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HOW TO DIFFERENTIATE A QUANTUM STOCHASTIC COCYCLE

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Abstract. Two new approaches to the infinitesimal characterisation of quantum stochastic cocycles are reviewed. The first concerns mapping cocycles on an operator space and demonstrates the role of Hölder continuity; the second concerns contraction operator cocycles on a Hilbert space and shows how holomorphic assumptions yield cocycles enjoying an infinitesimal characterisation which goes beyond the scope of quantum stochastic differential equations.

1. Introduction

The advent of the Hudson-Parthasarathy quantum stochastic calculus coincided with the beginning of Arveson's study of product systems of Hilbert spaces. The former is set in symmetric Fock space over an L^2-space of vector-valued functions ([24]); the latter has the continuous tensor product decomposition of such Fock spaces

$$\mathcal{F} = \mathcal{F}_{[0,t[} \otimes \mathcal{F}_{[t,\infty[}, \quad t \in \mathbb{R}_+,$$

as its paradigm example ([3]). At a conference during the 1986-7 Warwick Symposium on Operator Algebras, Arveson raised the following question. Noting that randomising a one-parameter unitary group $(U_x)_{x\in\mathbb{R}}$ on a Hilbert space $\mathfrak{h}$ using a Brownian motion $(B_t)_{t\geq 0}$:

$$V_t\xi : \omega \to U_{B_t(\omega)}\xi(\omega), \quad t \in \mathbb{R}_+, \xi \in \mathfrak{h} \otimes L^2(\Omega),$$

defines a family of unitaries $V = (V_t)_{t\geq 0}$ on $L^2(\Omega; \mathfrak{h}) = \mathfrak{h} \otimes L^2(\Omega)$ satisfying a cocycle identity with respect to the shift on Brownian paths:

$$V_{r+t} = V_r\sigma_r(V_t), \quad r, t \in \mathbb{R}_+,$$

he asked what other ways may such cocycles be generated — are they all of this type? Armed with various quantum martingale representation theorems ([21, 23]), the latter obtained in collaboration with Parthasarathy at the 1984-5 Warwick Symposium on Stochastic Differential Equations, Robin and I were able to provide an immediate answer. Our answer was a qualified 'yes', if the process of randomisation is broadened to involve *quantum Brownian motion* ([10]) along with

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the Poisson-type process derived from differential second quantisation, if adaptedness to the corresponding filtration of *operator algebras* is imposed, and if one assumes sufficient regularity for the cocycle. The setting is *Wiener space* $L^2(\Omega)$, with its semigroup of shifts inducing the semigroup $(\sigma_t)_{t\geq 0}$ on $B(L^2(\Omega))$, now called *CCR flow*. This is translated to symmetric Fock space $\mathcal{F}$ over $L^2(\mathbb{R}_+)$ via the Wiener-Segal-Itô isomorphism (see, for example [27]).

Theorem 1.1 ([22])**.** *Let $V = (V_t)_{t\geq 0}$ be a unitary quantum stochastic cocycle on $\mathfrak{h}$, and suppose that V is Markov-regular. Then there is a unique operator $F \in B(\mathfrak{h}\oplus\mathfrak{h})$ such that V satisfies the quantum stochastic differential equation*

$$dV_t = V_t d\Lambda_F(t),\ V_0 = I_{\mathfrak{h}\otimes\mathcal{F}}.$$

(All the terms used here will be defined in Section 2). By the quantum Itô formula, the coefficient F necessarily satisfies the Hudson-Parthasarathy unitarity conditions $q(F) = 0 = r(F)$, where

$$q(F) := F^* + F + F^*\Delta F \text{ and } r(F) := F + F^* + F\Delta F^*,$$

and Δ denotes the quantum Itô projection $\left[\begin{smallmatrix} 0 & \\ & I_{\mathfrak{h}} \end{smallmatrix}\right] \in B(\mathfrak{h}\oplus\mathfrak{h})$. In terms of its block matrix form and the component quantum stochastic integrators, the unitarity conditions on F read

$$F = \begin{bmatrix} iH - L^*L/2 & -L^*C \\ L & C - I_{\mathfrak{h}} \end{bmatrix},$$

with C being unitary, H selfadjoint and L arbitrary, and the quantum stochastic differential equation reads

$$dV_t = V_t\big(L\otimes I_{\mathcal{F}}\, dA^* + (C - I_{\mathfrak{h}})\otimes I_{\mathcal{F}}\, dN_t - L^*C\otimes I_{\mathcal{F}}\, dA_t + (iH - L^*L/2)\otimes I_{\mathcal{F}}\, dt\big),$$

with $(A^*_t)_{t\geq 0}$, $(N_t)_{t\geq 0}$ and $(A_t)_{t\geq 0}$ being respectively the *creation*, *preservation* (*number, exchange* or *guage*) and *annihilation processes*.

The result was proved as follows. Letting K denote the generator of the expectation semigroup of V, then K is bounded (by Markov-regularity) and

$$X := \Big(V_t - I_{\mathfrak{h}\otimes\mathcal{F}} - \int_0^t V_s(K\otimes I_{\mathcal{F}})ds\Big)_{t\geq 0}$$

defines a quantum martingale: $\mathbb{E}_s[X_t] = X_s$ $(s \leq t)$, satisfying $X_0 = 0$, which may be shown to be *regular* in the sense of Parthasarathy and Sinha ([41]) so that

$$X_t = \Lambda_t(G) = \int_0^t L'_t\, dA^* + \int_0^t (C'_t - I_{\mathfrak{h}\otimes\mathcal{F}})\, dN_t + \int_0^t M'_t\, dA_t,$$

where $G = \Big(\left[\begin{smallmatrix} 0 & M'_t \\ L'_t & C'_t - I_{\mathfrak{h}\otimes\mathcal{F}} \end{smallmatrix}\right]\Big)_{t\geq 0}$ for bounded processes L', M' and C'. The proof is completed by verifying that each of the processes

$$(V^*_t L'_t)_{t\geq 0},\ (V^*_t M'_t)_{t\geq 0} \text{ and } (V^*_t C'_t)_{t\geq 0}$$

is (a.e.) constant. If the cocycle is instead adapted to the filtration of a non-minimal variance quantum Brownian motion then the same result holds (with no preservation integral) with similar proof, but using the martingale representation theorem for martingales with respect to this filtration ([21]).

Journé then gave a qualified 'no' to Arveson's question. Along with an analysis of quantum stochastic cocycles V, assumed only to be strongly continuous, he gave an example to show that in general V will *not* satisfy a quantum stochastic differential equation. The essential point is that we may associate an operator L to V and, in favourable circumstances also an operator C, but the domains of L and L^*C may have insufficient intersection, thereby cheating us out of a dense domain for a coefficient of a quantum stochastic differential equation. In his example $\mathfrak{h} = L^2(\mathbb{R}_+)$, $C = I_{\mathfrak{h}}$ and L is the generator of the right-shift semigroup on $\mathfrak{h}$ ([25]).

Bradshaw considered the corresponding question for quantum stochastic mapping cocycles on a von Neumann algebra N. Using the quantum martingale representation theorem in a corresponding way to the argument sketched above, he was able to show that every Markov-regular, normal, injective, unital *-homomorphic cocycle on N is governed by a quantum stochastic differential equation ([7]).

Subsequent developments in the construction and analysis of quantum stochastic cocycles, up to the late nineties, are described in [27] and [18], both of which contain extensive bibliographies.

In this paper two new developments are described. The first, given in Section 3, is a direct and simple approach to differentiating mapping cocycles on an operator algebra, and more generally, on an operator space ([28]). The basic assumption here is that there is an *adjoint cocycle* and that both cocycles have locally Hölder continuous columns with exponent 1/2. This analysis extends to quantum stochastic cocycles in Banach space and in abstract operator space ([12]). The second, given in Section 4, is the case of contraction operator cocycles on a Hilbert space whose expectation semigroup is holomorphic ([30]). Here we go beyond quantum stochastic differential equations and yet still obtain a complete infinitesimal description of such cocycles.

The definitions and basic properties of quantum stochastic cocycles, for both (contraction operator) cocycles on a Hilbert space and (mapping) cocycles on an operator space, are given in Section 2. The latter requires *matrix spaces* over an operator space ([35]); the basic facts about these are given in that section.

Notations. For a set S and vector spaces U and V we write $F(S;V)$ for the linear space of functions from S to V, under pointwise operations, and $L(U;V)$ for the space of linear maps from U to V. For a vector-valued function $f : \mathbb{R}_+ \to V$ and subinterval I of $\mathbb{R}_+$, f_I denotes the function equal to f on I and zero outside I; for $c \in V$, $c_I : \mathbb{R}_+ \to V$ is defined in the same way, by viewing c as a constant function. Simple tensors $u \otimes \xi$ are usually abbreviated to $u\xi$. For a Hilbert space h we set $|\mathsf{h}\rangle = B(\mathbb{C};\mathsf{h})$ and, mindful of the Riesz-Fréchet Theorem, $\langle\mathsf{h}| = B(\mathsf{h};\mathbb{C})$, so that $|\mathsf{h}\rangle = \{|u\rangle : u \in \mathsf{h}\}$ and $\langle\mathsf{h}| = \{\langle u| : u \in \mathsf{h}\}$, where the -ket $|u\rangle$ maps $\lambda \in \mathbb{C}$ to λu, and the -bra $\langle u|$ maps $v \in \mathsf{h}$ to $\langle u, v\rangle$. The bra-/-ket notation is ampliated to the following useful *E-notations*:

$$E_\xi = I_{\mathfrak{h}} \otimes |\xi\rangle : \mathfrak{h} \to \mathfrak{h} \otimes \mathsf{h}, \text{ respectively } E^{\xi'} = I_{\mathfrak{h}} \otimes \langle\xi| : \mathfrak{h}' \otimes \mathsf{h}' \to \mathfrak{h}' \tag{1.1}$$

for vectors ξ and ξ' from Hilbert spaces h and h'. Ultraweak tensor products are denoted $\overline{\otimes}$ and purely algebraic tensor products by $\underline{\otimes}$. For an operator space V and Hilbert space h, $\iota^{\mathsf{V}}_{\mathsf{h}}$ denotes the ampliation $x \in \mathsf{V} \mapsto x \otimes I_{\mathsf{h}}$.

Fix now, and for the rest of the paper, two Hilbert spaces $\mathfrak{h}$ *and* k. Set $\widehat{\mathsf{k}} := \mathbb{C} \oplus \mathsf{k}$ and, for $c \in \mathsf{k}$, set $\widehat{c} := \binom{1}{c} \in \widehat{\mathsf{k}}$; also, for $f \in L^2(\mathbb{R}_+; \mathsf{k})$ set $\widehat{f}(t) := \widehat{f(t)}$. We often make the identification

$$\mathfrak{h} \otimes \widehat{\mathsf{k}} = \mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

and employ the *quantum Itô projection* $\Delta := \left[\begin{smallmatrix} 0 & \\ & I \end{smallmatrix}\right] \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) = B(\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}))$.

2. Quantum Stochastic Cocycles

For $0 \leq r < t \leq \infty$, the symmetric Fock space over $L^2([r, t[; \mathsf{k})$ is denoted by $\mathcal{F}_{[r,t[}$ and the identity operator on $\mathcal{F}_{[r,t[}$ by $I_{[r,t[}$; $\mathcal{F}_{\mathbb{R}_+}$ is abbreviated to $\mathcal{F}$. We use normalised exponential vectors:

$$\varpi(f) := e^{-\|f\|^2/2} \varepsilon(f), \quad f \in L^2(\mathbb{R}_+; \mathsf{k}),$$

where $\varepsilon(f)$ is the exponential vector $(1, f, (2!)^{-1/2} f^{\otimes 2}, \cdots) \in \mathcal{F}$. As is well-known, the family $\{\varepsilon(f) : f \in L^2(\mathbb{R}_+; \mathsf{k})\}$ is linearly independent and total in $\mathcal{F}$. The following considerable strengthening of the latter property, due to Parthasarathy-Sunder for one dimensional k and Skeide for general k, has proved very useful (see [27] for a proof). For a subset S of k define

$$\mathcal{E}_S := \operatorname{Lin}\{\varepsilon(f) : f \in \mathbb{S}_S\}$$

where $\mathbb{S}_S := \{f \in L^2(\mathbb{R}_+; \mathsf{k})$: f is an S-valued step function$\}$ (with the convention that we always take the right-continuous versions), and abbreviate $\mathcal{E}_{\mathsf{k}}$ to $\mathcal{E}$.

Proposition 2.1. *Let* T *be a subset of* k *containing* 0. *Then the following are equivalent*:

(i) T *is total in* k;
(ii) $\mathcal{E}_{\mathsf{T}}$ *is dense is* $\mathcal{F}$.

Write $W(f)$ ($f \in L^2(\mathbb{R}_+; \mathsf{k})$) for the unitary (*Weyl*) operator on $\mathcal{F}$ determined by

$$W(f)\varpi(g) = e^{-i \operatorname{Im}\langle f, g\rangle} \varpi(f + g), \quad g \in L^2(\mathbb{R}_+; \mathsf{k}),$$

R_t ($t \in \mathbb{R}_+$) for the unitary (*time-reversal*) operator on $\mathcal{F}$ determined by

$$R_t \varpi(g) = \varpi(r_t g) \text{ where } (r_t g)(s) = \begin{cases} g(t-s) & s \in [0, t[, \\ g(s) & s \in [t, \infty[, \end{cases} \quad (g \in L^2(\mathbb{R}_+; \mathsf{k})),$$

S_t ($t \in \mathbb{R}_+$) for the isometric (*shift*) operator on $\mathcal{F}$ determined by

$$S_t \varpi(g) = \varpi(s_t g) \text{ where } (s_t g)(r) = \begin{cases} 0 & r \in [0, t[, \\ g(r-t) & r \in [t, \infty[, \end{cases} \quad (g \in L^2(\mathbb{R}_+; \mathsf{k})),$$

σ_t ($t \in \mathbb{R}_+$) for the *-homomorphic (*shift*) operator on $B(\mathfrak{h} \otimes \mathcal{F})$ determined by

$$\big\langle u\varpi(f), \sigma_t(T) v\varpi(g)\big\rangle = \langle \varpi(f_{[0,t[}), \varpi(g_{[0,t[})\rangle \big\langle u\varpi(s_t^* f), T\, v\varpi(s_t^* g)\big\rangle,$$

($T \in B(\mathfrak{h} \otimes \mathcal{F})$, $f, g \in L^2(\mathbb{R}_+; \mathsf{k})$, $u, v \in \mathfrak{h}$), and $\mathbb{E}$ for the completely positive and contractive (*expectation*) map

$$\mathbb{E} = \operatorname{id}_{B(\mathfrak{h})} \overline{\otimes}\, \omega_{\varepsilon(0)} : B(\mathfrak{h} \otimes \mathcal{F}) \to B(\mathfrak{h}).$$

By a *quantum stochastic contraction cocycle on* $\mathfrak{h}$ *with noise dimension space* k, we mean a family of contraction operators $V = (V_t)_{t\geq 0}$ on $\mathfrak{h} \otimes \mathcal{F}$ satisfying

$$\begin{aligned}&s \mapsto V_s \text{ is strongly continuous,}\\ &V_t \in B(\mathfrak{h} \otimes \mathcal{F}_{[0,t[}) \otimes I_{[t,\infty[},\\ &V_{r+t} = V_r \sigma_r(V_t) \text{ and } V_0 = I_{\mathfrak{h}\otimes\mathcal{F}}, \quad r,t \in \mathbb{R}_+.\end{aligned}$$

Let $\mathbb{QS}_c\mathbb{C}(\mathfrak{h},\mathsf{k})$ denote the collection of such cocycles. For $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h},\mathsf{k})$,

$$P^{0,0} := \big(\mathbb{E}[V_t]\big)_{t\geq 0}$$

defines a contractive C_0-semigroup on $\mathfrak{h}$ called the *expectation semigroup* of V. The semigroup property follows from the identities

$$\mathbb{E} = \mathbb{E} \circ \mathbb{E}_r \quad \text{and} \quad \mathbb{E}[V_r(T \otimes I_{\mathcal{F}})] = \mathbb{E}[V_r]T, \quad r \in \mathbb{R}_+, T \in B(\mathfrak{h}),$$

for the conditional expectation maps

$$\mathbb{E}_r : B(\mathfrak{h} \otimes \mathcal{F}) \to B(\mathfrak{h} \otimes \mathcal{F}),\ X \mapsto \big(\mathrm{id}_{B(\mathfrak{h}\otimes\mathcal{F}_{[0,r[})} \,\overline{\otimes}\, \omega_{\varepsilon(0_{[r,\infty[})}\big)(X) \otimes I_{[r,\infty[}, \quad r \geq 0.$$

The continuity of $P^{0,0}$ corresponds precisely to the strong (equivalently, weak operator) continuity assumption on V ([37], Lemma 1.2). The expectation semigroup is just one of the family of *associated semigroups* of V:

$$P^{c,d} := \big((\mathrm{id}_{B(\mathfrak{h})} \,\overline{\otimes}\, \omega_{\varpi(c_{[0,t[}),\varpi(d_{[0,t[})})(V_t)\big)_{t\geq 0}, \quad c,d \in \mathsf{k},$$

which together determine V through the *semigroup representation* ([34], Proposition 6.2). In fact any family $\{P^{c,d} : c \in \mathsf{T}', d \in \mathsf{T}\}$ where T and T' are each total in k and contain 0 suffices, thanks to Proposition 2.1. This is important since, for example, if k is one-dimensional then one may take $\mathsf{T} = \{0,1\}$ and V is then determined by its expectation semigroup together with just three of its other associated semigroups.

Write $\mathbb{QS}_c\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h},\mathsf{k})$ for the subclass of *Markov-regular* QS contraction cocycles, that is those whose associated semigroups are all norm-continuous. Contractivity of V actually implies that Markov-regularity is equivalent to norm-continuity of just the expectation semigroup ([34], Theorem 6.6).

Example 2.2 (Weyl cocycles)**.** For $c \in \mathsf{k}$,

$$W^c := \big(I_{\mathfrak{h}} \otimes W(c_{[0,t[})\big)_{t\geq 0}$$

defines a Markov-regular QS contraction cocycle; $\mathbb{E}[W^c_t] = e^{-t\|c\|^2/2} I_{\mathfrak{h}}$.

Two useful constructions of new cocycles from old are as follows. Let $V \in \mathbb{QS}_c\mathbb{C}(\mathfrak{h},\mathsf{k})$. Then the *dual cocycle* $\widetilde{V}$ is given by

$$\widetilde{V}_t := \big(R_t V_t^* R_t\big)_{t\geq 0};$$

its associated semigroups are given, in terms of the associated semigroups $\{P^{c,d} : c,d \in \mathsf{k}\}$ of V, by $\widetilde{P}^{c,d}_t = (P^{d,c}_t)^*$; in particular, $\widetilde{V}$ is Markov-regular if and only if V is. The *associated cocycles* of V are defined by

$$V^{c,d} := \big((W^c_t)^* V_t W^d_t\big)_{t\geq 0}, \quad c,d \in \mathsf{k}.$$

It is easily verified that these are indeed QS contraction cocycles; they are all Markov-regular if V is, and the (c,d)-associated semigroup of V is precisely the expectation semigroup of the (c,d)-associated cocycle of V.

Example 2.3. Let $U=(U_x=e^{ixH})_{x\in\mathbb{R}}$ be a strongly continuous one-parameter unitary group on $\mathfrak{h}$ and let $(B_t)_{t\geq 0}$ be the standard Wiener process. Then

$$(V_tF)(\omega)=U_{B_t(\omega)}F(\omega),\quad F\in L^2(\Omega;\mathfrak{h})=\mathfrak{h}\otimes L^2(\Omega),$$

defines a family of unitaries $V=(V_t)_{t\geq 0}$ on $\mathfrak{h}\otimes L^2(\Omega)$ or, under the Wiener-Segal-Itô isomorphism, on $\mathfrak{h}\otimes\mathcal{F}$ (with $\mathsf{k}=\mathbb{C}$) which comprises a QS contraction cocycle with expectation semigroup $(e^{-tH^2/2})_{t\geq 0}$. This is the example highlighted by Arveson; it is Markov-regular if and only if H is bounded, in other words U is norm-continuous.

Processes other than Brownian motion may be used and multidimensions can easily be incorporated. Here we are interested in the general structure of subclasses of $\mathbb{QS}_{\mathrm{c}}\mathbb{C}(\mathfrak{h},\mathsf{k})$ and this involves *quantum* processes, as we have seen. For many examples of cocycles arising from quantum optics and classical probability see [6, 18, 36], and references therein.

So far we have only considered contraction operator cocycles on a Hilbert space. Given $V\in\mathbb{QS}_{\mathrm{c}}\mathbb{C}(\mathfrak{h},\mathsf{k})$,

$$k_t^V: x\in B(\mathfrak{h})\mapsto V_t(x\otimes I_{\mathcal{F}})V_t^*\in(\mathfrak{h}\otimes\mathcal{F}),\quad t\in\mathbb{R}_+,\tag{2.1}$$

defines a family of completely positive contractions $k^V=(k_t^V)_{t\geq 0}$ enjoying the cocycle relation

$$k_{r+t}=\widehat{k}_r\circ\sigma_r\circ k_t,\quad r,t\in\mathbb{R}_+,$$

where $\widehat{k}_r$ is the natural extension of k_r to a map

$$\operatorname{Ran}\sigma_r=B(\mathfrak{h})\otimes I_{[0,r[}\otimes B(\mathcal{F}_{[r,\infty[})\to B(\mathfrak{h})\otimes B(\mathcal{F}_{[0,r[})\otimes B(\mathcal{F}_{[r,\infty[})=B(\mathfrak{h}\otimes\mathcal{F}).$$

Note that the *induced cocycle* k^V is unital if and only if V is coisometric, and is homomorphic if V is partially isometric with $V_t^*V_t\in I_{\mathfrak{h}}\otimes B(\mathcal{F})$ $(t\geq 0)$, in particular if V is isometric. Partially isometric and projection-valued QS cocycles are analysed in [$\mathrm{W}_{1,2}$]. Note also that isometry for V is equivalent to unitality for $k^{\widetilde{V}}$.

In generalising the above class of cocycles, in particular to non-inner cocycles, it is convenient to drop the contractivity/boundedness condition. Fortunately the cocycle identity is easily expressible in terms of certain 'slices' of the maps $(k_t)_{t\geq 0}$ as follows:

$$\kappa_{r+t}^{f,g}=\kappa_r^{f,g}\circ\kappa_t^{S_r^*f,S_r^*g},\quad r,t\in\mathbb{R}_+,\tag{2.2}$$

where $\kappa_t^{f,g}(x):=\big(\mathrm{id}_{B(\mathfrak{h})}\,\overline{\otimes}\,\omega_{\xi,\eta}\big)\circ k_t$ with $\xi=\varpi(f_{[0,t[})$ and $\eta=\varpi(g_{[0,t[})$, and this requires only that each $k_t(x)$ is an operator whose domain includes sufficiently many vectors of the form $u\varpi(h)$ and that the resulting operators $\kappa_t^{f,g}(x)$ are bounded.

A *concrete operator space* is a closed subspace V of $B(\mathsf{H};\mathsf{K})$ for some Hilbert spaces H and K; we speak of V being an operator space *in* $B(\mathsf{H};\mathsf{K})$, or $B(\mathsf{H};\mathsf{K})$ being the *ambient* (*full operator*) *space* of V. The adjoint operator space $\mathsf{V}^\dagger$ is the operator space $\{x^*: x\in\mathsf{V}\}$ in $B(\mathsf{K};\mathsf{H})$, and the adjoint map of a linear map between operator spaces $\phi : \mathsf{V} \to \mathsf{W}$ is the map $\phi^\dagger : \mathsf{V}^\dagger \to \mathsf{W}^\dagger$, $x^* \mapsto \phi(x)^*$. A basic notion is that of *complete boundedness* for a linear map between operator spaces $\phi : \mathsf{V} \to \mathsf{W}$. This means that the norms of the matrix liftings

$$\phi^{(n)} : [v_{i,j}] \mapsto [\phi(v_{i,j})], \tag{2.3}$$

in which $M_n(\mathsf{V})$ has the norm inherited from $B(\mathsf{H};\mathsf{K})$, are uniformly bounded:

$$\|\phi\|_{\text{cb}} := \sup_{n\in\mathbb{N}} \|\phi^{(n)}\| < \infty.$$

There are several excellent texts on operator space theory ([5, 16, 42, 43]).

Now let V be an operator space in $B(\mathfrak{h};\mathfrak{h}')$. For any Hilbert spaces h and h', the h-h'-*matrix space over* V is defined as follows:

$$\mathsf{V} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}') := \big\{ T \in B(\mathfrak{h} \otimes \mathsf{h}; \mathfrak{h}' \otimes \mathsf{h}') : \forall_{\omega \in B(\mathsf{h};\mathsf{h}')_*} \ (\text{id}_{B(\mathfrak{h};\mathfrak{h}')} \overline{\otimes} \omega)(T) \in \mathsf{V} \big\}$$

([35]). Recalling the E-notations (1.1), a convenient characterisation, in terms of any total subsets S and S' of h and h', is as follows:

$$\mathsf{V} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}') = \big\{ T \in B(\mathfrak{h} \otimes \mathsf{h}; \mathfrak{h}' \otimes \mathsf{h}') : \forall_{\xi \in S, \xi' \in S'} \ E^{\xi'} T E_\xi \in \mathsf{V} \big\}.$$

Thus $\mathsf{V} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}')$ is an operator space in $B(\mathfrak{h} \otimes \mathsf{h}; \mathfrak{h}' \otimes \mathsf{h}')$.

Warning. For a C^*-algebra A, the operator space $\mathsf{A} \otimes_{\text{M}} B(\mathfrak{h})$ *need not* be a C^*-algebra (see [35] for an example).

The lifting of maps ϕ between operator spaces to maps between matrices over the operator spaces (2.3) extends to matrix spaces as follows. Let $\phi \in CB(\mathsf{V};\mathsf{W})$ for concrete operator spaces V and W, then there is a unique map

$$\phi \otimes_{\text{M}} \text{id}_{B(\mathsf{h};\mathsf{h}')} : \mathsf{V} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}') \to \mathsf{W} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}')$$

satisfying

$$E^{\xi'} (\phi \otimes_{\text{M}} \text{id}_{B(\mathsf{h};\mathsf{h}')})(T) E_\xi = \phi(E^{\xi'} T E_\xi), \quad T \in \mathsf{V} \otimes_{\text{M}} B(\mathsf{h};\mathsf{h}'), \xi \in \mathsf{h}, \xi' \in \mathsf{h}',$$

moreover, $\phi \otimes_{\text{M}} \text{id}_{B(\mathsf{h};\mathsf{h}')}$ is a completely bounded operator with cb-norm at most $\|\phi\|_{\text{cb}}$ ([35]). If $B(\mathsf{h};\mathsf{h}')$ is finite dimensional then ϕ need only be bounded for $\phi \otimes_{\text{M}} \text{id}_{B(\mathsf{h};\mathsf{h}')}$ to exist; in this case the matrix lifting is a bounded operator. To handle both situations conveniently we coined the term h-*boundedness* for a linear map $\phi : \mathsf{V} \to \mathsf{W}$, meaning bounded/completely bounded according as h is finite-dimensional/infinite-dimensional, and write h-$B(\mathsf{V};\mathsf{W})$ for this class of maps.

A *QS cocycle on* V *with noise dimension space* k *and exponential domain* $\mathcal{E}_\mathsf{T}$ (where T is a total subset of k containing 0) is a family $k = (k_t)_{t \geq 0}$ of linear maps

$$k_t : \mathsf{V} \to L(\mathfrak{h} \underline{\otimes} \mathcal{E}_\mathsf{T}; \mathfrak{h}' \otimes \mathcal{F}) \cong L\big(\mathcal{E}_\mathsf{T}; L(\mathfrak{h}; \mathfrak{h}' \otimes \mathcal{F})\big)$$

satisfying

$$E^{\varepsilon(f)} k_t(x) E_{\varepsilon(g)} \in \mathsf{V},$$

the *weak cocycle relation* (2.2) and the *adaptedness condition*

$$\langle u'\varepsilon(f), k_t(x) u \varepsilon(g) \rangle = \langle \varepsilon(f_{[t,\infty[}), \varepsilon(g_{[t,\infty[}) \rangle \langle u' \varepsilon(f_{[0,t[}), k_t(x) u \varepsilon(g_{[0,t[}) \rangle$$

($x \in \mathsf{V}$, $u \in \mathfrak{h}$, $u' \in \mathfrak{h}'$, $g \in \mathbb{S}_\mathsf{T}, f \in \mathbb{S}$, $t \in \mathbb{R}_+$). The collection of such cocycles is denoted $\mathbb{QSC}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ (we shall impose the appropriate continuity condition in

t shortly). The subclass of cocycles $k \in \mathbb{QSC}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ having an *adjoint cocycle* $k^\dagger \in \mathbb{QSC}(\mathsf{V}^\dagger : \mathcal{E}_{\mathsf{T}'})$, so that

$$E^\varepsilon k^\dagger(x^*) E_{\varepsilon'} = (E^{\varepsilon'} k(x) E_\varepsilon)^*, \quad x \in \mathsf{V}, \varepsilon \in \mathcal{E}_\mathsf{T}, \varepsilon' \in \mathcal{E}_{\mathsf{T}'}, t \in \mathbb{R}_+,$$

is denoted $\mathbb{QSC}^\ddagger(\mathsf{V} : \mathcal{E}_{\mathsf{T}'}, \mathcal{E}_\mathsf{T})$. A cocycle k has h-*bounded columns* if, in the notation $k_{t,\varepsilon} := k_t(\cdot)E_\varepsilon : \mathsf{V} \to L(\mathfrak{h}; \mathfrak{h}' \otimes \mathcal{F})$, it satisfies

- $k_{t,\varepsilon}(\mathsf{V}) \subset \mathsf{V} \otimes_\mathrm{M} |\mathcal{F}\rangle$, and
- $k_{t,\varepsilon}$ is h-bounded $\mathsf{V} \to \mathsf{V} \otimes_\mathrm{M} |\mathcal{F}\rangle$, $t \in \mathbb{R}_+, \varepsilon \in \mathcal{E}_\mathsf{T}$.

We say that a cocycle k is h-*bounded* if it satisfies the stronger conditions

- $k_t(\mathsf{V}) \subset \mathsf{V} \otimes_\mathrm{M} B(\mathcal{F})$, and
- k_t is h-bounded $\mathsf{V} \to \mathsf{V} \otimes_\mathrm{M} B(\mathcal{F})$, $t \in \mathbb{R}_+$.

Mapping cocycles $k \in \mathbb{QSC}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ also have associated semigroups:

$$\mathcal{P}^{c,d} := \big(E^{\varpi(c_{[0,t[})} k_t(\cdot) E_{\varpi(d_{[0,t[})} \big)_{t \geq 0}, \quad c \in \mathsf{k}, d \in \mathsf{T},$$

and, as for operator cocycles, the collection of associated semigroups determines the cocycle — thanks to Proposition 2.1. For a thorough investigation of the reconstructability of cocycles from compatible families of semigroups see [38], which was inspired by [1].

We now introduce the continuity condition which plays the central role in Section 3. Let $\mathbb{QSC}_{\mathsf{h}\text{-bHc}}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ denote the class of cocycles $k \in \mathbb{QSC}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ having h-bounded columns and such that

$$t \mapsto k_{t,\varepsilon} \text{ is Hölder } \frac{1}{2}\text{-continuous } \mathbb{R}_+ \to \mathsf{h}\text{-}B(\mathsf{V}; \mathsf{V} \otimes_\mathrm{M} |\mathcal{F}\rangle) \text{ at } t = 0,$$

and let $\mathbb{QSC}^\ddagger_{\mathsf{h}\text{-bHc}}(\mathsf{V} : \mathcal{E}_{\mathsf{T}'}, \mathcal{E}_\mathsf{T})$ denote the subclass

$$\Big\{ k \in \mathbb{QSC}^\ddagger(\mathsf{V} : \mathcal{E}_{\mathsf{T}'}, \mathcal{E}_\mathsf{T}) \cap \mathbb{QSC}_{\mathsf{h}\text{-bHc}}(\mathsf{V} : \mathcal{E}_\mathsf{T}) : k^\dagger \in \mathbb{QSC}_{\mathsf{h}\text{-bHc}}(\mathsf{V}^\dagger : \mathcal{E}_{\mathsf{T}'}) \Big\}.$$

We refer to elements of $\mathbb{QSC}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ as *Hölder cocycles*. The reason for highlighting this class of cocycle is hinted at in the observation (3.2) and fully justified in Theorem 3.1. In this connection, note the elementary estimate

$$\|W^c_{t,\varepsilon(0)} - W^c_{r,\varepsilon(0)}\| = \|c\|(t-r)^{1/2} + O(t-r), \quad c \in \mathsf{k}, \tag{2.4}$$

as $t \to r$ in a bounded interval.

We again call cocycles $k \in \mathbb{QSC}(\mathsf{V} : \mathcal{E}_\mathsf{T})$ *Markov-regular* (respectively, *cb-Markov-regular*) when each associated semigroup is norm-continuous (respectively, cb-norm-continuous), and we have the inclusions:

$$\mathbb{QSC}_{\mathsf{h}\text{-bHc}}(\mathsf{V} : \mathcal{E}_\mathsf{T}) \subset \mathbb{QSC}_\mathrm{Mreg}(\mathsf{V} : \mathcal{E}_\mathsf{T}), \text{ and } \mathbb{QSC}_\mathrm{cbHc}(\mathsf{V} : \mathcal{E}_\mathsf{T}) \subset \mathbb{QSC}_\mathrm{cbMreg}(\mathsf{V} : \mathcal{E}_\mathsf{T}).$$

Note that the prescription

$$k_t(|u\rangle) = V_t\big(|u\rangle \otimes I_\mathcal{F}\big) \tag{2.5}$$

($u \in \mathfrak{h}$, $\varepsilon \in \mathcal{E}$, $t \in \mathbb{R}_+$) defines a completely contractive mapping cocycle $k \in \mathbb{QSC}^\ddagger(|\mathfrak{h}\rangle : \mathcal{E}, \mathcal{E})$ for each contraction operator cocycles $V \in \mathbb{QS}_\mathrm{c}\mathbb{C}(\mathfrak{h}, \mathsf{k})$. The respective associated semigroups are related by

$$\mathcal{P}^{c,d}_t(|u\rangle) = |P^{c,d}_t u\rangle = P^{c,d}_t |u\rangle, \quad c, d \in \mathsf{k}, t \in \mathbb{R}_+.$$

The basic contraction mapping cocycles defined by

$$k_t(|u\rangle) = W_t^c\big(|u\rangle \otimes I_{\mathcal{F}}\big)$$

lie in $\mathbb{QSC}^{\ddagger}_{\text{cbHc}}(|\mathfrak{h}\rangle : \mathcal{E}, \mathcal{E})$, as follows from (2.4).

3. Hölder Continuous Cocycles

For this section we fix an operator space V *in* $B(\mathfrak{h};\mathfrak{h}')$. We first discuss the generation of QS cocycles on V by means of QS differential equations, and then we characterise various classes of cocycle so-generated. The main result (Theorem 3.1) is the converse to the existence theorem established in [35]. Further details and full proofs will apear in the forthcoming paper [28].

Let $\varphi \in F(\{1_{\mathbb{C}}\} \times \mathsf{T}; \widehat{\mathsf{k}}\text{-}B(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} |\widehat{\mathsf{k}}\rangle)$ for a total subset T of k containing 0. Then, for any total subset T' of k containing 0, the QS differential equation

$$dk_t = k_t \circ d\Lambda_\varphi(t) \quad k_0 = \iota_{\mathcal{F}}^{\mathsf{V}}, \tag{3.1}$$

has a unique $(\mathcal{E}_{\mathsf{T}'}, \mathcal{E}_{\mathsf{T}})$-weakly regular, $(\mathcal{E}_{\mathsf{T}'}, \mathcal{E}_{\mathsf{T}})$-weak solution, denoted k^φ.

Usually coefficients of QS differential equations are assumed to be *linear*: $\phi \in L(\widehat{\mathsf{D}}; \widehat{\mathsf{k}}\text{-}B(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} |\widehat{\mathsf{k}}\rangle)$ for a dense subspace D of k. However, bare QS differentiation of cocycles only yields coefficients φ which are functions defined on $\{1_{\mathbb{C}}\} \times \mathsf{T} := \{\widehat{c} : c \in \mathsf{T}\}$. As we shall see, linearity is recovered under the assumption that the cocycle also has a Hölder adjoint cocycle.

Here $(\mathcal{E}_{\mathsf{T}'}, \mathcal{E}_{\mathsf{T}})$-*weak solution* means

$$\begin{aligned}
&k_t \in L\big(\mathsf{V}; L(\mathfrak{h} \underline{\otimes} \mathcal{E}_{\mathsf{T}}; \mathfrak{h}' \otimes \mathcal{F})\big), E^{\varepsilon(f)} k_t(x) E_{\varepsilon(g)} \in \mathsf{V},\\
&s \mapsto \langle u'\varepsilon, k_s(x) u\varepsilon\rangle \text{ is continuous, and}\\
&\langle u'\varepsilon(f), k_t(x) u\varepsilon(g)\rangle = \langle u', xu\rangle \langle \varepsilon(f), \varepsilon(g)\rangle +\\
&\qquad\qquad \int_0^t ds \ \big\langle u'\varepsilon(f), k_s\big(E^{\widehat{f}(s)} \varphi_{\widehat{g}(s)}(x)\big) u\varepsilon(g)\big\rangle
\end{aligned}$$

$(u' \in \mathfrak{h}', f \in \mathbb{S}_{\mathsf{T}'}, x \in \mathsf{V}, u \in \mathfrak{h}, g \in \mathbb{S}_{\mathsf{T}})$; $(\mathcal{E}_{\mathsf{T}'}, \mathcal{E}_{\mathsf{T}})$-*weakly regular* means

$$s \mapsto E^{\varepsilon'} k_s(\cdot) E_\varepsilon \text{ is locally bounded } \mathbb{R}_+ \to B(\mathsf{V}), \quad \varepsilon \in \mathcal{E}_{\mathsf{T}}, \varepsilon' \in \mathcal{E}_{\mathsf{T}'}.$$

The unique solution enjoys the following further property:

$$k^\varphi \in \mathbb{QSC}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V} \colon \mathcal{E}_{\mathsf{T}}), \tag{3.2}$$

and if $\varphi(\{1_{\mathbb{C}}\} \times \mathsf{T}) \subset CB(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} |\widehat{\mathsf{k}}\rangle)$ then $k^\varphi \in \mathbb{QSC}_{\text{cbHc}}(\mathsf{V} \colon \mathcal{E}_{\mathsf{T}})$. Moreover, if φ has an adjoint map $\varphi^\dagger \in F\big(\{1_{\mathbb{C}}\} \times \mathsf{T}'; \widehat{\mathsf{k}}\text{-}B(\mathsf{V}^\dagger; \mathsf{V}^\dagger \otimes_{\mathrm{M}} |\widehat{\mathsf{k}}\rangle)\big)$ satisfying

$$E^{\widehat{c}} \varphi^\dagger_{\widehat{d}}(x^*) = \big(E^{\widehat{d}} \varphi_{\widehat{c}}(x)\big)^*, \quad x \in \mathsf{V}, c \in \mathsf{T}, d \in \mathsf{T}',$$

then $k^\varphi \in \mathbb{QSC}^{\ddagger}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V} \colon \mathcal{E}_{\mathsf{T}'}, \mathcal{E}_{\mathsf{T}})$ with adjoint cocycle $k^{\varphi^\dagger}$. In this case,

$$\varphi = \phi|_{\{1_{\mathbb{C}}\} \times \mathsf{T}}$$

for a unique map ϕ in $L^{\ddagger}_{\widehat{\mathsf{D}'}}\big(\widehat{\mathsf{D}};\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)$, the space of linear maps $\phi:\widehat{\mathsf{D}}\to\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)$ having an adjoint linear map $\phi^{\dagger}:\widehat{\mathsf{D}'}\to\widehat{\mathsf{k}}\text{-}B(\mathsf{V}^{\dagger};\mathsf{V}^{\dagger}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)$, where $\widehat{\mathsf{D}}=\mathbb{C}\oplus\mathsf{D}$, $\mathsf{D}=\operatorname{Lin}\mathsf{T}$ and similarly for D' and T', moreover

$$k^{\varphi}_t=k^{\phi}_t|_{\mathcal{E}_{\mathsf{T}}},\quad t\in\mathbb{R}_+.$$

The above facts are essentially contained in [35], supplemented by [32]; the minor modifications needed for the present generality are explained in [28].

The situation is summarised as follows. The map

$$F\big(\{1_{\mathbb{C}}\}\times\mathsf{T};\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)\to\mathbb{QSC}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}}),\ \varphi\mapsto k^{\varphi},$$

is injective and restricts to maps

$$\begin{aligned}
&F\big(\{1_{\mathbb{C}}\}\times\mathsf{T};CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)\to\mathbb{QSC}_{\mathrm{cbHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}})\\
&L^{\ddagger}_{\widehat{\mathsf{D}'}}\big(\{\widehat{\mathsf{D}};\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)\to\mathbb{QSC}^{\ddagger}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}'},\mathcal{E}_{\mathsf{T}})\\
&L^{\ddagger}_{\widehat{\mathsf{D}'}}\big(\{\widehat{\mathsf{D}};CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)\to\mathbb{QSC}^{\ddagger}_{\mathrm{cbHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}'},\mathcal{E}_{\mathsf{T}}),
\end{aligned}$$

where $\mathsf{D}=\operatorname{Lin}\mathsf{T}$ and $\mathsf{D}'=\operatorname{Lin}\mathsf{T}'$.

The following theorem extends Theorem 5.6 of [32], where V is assumed to be finite dimensional.

Theorem 3.1. *Each of the above four maps is bijective.*

The idea of the proof is to differentiate directly, in other words, to show that for $k\in\mathbb{QSC}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}})$ and $c\in\mathsf{T}$, the family

$$\varphi_{\widehat{c},t}:=t^{-1}\begin{bmatrix}E^{\varepsilon(0)}\\E^{1_{[0,t[}}\end{bmatrix}\big(k_{t,\varepsilon(c_{[0,t[})}-\iota^{\mathsf{V}}_{\varepsilon(c_{[0,t[})}\big)(\cdot)\quad(t>0)$$

converges in $\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)$ to a map $\varphi_{\widehat{c}}$, as $t\to0^+$, in the point-W.O. topology:

$$\langle\xi,\varphi_{\widehat{c},t}(x)\eta\rangle\to\langle\xi,\varphi_{\widehat{c}}(x)\eta\rangle,\quad x\in\mathsf{V},\xi\in\mathfrak{h}',\eta\in\mathfrak{h},$$

and then to show that the resulting map $\varphi\in F\big(\{1_{\mathbb{C}}\}\times\mathsf{T};\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big)$ satisfies $k=k^{\varphi}$ by verifying that the two (Markov-regular) cocycles k and k^{φ} have the same associated semigroup generators. The rest then follows easily. Here

$$\begin{aligned}E^{1_{[0,t[}}:=I_{\mathfrak{h}'\otimes\mathsf{k}}\otimes\langle1_{[0,t[}|&=I_{\mathfrak{h}'}\otimes\big(I_{\mathsf{k}}\otimes\langle1_{[0,t[}|\big)\\
&\in B(\mathfrak{h}'\otimes\mathcal{F};\mathfrak{h}'\otimes\mathsf{k})=B(\mathfrak{h}')\overline{\otimes}B(\mathcal{F};\mathsf{k}),\end{aligned}\quad(3.3)$$

by means of the inclusion $\mathsf{k}\otimes L^2(\mathbb{R}_+)=L^2(\mathbb{R}_+;\mathsf{k})\subset\mathcal{F}$, as one-particle subspace, and $\iota^{\mathsf{V}}_{\varepsilon}(x):=x\otimes|\varepsilon\rangle$ for $\varepsilon\in\mathcal{E}_{\mathsf{T}}$, $x\in\mathsf{V}$.

For bounded QS cocycles we can say more. Note first the natural inclusions

$$\mathsf{h}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}B(\widehat{\mathsf{k}}))\subset L\big(\widehat{\mathsf{k}};\mathsf{h}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}|\widehat{\mathsf{k}}\rangle)\big),$$

for any Hilbert space h, arising from the identification $\phi_{\widehat{c}}=\phi(\cdot)E_{\widehat{c}}$ $(c\in\mathsf{k})$.

Theorem 3.2. *Let* $k\in\mathbb{QSC}^{\ddagger}_{\widehat{\mathsf{k}}\text{-bHc}}(\mathsf{V}:\mathcal{E}_{\mathsf{T}'},\mathcal{E}_{\mathsf{T}})$ *be bounded, with locally uniform bounds. Then* $k=k^{\psi}|_{\mathcal{E}_{\mathsf{T}}}$ *for a unique map* $\psi\in\widehat{\mathsf{k}}\text{-}B(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}B(\widehat{\mathsf{k}}))$*. Moreover, if* $k\in\mathbb{QSC}^{\ddagger}_{\mathrm{cbHc}}(\mathsf{V}\colon\mathcal{E}_{\mathsf{T}'},\mathcal{E}_{\mathsf{T}})$ *then* $\psi\in CB(\mathsf{V};\mathsf{V}\otimes_{\mathrm{M}}B(\widehat{\mathsf{k}}))$.

The idea of the proof of this is to let $\phi \in L^{\ddagger}_{\widehat{\mathsf{D}'}}\big(\widehat{\mathsf{D}}; \widehat{\mathsf{k}}\text{-}B(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} |\widehat{\mathsf{k}}\rangle)\big)$ be the mapping arising from Theorem 3.1, to set

$$\tau = E^{\widehat{0}}\phi_{\widehat{0}}(\cdot),\ \chi = \Delta\phi_{\widehat{0}}(\cdot) \text{ and } \alpha = \big(\Delta\phi^{\dagger}_{\widehat{0}}(\cdot)\big)^{\dagger}$$

and to establish convergence, as $t \to 0^+$, of the (locally uniformly bounded) family

$$\nu_t := t^{-1}E^{1_{[0,t[}}k_t(\cdot)E_{1_{[0,t[}} \quad (t > 0)$$

to a map $\nu \in B(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\mathsf{k}))$, in the point-W.O. topology. The resulting map $\psi := \left[\begin{smallmatrix} \tau & \alpha \\ \chi & \nu - \iota \end{smallmatrix}\right]$, where $\iota = \iota^{\mathsf{V}}_{\mathsf{k}}$, lies in $B(\mathsf{V}; \mathsf{V} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ and satisfies the identity

$$E^{\widehat{c}}\psi(\cdot)E_{\widehat{d}} = E^{\widehat{c}}\phi_{\widehat{d}}(\cdot), \quad d \in \mathsf{D}, c \in \mathsf{D}';$$

the rest then follows easily.

The following two known results ([35, 34]) may be easily deduced from the above theorem.

Corollary 3.3. *Let $k \in \mathbb{QSC}_{\mathrm{Mreg}}(\mathsf{A} : \mathcal{E})$ be completely positive and contractive, on a C^*-algebra A. Then there is $\phi \in CB(\mathsf{A}; \mathsf{A} \otimes_{\mathrm{M}} B(\widehat{\mathsf{k}}))$ such that $k = k^{\phi}$.*

This is proved by judicious use of the operator Schwarz inequality which reveals that, under the given hypotheses, $k \in \mathbb{QSC}^{\ddagger}_{\mathrm{cbHc}}(\mathsf{A} : \mathcal{E}, \mathcal{E})$.

The precise form that ϕ must take for k^{ϕ} to be completely positive and contractive is given in [34]; this is a stochastic extension of the Christensen-Evans Theorem ([9]). A new proof of the form of ϕ, using global Schur-action semigroups on matrix spaces, is given in [39].

Corollary 3.4. *Let $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h}, \mathsf{k})$. Then there is a unique operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ such that $V = V^F$, that is V is the unique (weak, contractive) solution of the QS differential equation $dV_t = V_t d\Lambda_F(t)$, $V_0 = I_{\mathfrak{h}\otimes\mathcal{F}}$.*

This is deduced from Theorem 3.1 via the correspondences (2.5). The quantum Itô formula provides the following necessary and sufficient condition on an operator $F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}})$ for V^F to be contractive:

$$q(F) \leq 0 \text{ where } q(F) := F^* + F + F^*\Delta F, \tag{3.4}$$

equivalently, $r(F) \leq 0$ where $r(F) := F + F^* + F\Delta F^*$ ([33], Theorem 7.5). Thus $F \mapsto V^F$ defines a bijection

$$C_0(\mathfrak{h}, \mathsf{k}) := \{F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) : q(F) \leq 0\} \to \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h}, \mathsf{k}), \tag{3.5}$$

extending the bijection

$$\{F \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) : q(F) = 0 = r(F)\} \to \mathbb{QS}_{\mathrm{u}}\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h}, \mathsf{k})$$

given in Theorem 1.1, where $\mathbb{QS}_{\mathrm{u}}\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h}, \mathsf{k})$ denotes the collection of Markov-regular *unitary* cocycles.

In the following section we describe a significant extension of this result, to a class of *strongly continuous* operator cocycles, which necessarily goes beyond the realm of QS differential equations.

4. Holomorphic Cocycles

For this section we again fix Hilbert spaces $\mathfrak{h}$ *and* k. We introduce the class of holomorphic QS contraction cocycles on $\mathfrak{h}$ with noise dimension space k, and consider their infinitesimal characterisation. We then describe the connection to minimal quantum dynamical semigroups. The main results of this section are Theorems 4.5 and 4.11. Further details and full proofs will appear in the forthcoming paper [30].

Let $\mathfrak{X}(\mathfrak{h})$ denote the class of operators K on $\mathfrak{h}$ which are densely defined and dissipative: $\operatorname{Re}\langle u, Ku\rangle \leq 0$ ($u \in \operatorname{Dom} K$) with no dissipative extension, in other words *maximal dissipative*. It is well-known that such operators are precisely the generators of contractive C_0-semigroups on $\mathfrak{h}$ ([15], Theorem 6.4). We are interested in the collection $\mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$ of quadratic forms q on $\mathfrak{h}$ which are *accretive* and *semisectorial*:

$$\operatorname{Re} q[u] \geq 0 \text{ and } \big|\operatorname{Im} q[u]\big| \leq C\|u\|_+^2, \quad u \in \mathcal{Q},$$

for some $C \geq 0$, where

$$\mathcal{Q} := \operatorname{Dom} q \text{ and } \|u\|_+^2 := \big(\operatorname{Re} q[u] + \|u\|^2\big)^{1/2},$$

as well as being *densely defined* and *closed*:

$$\mathcal{Q} \text{ is dense in } \mathfrak{h} \text{ and } \mathcal{Q} \text{ is complete w.r.t. the norm } \|\cdot\|_+.$$

The term "semisectorial" is nonstandard, but avoids the potentially confusing term "continuous" favoured by some experts. "Sectorial", which is standard, is the strengthening $|\operatorname{Im} q[u]| \leq C \operatorname{Re} q[u]$ ($u \in \mathcal{Q}$) for some $C \geq 0$.

For $q \in \mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$ there is a unique operator K on $\mathfrak{h}$ satisfying

$$\operatorname{Dom} K = \Big\{u \in \mathcal{Q} : \text{ the (conjugate-linear) fnl. } v \in \mathcal{Q} \mapsto q(v,u) \text{ is bounded}\Big\},$$
$$\langle u, Ku\rangle = -q[u], \quad u \in \mathcal{Q},$$

where $\mathcal{Q} = \operatorname{Dom} q$ and $q(\ ,\ )$ is the sesquilinear form associated with $q[\cdot]$ via polarisation, moreover $K \in \mathfrak{X}(\mathfrak{h})$. The operators $K \in \mathfrak{X}(\mathfrak{h})$ that arise from forms $q \in \mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$ are precisely those generators of contractive C_0-semigroups which are semisectorial: $|\operatorname{Im}\langle u, Ku\rangle| \leq C\big(-\operatorname{Re}\langle u, Ku\rangle + \|u\|^2\big)^{1/2}$ ($u \in \operatorname{Dom} K$) for some $C \geq 0$. In this way we view $\mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$ as a subset of $\mathfrak{X}(\mathfrak{h})$.

By a *holomorphic contraction semigroup* on $\mathfrak{h}$ we mean a contractive C_0-semigroup $P = (P_t)_{t\geq 0}$ for which there is an angle $\theta \in]0, \pi/2]$ such that $(e^{-t}P_t)_{t\geq 0}$ extends to a contraction-valued holomorphic function $\Sigma_\theta \to B(\mathfrak{h})$, where Σ_θ denotes the open sector of the complex plane $\{z \in \mathbb{C} \setminus \{0\} : |\arg z| < \theta\}$.

Warning. Definitions in the literature vary. With our definition, the holomorphic contraction semigroups are precisely those whose generators are in $\mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$ (see e.g. [40]).

Definition 4.1. A cocycle $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}(\mathfrak{h}, \mathsf{k})$ is *holomorphic* if its expectation semigroup is holomorphic. Write $\mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ for the resulting class of cocycles.

For $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ we write γ^V for the form-generator of its expectation semigroup and $\mathcal{Q}^V$ for $\operatorname{Dom} \gamma^V$. The dual of a holomorphic QS contraction cocycle

V is holomorphic with

$$\gamma^{\widetilde{V}} = (\gamma^V)^* := u \mapsto \overline{\gamma^V[u]} \text{ and } \mathcal{Q}^{\widetilde{V}} = \mathcal{Q}^V.$$

Since norm-continuous semigroups are holomorphic, we have the inclusion

$$\mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k}) \supset \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{Mreg}}(\mathfrak{h}, \mathsf{k}). \tag{4.1}$$

Kato–type relative boundedness arguments for quadratic forms ([26]) yield our first two consequences of the holomorphic assumption.

Proposition 4.2. *Let $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$. Then the following hold:*

(a) *All of the associated semigroups of V are holomorphic and their form-generators have equal domain $\mathcal{Q}^V$.*

(b) *V is* nonsingular*, that is, with $E_{1_{[0,t[}} := \big(E^{1_{[0,t[}}\big)^* \in B(\mathfrak{h} \otimes \mathsf{k}; \mathfrak{h} \otimes \mathcal{F})$,*

$$t^{-1} E^{1_{[0,t[}} V_t E_{1_{[0,t[}} \to C^V \textit{ in the weak operator topology as } t \to 0^+,$$

for a contraction operator C^V on $\mathfrak{h} \otimes \mathsf{k}$.

Note that, in view of the identity $R_t E_{1_{[0,t[}} = E_{1_{[0,t[}}$, it follows that, for $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$,

$$C^{\widetilde{V}} = (C^V)^*.$$

We come now to the central definition of this section.

Notation. Let $\mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ denote the collection of quadruples $\mathbb{F} = (\gamma, L, \widetilde{L}, C - I)$, where $\gamma \in \mathfrak{X}_{\mathrm{hol}}(\mathfrak{h})$, L and $\widetilde{L}$ are operators from $\mathfrak{h}$ to $\mathfrak{h} \otimes \mathsf{k}$ with domain $\mathcal{Q} := \mathrm{Dom}\,\gamma$ and C is a contraction in $B(\mathfrak{h} \otimes \mathsf{k})$, satisfying

$$\|\Delta F \zeta\|^2 \leq 2 \,\mathrm{Re}\, \Gamma[\zeta] \tag{4.2}$$

for the operator ΔF and quadratic form Γ on $\mathfrak{h} \oplus (\mathfrak{h} \otimes \mathsf{k}) = \mathfrak{h} \otimes \widehat{\mathsf{k}}$ given by

$$\mathrm{Dom}\,\Gamma = \mathrm{Dom}\,\Delta F := \mathcal{Q} \oplus (\mathfrak{h} \otimes \mathsf{k}), \ \Delta F := \begin{bmatrix} 0 & 0 \\ L & C - I \end{bmatrix}, \text{ and}$$

$$\Gamma[\zeta] := \gamma[u] - \big(\langle \xi, Lu \rangle + \langle \widetilde{L} u, \xi \rangle + \langle \xi, (C - I)\xi \rangle\big) \ \text{ for } \zeta = \begin{pmatrix} u \\ \xi \end{pmatrix} \in \mathrm{Dom}\,\Gamma.$$

Remark 4.3. (i) The relation (4.2) contains the inequalities

$$\|Lu\|^2 \leq 2 \,\mathrm{Re}\, \gamma[u], \ \|\widetilde{L}u\|^2 \leq 2 \,\mathrm{Re}\, \gamma[u], \quad u \in \mathcal{Q}, \tag{4.3}$$

(ii) If γ is bounded, so that $\mathcal{Q} = \mathfrak{h}$, then L and $\widetilde{L}$ are bounded, $\Gamma[\zeta] = -\langle \zeta, F\zeta \rangle$ ($\zeta \in \mathfrak{h} \otimes \widehat{\mathsf{k}}$) where $F := \left[\begin{smallmatrix} K & M \\ L & C-I \end{smallmatrix}\right]$ in which $M = \widetilde{L}^*$ and K is the bounded operator associated to γ: $\langle u, Ku \rangle = -q[u]$ ($u \in \mathcal{Q}$), and the operator ΔF (derived from Γ) indeed equals the quantum Itô projection Δ composed with F; the constraint (4.2) on F is then equivalent to $F \in C_0(\mathfrak{h}, \mathsf{k})$, the class defined in (3.5). In this sense we have the inclusion

$$\mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k}) \supset C_0(\mathfrak{h}, \mathsf{k}).$$

We shall see that this matches up with the inclusion (4.1).

(iii) Clearly the form γ may be recovered from Γ, but so may each of the operators L, $\widetilde{L}$ and C — by polarisation. Thus Γ determines the quadruple $\mathbb{F}$, in particular it determines the operator ΔF.

(iv) In general, we are *not* here viewing the operator ΔF as a composition of Δ with some operator F.

Proposition 4.4. *Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$. Then the following hold.*

(a) *There are operators L^V and $\widetilde{L}^V$ from $\mathfrak{h} \to \mathfrak{h} \otimes \mathsf{k}$ with domain $\mathcal{Q}^V$ such that*

$$t^{-1} E^{1_{[0,t[}} V_t E_{\varepsilon(0)} u \to L^V u \text{ (weakly), and}$$
$$t^{-1} E^{1_{[0,t[}} V_t^* E_{\varepsilon(0)} u \to \widetilde{L}^V u \text{ (weakly), as } t \to 0^+, \quad u \in \mathcal{Q}^V.$$

(b) *The quadruple $\mathbb{F}^V := (\gamma^V, L^V, \widetilde{L}^V, C^V - I)$ belongs to $\mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$.*

(c) *$L^{\widetilde{V}} = \widetilde{L}^V$ so $\widetilde{L}^{\widetilde{V}} = L^V$ and thus*

$$\mathbb{F}^{\widetilde{V}} = \big((\gamma^V)^*, \widetilde{L}^V, L^V, (C^V)^* - I\big).$$

This is proved using abstract Itô integration in Fock space. Relative boundedness arguments with Yosida-type approximation, at the quadratic form level, now combine with the Markov-regular theory and the semigroup characterisation of QS cocycles to yield the central result.

Theorem 4.5. *The map $V \mapsto \mathbb{F}^V$ is a bijection from $\mathbb{QS}_c\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ to $\mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$.*

The inverse of this map is naturally denoted $\mathbb{F} \mapsto V^{\mathbb{F}}$ and referred to as the *quantum stochastic generation map* for holomorphic QS cocycles. It extends the bijection $F \mapsto V^F$ given in (3.5), by Remark 4.3 (ii).

What is achieved in the above result is the infinitesimal characterisation of a large class of QS contraction cocycles which includes all those whose expectation semigroup is symmetric, as well as the Markov-regular cocycles previously characterised. Whereas cocycles in the latter class are all governed by QS differential equations, it is not hard to construct examples in the former class which are not so-governed. Here is the simplest example.

Example 4.6. Let A be a nonnegative, selfadjoint, unbounded operator on $\mathfrak{h}$ with dense range, let $\mathsf{k} = \mathbb{C}$, let $P \in B(\mathfrak{h})$ be the orthogonal projection with range $\mathbb{C}v$ for a vector v in $\mathfrak{h} \setminus \mathrm{Dom}\, A$, and set $\mathbb{F} = \big(\|A \cdot\|^2/2, PA, -PA, 0\big)$. Then $\mathbb{F} \in \mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$, however the holomorphic QS contraction cocycle $V^{\mathbb{F}}$ is not governed by a QS differential equation on any exponential domain. The reason for this is that $(PA)^* = AP = 0|_{\{v\}^\perp}$, so that the domain on which V satisfies a QS differential equation ([37], Theorem 4.2) fails to be dense in $\mathfrak{h}$.

We may now put some flesh on Part (a) of Proposition 4.2.

Theorem 4.7. *Let $V \in \mathbb{QS}_c\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$, and let $\Gamma = \Gamma^V$ be the quadratic form on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$ associated with the QS generator $\mathbb{F}^V$ of V. Then, for each $c, d \in \mathsf{k}$, the quadratic form generator of the (c,d)-associated semigroup of V is given by*

$$\gamma_{c,d}[u] = \Gamma(u\widehat{c}, u\widehat{d}) + \chi(c,d)\|u\|^2, \quad u \in \mathcal{Q}^V,$$

where $\chi(c,d) := (\|c\|^2 + \|d\|^2)/2 - \langle c, d\rangle$.

This chimes with Remark 4.3 (ii) and also with (4.1) in [37], the corresponding identity for the case where V is governed by a QS diffierential equation with reasonable block matrix operator as coefficient.

The next result shows that to each holomorphic QS contraction cocycle on $\mathfrak{h}$ with noise dimension space k corresponds a holomorphic contraction semigroup on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$.

Theorem 4.8. *Let* $\mathbb{F} = (\gamma, L, \widetilde{L}, C - I) \in \mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ *and let* Γ *be the associated quadratic form on* $\mathfrak{h} \otimes \widehat{\mathsf{k}}$*, namely* $\operatorname{Dom}\Gamma = \mathcal{Q} \oplus (\mathfrak{h} \otimes \mathsf{k})$*, where* $\mathcal{Q} := \operatorname{Dom}\gamma$*, and*

$$\Gamma[\zeta] := \gamma[u] - \big(\langle \xi, Lu\rangle + \langle \widetilde{L}u, \xi\rangle + \langle \xi, (C - I)\xi\rangle\big), \quad \text{for } \zeta = \binom{u}{\xi} \in \operatorname{Dom}\Gamma.$$

Then $\Gamma \in \mathfrak{X}_{\mathrm{hol}}(\mathfrak{h} \otimes \widehat{\mathsf{k}})$.

Proof. Since $\mathcal{Q}$ is dense in $\mathfrak{h}$ and

$$\operatorname{Re}\Gamma[\zeta] \geq \frac{1}{2}\|\Delta F \zeta\|^2, \quad \zeta \in \operatorname{Dom}\Gamma = \mathcal{Q} \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

where $\Delta F := \left[\begin{smallmatrix} 0 & 0 \\ L & C-I \end{smallmatrix}\right]$, the quadratic form Γ is densely defined and accretive. The fact that Γ is also closed and semisectorial follows from the easily verified equivalence of the norms

$$\zeta \mapsto \big(\operatorname{Re}\Gamma[\zeta] + \|\zeta\|^2\big)^{1/2}, \text{ and}$$
$$\binom{u}{\xi} \mapsto \big(\operatorname{Re}\gamma[u] + \|u\|^2 + \|\xi\|^2\big)^{1/2}$$

on $\operatorname{Dom}\Gamma = \mathcal{Q} \oplus (\mathfrak{h} \oplus \mathsf{k})$, and the fact that the accretive form γ is closed and semisectorial itself. □

In view of Remark 4.3 (iii), it follows that there is a bijective correspondence between the collection $\mathbb{QS}_{\mathrm{c}}\mathbb{C}_{\mathrm{hol}}(\mathfrak{h}, \mathsf{k})$ and a class of holomorphic contraction semigroups on $\mathfrak{h} \otimes \widehat{\mathsf{k}}$. The latter semigroups are quite different from the global semigroups associated with cocycles considered in [38], all of which are specifically coordinate-dependent and enjoy a Schur-action.

We have not so far discussed the question of coisometry and unitarity of holomorphic QS contraction cocycles V. These questions are of interest for their (unitality, i.e. identity-preserving, and multiplicativity) implications for induced cocycles on $B(\mathfrak{h})$, and von Neumann subalgebras thereof. It is not hard to verify that a necessary condition for isometry is that equality holds in the inequality governing $\mathbb{F}^V$:

$$2\operatorname{Re}\Gamma^V[\zeta] = \|\Delta F^V \zeta\|^2, \quad \zeta \in \mathcal{Q} \oplus (\mathfrak{h} \otimes \mathsf{k}),$$

the holomorphic counterpart to the condition $q(F) = 0$ for Markov-regular cocycles. Whilst this condition is also sufficient in the case of Markov-regular cocycles, sufficient conditions are trickier in the holomorphic case. We quote a general result from [38] (see also [30]). Recall the definition (2.1).

Theorem 4.9. *Let* $V \in \mathbb{QS}_{\mathrm{c}}\mathbb{C}(\mathfrak{h}, \mathsf{k})$ *and let* $\{\mathcal{P}^{c,d} : c, d \in \mathsf{k}\}$ *be the associated semigroups of the induced cocycle* k^V *on* $B(\mathfrak{h})$*. Then, for any total subset* T *of* k *containing* 0*, the following are equivalent*:

(i) V *is coisometric* (*equivalently,* k^V *is unital*)*.*
(ii) $\mathcal{P}^{c,c}$ *is unital for all* $c \in \mathsf{T}$*.*

Thus, in case k is finite dimensional, (ii) need only involve the verification of unitality of $(1 + \dim \mathsf{k})$ semigroups.

The above result frames a theorem of Fagnola to the effect that, in case V satisfies a QS differential equation on a core for the generator of its expectation semigroup, unitality of just the expectation semigroup $\mathcal{P}^{0,0}$ of k^V suffices ([18], Theorem 5.23).

We next explain the connection to minimal quantum dynamical semigroups. By a *quantum dynamical semigroup* on $B(\mathfrak{h})$ is meant a semigroup $\mathcal{T} = (\mathcal{T}_t)_{t\geq 0}$ of normal, completely positive contractions on $B(\mathfrak{h})$ which is continuous in the point-ultraweak topology. Let $\mathfrak{X}^{(2)}(\mathfrak{h}, \mathsf{k})$ denote the collection of pairs (K, L) consisting of a C_0-semigroup generator $K \in \mathfrak{X}(\mathfrak{h})$ and operator L from $\mathfrak{h}$ to $\mathfrak{h} \otimes \mathsf{k}$ satisfying

$$\operatorname{Dom} L \supset \operatorname{Dom} K \text{ and } \|Lu\|^2 \leq -2 \operatorname{Re}\langle u, Ku\rangle, \quad u \in \operatorname{Dom} K.$$

For $x \in B(\mathfrak{h})$ define the quadratic form $\mathcal{L}(x) = \mathcal{L}_{(K,L)}(x)$ on $\mathfrak{h}$ by

$$\begin{aligned} &\operatorname{Dom} \mathcal{L}(x) := \operatorname{Dom} K, \\ &\mathcal{L}(x)[u] := \langle Ku, xu\rangle + \langle u, xKu\rangle + \langle Lu, (x \otimes I_{\mathsf{k}})Lu\rangle. \end{aligned}$$

A quantum dynamical semigroup $\mathcal{T}$ on $B(\mathfrak{h})$ is *minimal* for the pair $(K, L) \in \mathfrak{X}^{(2)}(\mathfrak{h}, \mathsf{k})$ if it satisfies:

(i) for all $x \in B(\mathfrak{h})$, $u \in \operatorname{Dom} K$ and $t \in \mathbb{R}_+$,

$$\langle u, \mathcal{T}_t(x)u\rangle = \langle u, xu\rangle + \int_0^t ds\ \mathcal{L}_{(K,L)}(\mathcal{T}_s(x))[u];$$

(ii) if $\mathcal{T}'$ is another quantum dynamical semigroup satisfying (i) then

$$\mathcal{T}'_t(x)) \geq \mathcal{T}_t(x) \text{ for all } x \in B(\mathfrak{h})_+, t \in \mathbb{R}_+.$$

Minimal quantum dynamical semigroups are the noncommutative counterparts of Feller's minimal solutions of the Fokker-Planck equation.

Theorem 4.10 ([14])**.** *Let $(K, L) \in \mathfrak{X}^{(2)}(\mathfrak{h}, \mathsf{k})$. Then there is a unique minimal quantum dynamical semigroup $\mathcal{T}^{(K,L)}$ associated with (K, L).*

Note that if $\mathcal{T}^{(K,L)}$ is *conservative*, that is unital: $\mathcal{T}_t^{(K,L)}(I) = I$ $(t \in \mathbb{R}_+)$, then $\mathcal{L}_{(K,L)}(I) = 0$:

$$\|Lu\|^2 + 2 \operatorname{Re}\langle u, Ku\rangle = 0, \quad u \in \operatorname{Dom} K.$$

In the spirit of the inclusion $\mathfrak{X}_{\text{hol}}(\mathfrak{h}) \subset \mathfrak{X}(\mathfrak{h})$ we may consider the subclass $\mathfrak{X}^{(2)}_{\text{hol}}(\mathfrak{h}, \mathsf{k})$ of $\mathfrak{X}^{(2)}(\mathfrak{h}, \mathsf{k})$ consisting of pairs (γ, L) where $\gamma \in \mathfrak{X}_{\text{hol}}(\mathfrak{h})$ and $(K, L) \in \mathfrak{X}^{(2)}(\mathfrak{h}, \mathsf{k})$, $-K$ being the operator corresponding to γ. In this case we write $\mathcal{T}^{(\gamma,L)}$ for the associated minimal semigroup on $B(\mathfrak{h})$. In view of the relation

$$\|Lu\|^2 \leq 2 \operatorname{Re} \gamma[u], \quad u \in \operatorname{Dom} \gamma,$$

for $\mathbb{F} = (\gamma, L, \widetilde{L}, C - I) \in \mathfrak{X}^{(4)}_{\text{hol}}(\mathfrak{h}, \mathsf{k})$, any such $\mathbb{F}$ truncates to a pair $(\gamma, L) \in \mathfrak{X}^{(2)}_{\text{hol}}(\mathfrak{h}, \mathsf{k})$. Conversely, any pair $(\gamma, L) \in \mathfrak{X}^{(2)}_{\text{hol}}(\mathfrak{h}, \mathsf{k})$ 'dilates' to a quadruple

$$\mathbb{F} = (\gamma, L, -C^* L, C - I) \in \mathfrak{X}^{(4)}_{\text{hol}}(\mathfrak{h}, \mathsf{k})$$

by choosing a unitary, or contraction, C in $B(\mathfrak{h};\mathsf{k})$; the choices $C = I$, respectively $C = 0$, or $C = -I$, being notable ones. The next result underpins these considerations.

Theorem 4.11. *Let* $V = V^{\mathbb{F}}$ *for* $\mathbb{F} = (\gamma, L, \widetilde{L}, C - I) \in \mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h},\mathsf{k})$. *Then the expectation semigroup of the induced cocycle on* $B(\mathfrak{h})$ *of the dual semigroup* $\widetilde{V} = V^{\widetilde{\mathbb{F}}}$ *coincides with the minimal semigroup associated with the truncation* $(\gamma, L) \in \mathfrak{X}^{(2)}_{\mathrm{hol}}(\mathfrak{h},\mathsf{k})$ *of* $\mathbb{F}$*, equivalently,*

$$\mathbb{E}\big[V_t^*(x \otimes I_{\mathcal{F}})V_t\big] = \mathcal{T}^{(\gamma,L)}(x), \quad x \in B(\mathfrak{h}), t \in \mathbb{R}_+.$$

This is the holomorphic counterpart to a corresponding result for QS contraction cocycles governed by a QS differential equation ([18], Theorem 5.22). The relationship between Fagnola's analysis — which uses finite particle vectors (rather than exponential vectors) in a crucial way, and builds on earlier work of Mohari, Sinha and others — and our analysis, remains an intriguing one.

To end I shall briefly describe a new tool for the construction and analysis of QS cocycles — the *QS Trotter product formula* ([29]). In brief what this achieves is the representation of the QS cocycle V^F, where F has the form

$$F = F_1 \boxplus F_2 := \begin{bmatrix} K_1 + K_2 & M_1 & M_2 \\ L_1 & C_1 - I_1 & \\ L_2 & & C_2 - I_2 \end{bmatrix} \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}) = B(\mathfrak{h} \oplus \mathfrak{h} \otimes \mathsf{k}_1 \oplus \mathfrak{h} \otimes \mathsf{k}_2)$$

with $\mathsf{k} = \mathsf{k}_1 \oplus \mathsf{k}_2$ and I_i denoting the identity operator on $\mathfrak{h} \otimes \mathsf{k}_i$ $(i = 1, 2)$, in terms of the cocycles V^{F_1} and V^{F_2}, where

$$F_i = \begin{bmatrix} K_i & M_i \\ L_i & C_i - I_i \end{bmatrix} \in B(\mathfrak{h} \otimes \widehat{\mathsf{k}}_i) = B(\mathfrak{h} \oplus \mathfrak{h} \otimes \mathsf{k}_i) \text{ for } i = 1, 2.$$

It is easily verified that, if $F_1 \in C_0(\mathfrak{h}, \mathsf{k}_1)$ and $F_2 \in C_0(\mathfrak{h}, \mathsf{k}_2)$, then $F \in C_0(\mathfrak{h}, \mathsf{k})$.

In the quantum control literature the composition $\boxplus$, known as the *concatenation product*, has application to quantum networks, particularly in combination with the *series product* ([20]). The latter corresponds to the Evans-Hudson perturbation formula ([17, 13, 19, 4]) specialised to the case where the 'free' QS flow is implemented by a (Markov-regular) QS unitary cocycle V; the perturbed QS flow is implemented by the unitary cocycle whose generator is (up to the choice of parameterisation) the series product of the stochastic generator of V and the perturbation coefficient.

The QS Trotter product formula extends to both mapping cocycles and to holomorphic contraction operator cocycles ([31]). In the latter case this is given by

$$\mathbb{F} = \mathbb{F}_1 \boxplus \mathbb{F}_2 = (\gamma, L, \widetilde{L}, C - I) \in \mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h},\mathsf{k})$$

where, for $(\gamma_i, L_i, \widetilde{L}_i, C_i - I_i) \in \mathfrak{X}^{(4)}_{\mathrm{hol}}(\mathfrak{h},\mathsf{k}_i)$ $(i = 1, 2)$,

$$\gamma = \gamma_1 + \gamma_2,\ L = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix},\ \widetilde{L} = \begin{bmatrix} \widetilde{L}_1 \\ \widetilde{L}_2 \end{bmatrix},\ C - I = \begin{bmatrix} C_1 - I_1 & \\ & C_2 - I_2 \end{bmatrix},$$

yielding $V^{\mathbb{F}}$ in terms of $V^{\mathbb{F}_1}$ and $V^{\mathbb{F}_2}$. The only constraint for forming this concatenation product is that the intersection $\mathrm{Dom}\,\gamma_1 \cap \mathrm{Dom}\,\gamma_2$ is dense in $\mathfrak{h}$ since the sum of two closed, accretive, semisectorial forms is closed, accretive and semisectorial, and we have the following identities, by Pythagoras:

$$\Gamma[\zeta] = \Gamma_1[\eta_1] + \Gamma_2[\eta_2] \quad \text{and} \quad \|\Delta F\zeta\| = \|\Delta F_1\eta_1\| + \|\Delta F_2\eta_2\|$$

where, for $\zeta = \binom{u}{\xi}$ and $\xi = \binom{\xi_1}{\xi_2}$, $\eta_1 := \binom{u}{\xi_1}$ and $\eta_2 := \binom{u}{\xi_2}$. In the case of mapping cocycles, the homomorphic property of cocycles so-constructed on an operator algebra is investigated in [11].

In all of this work, the key ingredients are Trotter products of the associated semigroups of the constituent cocycles.

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References

1. Accardi, L. and Kozyrev, S. V.: On the structure of Markovian flows, *Chaos Solitons and Fractals* **12** (2001) no. 14-15, 2639–2655.
2. Applebaum, D., Bhat, B. V. R., Kustermans, J., and Lindsay J. M.: *Quantum Independent Increment Processes, Vol. I: From Classical Probability to Quantum Stochastics,* (eds. U. Franz & M. Schürmann), Lecture Notes in Mathematics **1865**, Springer, Heidelberg, 2005.
3. Arveson, W.: Continuous analogues of Fock space, *Mem. Amer. Math. Soc.* **80** (1989), no. 409.
4. Belton, A. C. R., Lindsay, J. M., and Skalski, A. G.: Quantum Feynman-Kac perturbations, *in preparation*.
5. Blecher, D. P. and Le Merdy, C.: *Operator Algebras and their Modules—an Operator Space Approach,* Oxford University Press, Oxford, 2004.
6. Bouten, L., van Handel, R., and Silberfarb, L.: Approximation and limit theorems for quantum stochastic models with unbounded coefficients, *J. Funct. Anal.* **254** (2008) no. 12, 3123–3147.
7. Bradshaw, W. S.: Stochastic cocycles as a characterisation of quantum flows, *Bull. Sci. Math.* (2) **116** (1992), 1–34.
8. Chebotarev, A. M. and Fagnola, F.: Sufficient conditions for conservativity of minimal quantum dynamical semigroups, *J. Funct. Anal.* **153** (1998), 382–404.
9. Christensen, E. and Evans, D. E.: Cohomology of operator algebras and quantum dynamical semigroups, *J. London Math. Soc. (2)* **20** (1979) no. 2, 358–368.
10. Cockfoft, A. M. and Hudson, R. L.: Quantum mechanical Wiener processes, *J. Multivariate Analysis* **7** (1977) no. 1, 107–124.
11. Das, B., Goswami, D., and Sinha, K. B.: Trotter-Kato product formula for quantum stochastic flows, *Preprint*, 2010.
12. Das, B. K. and Lindsay, J. M.: Quantum stochastic analysis in Banach space, *in preparation.*
13. Das, P. K. and Sinha, K. B.: Quantum flows with infinite degrees of freedom and their perturbations, in: *Quantum Probability and Related Topics VI,"* (eds. L. Accardi & W. von Waldenfels), 109–123, World Scientific, Singapore, 1992.
14. Davies, E. B.: Quantum dynamical semigroups and the neutron diffusion equation, *Rep. Math. Phys.* **11** (1977) no. 2, 169–188.
15. Davies, E. B.: *One-Parameter Semigroups,* London Mathematical Society Monographs **15**, Academic Press, London, 1980.
16. Effros, E. G. and Ruan, Z-J.: *Operator Spaces,* Oxford University Press, 2000.
17. Evans, M. P. and Hudson, R. L.: Perturbations of quantum diffusions, *J. London Math. Soc. (2)* **41** (1990) no. 2, 373–384.

18. Fagnola, F.: Quantum Markov semigroups and quantum flows, *Proyecciones* **18** (1999) no. 3, 1–144.
19. Goswami, D., Lindsay, J. M., and Wills, S. J.: A stochastic Stinespring theorem, *Math. Ann.* **319** (2001) no. 4, 647–673.
20. Gough, J. and James, M. R.: The series product and its application to quantum feedforward and feedback networks, *IEEE Trans. Automat. Control* **54** (2009) no. 11, 2530–2544.
21. Hudson, R. L. and Lindsay, J. M.: A noncommutative martingale representation theorem for non-Fock quantum Brownian motion, *J. Funct. Anal.* **61** (1985) no. 2, 202–221.
22. Hudson, R. L. and Lindsay, J. M.: On characterizing quantum stochastic evolutions, *Math. Proc. Camb. Phil. Soc.* **102** (1987) no. 2, 363–369
23. Hudson, R. L., Linday, J. M., and Parthasarathy, K. R.: Stochastic integral representation of some quantum martingales in Fock space, in: *From Local Times to Global Geometry, Control and Physics,*' (ed. K.D. Elworthy), Pitman Research Notes in Mathematics **150**, 121–131, Longman Sci. Tech., Harlow, 1986.
24. Hudson, R. L. and Parthasarathy, K. R.: Quantum Itô's formula and stochastic evolutions, *Commun. Math. Phys.* **93** (1984) no. 3, 301–323.
25. Journé, J.-L.: Structure des cocycles markoviens sur l'espace de Fock, *Probab. Theory Related Fields* **75** (1987) no. 2, 291–316.
26. Kato, T.: *Perturbation Theory for Linear Operators,* Springer, Berlin, 1966.
27. Lindsay, J. M.: Quantum stochastic analysis — an introduction, in: [2].
28. Lindsay, J. M.: Hölder-continuous quantum stochastic cocycles on operator spaces, *in preparation.*
29. Lindsay, J. M. and Sinha, K.B.: A quantum stochastic Lie-Trotter product formula, *Indian J. Pure Appl. Math.* **41** (2010) no. 1, 313–325.
30. Lindsay, J. M. and Sinha, K. B.: Holomorphic quantum stochastic contraction cocycles, *in preparation.*
31. Lindsay, J. M. and Sinha, K. B.: Trotter product formulae for quantum stochastic cocycles, *in preparation.*
32. Lindsay, J. M. and Skalski, A. G.: Quantum stochastic differential equations, *J. Math. Ann. Appl.* **330** (2007) no. 2, 1093–1114.
33. Lindsay, J. M. and Wills, S. J.: Existence, positivity and contractivity for quantum stochastic flows with infinite dimensional noise, *Probab. Theory Related Fields* **116** (2000) no. 4, 505–543.
34. Lindsay, J. M. and Wills, S. J.: Markovian cocycles on operator algebras, adapted to a Fock filtration, *J. Funct. Anal.* **178** (2000) no. 2, 269–305.
35. Lindsay, J. M. and Wills, S. J.: Existence of Feller cocycles on a C^*-algebra, *Bull. London Math. Soc.* **33** (2001) no. 5, 613–621.
36. Lindsay, J. M. and Wills, S. J.: Construction of some quantum stochastic operator cocycles by the semigroup method, *Proc. Indian Acad. Sci.* (*Math. Sci.*) **116** (2006) no. 4, 519–529.
37. Lindsay, J. M. and Wills, S. J.: Quantum stochastic operator cocycles via associated semigroups, *Math. Proc. Camb. Phil. Soc.* **142** (2007) no. 3, 535–556.
38. Lindsay, J. M. and Wills, S. J.: Quantum stochastic cocycles and completely bounded semigroups on operator spaces, *Preprint*, 2010.
39. Lindsay, J. M. and Wills, S. J.: Quantum stochastic cocycles and completely bounded semigroups on operator spaces II, *in preparation.*
40. Ouhabaz, E. M.: *Analysis of Heat Equations on Domains,* London Mathematical Society Monographs **31**, Princeton University Press, Princeton, 2005.
41. Parthasarathy, K. R. and Sinha, K. B.: Stochastic integral representation of bounded quantum martingales in Fock space, *J. Funct. Anal.* **67** (1986) no. 1, 126–151.
42. Paulsen, V. I.: *Completely Bounded Maps and Operator Algebras,* Cambridge Studies in Advanced Mathematics **78**, CUP, Cambridge, 2002.
43. Pisier, G.: *Introduction to Operator Space Theory,* London Mathematical Society Lecture Note Series **294**, CUP, Cambridge, 2003.
44. Sinha, K. B. and Goswami, D.: *Quantum Stochastic Processes and Noncommutative Geometry,* Cambridge Tracts in Mathematics **169**, CUP, Cambridge, 2007.

45. Wills, S. J.: On the generators of quantum stochastic cocycles, *Markov Proc. Related Fields* **13** (2007) no. 1, 191—211.
46. Wills, S. J.: E-semigroups subordinate to CCR flows, *Commun. Stoch. Anal.* (this volume).

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