

PERMANENTAL PROCESSES

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ABSTRACT. This is a survey of results about permanental processes, real valued positive processes which are a generalization of squares of Gaussian processes. In a certain sense the symmetric positive definite function that determines a Gaussian process is replaced by a function that is not necessarily symmetric nor positive definite, but that nevertheless determines a stochastic process. This is a new avenue of research with very many open problems.

1. Introduction

Let T be an index set and $G = \{G(x); x \in T\}$ be a mean zero Gaussian process with covariance $U = \{U(x, y); x, y \in T\}$. The covariance U completely determines G . If instead of G we decided to consider its square, i.e., $G^2 := \{G^2(x); x \in T\}$, we would not find this difficult, since we could simply obtain information about G^2 from our extensive understanding of the properties of G . However, let us suppose that we were completely unaware of the theory of Gaussian processes. We then might begin our study of G^2 by considering its finite joint distributions, as given by their Laplace transforms, i.e.,

$$E\left(\exp\left(-\frac{1}{2}\sum_{i=1}^n\alpha_i G^2(x_i)\right)\right) = \frac{1}{|I + \alpha\Gamma|^{1/2}} \quad (1.1)$$

for all x_1, \dots, x_n in T , where I is the $n \times n$ identity matrix, α is the diagonal matrix with $(\alpha_{i,i} = \alpha_i)$, $\alpha_i \in R_+$ and $\Gamma = \{\Gamma(x_i, x_j)\}$ is an $n \times n$ matrix, that is symmetric and positive definite. (Of course, in truth, we do know the theory of Gaussian processes so we know that $\{\Gamma(x_i, x_j)\}$ is actually the covariance of the mean zero Gaussian vector, $(G(x_1), \dots, G(x_n))$. That is, $\Gamma = \{U(x_i, x_j)\}_{i,j=1}^n$.) Nevertheless let us continue to pretend that we do not know the theory of Gaussian processes. A natural problem for us to consider is to find necessary and sufficient conditions for G^2 to have continuous paths. Considering that the only unknown on the right-hand side of (1.1) is the matrix Γ , the answer would have to be given in terms of Γ .

Lets now suppose that we have obtained necessary and sufficient conditions for G^2 to have continuous paths. We might then ask whether we could relax the conditions that Γ is symmetric and positive definite in (1.1) and that the power

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of the determinant is $1/2$. For example we might ask when we can define a real valued positive stochastic process $\theta = \{\theta_x, x \in T\}$ with finite joint distributions that satisfy

$$E\left(\exp\left(-\frac{1}{2}\sum_{i=1}^n\alpha_i\theta_{x_i}\right)\right) = \frac{1}{|I + \alpha\Gamma|^\beta}, \quad (1.2)$$

where $\Gamma = \{\Gamma(x_i, x_j)\}_{i,j=1}^n$ is an $n \times n$ matrix and $\beta > 0$. Of course, aside from dimension 1 where θ_{x_1} has a gamma distribution, for $\beta > 0$, it is not at all clear whether (1.2) makes sense, i.e., the right hand side may not be a Laplace transform.

In 1997, D. Vere-Jones, [13], initiated the study of random vectors with Laplace transform given by (1.2). He also briefly considered stochastic processes for which (1.2) gives finite joint distributions. We call these random vectors and stochastic processes β -permanental processes.

In [13] Vere-Jones gives necessary and sufficient conditions in terms of Γ and α for (1.2) to exist, i.e., that the right-hand side is a Laplace transform. Unfortunately, at least to us, it seems impossible to verify them, except in a very important case that was observed by N. Eisenbaum and H. Kaspi [5]. This is the case when Γ is the potential density of a transient Markov process on T , or, equivalently, when all the finite dimensional matrices Γ are diagonally equivalent to M -matrices, or, equivalently, when all finite dimensional vectors $(\theta_{x_1}, \dots, \theta_{x_n})$ are infinitely divisible. None of these three mathematical concepts is generally considered by experts in Gaussian processes. One purpose of this review is to convince the reader that they should be.

Several researchers have been looking for something like permanental processes for a long time. The Dynkin Isomorphism Theorem, (DIT) relates the squares of Gaussian processes to the local times of Markov processes, which necessarily have symmetric 0-potential densities. An outstanding problem was to find a version of Dynkin's theorem when the 0-potential densities of the Markov processes are not symmetric. This is no doubt what really inspired Eisenbaum and Kaspi. They give a generalization of the DIT, or to be more precise, of a version of the DIT given in [6], in [5, Corollary 3.5]. Of course, the generalization of the process that is Gaussian squares, is the permanental process. The DIT enables one to use results about Gaussian processes to obtain results about the local times of associated Markov processes. This is what [11] is devoted to. In [12, Theorem 1.3] Theorem 3.2 in this paper is used to obtain a sufficient condition for the continuity of local times of Markov processes without requiring that they have symmetric 0-potential densities. This result with a slightly more restrictive hypothesis was obtained in [4] by a different method. We do not discuss Dynkin type isomorphism theorems further in this paper; our focus is on permanental processes. However, we advise any researcher who decides to work on permanental processes to study them.

Another fascinating property of some permanental processes is that they can be represented as the local times of loop soups associated with Markov processes. This was shown for permanental processes associated with symmetric Markov processes in [8], (see also [9]), and extended to permanental processes associated with more general Markov processes in [12].

In Section 2 we give Verre-Jones's definition of β permanents and his necessary and sufficient condition for the existence of β -permanental vectors. In Section 3, under the assumption that (1.2) defines a permanental process, we give sufficient conditions for them to be bounded or continuous. In Section 4 we summarize some of the results in [5] that show that β -permanental processes exist if their kernels are the zero potentials of transient Markov processes. (In which case the permanental processes are infinitely divisible.) Lévy processes killed the first time they hit zero, or at the end of and independent exponential time, have zero potentials that are kernels of permanental processes. In Section 5 we obtain these potentials of Lévy processes and thus get an idea of what the kernels of permanental processes, that are not the squares of Gaussian processes, look like. To be specific, in Section 5.1, we give the kernels of permanental processes that are generalizations of the squares of fractional Brownian motion. The study of permanental processes is just beginning. In Section 6 we list a few problems that interest us.

This paper is based on Eisenbaum and Kaspi's seminal paper [5] and on [12]. It is primarily a survey except that Remark 3.3, Lemmas 4.4–4.6 and Theorem 4.8 are new results. Also the process $\mathbf{FBMQ}^{\alpha,\beta}$ appears for the first time in this paper.

2. Existence of Permanental Processes

The square of a Gaussian process is a permanental process so there are plenty of examples. But we are interested here primarily in those that are not squares of Gaussian processes. To begin it is important to note that the right-hand side of (1.2) is not unique with respect to Γ . If D is any diagonal matrix with non-zero entries we have

$$|I + \alpha\Gamma| = |I + \alpha D^{-1}\Gamma D| = |I + \alpha D^{-1}\Gamma^T D|. \quad (2.1)$$

Γ is said to be diagonally equivalent to Γ' if $\Gamma' = D^{-1}\Gamma D$ for some diagonal matrix D with non-zero entries.

For a very large class of irreducible matrices Γ , it is known that these are the only sources of non-uniqueness; see [10]. On the other hand, in certain extreme cases, for example, if Γ_1 and Γ_2 are $n \times n$ matrices with the same diagonal elements and all zeros below the diagonal, then $|I + \alpha\Gamma_1| = |I + \alpha\Gamma_2|$. Therefore, a process that is the square of a Gaussian process can have a non-symmetric kernel. Nevertheless, as we show, there are many permanental processes that are not squares of Gaussian processes.

The first step in the definition of permanental processes is the definition of the β -permanent of a $n \times n$ matrix $B = \{b_{i,j}\}_{i,j=1}^n$, which we denote by $|B|_\beta$. It is

$$|B|_\beta = \sum_{\sigma} \beta^{m(\sigma)} b_{1,i_1} b_{2,i_2} \cdots b_{n,i_n}, \quad (2.2)$$

where the summation is taken over all distinct permutations

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ i_1 & i_2 & \cdots & i_n \end{pmatrix} \quad (2.3)$$

of the indices $1, 2, \dots, n$ and $m(\sigma)$ denotes the number of cycles into which σ can be decomposed. The name β -permanent is taken from the fact that $|B|_1$ is the permanent of B .

For $\beta > 0$ we define what it means for an $n \times n$ matrix B to be β positive definite. This is a rather complicated definition. To begin consider all multi-indices $\mathbf{k} = (k_1, \dots, k_n)$, (where the k_i are integers), and set $|\mathbf{k}| = k_1 + \dots + k_n$. For each \mathbf{k} , let $B(\mathbf{k})$ denote the $|\mathbf{k}| \times |\mathbf{k}|$ matrix with entries

$$\{B(\mathbf{k})\}_{i,j} = B_{p_i,p_j} \quad (2.4)$$

where $p_i = 1$ if $1 \leq i \leq k_1$, and $p_i = l$ if $k_{l-1} \leq i \leq k_l$, if $l = 2, \dots, n$. The matrix B is said to be β positive definite if for all multi-indices $|\mathbf{k}|$

$$|B(\mathbf{k})|_\beta \geq 0. \quad (2.5)$$

Clearly, to verify whether B is β positive definite one must check (2.5) for an infinite number of matrices.

In [13, Proposition 4.5] Verre-Jones gives necessary and sufficient conditions for (1.2) to be the Laplace transform of the vector $(\theta_{x_1}, \dots, \theta_{x_n})$ in terms of the modified resolvent matrix

$$\Gamma_r := \Gamma(I + r\Gamma)^{-1} \quad (2.6)$$

where $r \geq 0$, and Γ is the matrix in (1.2). Note that when Γ^{-1} exists, we have $\Gamma_r = (\Gamma^{-1} + rI)^{-1}$.

The next proposition is Proposition 4.5 in [13].

Proposition 2.1. *For (1.2) to represent the Laplace transform of a non-negative random vector it is necessary and sufficient that for all $r \geq 0$*

- (i) Γ_r exists and is β -positive definite.
- (ii) $\det(I + r\Gamma) > 0$.

Remark 2.2. Item (ii) is equivalent to: All the real, non-zero, eigenvalues of Γ are positive.

Verifying Proposition 2.1 (i), requires verifying (2.4) for an infinite number of matrices, which seems impossible unless all the entries of the matrix Γ_r are greater than or equal to zero. In [5, Theorem 3.1] Eisenbaum and Kaspi point out that this is the case for an important class of kernels Γ and that Proposition 2.1 (ii), also holds for these kernels. We take this up in Section 4. In the next section we assume that a permanental process exists and give sufficient conditions for it to be continuous.

3. Continuity of Permanental Processes

A key observation that allows us to obtain a ‘best possible’ sufficient condition for the continuity of permanental processes is that the bivariate marginals of 1/2-permanent processes are squares of bivariate normal random variables. We proceed to explain this.

For $n = 2$, (1.1) takes the form

$$\begin{aligned} E \left(\exp \left(-\frac{1}{2} (\alpha_1 \theta_x + \alpha_2 \theta_y) \right) \right) & \quad (3.1) \\ &= \frac{1}{|I + \alpha \Gamma|^{1/2}} = (1 + \alpha_1 \Gamma(x, x) + \alpha_2 \Gamma(y, y) \\ & \quad + \alpha_1 \alpha_2 (\Gamma(x, x) \Gamma(y, y) - \Gamma(x, y) \Gamma(y, x)))^{-1/2}. \end{aligned}$$

An obvious necessary condition for this to exist is that $|I + \alpha \Gamma| > 0$. Therefore, if we set $\alpha_2 = 0$ in (3.1) and take α_1 sufficiently large we see that for any $x \in T$

$$\Gamma(x, x) \geq 0, \quad (3.2)$$

and taking $\alpha_1 = \alpha_2$ sufficiently large, that

$$\Gamma(x, x) \Gamma(y, y) - \Gamma(x, y) \Gamma(y, x) \geq 0. \quad (3.3)$$

In addition, by [13, page 135, last line], a necessary condition for the existence of any β -permanental process is that for any pair $x, y \in T$

$$\Gamma(x, y) \Gamma(y, x) \geq 0. \quad (3.4)$$

This allows us to define

$$d(x, y) = 4\sqrt{2/3} \left(\Gamma(x, x) + \Gamma(y, y) - 2(\Gamma(x, y) \Gamma(y, x))^{1/2} \right)^{1/2}. \quad (3.5)$$

It follows from (3.2)–(3.4) that for any pair $x, y \in T$, the matrix

$$\begin{bmatrix} \Gamma(x, x) & (\Gamma(x, y) \Gamma(y, x))^{1/2} \\ (\Gamma(x, y) \Gamma(y, x))^{1/2} & \Gamma(y, y) \end{bmatrix} \quad (3.6)$$

is positive definite. Therefore we can construct a mean zero Gaussian vector $\{G(x), G(y)\}$ with covariance matrix

$$E(G(x)G(y)) = (\Gamma(x, y) \Gamma(y, x))^{1/2}. \quad (3.7)$$

Note that

$$(E(G(x) - G(y))^2)^{1/2} = \frac{\sqrt{3/2}}{4} d(x, y). \quad (3.8)$$

By (3.1) the Laplace transform of $\{\theta_x, \theta_y\}$ is the same as the Laplace transform of $\{G^2(x), G^2(y)\}$. This gives the following critical lemma:

Lemma 3.1. *Suppose that $\theta := \{\theta_x, x \in T\}$ is a 1/2-permanental process with kernel Γ as given in (1.1). Then for any pair x, y ,*

$$\{\theta_x, \theta_y\} \stackrel{L}{\leq} \{G^2(x), G^2(y)\} \quad (3.9)$$

where $\{G(x), G(y)\}$ is a mean zero Gaussian random variable with covariance matrix given by (3.7).

For $p \geq 1$, let $\psi_p(x) = \exp(x^p) - 1$ and $L^{\psi_p}(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \rightarrow R^1$ such that $E\psi_p(|\xi|/c) < \infty$ for some $c > 0$. $L^{\psi_p}(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$\|\xi\|_{\psi_p} = \inf \{c > 0 : E\psi_p(|\xi|/c) \leq 1\}. \quad (3.10)$$

We shall only be concerned with the cases $p = 1$ and 2.

It follows from Lemma 3.1 that

$$\|\theta_x^{1/2} - \theta_y^{1/2}\|_{\xi_2} = \||G_x| - |G_y|\|_{\xi_2} \leq \|G_x - G_y\|_{\xi_2} \leq Cd(x, y). \quad (3.11)$$

The proof of Theorem 3.2 below uses (3.11) and standard ideas used to prove the continuity of Gaussian processes along with some of the ideas in [1, 2].

Let (T, ρ) be a separable metric or pseudometric space. Let $B_\rho(t, u)$ denote the closed ball in (T, ρ) with radius u and center t . For any probability measure μ on (T, ρ) we define

$$J_{T, \rho, \mu}(a) = \sup_{t \in T} \int_0^a \left(\log \frac{1}{\mu(B_\rho(t, u))} \right)^{1/2} du. \quad (3.12)$$

We occasionally omit some of the subscripts T, ρ or μ , if they are clear from the context.

The function $d(x, y)$ may not be a metric or pseudometric on T . Nevertheless, we can still define the sets $B_d(s, u) = \{t \in T \mid d(s, t) \leq u\}$. We can then define $J_{T, d, \mu}(a)$ as in (3.12), for any probability measure μ on $\mathcal{B}(T, d)$, the σ -algebra generated by the sets $B_d(s, u)$.

Theorem 3.2. *Let $\theta = \{\theta_x : x \in T\}$ be a 1/2-permanent process with kernel Γ satisfying $\sup_{x \in T} \Gamma(x, x) < \infty$. Let D denote the d -diameter of T and assume that T is separable for d , and that there exists a probability measure μ on $\mathcal{B}(T, d)$ such that*

$$J_d(D) < \infty. \quad (3.13)$$

Then there exists a version $\theta' = \{\theta'_x, x \in T\}$ of θ such that for any $x_0 \in T$

$$\|\sup_{x \in T} \theta'_x\|_{\psi_1} \leq 4\|\theta'_{x_0}\|_{\psi_1} + C \left(\sup_{x \in T} \Gamma(x, x) \right) J_d^2(D), \quad (3.14)$$

where C is a constant.

If

$$\lim_{\delta \rightarrow 0} J_d(\delta) = 0, \quad (3.15)$$

there exists a version $\theta' = \{\theta'_x, x \in T\}$ of θ such that

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in T \\ d(s, t) \leq \delta}} |\theta'_s(\omega) - \theta'_t(\omega)| = 0, \quad a.s. \quad (3.16)$$

If (3.15) holds and

$$\lim_{\delta \rightarrow 0} \frac{J_d(\delta)}{\delta} = \infty, \quad (3.17)$$

then

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in T \\ d(s, t) \leq \delta}} \frac{|\theta'_s - \theta'_t|}{J_d(d(s, t)/2)} \leq 30 \left(\sup_{x \in T} \theta'_x \right)^{1/2} \quad a.s. \quad (3.18)$$

Continuity is not mentioned in Theorem 3.2 because we do not know whether d is a metric on T . If $d(x, y)$ is continuous on $T \times T$ and there exists a probability measure μ on T such that (3.15) holds then there exists a version $\theta' = \{\theta'_x : x \in T\}$ of θ that is continuous almost surely. In this case (3.18) gives a bound for the uniform modulus of continuity. It is not unreasonable to assume that $d(x, y)$ is

continuous on $T \times T$. We show in [12, Lemma 3.2] that when θ is continuous on T almost surely, $d(x, y)$ is continuous on $T \times T$.

It is easy to see why the term $\sup_{x \in T} (\theta'_x)^{1/2}$ appears on the left-hand side of (3.18). Consider a Gaussian process $\{G_x : x \in T\}$ with modulus of continuity $\omega(d(s, t))$. (Suppose that (T, d) is a metric space.) Then

$$\lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in T \\ d(s, t) \leq \delta}} \frac{|G_s^2 - G_t^2|}{\omega(d(s, t))} \leq \lim_{\delta \rightarrow 0} \sup_{\substack{s, t \in T \\ d(s, t) \leq \delta}} \frac{|G_s - G_t|}{\omega(d(s, t))} \left(2 \sup_{x \in T} G_x^2 \right)^{1/2} \quad a.s. \quad (3.19)$$

(With mild regularity conditions the inequality can be replaced by equality.)

Under the hypotheses of Theorem 3.2 an essentially optimal local modulus is also given for $\theta = \{\theta_x : x \in T\}$ in [12, Theorem 4.2].

We say that a metric or pseudometric d_1 dominates d on T if

$$d(x, y) \leq d_1(x, y), \quad \forall x, y \in T. \quad (3.20)$$

In the [12, Section 5] we give several natural metrics that dominate d . For example

$$\frac{1}{4} \inf_{x \in T} \Gamma^{1/2}(x, x) d(x, y) \leq \left(E(\theta_x - \theta_y)^2 \right)^{1/2} \leq \frac{3}{4} \sup_{x \in T} \Gamma^{1/2}(x, x) d(x, y). \quad (3.21)$$

If a metric d_1 dominates d and is such that (T, d_1) is separable and has finite diameter D then the results of Theorem 3.2 hold with d replaced by d_1 and the version θ' is uniformly continuous on (T, d_1) . (See [12, Corollary 1.2].)

Remark 3.3. We expect that Theorem 3.2 also holds for all β -permanental processes, $\beta > 0$, but we can't show this. What we can show is that all the statements in Theorem 3.2 and in the discussion following it hold for all β -permanental processes, $\beta > 0$, if $J_{T, \rho, \mu}(a)$ in (3.12) is replaced by

$$\mathcal{I}_{T, \rho, \mu}(a) = \sup_{t \in T} \int_0^a \left(\log \frac{1}{\mu(B_\rho(t, u))} \right) du. \quad (3.22)$$

This follows from the next lemma and corollary used in conjunction with the proof of [12, Theorem 1.1].

Lemma 3.4. *Let $\theta := \{\theta_x, x \in T\}$ be a β -permanental process, $\beta \leq 1/2$ with kernel Γ . Then for all $x, y \in T$ and $0 < \lambda < \infty$*

$$\left\| \theta_x - \theta_y \right\|_{\psi_1} \leq \frac{16(1-\beta)}{3} \left(\sup_{x \in T} \Gamma(x, x) \right) d(x, y). \quad (3.23)$$

Proof. Let $\Gamma(x, y)$ be the kernel of the 2-dimensional vector $\theta = (\theta_x, \theta_y)$. Let $G = (G_x, G_y)$ be the 2-dimensional Gaussian vector with covariance given by (3.7). By (3.1) we can assume that the kernel of θ is given by the matrix in (3.6) which we denote by Γ . It is well known that

$$E \left(\exp \left(-\frac{1}{2} (\alpha_x \theta_x + \alpha_y \theta_y) \right) \right) = \frac{1}{|I + \alpha \Gamma|^\beta}, \quad (3.24)$$

is the Laplace transform of vector in R_+^2 for all $\beta > 0$. (See Lemma 4.1 and Remark 4.2.) Denote this vector by $(\theta_{x, \beta}, \theta_{y, \beta})$. For $\beta < 1/2$ we have

$$(\theta_{x, \beta} - \theta_{y, \beta}) + (\theta'_{x, 1/2-\beta} - \theta'_{y, 1/2-\beta}) \stackrel{L}{=} G_x^2 - G_y^2 \quad (3.25)$$

where $(\theta'_{x,1/2-\beta}, \theta'_{y,1/2-\beta})$ is independent of $(\theta_{x,\beta}, \theta_{y,\beta})$, since (G_x^2, G_y^2) is the vector obtained from (3.24) when $\beta = 1/2$. Let $E_\theta E_{\theta'}$ denote expectation on the product space of $(\theta_{x,\beta}, \theta_{y,\beta}) \times (\theta'_{x,1/2-\beta}, \theta'_{y,1/2-\beta})$ and let $\|\cdot\|_p = (E_\theta E_{\theta'}|\cdot|^p)^{1/p}$, $p \geq 1$. We have

$$\begin{aligned} & \|(\theta_{x,\beta} - \theta_{y,\beta}) + (\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})\|_p & (3.26) \\ &= \left(E_\theta E_{\theta'} |(\theta_{x,\beta} - \theta_{y,\beta}) + (\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})|^p \right)^{1/p} \\ &\geq \left(E_\theta |(\theta_{x,\beta} - \theta_{y,\beta})| + E_{\theta'} (\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta}) \right)^{1/p} \\ &\geq \|\theta_{x,\beta} - \theta_{y,\beta} + E(\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})\|_p \\ &\geq \|\theta_{x,\beta} - \theta_{y,\beta}\|_p - |E_{\theta'}(\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})|. \end{aligned}$$

Consequently

$$\|\theta_{x,\beta} - \theta_{y,\beta}\|_p \leq \|G_x^2 - G_y^2\|_p + |E_{\theta'}(\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})|. \quad (3.27)$$

Using (3.24) we see that for any $\beta < 1/2$

$$\begin{aligned} |E_{\theta'}(\theta'_{x,1/2-\beta} - \theta'_{y,1/2-\beta})| &= (1-2\beta)|\Gamma(x, x) - \Gamma(y, y)| & (3.28) \\ &\leq (1-2\beta)\|G_x^2 - G_y^2\|_p, \end{aligned}$$

since for integers $p \geq 1$,

$$(E|G_x^2 - G_y^2|^p)^{1/p} \geq (|E(G_x^2 - G_y^2)|^p)^{1/p} = |\Gamma(x, x) - \Gamma(y, y)|. \quad (3.29)$$

Therefore

$$E(\theta_{x,\beta} - \theta_{y,\beta})^p \leq (2(1-\beta))^p E(G_x^2 - G_y^2)^p, \quad (3.30)$$

which implies that

$$\|\theta_x - \theta_y\|_{\psi_1} \leq 2(1-\beta)\|G_x^2 - G_y^2\|_{\psi_1} \quad (3.31)$$

which gives (3.23). \square

Corollary 3.5. *Let $\theta := \{\theta_x, x \in T\}$ be a β -permanental process, $\beta > 1/2$, with kernel Γ . Then for some integer $k \geq 1$*

$$\|\theta_x - \theta_y\|_{\psi_1} \leq \frac{16(k-\beta)}{3} \left(\sup_{x \in T} \Gamma(x, x) \right) d(x, y). \quad (3.32)$$

Proof. Set $\beta = k\beta'$ for some $\beta' \leq 1/2$. Then since

$$\|\theta_{x,\beta} - \theta_{y,\beta}\|_{\psi_1} \leq k\|\theta_{x,\beta'} - \theta_{y,\beta'}\|_{\psi_1}, \quad (3.33)$$

(3.32) follows from (3.23). \square

In general, given the existence of a $1/2$ -permanental process with kernel Γ , we do not know whether there exists any other β -permanental process with kernel Γ . If one does exist for rational β , the next theorem shows that it inherits continuity properties of the $1/2$ -permanental process.

We use the following inequality in the proof:

Lemma 3.6. [3, Theorem 1.1.5] *Let X_1, \dots, X_n be independent identically distributed random variables in a normed linear space. Then for all $1 \leq k \leq n$ and all $t > 0$*

$$P \left(\max_{1 \leq k \leq n} \left\| \sum_{j=1}^k X_j \right\| > t \right) \leq 9P \left(\left\| \sum_{j=1}^n X_j \right\| > \frac{t}{30} \right). \quad (3.34)$$

In particular

$$P(\|X_1\| > t) \leq 9P \left(\left\| \sum_{j=1}^n X_j \right\| > \frac{t}{30} \right). \quad (3.35)$$

Theorem 3.7. *Let $\theta = \{\theta_x : x \in T\}$ be a $1/2$ -permanental process with kernel Γ that has continuous paths on (T, d) almost surely. Let $\theta_\beta = \{\theta_{x,\beta} : x \in T\}$ be a β -permanental process with kernel Γ for any rational $\beta > 0$. Then θ_β has continuous paths on (T, d) almost surely.*

Proof. Let T' be a finite subset of T and $\|\cdot\| := \sup_{\substack{d(x,y) \leq \delta \\ x,y \in T'}} |\cdot|$. Suppose that $\beta = p/q$, for integers p and q . Then

$$\left\| \sum_{j=1}^q \left(\theta_{x,\beta}^{(j)} - \theta_{y,\beta}^{(j)} \right) \right\| \stackrel{\mathcal{L}}{=} \left\| \sum_{j=1}^{2p} \theta_{x,1/2}^{(j)} - \theta_{y,1/2}^{(j)} \right\|. \quad (3.36)$$

where the notation (j) denotes independent copies. By (3.35) and (3.36)

$$\begin{aligned} P(\|\theta_{x,\beta} - \theta_{y,\beta}\| > t) & \quad (3.37) \\ & \leq 9P \left(\left\| \sum_{j=1}^{2p} \theta_{x,1/2}^{(j)} - \theta_{y,1/2}^{(j)} \right\| > \frac{t}{30} \right) \\ & \leq 18pP \left(\|\theta_{x,1/2} - \theta_{y,1/2}\| > \frac{t}{60p} \right) \\ & = 18pP \left(\sup_{\substack{d(x,y) \leq \delta \\ x,y \in T}} |\theta_{x,1/2} - \theta_{y,1/2}| > \frac{t}{60p} \right). \end{aligned}$$

Since $\theta_{x,1/2}$ has continuous paths it follows from (3.37) that for any finite subset T' of T and $\epsilon > 0$ we can take δ sufficiently small so that

$$P(\|\theta_{x,\beta} - \theta_{y,\beta}\| > \epsilon) \leq \epsilon. \quad (3.38)$$

This implies that θ_β has continuous paths on (T, d) almost surely. \square

4. Examples of Permanental Vectors and Processes

The matrix A is said to be an M matrix if

- (1) $a_{i,j} \leq 0$ for all $i \neq j$.
- (2) A is nonsingular and $A^{-1} \geq 0$.

A diagonal matrix S with entries ± 1 is called a signature matrix.

Lemma 4.1. [5, Lemma 4.2] *Let $\{\Gamma_{i,j}\}$ be a real positive non-singular $n \times n$ matrix. There exists a positive infinitely divisible random vector $(\theta_1, \dots, \theta_n)$ such that for every $(\alpha_1, \dots, \alpha_n)$ in R_+^n*

$$E \left(\exp \left(-\frac{1}{2} \sum_{i=1}^n \alpha_i \theta_i \right) \right) = \frac{1}{|I + \alpha \Gamma|^{1/2}} \quad (4.1)$$

if and only if there exists a signature matrix S such that $S\Gamma^{-1}S$ is an M -matrix.

Remark 4.2. One can replace the $1/2$ in (4.1) by any $\beta > 0$. To see this denote the right hand side of (4.1) by $F(\alpha)$. A necessary and sufficient condition for a function $F(\alpha)$ to be the Laplace transform of an infinitely divisible random variable is that $\log F(\mathcal{S})$, where \mathcal{S} is a diagonal matrix with entries $(1 - s_i)t$, $t > 0$ and $0 < s_i \leq 1$, $i = 1, \dots, n$, has a power series expansion in (s_1, \dots, s_n) with positive coefficients for all t sufficiently large. (See, e.g., [11, Lemma 13.2.2].) In (4.1)

$$\log F(\mathcal{S}) = -(1/2) \log |I + \mathcal{S}\Gamma|. \quad (4.2)$$

Clearly, if this has a power series expansion in (s_1, \dots, s_n) with positive coefficients for all t sufficiently large, it does if $1/2$ is replaced by any $\beta > 0$.

Let $X = \{X_t, t \in R_+\}$ be a transient Markov process with state space E and potential density $u(x, y)$.

Theorem 4.3. [5, Theorem 3.1] *Let $u(x, y)$ be the potential density of a transient Markov process with state space E . Then for every $\beta > 0$ there exists a positive process $\{\theta_x, x \in E\}$, such that for every $(\alpha_1, \dots, \alpha_n)$ in R_+^n and $(x_1, \dots, x_n) \in E$,*

$$E \left(\exp \left(-\frac{1}{2} \sum_{i=1}^n \alpha_i \theta_{x_i} \right) \right) = \frac{1}{|I + \alpha U|^\beta} \quad (4.3)$$

where $U = \{u(x_i, x_j)\}_{i,j=1}^n$.

The proof of this theorem makes use of the fact that because $u(x, y)$ is the potential density of a transient Markov process, U^{-1} is an M -matrix. It is a property of M -matrices that all its real eigenvalues are positive. Consequently all the real eigenvalues of U are positive. In addition the matrix rU_r , see (2.6), is a resolvent matrix and hence it has only non-negative entries. In this case it is trivial to see that U_r is β -positive definite for all $\beta > 0$. Therefore, Theorem 4.3 follows from Proposition 2.1.

When a permanental process is determined by the potential of a transient Markov process we refer to it as an associated process. (This is ambiguous because the kernel of a permanental process is not unique.) So far, besides vectors with components that are squares of Gaussian vectors, the only permanental vectors and processes that we can show exist, are associated processes, and they have the additional property that they are infinitely divisible.

There is another property of associated processes that enables us to simplify the statement of Theorem 3.2. In general we don't know whether the function d in this theorem is a metric on T . We show in the next lemma that when the permanental process is an associated process d is a metric on T . In this case we

can add to the statement of Theorem 3.2 that the version θ' is continuous on (T, d) and that (3.18) gives a uniform modulus of continuity on (T, d) .

Lemma 4.4. *Let $\{\theta_x, x \in E\}$ be a β -permanental process associated with a transient Markov process with potential $u(x, y)$. Then*

$$\tilde{d}(x, y) = 4\sqrt{2/3} \left(u(x, x) + u(y, y) - 2(u(x, y)u(y, x))^{1/2} \right)^{1/2}. \quad (4.4)$$

is a metric on E .

The proof follows from the next two lemmas:

Lemma 4.5. *Let*

$$\mathcal{A} = \begin{pmatrix} u & a & b \\ a & v & c \\ b & c & w \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} u & \alpha_1 & \beta_2 \\ \alpha_2 & v & \gamma_1 \\ \beta_1 & \gamma_2 & w \end{pmatrix}. \quad (4.5)$$

where $\alpha_1\alpha_2 = a^2$, $\beta_1\beta_2 = b^2$ and $\gamma_1\gamma_2 = c^2$. Suppose that $\mathcal{A} \geq 0$. If \mathcal{B}^{-1} is an M matrix then \mathcal{A}^{-1} is an M matrix.

Proof. We first consider the case when $abc > 0$. We have

$$\mathcal{B}^{-1} = \frac{1}{\det \mathcal{B}} \begin{pmatrix} vw - c^2 & \beta_2\gamma_2 - \alpha_1w & \alpha_1\gamma_1 - \beta_2v \\ \beta_1\gamma_1 - \alpha_2w & uv - b^2 & \alpha_2\beta_2 - \gamma_1u \\ \alpha_2\gamma_2 - \beta_1v & \alpha_1\beta_1 - \gamma_2u & uv - a^2 \end{pmatrix} \quad (4.6)$$

Since \mathcal{B}^{-1} is an M matrix it's off diagonal terms are less than or equal to zero. This implies that

$$q_1 := \frac{aw}{bc} \geq 1, \quad q_2 := \frac{bv}{ac} \geq 1, \quad q_3 := \frac{cu}{ab} \geq 1. \quad (4.7)$$

We get this by considering the pairs $\mathcal{B}_{1,2}^{-1}, \mathcal{B}_{2,1}^{-1}$ and $\mathcal{B}_{1,3}^{-1}, \mathcal{B}_{3,1}^{-1}$ and $\mathcal{B}_{2,3}^{-1}, \mathcal{B}_{3,2}^{-1}$ and using the hypothesis that \mathcal{B}^{-1} is an M matrix. We now note that

$$\begin{aligned} \frac{\det \mathcal{A}}{abc} &= \left(\frac{uvw}{abc} - \frac{aw}{bc} \right) + \left(1 - \frac{bv}{ac} \right) + \left(1 - \frac{cu}{ab} \right) \\ &= (q_1q_2q_3 - q_1) + (1 - q_2) + (1 - q_3). \end{aligned} \quad (4.8)$$

One can check that on the range of (q_1, q_2, q_3) allowed in (4.7) $\det \mathcal{A} > 0$ unless two of the $q_i = 1, i = 1, 2, 3$. (Set $q_i = 1 + \epsilon_i, i = 1, 2, 3$ and observe that (4.8) is equal to $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_3 + \epsilon_1\epsilon_3 + \epsilon_1\epsilon_2\epsilon_3$.)

Without loss of generality suppose $q_1 = q_2 = 1$. In this case, when $abc > 0$, \mathcal{B}^{-1} can not be an M -matrix. We first note that if it is an M -matrix then $\{\mathcal{B}^{-1}\}_{1,2} = \{\mathcal{B}^{-1}\}_{2,1} = 0$. This holds since $\beta_2\gamma_2 \leq \alpha_1w$ and $\beta_1\gamma_1 \leq \alpha_2w$. Therefore, in order for $b^2c^2 = a^2w^2$, which is the case when $q_1 = 1$, we must have $\beta_2\gamma_2 = \alpha_1w$ and $\beta_1\gamma_1 = \alpha_2w$. A similar argument shows that argument shows that $\{\mathcal{B}^{-1}\}_{1,3} = \{\mathcal{B}^{-1}\}_{3,1} = 0$. In addition $\{\mathcal{B}^{-1}\}_{1,1} = 0$, since $q_1 = q_2 = 1$ implies that $vw = c^2$. Consequently when $q_1 = q_2 = 1$ the first row of $\mathcal{B}^{-1} = 0$. But this is impossible since an M -matrix is invertible.

We have now established that under the hypotheses of this theorem, when $abc > 0$, $\det \mathcal{A} > 0$.

Now suppose that $abc = 0$. It is easy to see from (4.6) that this implies that either two of these three numbers must be zero or all three must be zero. For example, suppose that $\alpha_1 = 0$. Then by looking at $\mathcal{B}_{1,2}^{-1}$ we see that either γ_2 or β_2 are equal to zero. Suppose $\alpha_1 = \beta_2 = 0$. Then

$$\begin{aligned} \det \mathcal{B} &= u(vw - c^2) + \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2 - v\beta_1\beta_2 - w\alpha_1\alpha_2 \\ &= u(vw - c^2). \end{aligned} \quad (4.9)$$

In addition, since in this case $a = b = 0$, $\det \mathcal{A} = u(vw - c^2)$. Therefore, since $\det \mathcal{B} > 0$ we have $\det \mathcal{A} > 0$. A similar argument shows that when $\alpha_1 = \gamma_2 = 0$ and \mathcal{B}^{-1} is an M -matrix, we also have $\det \mathcal{A} > 0$. If all three numbers a, b, c are equal to zero, $\det \mathcal{A} = uvw$ and so does $\det \mathcal{B}$, even though, once again, both \mathcal{B} and \mathcal{B}^{-1} can have non-zero off diagonal elements. In either case if \mathcal{B}^{-1} is an M -matrix, $\det \mathcal{A} > 0$.

Since $\det \mathcal{A} > 0$, we have

$$\mathcal{A}^{-1} = \frac{1}{\det \mathcal{A}} \begin{pmatrix} vw - c^2 & bc - aw & ac - bv \\ bc - aw & uw - b^2 & ab - cu \\ ac - bv & ab - cu & uv - a^2 \end{pmatrix}. \quad (4.10)$$

If \mathcal{A}^{-1} is not an M matrix then one of the off diagonal terms of \mathcal{A}^{-1} must be strictly positive. Without loss of generality we can suppose $aw < bc$. Then

$$\alpha_1\alpha_2w^2 < \beta_1\beta_2\gamma_1\gamma_2. \quad (4.11)$$

If this happens then one or both of the following inequalities must hold:

$$\alpha_1w < \beta_2\gamma_2, \quad \alpha_2w < \beta_1\gamma_1. \quad (4.12)$$

If either of these hold one of the off diagonal terms of \mathcal{B}^{-1} must be positive. Therefore, \mathcal{B}^{-1} is not an M matrix. This contradiction completes the proof. \square

Lemma 4.6. *Let $\{\theta_x, x \in R^1\}$ be a permanental process with kernel $\Gamma(x, y)$. Let*

$$d(x, y) := \left(\Gamma(x, x) + \Gamma(y, y) - 2(\Gamma(x, y)\Gamma(y, x))^{1/2} \right)^{1/2}. \quad (4.13)$$

Suppose that for any three different real numbers, x_1, x_2, x_3 , the 3×3 matrix $\tilde{\Gamma} := \{\Gamma(x_i, x_j)\}_{i,j=1,2,3}$ has an inverse that is an M -matrix. Then $d(x, y)$ is a metric on R^1 .

Proof. For any x_1, x_2, x_3 in R^1 it is easy to see that

$$d(x_1, x_2) \leq d(x_1, x_3) + d(x_3, x_2). \quad (4.14)$$

This follows because by Lemma 4.5 the determinant of

$$\begin{pmatrix} \Gamma(x_1, x_1) & (\Gamma(x_1, x_2)\Gamma(x_2, x_1))^{1/2} & (\Gamma(x_1, x_3)\Gamma(x_3, x_1))^{1/2} \\ (\Gamma(x_1, x_2)\Gamma(x_2, x_1))^{1/2} & \Gamma(x_2, x_2) & (\Gamma(x_3, x_2)\Gamma(x_2, x_3))^{1/2} \\ (\Gamma(x_1, x_3)\Gamma(x_3, x_1))^{1/2} & (\Gamma(x_3, x_2)\Gamma(x_2, x_3))^{1/2} & \Gamma(x_3, x_3) \end{pmatrix} \quad (4.15)$$

is strictly positive. The determinant of the principle 2×2 minor is also strictly positive by [12, (1.5)]. Therefore the matrix is positive definite and thus it is the covariance matrix of a Gaussian random variable which we denote by $(G_{x_1}, G_{x_2}, G_{x_3})$.

Since

$$d(x_i, x_j) = (E(G_{x_i} - G_{x_j})^2)^{1/2} \quad i, j = 1, 2, 3, \quad (4.16)$$

we get (4.14). \square

Remark 4.7. Note that $\det \mathcal{B} - \det \mathcal{A} = (\alpha_1 \beta_1 \gamma_1 - \alpha_2 \beta_2 \gamma_2)^2$ so it is possible to have $\det \mathcal{B} > 0$ and $\det \mathcal{A} \leq 0$, but not when \mathcal{B}^{-1} is an M -matrix.

4.1. Continuity of associated permanental processes. It follows from Theorem 4.3 that when $u(x, y)$, $x, y \in T$, is the potential density of a transient Markov process with state space T , a β -permanental process $\theta_\beta = \{\theta_{x,\beta} : x \in T\}$, with kernel u , exists for all $\beta > 0$. In addition, in this case, by Lemma 4.4, the function d in (3.5) is a metric on T . Using these facts we get the following extension of Theorem 3.2:

Theorem 4.8. *Let $\theta_\beta = \{\theta_{x,\beta}, x \in T\}$ be a β -permanental process associated with a transient Markov process with potential $u(x, y)$ satisfying $\sup_{x \in T} u(x, x) < \infty$. Let \tilde{d} be as given in (4.4) and let D denote the \tilde{d} diameter of T . Assume that there exists a probability measure μ on $\mathcal{B}(T, \tilde{d})$ such that*

$$J_{\tilde{d}}(D) < \infty. \quad (4.17)$$

Then there exists a version $\theta'_\beta = \{\theta'_{x,\beta}, x \in T\}$ of θ_β such that for any $x_0 \in T$

$$\| \sup_{x \in T} \theta'_x \|_{\psi_1} \leq C_\beta \left(\| \theta'_{x_0} \|_{\psi_1} + \sup_{x \in T} \Gamma(x, x) J_{\tilde{d}}^2(D) \right), \quad (4.18)$$

where C_β is a constant. If

$$\lim_{\delta \rightarrow 0} J_{\tilde{d}}(\delta) = 0, \quad (4.19)$$

there exists a version $\theta'_\beta = \{\theta'_{x,\beta}, x \in T\}$ of θ_β that is continuous on (T, \tilde{d}) .

If (4.19) holds and

$$\lim_{\delta \rightarrow 0} \frac{J_{\tilde{d}}(\delta)}{\delta} = \infty, \quad (4.20)$$

then

$$\lim_{\delta \rightarrow 0} \sup_{\substack{x, y \in T \\ \tilde{d}(x, y) \leq \delta}} \frac{|\theta'_{x,\beta} - \theta'_{y,\beta}|}{J_{\tilde{d}}(\tilde{d}(x, y)/2)} \leq C_\beta \left(\sup_{x \in T} \theta'_{x,\beta} \right)^{1/2} \quad a.s., \quad (4.21)$$

where C_β is a constant depending only on β .

Proof. Given β choose p so that $\beta \leq p/2$. We have

$$(\theta_{x,\beta} - \theta_{y,\beta}) + (\theta'_{x,p/2-\beta} - \theta'_{y,p/2-\beta}) \stackrel{\mathcal{L}}{=} \sum_{j=1}^p (\theta_{x,1/2}^{(j)} - \theta_{y,1/2}^{(j)}), \quad (4.22)$$

where θ_β and $\theta_{\beta'}$ are independent and where the notation (j) denotes independent copies. Let $\| \cdot \|$ be a pseudo norm on real valued functions on T . By (4.22),

$$E \| (\theta_{x,\beta} - \theta_{y,\beta}) + (\theta'_{x,p/2-\beta} - \theta'_{y,p/2-\beta}) \| \leq p E \| \theta_{x,1/2} - \theta_{y,1/2} \|. \quad (4.23)$$

Using the same argument we used in (3.26)–(3.30) we get

$$E \| \theta_{x,\beta} - \theta_{y,\beta} \| \leq 2(p - \beta) E \| \theta_{x,1/2} - \theta_{y,1/2} \|. \quad (4.24)$$

Similarly, we also note that starting with

$$\theta_{x,\beta} + \theta'_{x,p/2-\beta} \stackrel{\mathcal{L}}{=} \sum_{j=1}^p \theta_{x,1/2}^{(j)} \quad (4.25)$$

we can show that

$$E\|\theta_{x,\beta}\|_{\psi_1} \leq 2(p-\beta)E\|\theta_{x,1/2}\|_{\psi_1}. \quad (4.26)$$

Therefore, (4.18) follows from (3.14).

Using (4.24) we have

$$E \sup_{\substack{s,t \in T' \\ \tilde{d}(s,t) \leq \delta}} |\theta_{x,\beta} - \theta_{y,\beta}| \leq 2(p-\beta)E \sup_{\substack{s,t \in T \\ \tilde{d}(s,t) \leq \delta}} |\theta_{x,1/2} - \theta_{y,1/2}| \quad (4.27)$$

for all finite subsets T' of T . By (4.19) and (3.16) there exists a version $\theta'_\beta = \{\theta'_{x,\beta}, x \in T\}$ of θ_β

$$\lim_{\delta \rightarrow 0} E \sup_{\substack{s,t \in T' \\ \tilde{d}(s,t) \leq \delta}} |\theta'_{x,\beta} - \theta'_{y,\beta}| = 0 \quad (4.28)$$

This implies that θ'_β is continuous on (T, \tilde{d}) .

By (4.20) and (3.18) there exists a positive function $g(\delta)$, satisfying $\lim_{\delta \rightarrow 0} g(\delta) = 0$, such that

$$P \left(\sup_{\substack{s,t \in T \\ \tilde{d}(s,t) \leq \delta}} \frac{|\theta'_{s,1/2} - \theta'_{t,1/2}|}{J_{\tilde{d}}(\delta/2)} > 31 \left(\sup_{x \in T} \theta'_{x,1/2} \right)^{1/2} \right) < g(\delta). \quad (4.29)$$

Using the argument in (3.37) we get that there exists a constant C_β and a positive function $g'(\delta)$, satisfying $\lim_{\delta \rightarrow 0} g'(\delta) = 0$, such that

$$P \left(\sup_{\substack{s,t \in T \\ \tilde{d}(s,t) \leq \delta}} \frac{|\theta'_{s,\beta} - \theta'_{t,\beta}|}{J_{\tilde{d}}(\delta/2)} > C_\beta \left(\sup_{x \in T} \theta'_{x,1/2} \right)^{1/2} \right) < g'(\delta). \quad (4.30)$$

This implies that

$$\lim_{\delta \rightarrow 0} \sup_{\substack{x,y \in T \\ \tilde{d}(x,y) \leq \delta}} \frac{|\theta'_{x,\beta} - \theta'_{y,\beta}|}{J_{\tilde{d}}(\delta/2)} \leq C_\beta \left(\sup_{x \in T} \theta'_{x,\beta} \right)^{1/2} \quad a.s. \quad (4.31)$$

By [11, Lemma 7.1.6] this is equivalent to (4.21). \square

5. Permanent Processes Associated with Lévy Processes

It is interesting to see what kernels of permanent processes, that are not Gaussian squares, look like. We can do this by finding the kernels of permanent processes associated with Lévy processes. Let $X = \{X_t, t \in R_+\}$ be a Lévy process with characteristic function

$$Ee^{i\lambda X} = e^{-\psi(\lambda)t}. \quad (5.1)$$

We refer to ψ as the characteristic exponent of X .

We assume that X has local times $\{L_t^x, (x,t) \in R \times R_+\}$. Set

$$u_{T_0}(x,y) = E^x(L_{T_0}^y), \quad (5.2)$$

where T_0 is the first hitting time of X at zero. We consider the transient Markov process $\tilde{X} = \{\tilde{X}_t\}$ that is X killed at the first time it hits zero. The function $u_{T_0}(x, y)$ is the zero potential of \tilde{X} and thus is also the kernel of a permanental process.

Lemma 5.1.

$$u_{T_0}(x, y) = R(x, y) + H(x, y) \quad \text{and} \quad u_{T_0}(y, x) = R(x, y) - H(x, y) \quad (5.3)$$

where

$$R(x, y) = \frac{1}{\pi} \int_0^\infty \frac{(1 - \cos \lambda x - \cos \lambda y + \cos \lambda(x - y)) \operatorname{Re} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda \quad (5.4)$$

and

$$H(x, y) = -\frac{1}{\pi} \int_0^\infty \frac{(\sin \lambda x - \sin \lambda y - \sin \lambda(x - y)) \operatorname{Im} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda. \quad (5.5)$$

Proof. It follows from [11, Theorem 4.2.4], modified for non-symmetric Lévy processes, that for $x, y \in \mathbb{R}^1$

$$u_{T_0}(x, y) = \phi(x) + \phi(-y) - \phi(x - y), \quad (5.6)$$

where

$$\begin{aligned} \phi(x) &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{1 - e^{i\lambda x}}{\psi(\lambda)} d\lambda \\ &= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{(1 - e^{i\lambda x}) \psi(-\lambda)}{|\psi(\lambda)|^2} d\lambda \\ &= \frac{1}{2\pi} \int_0^\infty \frac{(1 - e^{i\lambda x}) \psi(-\lambda)}{|\psi(\lambda)|^2} d\lambda + \frac{1}{2\pi} \int_0^\infty \frac{(1 - e^{-i\lambda x}) \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda \\ &= \frac{1}{\pi} \operatorname{Re} \left(\int_0^\infty \frac{(1 - e^{i\lambda x}) \psi(-\lambda)}{|\psi(\lambda)|^2} d\lambda \right) \end{aligned} \quad (5.7)$$

Since $\overline{\psi(\lambda)} = \psi(-\lambda)$ we see that

$$\phi(x) = \frac{1}{\pi} \left(\int_0^\infty \frac{(1 - \cos \lambda x) \operatorname{Re} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda - \int_0^\infty \frac{\sin \lambda x \operatorname{Im} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda \right). \quad (5.8)$$

Therefore,

$$\frac{\phi(x) + \phi(-x)}{2} = \frac{1}{\pi} \int_0^\infty \frac{(1 - \cos \lambda x) \operatorname{Re} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda. \quad (5.9)$$

and

$$\frac{\phi(x) - \phi(-x)}{2} = -\frac{1}{\pi} \int_0^\infty \frac{\sin \lambda x \operatorname{Im} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda. \quad (5.10)$$

We write

$$u_{T_0}(x, y) = \frac{u_{T_0}(x, y) + u_{T_0}(y, x)}{2} + \frac{u_{T_0}(x, y) - u_{T_0}(y, x)}{2} \quad (5.11)$$

and

$$u_{T_0}(y, x) = \frac{u_{T_0}(x, y) + u_{T_0}(y, x)}{2} - \frac{u_{T_0}(x, y) - u_{T_0}(y, x)}{2}. \quad (5.12)$$

Set

$$R(x, y) = \frac{u_{T_0}(x, y) + u_{T_0}(y, x)}{2} \quad \text{and} \quad H(x, y) = \frac{u_{T_0}(x, y) - u_{T_0}(y, x)}{2}. \quad (5.13)$$

Using (5.6) and the following equations we get (5.3)–(5.5). \square

We now consider the transient Markov process $\bar{X} = \{\bar{X}_t\}$ that is X killed at ξ_α , an independent exponential time with mean $\alpha > 0$. We assume that X has local times $\{L_t^x, (x, t) \in R \times R_+\}$ and set

$$u^\alpha(x, y) = E^x \left(L_{\xi_\alpha}^y \right). \quad (5.14)$$

The function $u^\alpha(x, y)$ is the zero potential of \bar{X} and thus is also the kernel of a permanent process.

Lemma 5.2. *For $\alpha > 0$, the α -potential density*

$$u^\alpha(x, y) = R_\alpha(x, y) + H_\alpha(x, y) \quad \text{and} \quad u^\alpha(y, x) = R_\alpha(x, y) - H_\alpha(x, y), \quad (5.15)$$

where

$$R_\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda(x-y) \operatorname{Re}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda \quad (5.16)$$

and

$$H_\alpha(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda(x-y) \operatorname{Im}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda. \quad (5.17)$$

Proof. By (5.1)

$$\int_{-\infty}^\infty e^{i\lambda y} p_t(0, y) dy = e^{-t\psi(\lambda)}. \quad (5.18)$$

Therefore

$$p_t(0, y) = \frac{1}{2\pi} \int_{-\infty}^\infty e^{-i\lambda y} e^{-t\psi(\lambda)} d\lambda \quad (5.19)$$

and, similarly to (5.7)

$$\begin{aligned} u^\alpha(0, y) &= \int_0^\infty e^{-\alpha t} p_t(0, y) dt = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\lambda y}}{\alpha + \psi(\lambda)} d\lambda \\ &= \frac{1}{\pi} \operatorname{Re} \left(\int_0^\infty \frac{e^{i\lambda y} (\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda \right) \\ &= \frac{1}{\pi} \left(\int_0^\infty \frac{\cos \lambda y \operatorname{Re}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda \right. \\ &\quad \left. - \int_0^\infty \frac{\sin \lambda y \operatorname{Im}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda \right). \end{aligned} \quad (5.20)$$

Using the fact that $u^\alpha(x, y) = u^\alpha(0, y-x)$ we see that

$$\frac{u^\alpha(x, y) + u^\alpha(y, x)}{2} = \frac{1}{\pi} \int_0^\infty \frac{\cos \lambda(x-y) \operatorname{Re}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda. \quad (5.21)$$

and

$$\frac{u^\alpha(x, y) - u^\alpha(y, x)}{2} = \frac{1}{\pi} \int_0^\infty \frac{\sin \lambda(x-y) \operatorname{Im}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda, \quad (5.22)$$

which gives (5.15). \square

Remark 5.3. Denote R and H in (5.4) and (5.5) by R_0 and H_0 . Since

$$1 - \cos \lambda x - \cos \lambda y + \cos \lambda(x - y) = (1 - \cos \lambda x)(1 - \cos \lambda y) + \sin \lambda x \sin \lambda y \quad (5.23)$$

and

$$\cos \lambda(x - y) = \cos \lambda x \cos \lambda y + \sin \lambda x \sin \lambda y \quad (5.24)$$

we see that $R_\alpha(x, y)$ is positive definite for all $\alpha \geq 0$. Therefore they are the covariances of mean zero Gaussian processes $G_\alpha = \{G_\alpha(x), x \in \mathbb{R}^1\}$. It is easy to see that for $\alpha > 0$, G_α is a stationary Gaussian process and that G_0 , has stationary increments, and that $G_0(0) = 0$. Furthermore

$$E(G_0(x) - G_0(y))^2 = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos \lambda(x - y)) \operatorname{Re} \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda \quad (5.25)$$

and when $E(G^2(x)) = 1$,

$$E(G_\alpha(x) - G_\alpha(y))^2 = \frac{2}{\pi} \int_0^\infty \frac{(1 - \cos \lambda(x - y)) \operatorname{Re}(\alpha + \psi(\lambda))}{|\alpha + \psi(\lambda)|^2} d\lambda, \quad \forall \alpha > 0. \quad (5.26)$$

Since $R_\alpha(x, y)$ is positive definite for all $\alpha \geq 0$, it follows from (5.3) that $u_{T_0}(x, y)$ and $u^\alpha(x, y)$, $\alpha > 0$, are positive definite. This is easy to see since

$$u_{T_0}(x, y) = R_0(x, y) + H_0(x, y) \quad (5.27)$$

and $H_0(x, x) = 0$, and $H_0(x, y) = -H_0(y, x)$, and similarly for $u^\alpha(x, y)$.

Set

$$d_{G_\alpha}(x, y) = (E(G_\alpha(x) - G_\alpha(y))^2)^{1/2} \quad \alpha \geq 0. \quad (5.28)$$

We show in [12, Lemma 5.5] that this metric is equivalent to \tilde{d} in (4.4). Therefore it follows from Theorem 4.8, and [11, Chapter 6], that the denominator in (4.21) for a permanental processes with kernel $u_{T_0}(x, y)$ or $u^\alpha(x, y)$, $\alpha > 0$, is the same, up to a constant multiple, as the uniform modulus of continuity of Gaussian processes with increments variance d_{G_α} .

5.1. FBMQ $^{\alpha, \beta}$. There is a great deal of interest in fractional Brownian motion. Therefore, there should be considerable interest in permanental process for which the function $R(x, y)$ in (5.4) is the covariance of fractional Brownian motion. To study these processes we use Lemma 5.1 to find the kernel of the permanental process associated with the non-symmetric stable process of index $1 < \alpha + 1 < 2$. This process has a Lévy exponent of the form

$$\psi(\lambda) = c|\lambda|^{\alpha+1}(1 - i\beta \operatorname{sign}(\lambda) \tan((\alpha + 1)\pi/2)) \quad (5.29)$$

where c is an arbitrary constant which we set to 1, and $|\beta| \leq 1$. We refer to this permanental process as FBMQ $^{\alpha, \beta}$. We use this notation because when $\beta = 0$, $\psi(\lambda)$ is the Lévy exponent of the Lévy process associated with fractional Brownian motion of index α , (i.e. FBM). We add the Q , for quadratic, to denote the square of this process, as one does in the designation of the squared Bessel processes, (BESQ).

Lemma 5.4. *Let $X = \{X_t, t \in R_+\}$ be a Lévy process with characteristic given by (5.29). Let $u_{T_0}(x, y)$ be as given in (5.2) and write $u_{T_0}(x, y)$ and $u_{T_0}(y, x)$ as in (5.3). In this case*

$$R(x, y) = C_{\alpha, \beta} (|x|^\alpha + |y|^\alpha - |x - y|^\alpha), \quad (5.30)$$

and

$$\begin{aligned} H(x, y) & \quad (5.31) \\ &= \beta C_{\alpha, \beta} (\text{sign}(x)|x|^\alpha - \text{sign}(y)|y|^\alpha - \text{sign}(x - y)|x - y|^\alpha), \end{aligned}$$

where

$$C_{\alpha, \beta} = \frac{-\sin((\alpha + 1)\frac{\pi}{2})\Gamma(-\alpha)}{\pi(1 + \beta^2 \tan^2((\alpha + 1)\pi/2))} > 0. \quad (5.32)$$

Consequently

$$\begin{aligned} u_{T_0}(x, y) &= C_{\alpha, \beta} ((1 - \beta \text{sign}(x))|x|^\alpha \\ &+ (1 - \beta \text{sign}(-y))|y|^\alpha - (1 - \beta \text{sign}(x - y))|x - y|^\alpha). \end{aligned} \quad (5.33)$$

Proof. Let $c_{\alpha, \beta} = (1 + \beta^2 \tan^2((\alpha + 1)\pi/2))^{-1}$. Then

$$\frac{(1 - \cos \lambda x) \mathcal{R}e \psi(\lambda)}{|\psi(\lambda)|^2} = \frac{c_{\alpha, \beta} (1 - \cos \lambda x)}{|\lambda|^{\alpha+1}} \quad (5.34)$$

and

$$\frac{\sin \lambda x \mathcal{I}m \psi(\lambda)}{|\psi(\lambda)|^2} = -\frac{c_{\alpha, \beta} \beta \text{sign}(\lambda) \tan((\alpha + 1)\pi/2) \sin \lambda x}{|\lambda|^{\alpha+1}} \quad (5.35)$$

Therefore, by a change of variables

$$\begin{aligned} & \int_0^\infty \frac{\sin \lambda x \mathcal{I}m \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda \quad (5.36) \\ &= -c_{\alpha, \beta} \beta \tan((\alpha + 1)\pi/2) \int_0^\infty \frac{\sin \lambda x}{\lambda^{\alpha+1}} d\lambda \\ &= -c_{\alpha, \beta} \beta \tan((\alpha + 1)\pi/2) |x|^\alpha \text{sign}(x) \int_0^\infty \frac{\sin \lambda}{\lambda^{\alpha+1}} d\lambda \end{aligned}$$

and

$$\int_0^\infty \frac{(1 - \cos \lambda x) \mathcal{R}e \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda = c_{\alpha, \beta} |x|^\alpha \int_0^\infty \frac{(1 - \cos \lambda)}{\lambda^{\alpha+1}} d\lambda. \quad (5.37)$$

It follows from [7, 17.33.2] and an integration that

$$\int_0^\infty \frac{1 - e^{i\lambda}}{\lambda^{\alpha+1}} d\lambda = -\left(\sin\left((\alpha + 1)\frac{\pi}{2}\right) + i \cos\left((\alpha + 1)\frac{\pi}{2}\right)\right) \Gamma(-\alpha). \quad (5.38)$$

Combining (5.36)–(5.38) we see that

$$\frac{1}{\pi} \int_0^\infty \frac{(1 - \cos \lambda x) \mathcal{R}e \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda = C_{\alpha, \beta} |x|^\alpha \quad (5.39)$$

and

$$\frac{1}{\pi} \int_0^\infty \frac{\sin \lambda x \mathcal{I}m \psi(\lambda)}{|\psi(\lambda)|^2} d\lambda = \beta \text{sign}(x) C_{\alpha, \beta} |x|^\alpha. \quad (5.40)$$

Using (5.39) and (5.40) in (5.4) and (5.5) we get (5.30) and (5.31). \square

Remark 5.5. It follows from Remark 5.3 and [11, (7.186)] that when θ_x is $\mathbf{FBMQ}^{\alpha,\beta}$ then for any $T < \infty$

$$\lim_{\delta \rightarrow 0} \sup_{\substack{x, y \in [0, T] \\ d(x, y) \leq \delta}} \frac{|\theta_x - \theta_y|}{(|x - y|^\alpha \log 1/|x - y|)^{1/2}} \leq C_T \left(\sup_{x \in [0, T]} \theta_x \right)^{1/2} \quad a.s. \quad (5.41)$$

In addition by [12, Theorem 4.2] and [11, Example 7.6.6, (1)] for $x_0 \neq 0$

$$\lim_{\delta \rightarrow 0} \sup_{\hat{d}(x, x_0) < \delta/2} \frac{|\theta_x - \theta_{x_0}|}{(|x - x_0|^\alpha \log \log 1/|x - x_0|)^{1/2}} \leq C \theta_{x_0}^{1/2} \quad a.s. \quad (5.42)$$

and

$$\lim_{\delta \rightarrow 0} \sup_{\hat{d}(x, 0) < \delta/2} \frac{\theta_x}{|x|^\alpha \log \log 1/|x|} \leq C \quad a.s. \quad (5.43)$$

A very interesting unanswered question is: can we find \mathbf{FBMQ}^α for $1 \leq \alpha < 2$ other than the square of a fractional Brownian motion itself? We give some examples to explain what we have in mind. Let $\tilde{G}_\alpha = \{\tilde{G}_\alpha(x), x \in R_+\}$ be fractional Brownian motion with index $1 \leq \alpha < 2$, i.e. $\tilde{G}_\alpha(0) = 0$ and

$$\sigma^2(x - y) := E(\tilde{G}_\alpha(x) - \tilde{G}_\alpha(y))^2 = |x - y|^\alpha. \quad (5.44)$$

It follows that the covariance

$$\tilde{\Gamma}(x, y) := E(\tilde{G}_\alpha(x)\tilde{G}_\alpha(y)) = \frac{1}{2} (|x|^\alpha + |y|^\alpha - (|x - y|^\alpha)). \quad (5.45)$$

Consider the kernel

$$\mathcal{G}(x, y) = \begin{cases} \tilde{\Gamma}(x, y) + \epsilon \sigma(x - y) & x \geq y \\ \tilde{\Gamma}(x, y) - \epsilon \sigma(x - y) & x < y \end{cases} \quad (5.46)$$

where $\epsilon > 0$ is some small number. To simplify the notation set $\bar{d}(x, y) = d(x, y)/4\sqrt{2/3}$. We can write

$$\begin{aligned} \bar{d}^2(x, y) &= \mathcal{G}(x, x) + \mathcal{G}(y, y) - \mathcal{G}(x, y) - \mathcal{G}(y, x) \\ &\quad + \mathcal{G}(x, y) + \mathcal{G}(y, x) - 2(\mathcal{G}(x, y)\mathcal{G}(y, x))^{1/2} \\ &= \tilde{\Gamma}(x, x) + \tilde{\Gamma}(y, y) - 2\tilde{\Gamma}(x, y) + (\mathcal{G}^{1/2}(x, y) - \mathcal{G}^{1/2}(y, x))^2 \\ &\leq \sigma^2(x - y) + |\mathcal{G}(x, y) - \mathcal{G}(y, x)| \\ &\leq (1 + 2\epsilon)\sigma^2(x - y). \end{aligned} \quad (5.47)$$

Also we see from the second equality in (5.47) that $\bar{d}^2(x, y) \geq \sigma^2(x - y)$.

We do not know whether $\mathcal{G}(x, y)$ is the kernel of a permanental process. If it is, it would be a generalization of the square of fractional Brownian motion. It would be very useful to know whether such generalizations exist.

Here is a very simple example of the above. Consider (5.45) when $\alpha = 1$. This is the covariance of Brownian motion. Consider $\mathcal{G}(x, y)$ in this case for three values,

$0 < x < y < z$. The kernel obtained is

$$\mathcal{A}_\epsilon := \begin{pmatrix} x & x - \epsilon|x - y|^{1/2} & x - \epsilon|x - y|^{1/2} \\ x + \epsilon|x - y|^{1/2} & y & y - \epsilon|x - y|^{1/2} \\ x + \epsilon|x - y|^{1/2} & y + \epsilon|x - y|^{1/2} & z \end{pmatrix}. \quad (5.48)$$

When $\epsilon = 0$, \mathcal{A}_0 is the kernel of $\{B_x^2, B_y^2, B_z^2\}$ where $\{B_x, B_y, B_z\}$ are three values of standard Brownian motion. One can check that \mathcal{A}_0^{-1} is an M -matrix. This is not a surprise since we now know that Brownian motion is an associated process. (See e.g. [11, Lemma 2.5.1].) However, $\mathcal{A}_\epsilon^{-1}$ is not an M -matrix for any $\epsilon \neq 0$. It would be interesting to know whether \mathcal{A}_ϵ can be the kernel of a permanental vector.

Remark 5.6. One may consider kernels of the form (5.46) for any Gaussian process with stationary increments and try to answer the question, “Are they kernels of permanental processes?”

6. Some Problems

1. Necessary and sufficient conditions for the continuity of permanental processes. This question could be phrased more specifically as, “Is Theorem 3.2 a necessary and sufficient condition for the continuity of 1/2-permanental processes?” It is clear that Theorem 3.2 is best possible because it is a necessary and sufficient condition for the continuity of the square of a Gaussian process with continuous covariance; (see [12, Section 7.2]).

When one considers this question one is immediately aware of how many of the properties of Gaussian processes that are used in the proof of a necessary condition for continuity or boundedness are unknown for permanental processes. Let $\theta = \{\theta_x, x \in T\}$ be a 1/2-permanental process. If $\sup_{x \in T} \theta_x < \infty$, is $E(\sup_{x \in T} \theta_x < \infty)$? What about $\|\sup_{x \in T} \theta_x\|_{\psi_1}$? If the metric d in (3.5) of one permanental process is uniformly less than the metric d of another permanental process, does the continuity or boundedness of the process with the larger metric imply the continuity or boundedness of the process with the smaller metric? More generally, are there any analogues of Slepian’s Lemma for permanental processes? Gaussian processes often satisfy many important zero one laws. What zero one laws do permanental processes satisfy? All of these are interesting questions.

2. Independence of the components of a permanental vector. It follows from Lemma 3.1 that if $\theta := \{\theta_x, x \in T\}$ is a 1/2-permanental process with kernel Γ then for any pair x, y ,

$$\text{cov}(\theta_x, \theta_y) = 2\Gamma(x, y)\Gamma(y, x) \quad (6.1)$$

If $\Gamma(x, y)\Gamma(y, x) = 0$ it follows from Lemma 3.1 that θ_x and θ_y are independent. Another interesting question is, “If the elements of $\{\theta_x, x \in T\}$ are pairwise independent are they independent?”

Suppose that $T = \{1, 2, 3\}$ and the kernel of θ is

$$\begin{pmatrix} 1 & a & 0 \\ 0 & 1 & b \\ c & 0 & 1 \end{pmatrix}. \quad (6.2)$$

where $abc \neq 0$, and otherwise a, b, c are arbitrary.

By the general formula for the moments of β -permanental processes

$$\begin{aligned} E\left(\prod_{j=1}^n \theta_{x_j}\right) &= \text{Perm}_\beta\left(\{\Gamma(x_i, x_j)\}_{1 \leq i, j \leq n}\right) \\ &= \sum_{\pi} \alpha^{c(\pi)} \prod_{j=1}^n \Gamma(x_j, x_{\pi(j)}) \end{aligned} \quad (6.3)$$

where $c(\pi)$ is the number of cycles in the permutation π of $[1, n]$. (See [13, Proposition 4.2].) Therefore, if the kernel given by (6.2) defines a $1/2$ -permanental vector $(\theta_1, \theta_2, \theta_3)$, then we would have $E(\theta_1\theta_2\theta_3) = 1/8 + (1/2)abc$. Whereas, if $\theta_1, \theta_2, \theta_3$ were independent $E(\theta_1$

$\theta_2\theta_3) = 1/8$. On the other hand by (6.1) $\theta_1, \theta_2, \theta_3$ are pairwise independent.

The kernel in (6.2) and its transpose are the only possibilities for $\theta_1, \theta_2, \theta_3$ to be pairwise independent but not independent. So in this case the problem becomes, can this be the kernel of a $1/2$ -permanental vector. We can show that it can not. But in R_+^4 we can not answer this question. In R_+^4 we can have kernels of the form

$$\begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & b & e \\ 0 & 0 & 1 & f \\ d & 0 & 0 & 1 \end{pmatrix}. \quad (6.4)$$

3. Permanental vectors in R_+^3 . Given a 3×3 matrix Γ with components that satisfy (3.2)–(3.4), is it the kernel of a permanental process? If Γ is diagonally equivalent to the inverse of an M -matrix, the answer is yes. At this point we don't know of any examples of matrices without this property that are kernels of a permanental process except, of course, when the permanental process is the square of a Gaussian process. We think that there are many examples. The class of associated vectors that are Gaussian squares is a very small subset of all vectors that are Gaussian squares. Roughly speaking, they don't include any vectors that are squares of Gaussian processes that are smoother than Brownian motion. If, again, roughly speaking, one can always find permanental vectors with kernels that deviate slightly from the kernels of Gaussian squares that are infinitely divisible, one should also be able to find permanental vectors with kernels that deviate slightly from the kernels of Gaussian squares that are not infinitely divisible. (Here we are using infinitely divisible as a synonym for associated.)

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