admits the *compatible conditional expectations* $\mathsf{E}^t : \mathbb{L} \mapsto \mathbb{L}^t$ as positive projections such that $\mathsf{E}^s \circ \mathsf{E}^t = \mathsf{E}^s \ \forall\, r, t \in \mathbb{R}_+$ onto $\mathbb{L}^t = \mathbb{M}^t_\bigstar \subseteq \mathbb{L}$:

$$\mathsf{E}^s_\mu\left(1^t\right) = 1^s,\ \mathsf{E}^t_\mu\left(\mathrm{a}^*\mathrm{pa}\right) = \mathrm{a}^*\mathsf{E}^t_\mu(\mathrm{p})\,\mathrm{a} \quad \forall \mathrm{p} \in \mathbb{L}, \mathrm{a} \in \mathbb{B}^t.$$

Let $\mathcal{B}^t$ denote the $*$-algebras of *adapted* C-processes as continuous functions $\mathrm{Q} : r \to \mathrm{Q}(r) \in \mathbb{B}^r$ mapping $r \in [0, t[$ into the modular subalgebras $\mathbb{B}^r \subseteq \mathbb{B}^t$ such that $\mathcal{B}^s \preceq \mathcal{B}^t$ if $s < t$. Then the dual space $\mathcal{L} = \mathcal{B}^\star_\mu \equiv \mathcal{L}_\mu$ to $\mathcal{B} = \curlyvee_{t>0} \mathcal{B}^t$ with respect to the integral pairing

$$\langle \mathrm{Q} \cdot \mathrm{P} \rangle_\mu := \int \langle \mathrm{Q}(t) \cdot \mathrm{P}(t) \rangle \,\mathrm{d}t \quad \forall \mathrm{Q} \in \mathcal{B}, \mathrm{P} \in \mathcal{L}_\mu$$

consists of all locally integrable QS adapted processes $\mathrm{P}(t) \in \mathbb{L}^t$ dominated by $\mathrm{I}(t) = 1^t$.

An adapted QS process $\mathrm{Z} : t \mapsto \mathrm{Z}(t) \in \mathbb{L}^t$ is called L^1-process if $\mathrm{Z} \in \mathcal{L}$, *locally bounded* (*contractive*) if $\mathrm{Z}^*\mathrm{Z} \leq p\mathrm{I}$ for a positive $p\mathrm{I}$ (for $p = 1$). The process Z is called L^1-*martingale (supermartingale)* if

$$\mathsf{E}^r[\mathrm{Z}(t)] = \mathrm{Z}(r) \quad (\mathsf{E}^r[\mathrm{Z}(t)] \leq \mathrm{Z}(r)) \quad \forall r < t.$$

Absolutely continuous QS processes are described as the indefinite integrals $\mathrm{Y}(t) = \mathrm{Y}_0 + \int_0^t \mathrm{L}(s)\,\mathrm{d}s$ of L^1-process $\mathrm{L} \in \mathcal{L}$. They have finite variation norm $\int_0^t \|\mathrm{L}(s)\|_\star \,\mathrm{d}s$ for each $t < \infty$.

2.3. Quantum Itô semimartingales. We consider adapted *quantum Itô L^1-processes* Z defined as the *special semimartingales* by

$$\mathrm{Z}(t) - \mathrm{Z}(r) = \int_r^t \boldsymbol{K}(s) \centerdot \mathrm{d}\boldsymbol{B}(s) \equiv \imath^t_r(\boldsymbol{K}).$$

Here $\boldsymbol{K} = (\mathrm{K}_i)_{i \in \mathbb{J}_\circ}$ are adapted integrands indexed by a separable set $\mathbb{J}_\circ$ with an isolated point $\emptyset \in \mathbb{J}_\circ$ invariant under an involution $\star : \mathbb{J}_\circ \to \mathbb{J}_\circ$ such that $J^\star_\circ = J_\circ$ for $J_\circ = \mathbb{J}_\circ \setminus \emptyset$ and

$$\imath^t_r(\boldsymbol{K}) = \int_r^t \mathrm{K}_i(s)\,\mathrm{d}\mathrm{B}^i(s) + \int_r^t \mathrm{K}_\emptyset(s)\,\mathrm{d}s,$$

where $\mathrm{B}^i(t) \in \mathbb{L}^t$ are not necessarily canonical QS martingales with QS differentials forming a basis for an Itô algebra for each t. The finite variation part

$$\int_r^t \mathrm{K}_\emptyset(s)\,\mathrm{d}s := \int_r^t \epsilon^s(\mathrm{d}\mathrm{Z}(s)) \equiv \epsilon^t_r[\mathrm{Z}(t) - \mathrm{Z}(r)]$$

is given by the deterministic integrator $\mathrm{B}^{\emptyset}(t) = t1 = \mathrm{B}^{\emptyset}(t)^*$ defining it as absolutely continuous process

$$\epsilon_0^t [\mathrm{Z}](t) = \mathrm{Z}_0 + \int_0^t \mathrm{K}_{\emptyset}(s)\,\mathrm{d}s$$

for $\mathrm{K}_{\emptyset}(\cdot) :\in \mathcal{L}$. We assume that the martingale basis is $\star$-Hermitian in the sense

$$\mathrm{B}_i^{\star}(t) := \mathrm{B}_{i^{\star}}(t,)^* = \mathrm{B}_i(t) \quad \forall i \in J_{\circ}$$

2.4. Quantum stochastic covariation. We do not assume the independence of the increments $\mathrm{dB}_i(t)$ with $\mathbb{L}^t$ but shall assume the commutativity

$$\mathrm{Z}(t)\,\mathrm{dB}_i(t) = \mathrm{dB}_i(t)\,\mathrm{Z}(t) \quad \forall \mathrm{Z}(t) \in \mathbb{L}^t$$

as part of the adaptedness of $\mathrm{B}_i(t)$. This implies $\imath(\boldsymbol{K})^* = \imath(\boldsymbol{K}^{\star})$, where $\mathrm{K}^{\star}(t) = \mathrm{K}_{i^{\star}}(t)^*$, and therefore $\mathrm{Z}^*(t) = \mathrm{Z}_0^* + \imath_0^t(\boldsymbol{K}^{\star})$.

Moreover, we shall assume the *associativity*

$$[\mathrm{X} \cdot [\mathrm{Y} \cdot \mathrm{Z}]] = [[\mathrm{X} \cdot \mathrm{Y}] \cdot \mathrm{Z}]$$

of QS integrals in terms of *quantum stochastic covariation*

$$[\mathrm{Y} \cdot \mathrm{Z}]_0^t := \int_0^t [\mathrm{d}(\mathrm{YZ}) - (\mathrm{dY})\,\mathrm{Z} - \mathrm{Y}\,(\mathrm{dZ})](s)$$

such that it can be written in terms of an Itô product

$$(\boldsymbol{K} \centerdot \mathrm{d}\boldsymbol{B})(\boldsymbol{L} \centerdot \mathrm{d}\boldsymbol{B}) = (\boldsymbol{K} \cdot \boldsymbol{L}) \centerdot \mathrm{d}\boldsymbol{B}$$

of noncommuting $\mathrm{dY} = \boldsymbol{K} \centerdot \mathrm{d}\boldsymbol{B}$ and $\mathrm{dZ} = \boldsymbol{L} \centerdot \mathrm{d}\boldsymbol{B}$ as

$$[\mathrm{Y} \cdot \mathrm{Z}](t) = \int_0^t \mathrm{dYdZ} = \int_0^t (\boldsymbol{K} \cdot \boldsymbol{L}) \centerdot \mathrm{d}\boldsymbol{B}.$$

Thus, we assume that quantum Itô semimartingales form an integral $*$-algebra with respect to the bracket $[\mathrm{Y} \cdot \mathrm{Z}]$ given by an associative quantum Itô $\star$-algebra of the corresponding QS integrands $\boldsymbol{K}(t)$ defined as the QS derivatives $\boldsymbol{D}_{\mathrm{Z}}(t)$ of Z at t.[13].One can prove similar as in [17][18] that every integral Itô $\star$-algebra has also the canonical triangular block matrix representation in the B*-algebra $\mathfrak{b}(\mathcal{K})$ of a two-sided Hilbert-Schmidt module in place of the Hilbert space $\mathfrak{k}$ for the Lévy-Itô algebra at each time t.

3. Quantum Hidden Dynamics and QS Filtering

3.1. Quantum object-output stochastic processes. Let $\mathbb{M}_0^{t]} = \mathbb{M}^{\mathrm{o}} \bar{\otimes} \mathbb{M}_0^t$ denote the W*-product algebra generated on the Hilbert product $H_0^{t]} = \mathfrak{h} \otimes H^t$ by the present (quantum object) algebra $\mathbb{M}^{\mathrm{o}} \subseteq \mathcal{B}(\mathfrak{h})$, say the matrix algebra as operator algebra on $\mathfrak{h} = \mathbb{C}^d$, and the past (classical output) algebra $\mathbb{M}_0^t \subseteq \mathcal{B}(H_0^t)$, say generated by multiplication operators $Y_0^t : g_0^t \mapsto y_0^t g_0^t$ for adapted $\mathbb{C}$-valued functionals $y_0^t \in M_0^t$, $g_0^t \in H_0^t$ of classical Lévy process trajectories $\upsilon_0^t = \{\upsilon(r) : r \in [0, t)\}$ such that $M_0^t \subseteq L_{\mathsf{Q}}^{\infty}(\Upsilon, \mathfrak{B})$, $H_0^t \subseteq L_{\mathsf{Q}}^2(\Upsilon, \mathfrak{B})$ on the Lévy probability space $(\Upsilon, \mathfrak{B}, \mathsf{Q})$. The latter example has the split property $\mathbb{M}_0^{t+s} = \mathbb{M}_0^t \bar{\otimes} \mathbb{M}_t^s$ on $H_0^{t+s} = H_0^t \otimes H_t^s$ for every t and $s > 0$, and we shall assume this infinite divisibility not only for the

Abelian algebras $\mathbb{M}_0^t \simeq M_0^t$ but also for the general W*-algebras $\mathbb{M}_0^t$, say generated by the dual quantum Brownian or Lévy processes

$$\hat{e}_j(t) := \Lambda(t, \mathbf{e}_j) \equiv \mathrm{B}_j(t), \quad \check{e}^j(t) := \Lambda\left(t, \bar{\mathbf{e}}^j\right) \equiv \bar{\mathrm{B}}^j(t) \quad \forall j \in J_\circ$$

given by the canonical bases $\mathbf{e}_{J_\circ}$, $\tilde{\mathbf{e}}^{J_\circ}$ respectively for the dual noncommutative Itô algebras $\mathfrak{b}$ and $\widetilde{\mathfrak{b}}$.

Every QS-differentiable *adapted* process $\mathrm{Z}(t) \vdash \mathbb{M}_0^{t]} \otimes I_t$ has increments $\mathrm{Z}(t) - \mathrm{Z}(r) = \boldsymbol{K}_r^t \centerdot \boldsymbol{A}$ written in terms of $\mathrm{Z}\boldsymbol{I} := \mathbf{I} \otimes Z \equiv \boldsymbol{I}\mathrm{Z}$ as in the canonical QS integral form $\int_r^t (\boldsymbol{J}(s) - \mathrm{Z}(s)\boldsymbol{I}) \centerdot \mathrm{d}\boldsymbol{A}(s)$:

$$= \int_r^t (\mathrm{J}_\nu^\mu(s) - \mathrm{Z}(s)\delta_\nu^\mu)\,\mathrm{dA}_\mu^\nu(s) \equiv \Lambda_r^t(\mathbf{J}_\mathrm{Z} - \mathrm{Z}\mathbf{I})$$

of the *QS derivative* $\boldsymbol{K}(t) = \boldsymbol{D}_\mathrm{Z}^t$ defining the *QS germ-matrix* $\mathbf{J}_\mathrm{Z}^t = \mathbf{Z}(t) + \boldsymbol{D}_\mathrm{Z}^t \boldsymbol{I}$, where $\mathbf{Z} := \mathbf{I} \otimes \mathrm{Z} \equiv \mathrm{Z}\mathbf{I}$ is the ampliation $[\mathrm{Z}\delta_\nu^\mu]$ of Z. The QS dynamics is described by a hemigroup

$$\widetilde{\alpha}_{t_0}(t) = \widetilde{\alpha}_{t_0}(r) \circ \widetilde{\alpha}_r(t) \quad \forall t_0 < r < t$$

of QS monomorphic transformations $\widetilde{\alpha}_{t_0}(t) : \mathbb{M}_0^{t]} \bar{\otimes} \mathbb{N}_r \to \mathbb{M}_0^{r]} \bar{\otimes} \mathbb{N}_r$

$$\widehat{\mathrm{Z}}_r(t) = \mathrm{V}_r(t)\,\mathrm{Z}(t)\,\mathrm{V}_r(t)^* \equiv \widetilde{\alpha}_r(t, \mathrm{Z}(t)),$$

where $\mathrm{V}_r(t) : H_0^{t]} \otimes F_t \to H_0^{r]} \otimes F_r$ is a unitary or an isometric evolution $\mathrm{V}_r(t) = \mathrm{I} + \Lambda_r^t(\mathrm{V}\mathbf{L})$ such that $\widetilde{\alpha}_r(\mathrm{Z}^*\mathrm{Z}) = \widetilde{\alpha}_r(\mathrm{Z})^* \widetilde{\alpha}_r(\mathrm{Z})$ for any $\mathrm{Z} \in \mathbb{M}_0^{t]}$. We shall call the isometric W*-monomorphisms $\widetilde{\alpha}_{t_0}(t)$ *semimorphisms* as they preserve all algebraic operations except,.maybe, the identity $\mathrm{I}_0^{t]} \in \mathbb{M}_0^{t]}$ which is mapped into the decreasing orthoprojectors $\mathrm{P}_r(t) = \mathrm{V}_r(t)\,\mathrm{V}_r(t)^* \geq \mathrm{P}_r^{t+s]}$ $\forall s > 0$ of the quantum object survival from r up to t which is said to be *stable* only if $\mathrm{P}_r(t) = \mathrm{I}_0^{r]}$ for all t and $r < t$.

The input – output semiunitary transformations $\mathrm{V} = \{\mathrm{V}_r(t) : r \leq t\}$ form a linear *hemigroup*

$$\mathrm{V}_{t_0}(t) = \mathrm{V}_{t_0}(r) \circ \mathrm{V}_r(t) \quad \forall t_0 < r < t.$$

as the representation of a small category of the totally ordered set $\mathbb{R}_+$. Since the seminal paper [22] it has been main object of study in quantum stochastics as a resolving family for the Hudson-Parthasarathy (HP) QS differential equation

$$\mathrm{dV}(t) = \mathrm{V}(t)\,\Lambda(\mathrm{d}t, \mathbf{S} - \mathbf{I})\,\mathrm{V}(t) = \boldsymbol{L}(t) \centerdot \boldsymbol{A}(\mathrm{d}t)$$

written in the canonical Hudson-Parthasarathy basis $\boldsymbol{A} = \left[\mathrm{A}_\mu^\nu\right]_{\mu=-,\circ}^{\nu=+,\circ}$ as

$$\mathrm{dV}(t) = \mathrm{V}(t)\left(\mathrm{L}_+^- \mathrm{d}t + \mathrm{L}_\circ^- \mathrm{dA}_-^\circ + \mathrm{L}_\circ^\circ \mathrm{dA}_\circ^\circ + \mathrm{L}_+^\circ \mathrm{dA}_\circ^+\right).$$

Here $\boldsymbol{L} = [\mathrm{L}_\kappa^\iota]_{\kappa=+,\circ}^{\iota=-,\circ} \equiv \mathbf{L}\boldsymbol{I}$ is the quadruple of QS logarithmic derivatives defining the QS germ $\mathbf{G}(t) = \mathrm{V}(t)\,\mathbf{S}(t) \equiv \mathbf{J}_\mathrm{V}^t$ for the Hemigroup V by $\mathbf{S} = \mathbf{I} + \mathbf{L}$. The following theorem gives necessary HP unitarity conditions in a compact germ form. They are also sufficient for the existence and uniqueness of the semiunitary solutions under the appropriate integrability conditions [11][12] for the operator-valued adapted function $\mathbf{L}(t)$.

Theorem 3.1. *The solution of HP equation is unitary,* $\mathrm{V}^* = \mathrm{V}^{-1}$ *(isometry,* $\mathrm{V}^*\mathrm{V} = \mathrm{I}$*) iff the germ is pseudo-unitary,* $\mathbf{G}^\ddagger = \mathbf{G}^{-1}$ *(pseudo-isometry* $\mathbf{G}^\ddagger\mathbf{G} = \mathbf{I}$*), i.e. iff the logarithmic germ* $\mathbf{S}(t) = \mathrm{V}(t)^* \mathbf{J}_{\mathrm{V}}^t$ *is pseudo-unitary (pseudo-isometry). This implies the relations*

$$\mathrm{S}_\circ^{\circ *}\mathrm{S}_\circ^\circ = \mathrm{I}_\circ^\circ, \quad \mathrm{S}_\circ^{\circ *}\mathrm{S}_+^\circ + \mathrm{S}_\circ^{-*} = 0, \quad \mathrm{S}_+^{\circ *}\mathrm{S}_\circ^\circ + \mathrm{S}_\circ^- = 0, \quad \mathrm{S}_+^- + \mathrm{S}_+^{\circ *}\mathrm{S}_+^\circ + \mathrm{S}_+^{-*} = 0.$$

for $\mathrm{S}_\circ^\circ = \mathrm{I}_\circ^\circ + \mathrm{L}_\circ^\circ$, $\mathrm{S}_+^\circ = \mathrm{L}_+^\circ$, $\mathrm{S}_\circ^- = \mathrm{L}_\circ^-$, $\mathrm{S}_+^- = \mathrm{L}_+^-$.

3.2. Quantum stochastic flows and evolution equation. The QS Heisenberg equation for the maximal W*-algebra $\mathbb{N}_t^s = \mathcal{B}(F_t^s)$ is generated by the vacuum noise increments $\mathrm{A}_\mu^\nu(t') - \mathrm{A}_\mu^\nu(t)$, $t' \in [t, t^s)$ of the canonical HP basis $\mathrm{B}_\mu^\nu = \mathrm{A}_\mu^\nu$. It may actually be driven by a smaller Itô $\star$-subalgebra than $\mathfrak{b}(\mathfrak{k})$, corresponding to a product W*-algebra $\mathbb{N}_0^t \subseteq \mathcal{B}(F_t^s)$ of quantum noise on Fock space $F_t^s = \Gamma(K_t^s)$ for $K_t^s = L_{\mathfrak{k}}^2(t, t^s]$, say generated by the classical Langevin forces $\breve{f}_k(t')$ given by an Abelian Itô $\star$-subalgebra for each $t^s = t + s > t$ on F_t^s. It is usually described by the Langevin QS equation

$$\mathrm{d}\widehat{\mathrm{Z}}_r(t) = \Lambda\left(\mathrm{d}t, \boldsymbol{\gamma}_r\left(t, \mathbf{J}_{\mathrm{Z}}^t\right) - \widehat{\mathbf{Z}}_r(t)\right), \;\; \widehat{\mathrm{Z}}_r(r) = \mathrm{Z}(r).$$

where the QS germ $\boldsymbol{\gamma}(t) = \mathbf{J}_{\widetilde{\alpha}}^t = \widetilde{\alpha}(t) \circ \boldsymbol{\varsigma}$ for $\widetilde{\alpha}$ with $\boldsymbol{\varsigma}(\mathbf{J}) = \mathbf{SJS}^\ddagger$,

$$\boldsymbol{\gamma}_r\left(t, \mathbf{J}_{\mathrm{Z}}^t\right) = \mathbf{G}_r(t)\,\mathbf{J}_{\mathrm{Z}}^t\mathbf{G}_r(t)^\ddagger = \widehat{\boldsymbol{\varsigma}}_r\left(t, \widehat{\mathbf{J}}_{\mathrm{Z}}^t\right),$$

is obtained by applying quantum Itô product formula

$$\mathrm{d}(\mathrm{VZV}^*) = \Lambda\left(\mathrm{d}t, \mathbf{GJ}_{\mathrm{Z}}^t\mathbf{G}^\ddagger - \mathbf{VZV}^\ddagger\right).$$

Theorem 3.2. [14] *The solution* $\widehat{\mathrm{Z}}(t) = \widetilde{\alpha}(t, \mathrm{Z}(t))$ *of the QS Heisenberg equation is homomorphic (unital,* $\widetilde{\alpha}_r(t, \mathrm{I}) = \mathrm{I}$*) iff all germs* $\boldsymbol{\gamma}_r(t)$ *are pseudo-homomorphic (unital):*

$$\boldsymbol{\gamma}_r\left(t, \mathbf{J}^\ddagger\mathbf{J}\right) = \boldsymbol{\gamma}_r(t, \mathbf{J})^\ddagger\,\boldsymbol{\gamma}_r(t, \mathbf{J}), \;\; (\boldsymbol{\gamma}_r(t, \mathbf{I}) = \mathbf{I}).$$

In particular, $\mathbf{J}_{\mathrm{X}}^t = \mathbf{I}^t\mathtt{x} \equiv \mathbf{X}(t)$ *gives Langevin equation, and the case* $\mathbf{J}_{\mathrm{Y}}^{t]} = \mathbf{J}_{\mathrm{Y}}^t\mathtt{1}$ *with* $\boldsymbol{\gamma}_r\left(t, \mathbf{J}_{\mathrm{Y}_t}^t\right) = \widehat{\mathrm{Y}}_r(t)\,\mathbf{I} + \boldsymbol{\gamma}_r\left(t, \mathbf{D}_{\mathrm{Y}}^t\right)$ *gives the output equation:*

$$\mathrm{d}\widehat{\mathrm{X}} = \Lambda\left(\mathrm{d}t, \widehat{\boldsymbol{\varsigma}}\left(t, \widehat{\mathbf{X}}\right) - \widehat{\mathbf{X}}(t)\right), \; \mathrm{d}\widehat{\mathrm{Y}} = \Lambda\left(\mathrm{d}t, \widehat{\boldsymbol{\varsigma}}\left(t, \mathbf{D}_{\widehat{\mathrm{Y}}}^t\right)\right).$$

3.3. Quantum quasi-Markov hemigroups and master equation. Let $\mathbb{M}^{t]} = \mathbb{A}^{t]} \vee \mathbb{B}^{t]}$ be generated by the input W*-algebras $\mathbb{A}^{t]} = \mathbb{A}^{\mathrm{o}}\bar{\otimes}\mathbb{A}^t$ on the Hilbert spaces $H^{t]} = \mathfrak{h} \otimes H^t$ with an initial cyclic state-vector $\psi_0^{t]} = \eta \otimes \psi_0^t$ and an output modular algebras $\mathbb{B}^{t]} = \mathbb{B}^{\mathrm{o}}\bar{\otimes}\mathbb{B}^t$ from the commutants $\widetilde{\mathbb{A}^{t]}} = \mathrm{J}^*\mathbb{A}^{t]}\mathrm{J}$ of $\mathbb{A}^{t]}$ on $H^{t]}$. We assume that $(\mathbb{A}^t, \mathbb{B}^t)$ is increasing W*-product system on the components $H^t = F_0^t$ of Fock space $F_0 = F_0^t \otimes F_t$ and take the vacuum $\delta_\emptyset^t \in F_0^t$ as an initial product state vector ψ_0^t separating $\mathbb{B}^t$. A QS adapted dynamics $\widetilde{\alpha}_r(t) : \mathbb{M}^{t]} \to \mathbb{M}^{r]}\bar{\otimes}\mathbb{N}_r$ is called $\mathbb{A}^{t]}$-*Markov* on vacuum state vectors $\delta_\emptyset \in F_t$ for the decreasing future quantum noise W*-algebras $\mathbb{N}_t \subseteq \mathcal{B}(F_t)$ if

$$\phi_t(t^s, \mathrm{Z}) = \mathrm{E}_t(t^s)\,\widetilde{\alpha}_t(t^s, \mathrm{Z})\,\mathrm{E}_t(t^s) \in \mathbb{A}^{t]} \;\forall t > r, \; \mathrm{Z} \in \mathbb{A}^t\bar{\otimes}\mathbb{M}_t^{s]}, \tag{3.1}$$

where $\mathrm{E}_t(t^s) = \delta_\emptyset^{s*}$ is the vacuum projection of $H^{t]} \otimes F_t^s$ onto $H^{t]}$ as Hermitian adjoint to the injection $\delta_\emptyset^s : \psi^{t]} \mapsto \psi^{t]} \otimes \delta_\emptyset^s$ of $\psi_t \in H^{t]}$ into $H^{t]} \otimes F_t^s$. Then the family of CP maps $\phi_t(t^s)$ is output compatible, $\left[\phi_t(t^s, \mathrm{Z}), \mathbb{B}^{t]}\right] = 0$, and can be lifted to a hemigroup of CP maps $\mathbb{M}^{t+s]} \to \mathbb{M}^{t]}$ satisfying the $\mathbb{B}^t$-modularity condition

$$\phi_t(t^s, \mathrm{Y}_t \mathrm{Z} \mathrm{Y}_t^*) = \mathrm{Y} \phi_t(t^s, \mathrm{Z}) \mathrm{Y}^* \;\; \forall \mathrm{Y} \in \mathbb{B}^t, \mathrm{Z} \in \mathbb{M}^{t+s]}, \mathrm{Y}_t = \mathrm{Y} \otimes \mathrm{I}_t^{s]}.$$

It is resolving family for the generalized Lindblad equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_r(t, \mathrm{Z}(t)) = \phi_r\left(t, \lambda\left(t, \mathbf{J}_\mathrm{Z}^t\right)\right), \; \phi_r(r, \mathrm{Z}) = \mathrm{Z} \in \mathbb{M}^{r]}. \tag{3.2}$$

defined by the form-generator $\lambda(t, \mathbf{J}) := \varsigma_+^-(t, \mathbf{J})$ commuting with $\mathbb{B}^{t]}$ on the germs $\mathbf{J} \in \mathfrak{G}^\circ \bar{\otimes} \mathbb{A}^t$ of the process $\mathrm{Z}(t^s) \in \mathbb{A}^t \bar{\otimes} \mathbb{M}_t^{s]}$ at each t. Here $\mathfrak{G}^\circ = \mathbb{M}^\circ \bar{\otimes} \mathfrak{g}$ is the initial germ algebra defined as the $\mathbb{M}^\circ$-envelope of the $\star$-semigroup $\mathfrak{g} = \mathbf{1} \oplus \mathfrak{m}$ unitizing the Itô $\star$-algebra $\mathfrak{M}(t) = \mathfrak{a} \vee \mathfrak{b} \equiv \mathfrak{m}$ of the mutually commutative Itô subalgebras $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{b}(\mathfrak{k})$ generating respectively $\mathbb{A}^t = \imath_0^t(\mathfrak{A}^\otimes)$ and $\mathbb{B}^t = \imath_0^t(\mathfrak{B}^\otimes)$ by multiple QS integrals $\imath_0^t(\mathfrak{M}^\otimes)$ of $\mathfrak{M}^\otimes = \oplus_{n=0}^\infty \mathfrak{M}^{\otimes n}$ as defined for $\mathfrak{M} = \int_{t>0}^\oplus \mathfrak{M}(t)\,\mathrm{d}t$ in [11][12]. Every such completely bounded generator $\lambda(t)$ can be uniquely lifted to the $\mathbb{B}^{t]}$-modular map $\mathfrak{G}^\circ \bar{\otimes} \mathbb{M}^{t]} \to \mathbb{M}^{t]}$.

Proposition 3.3. *Let $\mathbf{J}(t) = \mathbf{J}_\mathrm{Z}^t$ be the germ of a QS process $\mathrm{Z} : t \mapsto \mathbb{M}_t$ with $\mathrm{J}_+^-(t) = 0$ for all t. Then the $\mathbb{A}^{t]}$-Markov generator $\lambda(t)$ for a QS hemigroup of $\widetilde{\alpha}_r(t, \mathrm{Z}(t)) = \mathrm{V}_r(t)\mathrm{Z}(t)\mathrm{V}_r(t)^*$ has the generalized Lindblad structure $\lambda(t, \mathbf{J}) = \left(\mathbf{S}\mathbf{J}\mathbf{S}^\ddagger\right)_+^-$ on $\mathfrak{G}^\circ \bar{\otimes} \mathbb{M}^{t]}$ written in terms of the logarithmic QS derivatives $\mathrm{L}_\nu = \mathrm{S}_\nu^-$ of the QS evolution $\mathrm{V}_r(t)$ as*

$$\lambda\left(t, \mathbf{J}_\mathrm{Z}^t\right) = \left(\mathrm{L}_+\mathrm{Z} + \mathrm{Z}\mathrm{L}_+^* + \mathrm{L}_\circ \mathrm{J}_+^\circ + \mathrm{L}_\circ \mathrm{J}_\circ^\circ \mathrm{L}_\circ^* + \mathrm{J}_\circ^- \mathrm{L}_\circ^*\right)(t) \;\; . \tag{3.3}$$

Note that the condition $\mathrm{J}_+^- = 0$ corresponds to the property of $\mathrm{Z}(t)$ to be a local vacuum martingale $\mathrm{M}(t)$: $\mathrm{E}_r(t)\mathrm{M}(t)\mathrm{E}_r(t) = \mathrm{M}(r)$, and $\lambda(t, \mathbf{J}_\mathrm{Z}^t) = \lambda(t, \mathbf{J}_\mathrm{M}^t) + \mathrm{J}_+^-(t)$ for any vacuum semimartingale $\mathrm{Z}(t) = \mathrm{M}(t) + \int_0^t \mathrm{J}_+^- \mathrm{d}r$. In particular, if $\mathbf{D} = \mathbf{J} - \mathbf{X} = 0$, λ reads simply as the Lindbladian on $\mathrm{X} = \mathrm{I}\mathtt{x}$,

$$\lambda(\mathbf{X}) = \mathrm{L}_+\mathrm{X} + \mathrm{X}\mathrm{L}_+^* + \mathrm{L}_\circ \mathrm{X} \mathrm{L}_\circ^* \equiv \lambda(\mathtt{x}). \tag{3.4}$$

However, it defines the $\mathbb{A}^\circ$-Markov dynamics only if $\lambda(t, \mathbf{J}) \in \mathbb{A}^\circ$ for $\mathbf{J} \in \mathfrak{G}^\circ$, e.g. if $\mathrm{L}_\nu(t) = \mathrm{I}\mathrm{L}_\nu(t)$ with $\mathtt{L}_\nu(t) \in \mathbb{A}^\circ$.

For the output processes $\mathrm{Z}(t) = \mathrm{Y}_t \mathtt{1}$ having the QS derivative $\mathbf{D} = \mathtt{1}\mathbf{b}(t)\mathrm{Y}_t$ with $\mathbf{b}(t) \in \mathbb{B} \; \forall \; t \geq 0$,

$$\lambda\left(\mathbf{J}_{\mathrm{Y}_t}^t\right) = \left(\lambda(\mathtt{1}) + \mathrm{L}_\circ b_+^\circ + \mathrm{L}_\circ b_\circ^\circ \mathrm{L}_\circ^* + b_\circ^- \mathrm{L}_\circ^* + b_+^-\right)\mathrm{Y}_t.$$

Here $\mathrm{Y}_t \in \mathbb{B}^{t]}$ is a local QS semimartingale, e.g. a local QS martingale characterized by $b_+^- = 0$, say $\mathrm{Y}_t = \mathrm{W}_t(\mathbf{b})\,\mathtt{y}$ with any $\mathtt{y} \in \mathbb{B}$ of zero expectation and a QS exponential martingale $\mathrm{W}_t(\mathbf{b}) \in \mathbb{B}^t$ as the normal-ordered Weyl exponents $\mathrm{W}_t(\mathbf{b}) =: \exp[\Lambda_0^t(\mathbf{b})] :$. The compatibility of $\lambda(\mathbf{J})$ with $\mathbb{B}^{t]}$ on the QS logarithmic germs $\mathbf{J} = \mathbf{J}_\mathrm{Y}^t \mathrm{Y}_t^{-1} = \mathbf{1} + \mathbf{b}(t)$ is ensured by $\mathrm{L}_\nu \in \mathbb{A}^{t]}$.

In the following we assume that $\mathfrak{b} = \widetilde{\mathfrak{a}}$ is dual to a modular Itô B*-algebra in the standard representation $\mathfrak{a} \subseteq \mathfrak{b}(\mathfrak{k})$.

3.4. Filtering of quantum hidden quasi-Markov dynamics. Let us consider a QS hemigroup $\{\check{\mathrm{F}}_r(t)\}$ of propagators $H^{r]}\otimes F_r \to H^{t]}\otimes F_t$ satisfying the QS evolution equation

$$\mathrm{d}\check{\mathrm{F}}_r(t) = \check{\mathrm{F}}_r(t)\left(\mathrm{L}_+(t)\,\mathrm{d}t + \mathrm{L}_k(t)\,\mathrm{d}\check{e}^k_t\right) = \Lambda\left(\mathrm{d}t, \check{\mathrm{F}}_r(t)\,\mathbf{L}(t)\right) \quad \check{\mathrm{F}}_r(r) = \mathrm{I}^{r]}. \tag{3.5}$$

with $\mathrm{L}_k = \mathrm{L}_\circ e^\circ_k$ and $\check{e}^k(t) = \Lambda\left(t, \bar{\mathbf{e}}^k\right)$. Applying the QS Itô formula to

$$\check{\phi}_r(t, \mathrm{Z}(t)) = \check{\mathrm{F}}_r(t)\,\mathrm{Z}(t)\,\check{\mathrm{F}}_r(t)^*,$$

we obtain a QS evolution equation

$$\mathrm{d}\check{\phi}(t, \mathrm{Z}(t)) = \Lambda\left(\mathrm{d}t, \check{\phi}\left(t, \boldsymbol{\zeta}\left(t, \mathbf{J}^t_{\mathrm{Z}}\right) - \mathbf{X}(t)\right)\right) \quad \mathrm{Z}(t) \in \mathbb{M}^{t]} \tag{3.6}$$

describing in terms of $\boldsymbol{\zeta}(t, \mathbf{J}) = \mathbf{C}(t)\,\mathbf{J}\mathbf{C}(t)^{\ddagger}$ a QS *entangling process*

$$\check{\phi}(t) : \mathbb{M}^{t]} \to \mathbb{M}^{\mathrm{o}}\bar{\otimes}\mathbb{A}^t_\star$$

by the hemigroup $\{\check{\phi}_r(t)\}$ of *simple* CP transformations

$$\check{\phi}_r(t) : \mathbb{M}^{t]} \to \mathbb{M}^{r]}\bar{\otimes}\mathbb{A}^{t-r}_{r\star}.$$

They are defined by the logarithmic QS germ $\mathbf{C}(t) = \mathbf{I} + \mathbf{L}(t)$ of $\check{\mathrm{F}}_r(t)$ given by the logarithmic QS derivative

$$\mathbf{L}(t) = \mathrm{L}_+(t)\,\mathbf{d} \; + \sum_{k\in J_\circ} \mathrm{L}_k(t)\,\bar{\mathbf{e}}^k. \tag{3.7}$$

In general the QS transformations $\check{\phi}_r(t)$ map $\mathbb{M}^{t]}$ not into a W*-algebra but into the kernels [15] which are affiliated to $\mathbb{M}^{r]}\bar{\otimes}\mathbb{A}^{t-r}_r$ since, as it follows from the next Lemma, they satisfy the *quasi-Markovianity* condition

$$\phi_t(t^s, \mathrm{Z}) = \mathrm{E}_t(t^s)\,\check{\phi}_t(t^s, \mathrm{Z})\,\mathrm{E}_t(t^s) \in \mathbb{A}^{t]}\ \forall t > r,\ \mathrm{Z} \in \mathbb{A}^t\bar{\otimes}\mathbb{M}^{s]}_t$$

following from the condition (3.1) for $\widetilde{\alpha}_r(t, \mathrm{Z})$. This quasi-Markovianity, defined as the Markovianity with respect to an increasing algebras $\mathbb{A}^{t]}$ in place of a single object algebra $\mathbb{A}^{\mathrm{o}}$, and also the uniqueness of the solutions to (3.5), it is sufficient to assume that only the operator-functions $\mathrm{L}_+(t)$ and $\mathrm{L}_\circ(t)$ are respectively locally L^1 and L^2-integrable [11][14] with values in $\mathbb{A}^{t]}$, or $\mathbb{A}^{t]}$-adapted $\mathrm{L}_\nu(t) \vdash \mathbb{A}^{t]}$ as affiliated to $\mathbb{A}^{t]}$.

Lemma 3.4. *Let the $\mathbb{A}^{t]}$-Markov hemigroup $\{\phi_r(t)\}$ be defined as the unique solution of the generalized Lindblad equation (3.2) on the increasing $\mathbb{M}^{t]}$ with the generator (3.3) $\lambda(\mathbf{J}) = \varsigma^-_+(\mathbf{J})$. Then it is unraveled by the $\mathbb{A}^{t]}$-Markov QS hemigroup of simple CP transformation $\check{\phi}_r(t)$ in the sense that it is the conditional expectation*

$$\phi_r(t, \mathrm{Z}(t)) = \epsilon_r\left[\check{\phi}_r(t, \mathrm{Z}(t))\right] := \mathrm{E}_r\check{\phi}_r(t, \mathrm{Z}(t))\,\mathrm{E}_r$$

corresponding to the vacuum-induced white noise thermostates $\epsilon_r[\mathrm{X}] = \delta^{r*}_\emptyset \mathrm{X}\delta^r_\emptyset$ on the algebras $\mathbb{A}_r$ generated by the independent increments of the input quantum Lévy noise $\check{\boldsymbol{e}}(t) - \check{\boldsymbol{e}}(r)$.

Proof. Since $\mathrm{E}_r(t)\,\mathrm{E}_t = \mathrm{E}_r = \mathrm{E}_t\mathrm{E}_r(t)$, the vacuum expectation $\epsilon_r = \epsilon_r(t)\circ\epsilon_t$ of $\mathrm{d}\check{\phi}_r(\mathrm{Z})(t)$ is defined by the expectation $\check{\phi}_r\left(t,\zeta_+^-\left(\mathbf{J}_\mathrm{Z}^t\right)\right)\mathrm{d}t$ of $\mathrm{d}\check{\phi}_r(\mathrm{Z})(t)$ written by modularity property as

$$\begin{aligned}\epsilon_t\left[\mathrm{d}\check{\phi}_r(\mathrm{Z})\right](t) &= \check{\mathrm{F}}_r(t)\,\epsilon_t\left[\Lambda\left(\mathrm{d}t,\boldsymbol{\zeta}\left(t,\mathbf{J}_\mathrm{Z}^t\right)-\mathbf{Z}(t)\right)\right]\check{\mathrm{F}}_r(t)^* \\ &= \check{\phi}_r\left(t,\zeta_+^-(t,\mathbf{J})\right)\mathrm{d}t,\end{aligned}$$

where $\zeta_+^-(\mathbf{J})\,\mathrm{d}t = \epsilon_t\left[\Lambda\left(\mathrm{d}t,\mathbf{C}(t)\,\mathbf{J}\mathbf{C}(t)^\ddagger\right)\right]$ with $\mathrm{J}_+^- = 0$ is a generalized Lindblad generator

$$(\boldsymbol{\zeta}(\mathbf{J})-\mathbf{Z})_+^- = \mathrm{C}_+^-\mathrm{Z}+\mathrm{Z}\mathrm{C}_+^{-*}+\mathrm{C}_\circ^-\mathrm{J}_+^\circ+\mathrm{C}_\circ^-\mathrm{J}_\circ^\circ\mathrm{C}_\circ^{-*}+\mathrm{J}_\circ^-\mathrm{C}_\circ^{-*} \equiv \zeta_+^-(\mathbf{J})\,. \tag{3.8}$$

In fact, since $\mathrm{C}_+^- = \mathrm{L}_+$ and $\mathrm{C}_\circ^- = \sum_k \mathrm{L}_k(t)\,\bar{e}_\circ^k = \mathrm{L}_\circ$ due to the completeness relation $e_k^\circ\bar{e}_\circ^k \equiv \mathrm{I}_\circ^\circ$ with $\bar{e}_\circ^k$, this $\zeta_+^-(t)$ coincides with the generator $\lambda(t) = \varsigma_+^-(t)$ of the $\mathbb{A}^{t]}$-Markov backward master equation vacuum induced by the isometric or unitary evolution $\mathrm{V}_r(t)$ on $\mathbb{M}^{t]}$. Thus, expectation $\epsilon_r\circ\check{\phi}_r(t)$ of the QS bracket evolution $\check{\phi}_r(t)$ driven by the white thermonoise $\check{\boldsymbol{e}}(t)$ must coincide by the uniqueness argument with the $\mathbb{A}^{t]}$-Markov evolution $\epsilon_r\circ\widetilde{\alpha}_r(t)$ vacuum induced by the Heisenberg evolution $\widetilde{\alpha}_r(t)$. □

The following theorem defines the structure of the generators for QS master equations induced by the general QS dynamics with respect to any output quantum Lévy process given by a modular Itô B*-algebra. This result extends the semiquantum filtering theory [14] determined by any classical output Lévy process to fully quantum case. The noncommutative filtering in the $\mathbb{A}^\mathrm{o}$-Markovian case was outlined for quantum finite-dimensional Wiener temperature noise corresponding to a Wiener-Itô modular algebra in [10].

Theorem 3.5. *Let $\mathfrak{b}$ be a modular output Lévy-Itô B^*-algebra with the symmetric basis $\mathbf{e}_{J_\bullet}$ not necessarily complete in $\widetilde{\mathfrak{a}}$. Then the $\mathbb{A}^{t]}$-Markov QS evolution*

$$\check{\phi}(t,\mathrm{x}) = \check{\mathrm{F}}_0(t)\,\mathrm{x}\check{\mathrm{F}}_0(t)^*\,,\quad t\geq 0$$

restricted to the object algebra $\mathbb{A}^\mathrm{o}$ and filtered with respect to $\mathbb{B}^t$ satisfies the Heisenberg QS equation

$$\mathrm{d}\check{\phi}_r(t,\mathrm{x})+\check{\phi}(t,\kappa(t,\mathrm{x}))\,\mathrm{d}t = \sum_{i,k\in J_\bullet}\check{\phi}\left(t,\kappa_j^\varepsilon(t,\mathrm{x})\right)\mathrm{d}\check{e}^j(t)\ \ , \tag{3.9}$$

where $\kappa(\mathrm{x}) = \mathrm{K}\mathrm{x}+\mathrm{x}\mathrm{K}^*-\sum_{i,k\in J_\circ}\mathrm{L}_i\mathrm{x}\bar{\sigma}^{i,k}\hat{\rho}\mathrm{L}_k^\star = -\lambda(\mathrm{x})$ defines the Lindbladian (3.4) in the Hermit-symmetric basis with $\mathrm{K} = -\mathrm{L}_+$, $\mathrm{L}_k^\star = \mathrm{L}_{k^\star}^*$ and

$$\kappa_j^\varepsilon(t,\mathrm{x}) = \mathrm{L}_j(t)\,\mathrm{x}+\mathrm{x}\mathrm{L}_j^\star(t)+\sum_{i,k\in J_\circ}\mathrm{L}_i(t)\,\mathrm{x}\bar{\varepsilon}_j^{i,k}\mathrm{L}_k^\star(t)\,,$$

are the fluctuating coefficients $\boldsymbol{\kappa} = \left(\kappa_j^\varepsilon\right)_{j\in J_\circ}$ with $\bar{\varepsilon}_j^{i,k} = \varepsilon_{k,i}^j$.

Proof. By QS product Itô formula applied to $\check{\mathrm{F}}_0(t)\,\mathrm{x}\check{\mathrm{F}}_r(t)^*$ we obtain the equation (3.6) with $\mathrm{Z}(t) = \mathrm{I}^t\mathrm{x}$ and $\mathbf{J}_\mathrm{Z}^t = \mathrm{Z}(t)\,\mathbf{I}$. Taking for these the element (3.8) as vacuum conditional expectation with $\mathbf{C} = \mathbf{I}+\mathbf{L}$ given by (3.7) we obtain

$$\check{\kappa}_+^-(\mathrm{x}) = \mathrm{L}_+\mathrm{x}+\mathrm{x}\mathrm{L}_+^*+\mathrm{L}_i\mathrm{x}\left(\bar{\mathbf{e}}^i\bar{\mathbf{e}}^k\right)_+^-\hat{\rho}\mathrm{L}_k^\star = -\kappa(\mathrm{x})\,,$$

since $\left(\bar{\mathbf{e}}^i\bar{\mathbf{e}}^k\right)_+^- = \bar{\sigma}^{i,k}$, and by $\bar{\mathbf{e}}^i\bar{\mathbf{e}}^k = \bar{\varepsilon}_j^{i,k}\bar{\mathbf{e}}^j$ we obtain

$$\breve{\kappa}\left(\mathtt{x}\right)_j \bar{\mathbf{e}}^j = \mathrm{L}_i\bar{\mathbf{e}}^i\mathtt{x} + \mathtt{x}\bar{\mathbf{e}}^k\mathrm{L}_k^\star + \mathrm{L}_i\mathtt{x}\bar{\mathbf{e}}^i\bar{\mathbf{e}}^k\mathrm{L}_k^\star = \kappa_j^\varepsilon\left(\mathtt{x}\right)\bar{\mathbf{e}}^j$$

□

4. Unraveling L-Dynamics and the General Master Equation

4.1. Noncommutative Girsanov transformation. Let $\mathbb{D}^s \subseteq \mathbb{D}^t$, $s \leq t$ be increasing unital $*$-subalgebras invariant also with respect to the left and right involutions $\dagger_\nu$, $\dagger_\nu^*$ for a reference weight ν on $\mathbb{D} = \cup\mathbb{D}^t$ and let $\mathtt{S}\left(t\right) = \mathtt{s}_\nu^t \in \mathbb{L}_\nu^t$ be a positive martingale normalized as $\nu\left(\mathtt{s}_\nu^t\right) = 1$, defining a state $\varsigma\left(\mathtt{d}\right) = \nu\left(\mathtt{s}\cdot\mathtt{d}\right)$ on $\mathbb{D}$. If β is an adapted $*$-morphism $\beta : \mathbb{D}^t \to \mathbb{B}^t$ of $\mathbb{D}$ into $\mathbb{B}$, then the adapted process $\mathrm{Z}\left(t\right) = \beta\left(\mathtt{x}^t\right) \equiv \mathrm{z}^t$ may not be a martingale in $(\mathbb{B},\mu)$ even if it is given by a martingale $t \mapsto \mathtt{x}^t \in \mathbb{D}^t$ defined in $(\mathbb{D},\nu)$ as

$$\left\langle \mathtt{q}|\mathtt{x}^t\right\rangle_\nu^t := \nu\left(\mathtt{q}^*\cdot\mathtt{x}^t\right) = \nu\left(\mathtt{q}^*\cdot\mathtt{x}^s\right) \equiv \left\langle \mathtt{q}|\epsilon^s\left(\mathtt{x}^t\right)\right\rangle_\nu^s \quad \forall s \leq t, \mathtt{q} \in \mathbb{D}^s.$$

It can be equivalently described by the process $X\left(t\right) = \mathtt{x}^t$ in $(\mathbb{D},\nu)$ with respect to the generalized *Cameroon-Martin density*

$$\mathtt{m}_\nu^t = \mu_\nu^\beta\left(1_\mu^t\right) \in \mathbb{L}_\nu^t$$

as a positive martingale defined by the comorphism $\mu_\nu^\beta = \varphi_\nu \circ \beta_\varphi^\star = \beta_\nu^\star$, where φ_ν is the RN derivative of $\varphi = \mu\circ\beta$ with respect to ν and $\beta_\varphi^\star = \mu_\varphi^\beta$. Any other state $\varsigma : \mathbb{D} \to \mathbb{C}$ described in $(\mathbb{D},\nu)$ by a positive normalized martingale density $\mathrm{r}_\mu^t \in \mathbb{B}^t$ can also be represented by a density $\mathtt{s}_\nu^t = \beta_\nu^\star\left(\mathrm{r}_\mu^t\right)$ as

$$\varsigma\left(\mathtt{x}^t\right) = \mu\left(\beta\left(\mathtt{x}^t\right)\cdot\mathrm{r}_\mu^t\right) = \varphi\left(\mathtt{x}^t\cdot\beta_\varphi^\star\left(\mathrm{r}_\mu^t\right)\right) = \nu\left(\mathtt{x}^t\cdot\mathtt{s}_\nu^t\right) .$$

The noncommutative Girsanov theorem makes the process $\mathrm{Z}\left(t\right)$ into a $(\mathbb{B},\varsigma)$-martingale with respect to a transformed state ς. If $\mathrm{Z}\left(t\right) = \beta\left(\mathtt{x}^t\right)$ given by an injective transformation of a martingale $\mathtt{x}^t$ in the modular $*$-algebra $(\mathbb{D},\varsigma) \subseteq (\widetilde{\mathbb{A}},\varpi)$, where $\varsigma = \rho\circ\beta = \varpi|\mathbb{D}$ for $\rho = \varpi\circ\widetilde{\alpha}$ given by a surjective (quasi or semi) coexpectation $\widetilde{\alpha}$ inverting β as the predual to a (quasi or semi) conditional expectation $\widetilde{\alpha}_\rho^\star$. It is described by the positive martingale $\hat{\rho}\left(t\right) = \widetilde{\alpha}_\mu^\star\left(1^t\right) \equiv \mathrm{r}_\mu^t$ in $(\mathbb{B},\mu)$ as the density of ρ with respect to μ such that

$$\mu\left(\mathrm{Q}\left(t\right)\mathrm{Z}\left(t\right)\cdot\hat{\rho}\left(t\right)\right) = \left\langle\hat{\rho}\left(t\right)|\mathrm{Q}\left(t\right)\mathrm{Z}\left(t\right)\right\rangle_\mu = \varpi\left(\widetilde{\alpha}\left(\mathrm{Q}\left(t\right)\right)\mathtt{x}^t\right) \ \forall \mathrm{Q}\left(t\right) \in \mathbb{B}^t.$$

Any other state φ on $\mathbb{D}$ defined by the martingale density $\mathtt{q}_\varpi^t \in \widetilde{\mathbb{A}}^t$ is also represented in $(\mathbb{B},\mu)$:

$$\varphi\left(\mathtt{x}^t\right) = \mu\left(\beta(\mathtt{x}^t)\cdot\widetilde{\alpha}_\mu^\star(\mathtt{q}_\varpi^t)\right) = \varpi\left(\mathtt{x}^t\cdot\mathtt{q}_\varpi^t\right) .$$

If $\mu = \varpi\circ\widetilde{\alpha} = \rho$, this can be simply written in the form of the transformation identity

$$\left\langle\overline{\alpha}\left(\mathtt{q}_\varpi^t\right)|\beta\left(\mathtt{x}^t\right)\right\rangle_\rho = \left\langle\mathtt{q}^t|\mathtt{x}^t\right\rangle_\varpi . \ \forall \mathtt{x}^t \in \mathbb{D}^t, \mathtt{q}^t \in \widetilde{\mathbb{A}}^t.$$

4.2. Standard Itô pairing and marginalization. Let $\mathbb{A}$ be a C*-algebra with a vector state $\omega(\mathrm{A}) = (\upsilon|\mathrm{A}\upsilon) \equiv \upsilon^* \mathrm{A}\upsilon$ on the Hilbert space H generated by $\mathbb{A}\upsilon$ and $\mathbb{B} \subseteq \mathbb{A}'$ be a modular subalgebra of $\mathcal{B}(H)$ commuting with $\mathbb{A}$, with the dual subalgebra $\widetilde{\mathbb{B}} \subseteq \mathbb{A}$ represented on the Hilbert subspace $H_\circ \subseteq H$ generated by $\mathbb{A}'\upsilon$. Let $(\mathfrak{a}, l)$ be a B*-Itô $\star$-algebra with

$$l(a) = (\mathbf{v}_\emptyset|\mathbf{a}\mathbf{v}_\emptyset) \equiv \mathbf{v}_\emptyset^\ddagger \mathbf{a}\mathbf{v}_\emptyset = a_+^-$$

in the canonical matrix representation $\mathfrak{a} \subseteq \mathfrak{b}(\mathfrak{k})$ of the Hilbert space $\mathfrak{k}$ generated by the action of $\mathfrak{a}$ on the vector $\mathbf{v}_\emptyset = (\delta_+^\mu)$, and $\mathfrak{b} \subseteq \mathfrak{a}'$ be a modular Itô subalgebra of the commutant Itô algebra $\mathfrak{a}' = \{c_+^- \mathbf{d} + \mathbf{c}_\circ\}$ with the dual subalgebra $\widetilde{\mathfrak{b}} \subseteq \mathfrak{b}(\mathfrak{k}_\circ)$ of $\mathrm{J}^*\mathfrak{a}'\mathrm{J} = \mathfrak{a}_+^-\mathbf{d} + \mathfrak{a}_\circ$ represented in $\mathfrak{b}(\mathfrak{k}_\circ)$ of the Hilbert subspace $\mathfrak{k}_\circ \subseteq \mathfrak{k}$ generated by $\mathfrak{a}'$ on $\mathbf{v}_+$. We denote by $\mathfrak{R} = \mathfrak{r}\bar{\otimes}\mathbb{A}$ and $\mathfrak{S} = \mathfrak{s}\bar{\otimes}\mathbb{B}$ the germ algebras given respectively by $\mathfrak{r} = 1 \oplus \mathfrak{a}$ and $\mathfrak{s} = 1 \oplus \mathfrak{b}$. represented in $\mathfrak{b}(\mathfrak{k})$ by matrices $\mathbf{I} + \mathbf{a}$ and $\mathbf{I} + \mathbf{b}$ with $\mathbf{a} \in \mathfrak{a}$, $\mathbf{b} \in \mathfrak{b}$ such that $\mathfrak{S} \subseteq \mathfrak{R}'$. Let us define the standard $(\mathfrak{R}, \mathfrak{R}_\mathsf{T})$-*pairing*

$$\langle \mathbf{R}, \mathbf{S} \rangle := (\mathbf{v}_+|\mathbf{R}\mathbf{S}\mathbf{v}_+) = (\upsilon|\mathrm{R}_\circ^- \mathrm{S}_+^\circ \upsilon) \equiv \langle \mathrm{R}_\circ^-, \mathrm{S}_+^\circ \rangle \quad \forall \mathbf{R} \in \mathfrak{R}, \mathbf{S} \in \mathfrak{R}_\mathsf{T}, \tag{4.1}$$

induced by the state $l \otimes \omega$, where $\mathbf{v}_+^\ddagger = (\upsilon^*, 0, 0)$, by extending this bilinear from $\mathbf{S} \in \mathfrak{R}'$ on the completion $\mathfrak{R}_\mathsf{T} = \mathfrak{R}'_\star$ of the commutant $\mathfrak{R}'$ represented on $\mathfrak{k}_\circ \otimes H_\circ$ as the dual space to the normed B*-algebra $\mathfrak{R}$. Note the decomposition $\mathfrak{R}_\mathsf{T} = \mathbb{A}_\mathsf{T} \oplus \dot{\mathfrak{R}}_\mathsf{T}$ corresponding to $\mathfrak{R} = \mathfrak{R}_+^- \oplus \mathfrak{R}_\circ$ such that $\langle \mathbf{R}, \mathbf{S} \rangle$ for $\mathbf{R} = \mathrm{R}_+^- \mathbf{d} + \mathbf{R}_\circ$ and $\mathbf{S} = \mathrm{Y}\mathbf{I} + \dot{\mathbf{S}}$ can be written as

$$\left\langle \mathbf{R}_\circ, \dot{\mathbf{S}} \right\rangle + \langle \mathrm{R}_+^-, \mathrm{Y} \rangle = \langle \mathrm{X}, \mathrm{P}_+^- \rangle + \left\langle \dot{\mathbf{R}}_\circ, \dot{\mathbf{S}}^\circ \right\rangle + \langle \mathrm{R}_+^-, \mathrm{Y} \rangle = \langle \mathrm{X}, \mathrm{P}_+^- \rangle + \left\langle \dot{\mathbf{R}}, \mathbf{S}^\circ \right\rangle.$$

Here $\mathbf{R}_\circ = \mathrm{X} \oplus \dot{\mathbf{R}}_\circ \in \mathfrak{R}_\circ$ is the germ-martingale, $(\mathbf{R}_\circ)_+^- = 0$, of X, uniquely defined by the decomposition $\mathbf{S} = \mathrm{S}_+^- \mathbf{d} + \mathbf{S}^\circ$.

It can be easily seen that the standard pairing has the modularity property

$$\left\langle \mathbf{C}\mathbf{R}\mathbf{C}^\ddagger, \mathbf{S} \right\rangle = \left(\mathbf{v}_+|\mathbf{C}\mathbf{R}\mathbf{S}\mathbf{C}^\ddagger \mathbf{v}_+ \right) = \langle \mathbf{R}, \mathbf{C}'\mathbf{S}\mathbf{C}'^* \rangle \quad \forall \mathbf{S} \in \mathfrak{R}_\mathsf{T}, \mathbf{R} \in \mathfrak{R}_\star^\bigstar$$

for any $\mathbf{C} \in \mathfrak{R}$ having a *transposed* $\mathbf{C}' \in \mathfrak{R}'$ uniquely defined by

$$\mathbf{v}_+^\ddagger \mathbf{R}\mathbf{C}' = \mathbf{v}_+^\ddagger \mathbf{C}\mathbf{R} = \mathbf{v}_+^\ddagger \mathbf{C}'\mathbf{R} \quad \forall \mathbf{R} \in \mathfrak{R}$$

as $\mathbf{C}' := \mathrm{J}^* \mathbf{C}^\dagger \mathrm{J} \equiv \mathbf{C}^{\flat\ddagger}$ by the unitary conjugation $\mathrm{J}^* = \mathrm{J}^{-1}$. Here $\dagger$ denotes the left involution $(\mathbf{X} \cdot \mathbf{C}^\dagger \mathbf{Z}) = \left(\mathbf{X}\mathbf{L}^\ddagger \cdot \mathbf{Z} \right)$ with respect to the *modular pairing* $(\mathbf{X} \cdot \widetilde{\mathbf{Y}}) = \langle \mathbf{X}, \mathbf{Y} \rangle$ for $\widetilde{\mathbf{X}} = \overline{\mathbf{X}}^\ddagger, \mathbf{Y} \in \mathfrak{R}'$, which is symmetric due to $\mathrm{J}\mathbf{X}\mathrm{J}^* = \overline{\mathbf{X}} = \mathrm{J}^*\mathbf{X}\mathrm{J}$:

$$(\mathbf{X} \cdot \mathbf{Z}) := (\overline{\mathbf{X}}\mathbf{v}_+|\mathbf{Z}\mathbf{v}_+) = (\overline{\mathbf{Z}}\mathbf{v}_+|\mathbf{X}\mathbf{v}_+) \equiv (\mathbf{Z} \cdot \mathbf{X}) \quad \forall \mathbf{X}, \mathbf{Z} \in \mathfrak{R}.$$

A positive normalized element $\hat{\rho} \in \mathbb{A}_\mathsf{T} := \mathbb{A}'_\bigstar$ is called the *covariant density* of the *regular state* $\rho(\mathrm{X}) = \langle \mathrm{X}, \hat{\rho} \rangle$ on $\mathbb{A}$, which is *absolutely continuous* with respect to ω in the sense $\omega(\mathrm{A}^*\mathrm{A}) = 0 \Rightarrow \rho(\mathrm{A}^*\mathrm{A}) = 0$. A *simple conditionally CP transformation* $\lambda(\mathbf{R}) = (\mathbf{C}\mathbf{R}\mathbf{C})_+^-$ from $\mathfrak{R}$ into $\mathbb{A}$ such that $\lambda(\widetilde{\mathfrak{S}}) \subseteq \overline{\mathbb{B}} := \mathbb{B}_\mathsf{T}^\bigstar$ transforms $\mathbb{B}_\mathsf{T}$ into $\mathfrak{S}_\star$ by

$$\langle \mathbf{R}, \lambda_\mathsf{T}(\hat{\rho}) \rangle = \rho \circ \lambda(\mathbf{R}) = \langle \lambda(\mathbf{R}), \hat{\rho} \rangle \quad \forall \mathbf{R} \in \mathfrak{R}, \hat{\rho} \in \mathbb{B}_\mathsf{T}$$

if it is given by the transposable $\mathbf{C} \in \mathfrak{R}$. One can show that λ_{T} is also conditionally CP such that each $\lambda_{\mathsf{T}}(\hat{\rho})$ is also conditionally absolutely continuous with respect to $l \otimes \omega$. It is defined on $\mathfrak{R}_{\circ}$ by the decomposition $\lambda_{\mathsf{T}}(\hat{\rho}) = \hat{\rho}\mathbf{I} + \varkappa(\hat{\rho}) \in \dot{\mathfrak{S}}_{\star}$, where

$$\varkappa(\hat{\rho}) = \mathbf{L}'\hat{\rho} + \hat{\rho}\mathbf{L}'^{\ddagger} + \mathbf{L}'\hat{\rho}\mathbf{L}'^{\ddagger} \tag{4.2}$$

in terms of $\mathbf{L} = \mathbf{C} - \mathbf{I}$ is decomposed as $\varkappa(\hat{\rho}) = \varkappa^{\emptyset}(\hat{\rho})\,\mathbf{d} + \varkappa^{\circ}(\hat{\rho})$.

4.3. Dual reduced and unravelling QS L-dynamics. Let $\mathbb{A}^t$ be the increasing W*-algebras generated by quantum Itô-Lévy noises $\check{e}^j(r)$ for $r \in [0, t)$ and $j \in J$ on Fock space $F = \Gamma\left(L^2_{\mathfrak{k}}\right)$. The modular subalgebras $\widetilde{\mathbb{B}}^t = \mathrm{J}^*\mathbb{B}^t\mathrm{J}$ are defied on $F_{\circ} = \Gamma\left(L^2_{\mathfrak{k}_{\circ}}\right)$ as the dual to $\mathbb{B}^t$ generated by $\hat{e}_i = \Lambda(\mathbf{e}_i)$ for $i \in J_{\bullet} \subseteq J_{\circ}$ with the Weyl operator basis $\{\mathrm{W}_t(\mathbf{b}) : \mathbf{b} \in \mathfrak{b}\}$, where $\mathfrak{b}$ is the modular Itô algebra spanned by $\mathbf{e}_{\emptyset}$ and $\mathbf{e}_{J_{\bullet}}$. It is given by the solutions to

$$\mathrm{dW}_t(\mathbf{b}) = \mathrm{W}_t(\mathbf{b})\,\Lambda(\mathrm{d}t, \mathbf{b}), \quad \mathrm{W}_0(\mathbf{b}) = \mathrm{I}.$$

This solution can be written in the normal form in terms of the logarithmic matrix $l^{\circ}_{\circ} = \ln(\mathrm{I}^{\circ}_{\circ} + b^{\circ}_{\circ})$ as.

$$\mathrm{W}_t(\mathbf{b}) = \mathrm{e}^{\int_0^t b^i_+(r)\mathrm{dA}^+_i}\mathrm{e}^{\int_0^t l^i_k(r)\mathrm{dA}^k_i}\mathrm{e}^{\int_0^t b^-_k(r)\mathrm{dA}^k_-}\mathrm{e}^{\int_0^t b^-_+(r)\mathrm{d}r}.$$

If the QS dynamics is a $\mathbb{A}^{t]}$-Markov process on the state vector $\upsilon = \eta \otimes \delta_{\emptyset} \in \mathfrak{h} \otimes F$ with the vacuum $\delta_{\emptyset} \in F_t$, the dual state dynamics $\varphi_t(r)$ form a forward hemigroup defied on the predual space $\mathbb{L} = \mathbb{A}^{r]}_{\mathsf{T}}$ by the CP maps $\phi_r(t)_{\mathsf{T}}$:

$$\langle \mathrm{X}_t, \varphi_t(r, \hat{\rho})\rangle_t := \langle \phi_r(t, \mathrm{X}), \hat{\rho}\rangle_r \equiv \left\langle \mathrm{X}_t, \phi_r(t)_{\mathsf{T}}\right\rangle_t \quad \forall \mathrm{X} \in \mathbb{A}^{t]},\ \hat{\rho} \in \mathbb{A}^{r]}_{\mathsf{T}}.$$

Here and below we shall take the vacuum $(\mathbb{A}^{t]}, \mathbb{A}^{t]}_{\mathsf{T}})$-pairings

$$\langle \mathrm{X}, \hat{\rho}\rangle := \left\langle \eta \otimes \delta^t_{\emptyset} | \mathrm{X}\hat{\rho}\left(\eta \otimes \delta^t_{\emptyset}\right)\right\rangle \equiv \langle \mathrm{X}, \hat{\rho}\rangle_t\,,\ \forall \mathrm{X} \in \mathbb{A}^{t]}, \hat{\rho} \in \mathbb{A}^{t]}_{\mathsf{T}}.$$

Let $\mathbb{B}^{t]}_{\star}$ denote the preduals to the increasing W*-subalgebras $\overline{\mathbb{B}}^{t]} = \overline{\mathbb{B}}^{\mathrm{o}}\bar{\otimes}\overline{\mathbb{B}}^{t}$ of $\mathbb{A}^{t]}$ generated an initial modular subalgebra $\widetilde{\mathbb{B}}^{\mathrm{o}} \subseteq \mathbb{A}^{\mathrm{o}}$ and the Weyl basis $\mathrm{W}_t(\mathbf{a})$, $\mathbf{a} \in \widetilde{\mathfrak{b}} \subseteq \mathfrak{a}$. If the QS dynamics is also Markov with respect to $\overline{\mathbb{B}}^{t]} \subseteq \mathbb{A}^{t]}$ for all t, the dual dynamics can be reduced to a hemigroup of $\vartheta_t(r) : \mathbb{B}^{r]}_{\star} \to \mathbb{B}^{t]}_{\star}$. It is given by the marginalization ι_{T} of the reference $\lambda = \omega|\mathbb{A}$ to $\overline{\mu} = \omega|\overline{\mathbb{B}}$ defining the CP maps $\vartheta_t(r) \circ \iota_{\mathsf{T}} = \iota_{\mathsf{T}} \circ \varphi_t(r)$ as the preduals to the restrictions

$$\vartheta^{\mathsf{T}}_r(t) := \phi_r(t)\,|\overline{\mathbb{B}}^{t]} \equiv \vartheta_t(r)^{\mathsf{T}}\,.$$

In particular, the reduced evolution $\hat{\rho}_t = \vartheta_t(\hat{\varrho})$ from the initial state ϱ on $\overline{\mathbb{B}}^{\mathrm{o}} \subseteq \mathbb{A}^{\mathrm{o}}$ is defined by

$$\langle \mathrm{X}_t, \vartheta_t(\hat{\varrho})\rangle_{\mu} = \langle \phi_0(t, \mathrm{X}_t), \hat{\varrho}\rangle_{\lambda}\,, \quad \forall \mathrm{X}_t \in \overline{\mathbb{B}}^{t]},\ \hat{\varrho} \in \overline{\mathbb{B}}^{\mathrm{o}}_{\star}.$$

The reduced states $\hat{\rho}_t$ can be described on $\mathrm{X}_t = \mathrm{W}_t(\mathbf{a})\,\mathtt{x}$ with $\mathtt{x} \in \overline{\mathbb{B}}$ by the characteristic function

$$C_t(\mathbf{a}, \mathrm{x}) = \langle \mathrm{W}_t(\mathbf{a})\,\mathrm{x}, \hat{\rho}_t\rangle = \langle \phi_0(t, \mathrm{X}_t), \hat{\varrho}\rangle$$

satisfying the equation

$$\mathrm{d}C_t(\mathbf{a}, \mathrm{x}) = \mathrm{d}\,\langle \mathrm{X}_t(\mathbf{a}), \hat{\rho}_t\rangle \quad \text{for} \quad C_t(\mathbf{a}, \mathrm{x}) = \langle \mathrm{X}_t(\mathbf{a}), \hat{\rho}_t\rangle\,.$$

Note that since $\vartheta_t(r)$ mapping $\mathbb{B}_\star^{r]}$ into the increasing state spaces $\mathbb{B}_\star^{t]}$, they can not satisfy a deterministic evolution equation which should involve the increments $\mathrm{d}\hat{e}_{J_\bullet}$ generating $\mathrm{d}\mathbb{B}_\star^t$. Thus they should satisfy a QS evolution equation $\mathrm{d}\hat{\rho}_t = \varkappa(\hat{\rho}) \cdot \mathrm{d}\hat{\mathbf{e}}$ with $\mathrm{d}\hat{e}_\emptyset = \Lambda(\mathrm{d}t, \mathbf{d}) = \mathrm{d}t\mathrm{I}$ and the innovation $\{\mathrm{d}\hat{e}_j : j \in J_\bullet\}$ such that by Itô formula

$$\langle \mathrm{dX}_t, \hat{\rho}_t \rangle + \langle \mathrm{X}_t, \mathrm{d}\hat{\rho}_t \rangle + \langle \mathrm{dX}_t, \mathrm{d}\hat{\rho}_t \rangle = \mathrm{d}C_t(\mathbf{a}, \mathrm{x}).$$

It is characterized on $\mathrm{X}_t = \mathrm{W}_t(\mathbf{a})\,\mathrm{x}$ with $\mathbf{a} \in \widetilde{\mathfrak{b}}$, $\mathrm{x} \in \widetilde{\mathbb{B}}^{\mathrm{o}}$ by the equation

$$\mathrm{d}C_t(\mathbf{a}, \mathrm{x}) = \mathrm{d}\langle \mathrm{X}_t(\mathbf{a}), \hat{\rho}_t \rangle \ \text{ for } \ C_t(\mathbf{a}, \mathrm{x}) = \langle \mathrm{X}_t(\mathbf{a}), \hat{\rho}_t \rangle,$$

where $\mathrm{d}\langle \mathrm{X}_t(\mathbf{a}), \hat{\rho}_t \rangle = 0$ in the martingale case $a_+^- = 0$, $\langle \mathrm{X}_t, \mathrm{d}\hat{\rho}_t \rangle = \left\langle \mathrm{X}_t, \varkappa^\emptyset(\hat{\rho}_t) \right\rangle \mathrm{d}t$ and

$$\langle \mathrm{dX}_t, \mathrm{d}\hat{\rho}_t \rangle = \sum_{i \in J_\circ} \left\langle \mathrm{X}_t \mathbf{a}, \varkappa^i(\hat{\varrho})\,\mathbf{e}_i \right\rangle \mathrm{d}t \equiv \langle \mathrm{X}_t \mathbf{a}, \varkappa^\circ(\hat{\varrho}) \rangle\,\mathrm{d}t \ \ \forall \mathbf{a} \in \widetilde{\mathfrak{b}}. \tag{4.3}$$

4.4. Noncommutative QS master equation. The following theorem defines the structure of the generators for QS master equations induced by the general QS dynamics with respect to any output quantum Lévy process given by a modular Itô B*-algebra. This result extends the semi-quantum unraveling dynamics [5] determined by any classical output Lévy process to the quantum Levy-Itô temperature process. The proof of the theorem in the general case is similar to the proof of the semi-quantum filtering theorem [7] for the commutative processes $\hat{e}_i = \check{e}_{i^\star}$ but noncommutative initial algebra $\mathbb{B}$. It was outlined in [10] for fully quantum diffusive case of noncommuting self-adjoint Wiener processes $\hat{e}_i = \hat{e}_i^*$ corresponding to quantum finite-dimensional Wiener-Itô nilpotent in second order modular algebra.

Theorem 4.1. ***(Main)*** *Let $\{\vartheta_r^\intercal(t)\}$ be the quantum hemigroup dynamics on the increasing W*-subalgebras $\overline{\mathbb{B}}^{t]} \subseteq \mathbb{A}^{t]}$ generated by $\widetilde{\mathbb{B}}^{\mathrm{o}} \subseteq \mathbb{A}^{\mathrm{o}}$ and the modular Lévy-Itô B*-algebra $\widetilde{\mathfrak{b}} \subseteq \mathfrak{a}$ with a symmetric basis $\{\mathbf{d}, \bar{\mathbf{e}}_i : i \in J_\bullet\}$ indexed by $J_\bullet = J_\bullet^\star \subseteq J_\circ$. Assume that it is induced by the hemigroup $\{\phi_r(t)\}$ having the generator (3.3) with $\mathbb{A}^{t]}$-adapted transposable coefficients $\mathrm{L}_+^\iota(t)$. Then the reduced states $\hat{\rho}(t) = \vartheta_t(\hat{\rho}) \in \mathbb{B}_\star^{t]}$ satisfy the QS stochastic equation*

$$\mathrm{d}\hat{\rho}(t) + \varkappa(t, \hat{\rho}(t))\,\mathrm{d}t = \sum_{j \in J_\bullet} \varkappa^j(t, \hat{\rho}(t))\,\mathrm{d}\hat{e}_j(t) \ \ \forall \hat{\rho}(0) \in \mathbb{B}_\star^{\mathrm{o}} \tag{4.4}$$

with QS modular noises $\hat{e}^j(t) = \Lambda\left(t, \mathbf{e}^j\right)$ and fluctuating coefficients given by

$$\varkappa^j(t, \hat{\rho}) = L^{j^\star}(t)\,\hat{\rho} + \hat{\rho}L^{j*}(t) + \sum_{i,k \in J_\circ} L^{k^\star}(t)\,\varepsilon_{i,k}^j \hat{\rho} L^{k*}(t).$$

including $\varkappa^\emptyset = -\varkappa$ with $L^\emptyset = -K = \mathrm{L}_+^{-\prime}$ and $L^i = (\mathbf{L}')_\circ^- \bar{e}_\circ^{i*} = \mathrm{L}_+^{\circ\prime} \bar{e}_\circ^{i*}$.

Proof. The predual generator $\varkappa : \mathbb{B}_\star^{t]} \to \dot{\mathfrak{S}}^{t]}$ for the generalized Lindbladian written for an $\mathbb{A}^{t]}$-martingales $\mathrm{Z} = \mathrm{X}$ in the form (3.3) is found by modularity in (4.2) for the $\mathbb{A}^{t]}$-adapted transposable $\mathbf{C} = \mathbf{I} + \mathbf{L}$ such that $\mathbf{C}'(t)$ commutes with $\mathfrak{R}^{t]}$. This allows the expansions

$$\mathbf{L}' = \sum_{i \in J_\circ} \mathbf{e}_{i^\star} L^i, \ \ \mathbf{L}'^\ddagger = \sum_{i \in J_\circ} \mathbf{e}_i L^{i*},$$

where $L^i = \mathrm{L}_+^{\circ\prime}\bar{e}_\circ^{i*}$, $L^{i*} = \bar{e}_\circ^i \mathrm{L}_+^{\circ\flat}$ are determined by the completeness as

$$\left\langle \mathrm{X}^*\bar{\mathbf{e}}^j, \mathbf{L}'^{\ddagger}\right\rangle = \left(\mathrm{X}\upsilon|\left(\bar{\mathbf{e}}^j\mathbf{L}'^{\ddagger}\right)_+^-\upsilon\right) = \left(\bar{\mathbf{e}}^j\mathbf{e}_i\right)_+^-\left(\mathrm{X}\upsilon|L^{i*}\upsilon\right)$$
$$= \pi_i^j\bar{e}_\circ^i\left(\mathrm{X}\upsilon|\mathrm{L}_+^{\circ\flat}\upsilon\right) = \left(\mathrm{X}\upsilon|L^{j*}\upsilon\right)$$

since $\pi_j^i = \left(\bar{\mathbf{e}}^i\mathbf{e}_j\right)_+^- = \bar{e}_\circ^i e_j^\circ$ is either δ_j^i or a an idempotent kernel such that $\pi_j^i\bar{e}_\circ^j = \bar{e}_\circ^i$ in the case of overcompleteness of the basis $\left\{\bar{e}_\circ^i\right\}$ in $\mathfrak{k}_\circ^*$. In the similar way we obtain $L^{j^\star} = \left(\mathbf{L}'\bar{\mathbf{e}}^j\right)_+^-$ and

$$\left\langle \mathrm{X}^*\bar{\mathbf{e}}^j, \mathbf{L}'\hat{\rho}\mathbf{L}'^{\ddagger}\right\rangle = \left(\mathrm{X}\upsilon|\left(\mathbf{L}'\hat{\rho}\bar{\mathbf{e}}^j\mathbf{L}'^{\ddagger}\right)_+^-\upsilon\right)$$
$$= \left(\mathbf{e}_i\bar{\mathbf{e}}^j\mathbf{e}_k\right)_+^-\left(\mathrm{X}\upsilon|L^{i^\star}\hat{\rho}L^{k*}\upsilon\right) = \left(\mathrm{X}\upsilon|L^{i^\star}\varepsilon_{i,k}^j\hat{\rho}L^{k*}\upsilon\right).$$

since $\left(\mathbf{e}_i\bar{\mathbf{e}}^j\mathbf{e}_k\right)_+^- = \left(\bar{\mathbf{e}}^j\mathbf{e}_i\mathbf{e}_k\right)_+^- = \varepsilon_{i,k}^m\left(\bar{\mathbf{e}}^j\mathbf{e}_m\right)_+^- = \pi_m^j\varepsilon_{i,k}^m = \varepsilon_{i,k}^m$. Therefore,

$$\varkappa^j\left(\hat{\rho}\right) = L^{j^\star}\hat{\rho} + \hat{\rho}L^{j*} + L^{i^\star}\varepsilon_{i,k}^j L^{k*}$$

in (4.2) determined for $j \in J_\bullet$ by (4.3) for any $\mathrm{X} \in \mathbb{B}^{t]}$ and $\mathbf{a} \in \widetilde{\mathfrak{b}}$ spanned by $\bar{\mathbf{e}}^{J_\bullet}$. This formula also determines $\varkappa^\emptyset\left(\hat{\rho}\right)$ with $\varepsilon_{i,k}^\emptyset = \sigma_{i.k}$ by

$$\left(\mathrm{X}_t\upsilon|\varkappa^\emptyset\left(\hat{\rho}_t\right)\upsilon\right) = \left(\mathrm{X}_t\upsilon|\left(\mathbf{L}'\hat{\rho} + \hat{\rho}\mathbf{L}'^{\ddagger} + \mathbf{L}'\hat{\rho}\mathbf{L}'^{\ddagger}\right)_+^-\upsilon\right)$$

with $\left(\mathbf{L}'\hat{\rho}\right)_+^- = L^\emptyset\hat{\rho}$, $\left(\hat{\rho}\mathbf{L}'^{\ddagger}\right)_+^- = \hat{\rho}L^{\emptyset*}$ given by $L^\emptyset = \left(\mathbf{L}'\right)_+^- = \mathrm{L}_+^{-\prime}$. This complets the proof. □

Note that if $\mathbf{L}\left(t\right) \in \widetilde{\mathfrak{S}}^{t]}$ corresponding to the span $L^\circ = e_i^\circ L^i$ of $\left(\mathbf{L}'\right)_+^\circ$ by the family $e_{J_\bullet}^\circ$ of Itô basis $\mathbf{e}_{J_\bullet}$ with $\mathbb{B}^{t]}$-adapted $L^i\left(t\right)$, the $\mathbb{B}^{t]}$-Markov QS evolution (4.4) is completely unravelled by the hemigroup of QS propagators $\hat{V}_r\left(t\right)$ satisfying the *QS unraveling equation*

$$\mathrm{d}\hat{V}_r\left(t\right) + \hat{V}_r\left(t\right)K\left(t\right)\mathrm{d}t = \hat{V}_r\left(t\right)L^i\left(t\right)\mathrm{d}\hat{e}_i, \quad \hat{V}_r\left(t\right) = \mathrm{I}.$$

This can be easily seen by applying the Itô formula to the solution of (4.4) for $t > r$ with $\hat{\rho}\left(r\right) = \hat{\rho}$ in the form $\vartheta_t\left(t, \hat{\rho}\right) = \hat{V}_r\left(t\right)\hat{\rho}\hat{V}_r\left(t\right)^*$. In particular, this is the case if $\mathbf{e}_{J_\bullet} = \mathbf{e}_{J_\circ}$ is complete basis in the maximal modular Itô algebra $\mathfrak{a}$ generating $\mathfrak{k}_\circ$ corresponding to $\mathbb{A}^{t]}$-Markovianity of the dual dynamics $\vartheta^\intercal = \phi$. Such unraveling is predetermined by hemigroup Markovianity on the W*-algebras $\overline{\mathbb{B}}^{t]}$. If the basis $\mathbf{e}_{J_\bullet}$ of $\mathfrak{b}$ is not complete in $\mathfrak{k}_\circ$, the QS dynamics ϑ may not completely unravel the quantum Markov dynamics $\vartheta^\intercal$. However it can be extended to an $\mathbb{A}^{t]}$-Markov unravelled dynamics ϕ. by a choice of the complementary basis $\mathbf{e}_{J_\circ\setminus J_\bullet}$. The extended unravelling evolution satisfies the QS equation of the same form as (3.9) with fluctuating coefficients $\varkappa^j$ indexed by the complete set $J_\circ = J$. Note that such ravelling is not unique and can be chosen classical by taking a commutative Itô algebra $\mathfrak{c} \subset \mathfrak{b}\left(\mathfrak{k}_\bullet^\perp\right)$ which is always modular with the basis $\bar{\mathbf{e}}_j = \mathbf{e}_{j^\star}$ in the orthogonal complement $\mathfrak{k}_\bullet^\perp = \mathfrak{k} \ominus \mathfrak{k}_\bullet$ corresponding to the classical Lévy-Itô noise $\check{e}_j = \hat{e}_{j^\star}$ for $j \in J \setminus J_\bullet$ independent of $\check{e}_{J_\bullet} = \overline{\hat{e}_{J_\bullet}}$. The incompletely unravelled

$\overline{\mathbb{B}}^{t]}$-Markov dynamics $\vartheta^{\intercal}$ is then represented as a conditional expectation of $\mathbb{A}^{t]}$-Markov dynamics ϕ over the noise indexed by $J_{\circ} \setminus J_{\bullet}$. which has obviously the same form as (3.9) $\phi_r(t, \mathrm{Z}) = \epsilon_r[\vartheta_r^{\intercal}(t, \mathrm{Z})]$.

If ϕ is unital, the density operators $\hat{\rho}(t)$ are normalized on $\overline{\mathbb{B}}^{t]}$ to the positive martingale $\hat{\pi}(t) = \langle \mathrm{I}, \hat{\rho}(t)\rangle$ describing the statistics of the complementary processes $\mathtt{y}_j(t) = \mathrm{V}_0(t)\,\hat{e}_{\circ}(t)\,\mathrm{V}_0(t)^*$, otherwise $\hat{\pi}(t)$ is supermartingale normalized to the probability $\pi(t) = \langle\hat{\pi}(t)\rangle_{\emptyset}$ of the object survival up to t.

5. Appendix. Modualar Algebras and L-Transformations

5.1. Modular algebra and its dual $\mathbb{L}$. Let $(\mathbb{A}, \lambda)$ be a pre-C*-algebra with a *reference weight* as a faithful positive liner functional $\lambda : \mathbb{A} \to \mathbb{C}$ in the sense

$$\lambda(\mathrm{p}) \geq 0 \;\forall \mathrm{p} \geq 0, \quad \lambda(\mathrm{p}) = 0 \Leftrightarrow \mathrm{p} = 0,$$

on the positive elements $\mathrm{p} = \mathrm{aa}^*$ $\forall \mathrm{a} \in \mathbb{A}$. Such algebra is called λ-*modular* if it is invariant with respect to the left *modular involution* $\dagger_-$ making sense in

$$\lambda(\mathrm{pq}^*) = \lambda\left(\mathrm{q}^{\dagger_-}\mathrm{p}\right), \quad \lambda(\mathrm{q}^*\mathrm{p}) = \lambda\left(\mathrm{pq}^{\dagger_-^*}\right) \quad \forall \mathrm{p}, \mathrm{q} \in \mathbb{A}$$

with the *right modular* involution defined as $\dagger_-^* = * \dagger_- *$. One can use this to define the left $\langle \mathrm{q}|\mathrm{p}\rangle_- = \lambda\left(\mathrm{q}^{\dagger_-}\mathrm{p}\right)$ or right $\langle \mathrm{q}|\mathrm{p}\rangle_+ = \lambda\left(\mathrm{pq}^{\dagger_-^*}\right)$ modular pairings such that $\langle \mathrm{p}^*|\mathrm{q}\rangle_- = \lambda(\mathrm{qp}) = \langle \mathrm{q}^*|\mathrm{p}\rangle_+$, but usually a symmetric pairing

$$\langle \mathrm{q}^*|\mathrm{p}\rangle \equiv \lambda(\mathrm{q}\cdot\mathrm{p}) = \lambda(\mathrm{q}\cdot\mathrm{p}) \equiv \langle \mathrm{p}^*|\mathrm{q}\rangle$$

is preferred for use, satisfying the modular conditions

$$\langle \mathrm{aq}|\mathrm{p}\rangle = \left\langle \mathrm{q}|\mathrm{a}^{\dagger}\mathrm{p}\right\rangle, \quad \langle \mathrm{qa}|\mathrm{p}\rangle = \left\langle \mathrm{q}|\mathrm{pa}^{\dagger^*}\right\rangle \quad \forall \mathrm{a}, \mathrm{q}, \mathrm{p} \in \mathbb{A}$$

with respect to another left involution $\dagger$ and $\dagger^* = * \dagger *$ as the adjoint to $\dagger$. The symmetric pairing inducing the weight by $\lambda(1\cdot\mathrm{q}) = \lambda(\mathrm{q}) = \lambda(\mathrm{q}\cdot 1)$ on $\mathbb{A}$ is defined by this involution as

$$\langle \mathrm{q}|\mathrm{p}\rangle \equiv \langle \mathrm{q}|\mathrm{p}\rangle_{\lambda} := \lambda(\mathrm{q}\alpha(\mathrm{p})) \equiv \lambda(\mathrm{q}\cdot\mathrm{p}),$$

where α is a root of the modular automorphism $\delta_{\lambda} = \alpha\circ\alpha$ of $\mathbb{A}$ uniquely determined by the polar decomposition $\mathrm{a}^{\dagger} = \alpha(\mathrm{a})^*$ in the case of the positive definite pairing $\langle \mathrm{a}|\mathrm{a}\rangle \geq 0$ for all $\mathrm{a} \in \mathbb{A}$.

If λ is tracial in the sense $\lambda(\mathrm{a}^*\mathrm{a}) = \lambda(\mathrm{aa}^*)$, as it is always in the commutative case, the left involution $\dagger$ coincides with the right involution $\dagger^*$ and for the positive-definite pairing is identical to $*$, but in general it depends on λ: $\dagger = * \circ \delta_{\lambda}^{\frac{1}{2}} \equiv \dagger_{\lambda}$.

5.2. Regular states and Radon-Nikodym densities. The noncommutative L^1-*space* $\mathbb{L} = \mathbb{L}_{\lambda}$ is defined by the L^1-completion $\mathbb{A}_{\star} = \mathbb{A}_{\lambda}^{\star}$ of the modular $*$-algebra $\mathbb{A}$ with respect to the dual norm

$$\|\mathrm{b}\|_{\star} := \sup_{\|\mathrm{a}\|\leq 1} |\langle \mathrm{a}|\mathrm{b}\rangle_{\lambda}| = \frac{1}{2}\sup_{\|\mathrm{a}\|\leq 1} \lambda(\mathrm{a}^*\mathrm{b} + \mathrm{b}^*\mathrm{a}) \equiv \lambda(|\mathrm{b}|_{\lambda})$$

Without changing notation we can extend by weak continuity the bilinear form $\lambda(\mathrm{b}\cdot\mathrm{a})$ to the natural pairing $\mathbb{M}\times\mathbb{L}\to\mathbb{C}$ of the the norm-adjoint algebra $\mathbb{M}=\mathbb{L}^{\bigstar}$ as the weak$^{\bigstar}$ closure of $\mathbb{A}\equiv\mathbb{L}^{\star}\subseteq\mathbb{M}$ equipped with

$$\|\mathrm{a}\|_{\star}^{\bigstar}=\inf\{c:\mathrm{a}^{*}\mathrm{a}\leq c1\}\equiv\|\mathrm{a}\|.$$

Note that $\mathbb{M}$ is unital $*$-algebra called W*-algebra as having the preadjoint Banach space $\mathbb{L}=\mathbb{M}_{\bigstar}$. It is isomorphic to a von Neumann operator algebra, the weak closure of the GNS representation of $\mathbb{B}$ associated with λ. Both $\mathbb{L}$ and $\mathbb{M}$ are two-sided $\mathbb{A}$-modules and $\mathbb{L}^{*}=\mathbb{L}$, $\mathbb{M}^{*}=\mathbb{M}$ with respect to the left and right involutions such that for any

$$\langle\mathrm{bqb}^{*}|\mathrm{p}\rangle=\langle\mathrm{q}|\mathrm{apa}^{*}\rangle\quad\forall\mathrm{q}\in\mathbb{M},\mathrm{p}\in\mathbb{L},\mathrm{b}=\mathrm{a}^{\star}\in\mathbb{A}$$

A linear positive functional $\pi:\mathbb{A}\to\mathbb{C}$ of the form

$$\pi(\mathrm{a})=\lambda(\mathrm{a}\cdot\mathrm{p})=\langle\mathrm{p}|\mathrm{a}\rangle_{\lambda}\quad\forall\mathrm{a}\in\mathbb{A},$$

defining by a positive normalized $\mathrm{p}\in\mathbb{L}^{+}$, $\lambda(\mathrm{p})=1$ an expectation on $\mathbb{A}$, is referred as the *regular state,* and as the *normal* regular state when it is extended on $\mathbb{M}=\mathbb{L}^{\bigstar}$. If $\mathbb{A}=\mathbb{B}/\mathbb{I}$ is a quotient algebra of a $*$-algebra $\mathbb{B}\succeq\mathbb{A}$ with respect to a $\star$-ideal $\mathbb{I}\subseteq\mathbb{B}$, and $\mu:\mathbb{B}\to\mathbb{C}$ be weight such that $\mu(\mathrm{b}^{*}\mathrm{b})=0\Rightarrow\mathrm{b}\in\mathbb{I}$, then the induced state $\rho=\pi\circ\gamma$ is *absolutely continuous* with respect to μ in the sense

$$\mu(\mathrm{b}^{*}\mathrm{b})=0\Rightarrow\pi(\mathrm{a}^{*}\mathrm{a})=0\quad\forall\mathrm{b}\in\mathrm{a},\mathrm{a}\in\mathbb{A}.$$

It has *density* $\mathrm{r}=\lambda_{\mu}^{\gamma}(\mathrm{p})\in\mathbb{B}_{\star}$ given by the 'Radon-Nikodym' (RN) *derivative* $\lambda_{\mu}^{\gamma}=\gamma_{\mu}^{\star}$ as a CP contraction $\gamma_{\mu}^{\star}:\mathbb{A}_{\lambda}^{\star}\to\mathbb{B}_{\mu}^{\star}$ well defined for the regular $\mathbb{I}$ (i.e. adjointable quotient map $\gamma(\mathrm{b})=\mathrm{b}+\mathbb{I}$.) as a comorphism in the next section. The state π id is said to be *dominated* by μ if $\pi(\mathrm{a}^{*}\mathrm{a})\leq t\mu(\mathrm{b}^{*}\mathrm{b})$ for all a and $\mathrm{b}\in\mathrm{a}$ and a positive $t\in\mathbb{R}$. Every ρ absolutely continuous with respect to a regular μ is also regular, and $\mathrm{r}\leq t1$ for a $t>0$ iff it is dominated by μ.

5.3. Comorphisms and quasimorphisms on $\mathbb{L}$. Let $\alpha:\mathbb{D}\to\mathbb{A}$ be a regular $*$-morphism of a modular algebra $(\mathbb{D},\nu)$ into the modular algebra $(\mathbb{A},\lambda)$ such that there exists a dual map $\lambda_{\nu}^{\alpha}:\mathbb{L}\to\mathbb{L}_{\nu}$ called the *comorphism*, uniquely defined by $\lambda_{\nu}^{\alpha}=\alpha_{\nu}^{\star}$ in

$$\langle\mathrm{d}|\alpha_{\nu}^{\star}(\mathrm{p})\rangle_{\nu}=\langle\alpha(\mathrm{d})|\mathrm{p}\rangle\quad\forall\mathrm{p}\in\mathbb{A}_{\lambda}^{\star},\mathrm{d}\in\mathbb{D}.$$

It is easy to see that the comorphism, intertwining the left involution $\dagger_{\nu}$ with $\dagger=\dagger_{\lambda}$, satisfies the α-modularity property

$$\lambda_{\nu}^{\alpha}\left(\alpha(\mathtt{d})^{\dagger}\mathrm{p}\right)=\mathtt{d}^{\dagger_{\nu}}\lambda_{\nu}^{\alpha}(\mathrm{p})\quad\forall\mathrm{p}\in\mathbb{L},\mathtt{d}\in\mathbb{D}$$

and $\lambda_{\nu}^{\alpha}(\mathrm{pa})=\lambda_{\nu}^{\alpha}(\mathrm{p})\mathtt{d}$ for any $\mathrm{a}=\alpha(\mathtt{d})$. Therefore λ_{ν}^{α} is a CP map and $\lambda_{\nu}^{\alpha}(\mathbb{L})\mathbb{I}_{\alpha}=\mathbb{O}$ such that it cannot be unital for the nontrivial $*$-ideal

$$\mathbb{I}_{\alpha}:=\{\mathtt{d}\in\mathbb{D}:\alpha(\mathtt{d})=0\}\equiv\alpha^{-1}(0).$$

If $\widetilde{\alpha}:\mathbb{A}\to\mathbb{D}$ is an injection inverted by α and having the $*$-modularity property

$$\widetilde{\alpha}\left(\alpha(\mathtt{d})\,\mathrm{a}\alpha(\mathtt{d})^{*}\right)=\mathtt{d}\widetilde{\alpha}(\mathrm{a})\,\mathtt{d}^{*}\quad\forall\mathrm{a}\in\mathbb{A},\mathtt{d}\in\mathbb{D},$$

then $\widetilde{\alpha}(\mathbb{A})\mathbb{I}_{\alpha}=\mathbb{O}$ and therefore $\widetilde{\alpha}$ is also $*$-morphism. The induced weight $\varphi=\lambda\circ\alpha$ admits a conditional expectation $\varphi\circ\epsilon=\varphi$ given on $\mathbb{D}$ by the $*$-projection $\epsilon=\widetilde{\alpha}\circ\alpha$ onto the $*$-subalgebra $\widetilde{\alpha}(\mathbb{A})\subseteq\mathbb{D}$ isomorphic to $\mathbb{D}/\mathbb{I}_{\alpha}\simeq\mathbb{A}$, and $\lambda_{\nu}^{\alpha}=\varphi_{\nu}\circ\lambda_{\varphi}^{\alpha}$

is given by the extension $\lambda_{\varphi}^{\alpha}$ of $\widetilde{\alpha}$ from $\mathbb{A}$ onto $\mathbb{L} = \mathbb{A}_{\lambda}^{\star}$ and the RN derivative φ_{ν} of the induced weight φ with respect to ν on $\mathbb{D}$. Note that $\widetilde{\alpha}(\mathbb{A})\mathbb{I}_{\alpha} = \mathbb{O}$ and therefore $\widetilde{\alpha}$ cannot be unital if $\mathbb{I}_{\alpha} \neq \mathbb{O}$.

More generally, a completely positive $*$-contraction $\widetilde{\alpha} : \mathbb{B} \to \widetilde{\mathbb{A}}$ quasiinverse in the sense $\beta \circ \widetilde{\alpha} \circ \beta = \beta$ to a $*$-morphism β of $\mathbb{D} \subseteq \widetilde{\mathbb{A}}$ into $\mathbb{B}$ is called *quasimorphism* (*quasiexpectation inverting* β) if it satisfies the β-modularity condition

$$\widetilde{\alpha}(\mathrm{b}\beta(\mathtt{d})) = \widetilde{\alpha}(\mathrm{b})\,\mathtt{d} \quad \forall \mathrm{b} \in \mathbb{B}, \mathtt{d} \in \mathbb{D}$$

in a modular $*$-algebra $(\widetilde{\mathbb{A}}, \varpi)$ (and inverts β). Every quasimorphism on $(\mathbb{B}, \mu)$ defines a positive projection $\epsilon = \widetilde{\alpha} \circ \beta$ which is almost unital in the sense that $\mathtt{e} = \epsilon(1)$ is the maximal $*$-idempotent in $\mathbb{D}$ such that $\mathtt{e}\mathbb{I}_{\beta} = \mathbb{O}$ since $\mathbb{I}_{\beta} := \beta^{-1}(0) = \mathtt{e}^{\perp}\mathbb{D}$ for $\mathtt{e}^{\perp} = \mathtt{1} - \mathtt{e}$ in the unital $\mathbb{D}$. It induces a new weight $\rho = \varpi \circ \widetilde{\alpha}$ on $\mathbb{B}$ and the reduced weight $\varsigma = \varrho \circ \beta = \varpi \circ \epsilon$ on $\mathbb{D}$ compatible with the projection ϵ such that the modular involution $\dagger_{\rho}$ leaves the subalgebra $\epsilon(\mathbb{D}) \subseteq \widetilde{\mathbb{A}}$ invariant as $\epsilon(\mathtt{d})^{\dagger_{\varsigma}} = \epsilon\left(\mathtt{d}^{\dagger_{\varsigma}}\right)$ for all $\mathtt{d} \in \mathbb{D}$ due to $\beta\left(\mathtt{d}^{\dagger_{\varsigma}}\right) = \beta(\mathtt{d})^{\dagger_{\varrho}}$ and modularity of $\widetilde{\alpha}$. Note that if $\widetilde{\alpha}(\mathbb{B}) \subseteq \mathbb{D}$, then the modularity condition follows since $\pi = \beta \circ \widetilde{\alpha}$ is also a positive projection onto $\beta(\mathbb{D}) \sim \widetilde{\alpha}(\mathbb{B})$. Such quasimorphism inverting β is called *conditional expectation* with respect to β.

A quasimorphism $\widetilde{\alpha}$ uniquely defines the comorphism $\beta_{\nu}^{\star} = \varsigma_{\nu} \circ \beta_{\varsigma}^{\star}$ by $\beta_{\varsigma}^{\star}|\mathbb{B} = \epsilon_{\varsigma}^{\star} \circ \widetilde{\alpha}$ iff $\overline{\alpha} \circ \epsilon = \beta$, where $\overline{\alpha} = \widetilde{\alpha}_{\rho}^{\star}|\widetilde{\mathbb{A}}$ is defined by the dual CP map $\widetilde{\alpha}_{\rho}^{\star}$. In the most important case $\epsilon = \iota$ of injective β this defines $\beta_{\varsigma}^{\star}$ on $\mathbb{B}$ by *marginalization* $\iota_{\varsigma}^{\star} \circ \widetilde{\alpha}$ iff $\overline{\alpha}|\mathbb{D} = \beta$.

5.4. Semiconditioning and semi-Markovianity. A comorphism $\widetilde{\alpha}_{\rho}^{\star}$ for $\rho = \varpi \circ \widetilde{\alpha}$ as a modular map

$$\widetilde{\alpha}_{\rho}^{\star}\left(\mathtt{d}^{\dagger_{\varpi}}\mathtt{p}\right) = \beta(\mathtt{d})^{\dagger_{\rho}}\,\widetilde{\alpha}_{\rho}^{\star}(\mathtt{p}) \quad \forall \mathtt{p} \in \widetilde{\mathbb{A}}_{\varpi}, \mathtt{d} \in \mathbb{D}$$

inverting a comorphism $\beta_{\varsigma}^{\star}$ for $\varsigma = \rho \circ \beta$ is called the *conditional expectation* if it coincides on $\mathbb{D} \subseteq \widetilde{\mathbb{A}}$ with the $*$-morphism $\beta : \mathbb{D} \to \mathbb{B}$. This is equivalent to the invariance $\dagger_{\varsigma} = \dagger_{\varpi}|\mathbb{D}$ of the $\dagger_{\varsigma}$-domain $\mathbb{D}$ for $\varsigma = \varpi \circ \epsilon$ with respect to the modular involution $\dagger_{\varpi}$ on $\widetilde{\mathbb{A}}$. Then the composition $\mathsf{E} = \beta_{\varsigma}^{\star} \circ \widetilde{\alpha}_{\rho}^{\star} = \iota_{\varsigma}^{\star}$, given by the comorphism $\beta_{\varsigma}^{\star}$ and $\widetilde{\alpha}_{\rho}^{\star}$, is a positive projection $\mathbb{L}_{\varpi} \to \mathbb{L}_{\varsigma}$ satisfying the modularity condition

$$\mathsf{E}(\mathtt{d}^{*}\mathtt{pd}) = \mathtt{d}^{*}\mathsf{E}(\mathtt{p})\,\mathtt{d} \quad \forall \mathtt{p} \in \mathbb{L}_{\varpi}, \mathtt{d} \in \mathbb{D}.$$

The restriction $\mathsf{E}|\widetilde{\mathbb{A}} = \beta^{\star} \circ \overline{\alpha}$, extended by continuity on weak closure $\mathbb{M}$ of $\widetilde{\mathbb{A}}$, is the usual conditional expectation as a normal positive projection ϵ intertwining the left involutions $\dagger_{\varpi}$ and $\dagger_{\varsigma}$. It satisfies the modularity $*$-condition

$$\epsilon(\mathtt{d}^{*}\mathtt{qd}) = \mathtt{d}^{*}\epsilon(\mathtt{q})\,\mathtt{d} \;\; \forall \mathtt{q} \in \widetilde{\mathbb{A}}, \mathtt{d} \in \mathbb{D}.$$

This all remains true if $\widetilde{\alpha}_{\rho}^{\star}$ is *quasi-comorphism* defined as a CP modular contraction quasiinverse to a comorphism $\beta_{\varsigma}^{\star}$. Note that positivity with modularity implies CP property if the algebra $\mathbb{B}$ is generated by a central Abelian $*$-subalgebra $\mathbb{A} \subseteq \mathbb{B}$ and $\beta(\mathbb{D})$ since every positive quasimorphism on $\mathbb{A}$ is by Naimark's theorem CP. Such quasi- are referred as semi- (morphisms, comorphisms and conditional expectations).

If $\widetilde{\alpha}_1 : \mathbb{B}_1 \to \widetilde{\mathbb{A}}$ is a conditional coexpectation inverting $\beta_1 : \mathbb{D}_1 \to \mathbb{B}_1$, the composition $\phi = \overline{\alpha} \circ \widetilde{\alpha}_1$ is obviously a contractive CP map $\mathbb{B}_1 \to \mathbb{B}$ defining a CP contractive kernel

$$\kappa\left(\mathtt{d}_1^*, \mathrm{a}_1, \mathtt{d}_1\right) = \phi\left(\beta_1\left(\mathtt{d}\right)^* \mathrm{a}_1 \beta_1\left(\mathtt{d}_1\right)\right)$$

where a_1 is from a central $*$-subalgebra $\mathbb{A}_1 \subseteq \mathbb{B}_1$.

A contractive CP kernel $\kappa : \mathbb{D}_1 \times \mathbb{A}_1 \times \mathbb{D}_1 \to \mathbb{L}_\varrho$ is called semi-Markov with respect to $\beta : \mathbb{D} \to \mathbb{B}$ if its range is generated by $\mathbb{B}_0 = \beta\left(\mathbb{D}\right)$ and an abelian $\mathbb{A}_0 \subseteq \mathbb{B}$ commuting with $\mathbb{B}_0$, and is called Markov if it is unital and its range is in the von Neumann algebra generated only by $\mathbb{B}_0$. Every (semi) Markov kernel is a composition of a coexpectation $\widetilde{\alpha}_1$ on an algebra $\mathbb{B}_1$ generated by $\mathbb{B}_1 = \beta_1\left(\mathbb{D}_1\right)$ commuting with the central $\mathbb{A}_1 \subseteq \mathbb{B}_1$ and a (semi) conditional expectation $\overline{\alpha}$ with β and β_1 defined on respective modular $*$-subalgebras $\mathbb{D}, \mathbb{D}_1 \subseteq \widetilde{\mathbb{A}}$. The hemigroups of (semi) Markov maps $\phi_{t_0}\left(t\right) : \mathbb{B}_t \to \mathbb{B}_{t_0}$ give a weak description of (semi) Markov quantum stochastic processes as defined in [1][2].

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