Communications on Stochastic Analysis
Vol. 5, No. 1 (2011) 121-133

Serials Publications
www.serialspublications.com

ON FRACTIONAL ORNSTEIN-UHLENBECK PROCESSES

TERHI KAARAKKA AND PAAVO SALMINEN

Abstract. In this paper we study Doob's transform of fractional Brownian motion (FBM). It is well known that Doob's transform of standard Brownian motion is identical in law with the Ornstein-Uhlenbeck diffusion defined as the stationary solution of the (stochastic) Langevin equation where the driving process is a Brownian motion. It is also known that Doob's transform of FBM and the process obtained from the Langevin equation with FBM as the driving process are different. However, also the first one of these can be described as a solution of a Langevin equation but now with some other driving process than FBM. We are mainly interested in the properties of this new driving process denoted $Y^{(1)}$. We also study the solution of the Langevin equation with $Y^{(1)}$ as the driving process. Moreover, we show that the covariance of $Y^{(1)}$ grows linearly; hence, in this respect $Y^{(1)}$ is more like a standard Brownian motion than a FBM. In fact, it is proved that a properly scaled version of $Y^{(1)}$ converges weakly to Brownian motion.

1. Introduction

It is well known that the Ornstein-Uhlenbeck diffusion $U = \{U_t\,;\, t \geq 0\}$ can be constructed as the unique strong solution of the Langevin SDE

$$dU_t = -\alpha U_t\, dt + dB_t, \tag{1.1}$$

where $\alpha > 0$ and $B = \{B_t : t \geq 0\}$ is a standard Brownian motion initiated from 0. The solution of (1.1) can be expressed as

$$U_t = \mathrm{e}^{-\alpha t}\left(x + \int_0^t \mathrm{e}^{\alpha s}\, dB_s\right), \tag{1.2}$$

where x is the (random) initial value of U (independent of B).

Letting $B^{(-)} = \{B_t^{(-)} : t \geq 0\}$ be another standard Brownian motion initiated from 0 and independent of B. Introduce for $t \in \mathbf{R}$

$$\widehat{B}_t := \begin{cases} B_t\,, & t \geq 0, \\ B_{-t}^{(-)}\,, & t \leq 0. \end{cases}$$

It is easily seen that

$$\xi := \int_{-\infty}^0 \mathrm{e}^{\alpha s}\, d\widehat{B}_s$$

Received 2010-1-18.
2000 *Mathematics Subject Classification.* 60G15, 60H05, 60G18.
Key words and phrases. Fractional Brownian motion, fractional Ornstein-Uhlenbeck process, long range dependence, short range dependence, covariance kernel, weak convergence.

is a normally distributed random variable with mean 0 and variance $1/(2\alpha)$.

Choosing $x = \xi$ in (1.2) we may write the process with $t \in \mathbf{R}$ as

$$U_t = \mathrm{e}^{-\alpha t} \int_{-\infty}^{t} \mathrm{e}^{\alpha s}\, d\widehat{B}_s$$

and this describes a stationary solution of (1.1).

There is also another well known construction of the Ornstein-Uhlenbeck diffusion. This is due to Doob [5] and expresses the stationary Ornstein-Uhlenbeck diffusion U (with time axis the whole $\mathbf{R}$) as a deterministic time change of a standard Brownian motion:

$$U_t = \mathrm{e}^{-\alpha t} B_{a_t}, \quad t \in \mathbf{R}, \tag{1.3}$$

where $\alpha > 0$ and $a_t := \mathrm{e}^{2\alpha t}/2\alpha$. The covariance of U is easily obtained from (1.3)

$$\mathbf{E}\,(U_t\, U_s) = \frac{1}{2\alpha}\,\mathrm{e}^{-\alpha(t-s)}, \quad t \geq s. \tag{1.4}$$

In this note we study fractional Ornstein-Uhlenbeck processes. These are processes constructed as U above but now the Brownian motion is replaced with the fractional Brownian motion (FBM). It is known that the process obtained as the solution of the Langevin SDE with FBM as the driving process does not coincide with the process obtained as Doob's transform of FBM. In Cheridito et al. [3] it is proved that the covariance of the former one behaves like the covariance of the increment process of FBM. In particular, if the Hurst parameter H is bigger than $1/2$ the process is long range dependent. On the other hand, the covariance of Doob's transform[1] of FBM decays exponentially and, hence, the process is short range dependent for all values of $H \in (0,1)$. Our main contribution in this paper is to extract from Doob's transform the driving process, to study its properties and use it in the Langevin SDE to generate new kind of fractional Ornstein-Uhlenbeck processes.

In the next section we discuss the basic properties of FBM important for our purposes. To make the paper more readable, we also recall some results from [3]. In the main section of the paper the new driving process is constructed and the solution of the associated Langevin SDE is introduced. The covariance of the driving process and also the covariance of the solution have kernel representations in case $H > 1/2$. It is proved then that the driving process and the solution are short range dependent. Moreover, it is seen that it is possible to scale the driving process so that it converges weakly to a Brownian motion as the scaling parameter tends to infinity.

2. Preliminaries

2.1. Fractional Brownian motion. Let $Z = \{Z_t : t \geq 0\}$ be *a fractional Brownian motion,* FBM, with self-similarity (or Hurst) parameter $H \in (0,1)$, that is, Z is a centered Gaussian process with the covariance function

$$\mathbf{E}(Z_t\, Z_s) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{2.1}$$

[1] In [3] this transform is called Lamperti's transform (see Lamperti [9]).

Notice that

$$\mathbf{E}(Z_0^2) = 0 \quad \text{and} \quad \mathbf{E}(Z_1^2) = 1,$$

and, hence, in particular $Z_0 = 0$. Using Kolmogorov's continuity criterion it can be proved that Z has a continuous version; therefore, we take Z to be continuous. In fact, Z is locally Hölder continuous of exponent α for all $\alpha < H$.

Fractional Brownian motion is H-self-similar in the sense

$$\{Z_{\alpha t} : t \geq 0\} \stackrel{d}{=} \{\alpha^H Z_t : t \geq 0\} \quad \text{for all } \alpha > 0, \tag{2.2}$$

where $\stackrel{d}{=}$ means that the right hand side and the left hand side are identical in law. This follows from (2.1) because the covariance function determines a mean zero Gaussian distribution uniquely. Moreover, from (2.1), for $t_2 > t_1 > s_2 > s_1$

$$\begin{aligned} &\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) \\ &\qquad = \frac{1}{2}\left((t_2 - s_1)^{2H} - (t_1 - s_1)^{2H} - (t_2 - s_2)^{2H} + (t_1 - s_2)^{2H}\right), \end{aligned} \tag{2.3}$$

and, consequently, the increments of Z are

- positively correlated if $H > 1/2$,
- negatively correlated if $H < 1/2$.

Consider now the increment process of Z defined as

$$I_Z := \{Z_{n+1} - Z_n \,:\, n = 0, 1, 2, \dots\}.$$

It is easily seen that I_Z is a stationary second order stochastic process and, from (2.3),

$$\rho_{I_Z}(n) := \mathbf{E}\left(Z_1(Z_{n+1} - Z_n)\right) = H(2H-1)n^{-2(1-H)} + O(n^{2H-3}). \tag{2.4}$$

Next we recall the following definition (see Beran [1] p. 6 and 42).

Definition 2.1. Let $X = \{X_n : n = 0, 1, 2, \dots\}$ be a stationary second order stochastic process with mean zero and set $\rho_X(n) := \mathbf{E}\left(X_i X_{i+n}\right)$, where i is an arbitrary non-negative integer (by stationarity, $\rho_X(n)$ does not depend on i). Then X is called

(1) *long range dependent* if there exist $\alpha \in (0,1)$ and a constant $C > 0$ such that $\lim_{n\to\infty} \rho_X(n)/(C\, n^{-\alpha}) = 1$,

(2) *short range dependent* if $\lim_{k\to\infty} \sum_{n=0}^{k} \rho_X(n)$ exists.

From Definition 2.1 and formula (2.4) it follows that the increment process I_Z of the fractional Brownian motion Z is

- long range dependent if $H > 1/2$,
- short range dependent if $H < 1/2$.

2.2. Fractional Ornstein-Uhlenbeck processes of the first kind. We replace now the Brownian motion B in (1.1) with the fractional Brownian motion Z, and consider the SDE

$$dU_t^{(Z,\alpha)} = -\alpha U_t^{(Z,\alpha)}\, dt + dZ_t. \tag{2.5}$$

Analogously with (1.2), the solution can be expressed as

$$U_t^{(Z,\alpha)}(x) = \mathrm{e}^{-\alpha t}\left(x + \int_0^t \mathrm{e}^{\alpha s}\, dZ_s\right) \tag{2.6}$$

with some (random) initial value x. The stochastic integral exists pathwise as a Riemann-Stieltjes integral (see Cheridito et al. [3]) and it holds

$$\int_0^s \mathrm{e}^{\alpha u}\, dZ_u = \mathrm{e}^{\alpha s} Z_s - \int_0^s \alpha \mathrm{e}^{\alpha u} Z_u\, du. \tag{2.7}$$

Furthermore, we introduce $\widehat{Z}$, a two-sided fractional Brownian motion through 0, and consider

$$\xi := \int_{-\infty}^0 \mathrm{e}^{\alpha s}\, d\widehat{Z}_s. \tag{2.8}$$

To see that ξ is well-defined notice first that we may extend (2.7) for negative values on s

$$\int_s^0 \mathrm{e}^{\alpha u}\, d\widehat{Z}_u = -\mathrm{e}^{\alpha s}\widehat{Z}_s - \int_s^0 \alpha \mathrm{e}^{\alpha u}\widehat{Z}_u\, du. \tag{2.9}$$

To prove that the limit of the r.h.s. of (2.9) as $s \to -\infty$ exists we remark that

$$\left\{ Z_t^{(o)} := t^{2H}\widehat{Z}_{-1/t},\, t > 0 \right\}$$

is a centered Gaussian process with the same covariance kernel as FBM. Consequently, $\left\{ Z_t^{(o)},\, t > 0 \right\}$ is identical in law with $\left\{ \widehat{Z}_{-t},\, t > 0 \right\}$ and it holds

$$\lim_{t\to 0+} Z_t^{(o)} = \lim_{t\to 0+} \widehat{Z}_{-t} = 0 \quad \text{a.s.}$$

Consequently

$$\lim_{s\to -\infty} \frac{\widehat{Z}_s}{|s|^{2H}} = \lim_{t\to 0+} t^{2H}\widehat{Z}_{-1/t} = \lim_{t\to 0+} Z_t^{(o)} = 0. \tag{2.10}$$

Hence the limit of the r.h.s. of (2.9) exists and ξ is well-defined. The fact

$$\lim_{s\to -\infty} \frac{\widehat{Z}_s}{|s|^{2H}} = 0$$

can also be seen as the strong law of large numbers of FBM and proved via the Borel-Cantelli lemma. For other proofs that ξ is well-defined, we refer to Garrido-Atienza et al. [6] and Maslowski and Schmalfuss [10]. Taking in (2.6) $x = \xi$ we write the solution in the form

$$U_t^{(Z,\alpha)} = \mathrm{e}^{-\alpha t} \int_{-\infty}^t \mathrm{e}^{\alpha s}\, d\widehat{Z}_s. \tag{2.11}$$

Since the increments of Z are stationary and the stochastic integral is a Riemann-Stieltjes integral it follows that the process $U^{(Z,\alpha)}$ is stationary. The stationary probability distribution, i.e., the distribution of ξ, is normal with mean 0 and variance (see Cheridito et al. [3])

$$\frac{\Gamma(2H+1)\sin(\pi H)}{\pi}\, \alpha^{-2H} \int_0^{+\infty} \frac{|x|^{1-2H}}{1+x^2}\, dx.$$

In case $H = 1/2$, the variance equals $1/2\alpha$, as it should.

Definition 2.2. The process $U^{(Z,\alpha)}$ given in (2.11) is called the *stationary fractional Ornstein-Uhlenbeck process of the first kind*.

Next we recall the asymptotic formula for the covariance of $U^{(Z,\alpha)}$ taken from [3] Theorem 2.3., which is then applied to derive the range dependence properties of $U^{(Z,\alpha)}$.

Proposition 2.3. *Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$ and $N = 1, 2, \ldots$. Then for fixed $s \in \mathbb{R}$ and $t \to \infty$,*

$$\mathbf{E}(U_s^{(Z,\alpha)} U_{t+s}^{(Z,\alpha)}) = \frac{1}{2} \sum_{n=1}^{N} \alpha^{-2n} \left(\prod_{k=0}^{2n-1} (2H - k) \right) t^{2H-2n} + O(t^{2H-2N-2}). \qquad (2.12)$$

Proposition 2.4. *The stationary sequence $\{U_n^{(Z,\alpha)} : n = 1, 2, \ldots\}$ (and, equivalently, the process $U^{(Z,\alpha)}$) is long range dependent when $H > 1/2$, and short range dependent when $H < 1/2$.*

Proof. The leading term of the sum in (2.12) is of the order t^{2H-2}. Consequently,

$$\sum_{n=0}^{\infty} |\rho_{U^{(Z,\alpha)}}(n)| = \sum_{n=0}^{\infty} |\mathbf{E}(U_i^{(Z,\alpha)} U_{i+n}^{(Z,\alpha)})| \simeq \sum_{n=0}^{\infty} n^{2H-2},$$

which, by Definition 2.1, gives the claim. □

3. Fractional Ornstein-Uhlenbeck Processes of the Second Kind

3.1. Definition and some basic properties. In this section we derive from Doob's transform of Z a Gaussian process with stationary increments. This process is used as the driving process in the Langevin SDE. In this way we construct a new family of Gaussian processes which we call fractional Ornstein-Uhlenbeck processes of the second kind. This terminology can be justified by observing that in the standard Brownian case, i.e., $H = 1/2$, these processes coincide with the Ornstein-Uhlenbeck diffusions; as also do the fractional Ornstein-Uhlenbeck processes of the first kind introduced in Definition 2.2.

Doob's transform of Z is the process given by

$$X_t^{(D,\alpha)} := \mathrm{e}^{-\alpha t} Z_{a_t}, \quad t \in \mathbf{R}, \qquad (3.1)$$

where $\alpha > 0$ and $a_t := a(t, H) := H\,\mathrm{e}^{\alpha t/H}/\alpha$. The covariance of X can be computed from (2.1). Indeed, for $t > s$ we have

$$\mathbf{E}(X_t^{(D,\alpha)} X_s^{(D,\alpha)}) = \frac{1}{2} \left(\frac{H}{\alpha} \right)^{2H} \left(\mathrm{e}^{\alpha(t-s)} + \mathrm{e}^{-\alpha(t-s)} - \mathrm{e}^{\alpha(t-s)} \left(1 - \mathrm{e}^{-\frac{\alpha(t-s)}{H}} \right)^{2H} \right). \qquad (3.2)$$

Since $X^{(D,\alpha)}$ is a Gaussian process it follows herefrom that it is stationary. In particular, using the self-similarity property of the fractional Brownian motion (see (2.2)) it is seen that $X_t^{(D,\alpha)}$ is for all t normally distributed with mean 0 and variance $(H/\alpha)^{2H}$.

Proposition 3.1. *The stationary process* $\{X_t^{(D,\alpha)} : t \in \mathbf{R}\}$ *is, for all* $H \in (0,1)$, *short range dependent.*

Proof. Formula (3.2) yields for a fixed s as $t \to \infty$

$$\mathbf{E}(X_t^{(D,\alpha)} X_s^{(D,\alpha)}) = O\left(\exp\left(-\alpha\, t(1-H)/H\right)\right), \tag{3.3}$$

and this implies the result. □

Consider now the process $Y^{(\alpha)}$ defined via

$$Y_t^{(\alpha)} := \int_0^t \mathrm{e}^{-\alpha s} dZ_{a_s}, \tag{3.4}$$

where the integral is a (pathwise) Riemann-Stiltjes integral (cf. Section 2.2). It is possible to represent $Y_t^{(\alpha)}$ as Volterra process w.r.t the Brownian motion, by using e.g. L. Decreusefond and A.S. Üstünel [4]. In case $H = 1/2$, $Y^{(\alpha)}$ is, for all α, by Lévy's theorem a standard Brownian motion. Using $Y^{(\alpha)}$ the process $X^{(D,\alpha)}$ can be viewed as the solution of the equation

$$dX_t^{(D,\alpha)} = -\alpha X_t^{(D,\alpha)} dt + dY_t^{(\alpha)}, \tag{3.5}$$

with the random initial value $X_0^{(0,\alpha)} = Z_{a_0} = Z_{\frac{H}{\alpha}} \sim N(0, \left(\frac{H}{\alpha}\right)^{2H})$.

Proposition 3.2. *For all* $\alpha > 0$, *we have*

$$\{\alpha^H Y_{t/\alpha}^{(\alpha)} : t \geq 0\} \overset{d}{=} \{Y_t^{(1)} : t \geq 0\}. \tag{3.6}$$

Moreover, the process $Y^{(\alpha)}$ *has stationary increments.*

Proof. Integrating by parts we obtain

$$Y_t^{(\alpha)} = \int_0^t \mathrm{e}^{-\alpha s} dZ_{a_s} = \mathrm{e}^{-\alpha t} Z_{a_t} - Z_{a_0} + \alpha \int_0^t \mathrm{e}^{-\alpha s} Z_{a_s}\, ds \tag{3.7}$$

Using (2.2) – the self-similary property of FBM – the claimed identity in law (3.6) follows from (3.7). Moreover, the equality

$$\mathbf{E}\left(\left(Y_{t_2}^{(\alpha)} - Y_{t_1}^{(\alpha)}\right)\left(Y_{s_2}^{(\alpha)} - Y_{s_1}^{(\alpha)}\right)\right) = \mathbf{E}\left(\left(Y_{t_2+h}^{(\alpha)} - Y_{t_1+h}^{(\alpha)}\right)\left(Y_{s_2+h}^{(\alpha)} - Y_{s_1+h}^{(\alpha)}\right)\right)$$

holds for $t_2 > t_1 > s_2 > s_1 > 0$ and $h > 0$ again by the self similarity of FBM and exploiting (3.7). Consequently, the increments of $Y^{(\alpha)}$ are stationary. □

Inspired by Proposition 3.2, we consider the Langevin SDE with $Y^{(1)}$ as the driving process:

$$dU_t^{(D,\gamma)} = -\gamma U_t^{(D,\gamma)} dt + dY_t^{(1)}, \quad \gamma > 0. \tag{3.8}$$

The solution can be expressed (cf. (2.11)) as

$$U_t^{(D,\gamma)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^t \mathrm{e}^{\gamma s}\, d\widehat{Y}_s^{(1)} = \mathrm{e}^{-\gamma t} \int_{-\infty}^t \mathrm{e}^{(\gamma-1)s} dZ_{a_s}, \quad \gamma > 0, \tag{3.9}$$

where $\widehat{Y}^{(1)}$ stands for the two sided $Y^{(1)}$ process and $\alpha = 1$ in a_t. To show that a stochastic integral term makes sense also for $\gamma \in (0,1]$ recall first that for all $\beta < H$

$$\lim_{s \to 0} Z_s / |s|^\beta = 0 \quad \text{a.s.} \tag{3.10}$$

because Z is Hölder continuous of order $\beta < H$. Next for $T < 0$ using partial integration

$$\int_T^s \mathrm{e}^{(\gamma-1)u}\, dZ_{a_u} = \mathrm{e}^{(\gamma-1)s} Z_{a_s} - \mathrm{e}^{(\gamma-1)T} Z_{a_T} - (\gamma-1)\int_T^s \mathrm{e}^{(\gamma-1)u} Z_{a_u}\, du,$$

and by (3.10) the right hand side has a well defined limit as $T \to -\infty$.

Since the increment process of $Y^{(1)}$ is stationary it follows that $U^{(D,\gamma)}$ is stationary and, therefore, we have well justified the following

Definition 3.3. The process $U^{(D,\gamma)}$ defined in (3.9) or, equivalently, via the SDE (3.8) is called *the fractional Ornstein-Uhlenbeck process of the second kind.*

We conclude this section by characterizing the Hölder continuity of $Y^{(\alpha)}$ and $U^{(D,\gamma)}$. The result holds for more general stochastic integrals with respect to Z (see Zähle [12]), but the following simple proof in our special case is perhaps worthwhile to present here.

Proposition 3.4. *The sample paths of $Y^{(\alpha)}$ and $U^{(D,\gamma)}$ are (locally) Hölder continuous of order $\beta < H$.*

Proof. From (3.5) we have

$$Y_t^{(\alpha)} = X_t^{(D,\alpha)} - X_0^{(D,\alpha)} + \int_0^t \alpha X_s^{(D,\alpha)}\, ds. \tag{3.11}$$

Consequently, $t \mapsto Y_t^{(\alpha)}$ is continuous and the Hölder continuity properties of $Y^{(\alpha)}$ and $X^{(D,\alpha)}$ are the same. Hence, let $T > 0$ be given and consider for $s, t < T$ and $\beta > 0$

$$\frac{\left|X_t^{(D,\alpha)} - X_s^{(D,\alpha)}\right|}{|t-s|^{\beta}} = \frac{|\mathrm{e}^{-\alpha t} Z_{a_t} - \mathrm{e}^{-\alpha s} Z_{a_s}|}{|t-s|^{\beta}} \le K_T \frac{|Z_{a_t} - Z_{a_s}|}{|a_t - a_s|^{\beta}} + C_T,$$

where K_T and C_T are (random) constants which do not depend on s and t. The claim follows now from the fact that the paths of FBM are (locally) Hölder continuous of order $\beta < H$. Similarly, for the process $U^{(D,\gamma)}$ we have, e.g., for $t > 0$

$$U_t^{(D,\gamma)} - \mathrm{e}^{-\gamma t} U_0^{(D,\gamma)} = Y_t^{(1)} - \gamma \mathrm{e}^{-\gamma t} \int_0^t \mathrm{e}^{\gamma s}\, Y_s^{(1)}\, ds,$$

and it follows that also $U^{(D,\gamma)}$ is Hölder continuous of order $\beta < H$. □

3.2. Kernel representations of covariances and short range dependence. We make now the following assumption valid throughout the rest of the paper

$$1/2 < H < 1.$$

In this case, as is easily checked, the covariance of the fractional Brownian motion has for $t_2 > t_1$ and $s_2 > s_1$ the kernel representation

$$\mathbf{E}\left((Z_{t_2} - Z_{t_1})(Z_{s_2} - Z_{s_1})\right) = \int_{t_1}^{t_2} \int_{s_1}^{s_2} H(2H-1)|u-v|^{2H-2}\, du\, dv.$$

In the next proposition we derive an analogous representation for the process $Y^{(1)}$. The result is formulated for all values of $\alpha > 0$.

Proposition 3.5. *The covariance of* $Y^{(\alpha)}$ *with* $1/2 < H < 1$ *has the kernel representation*

$$\mathbf{E}\left(\left(Y^{(\alpha)}_{t_2} - Y^{(\alpha)}_{t_1}\right)\left(Y^{(\alpha)}_{s_2} - Y^{(\alpha)}_{s_1}\right)\right) = C(\alpha,H)\int_{t_1}^{t_2}\int_{s_1}^{s_2} \frac{\mathrm{e}^{-\alpha(1-H)(u-v)/H}}{|1-\mathrm{e}^{-\alpha(u-v)/H}|^{2(1-H)}}\,du\,dv, \tag{3.12}$$

where $t_2 > t_1$, $s_2 > s_1$, *and*

$$C(\alpha,H) := H(2H-1)\left(\frac{\alpha}{H}\right)^{2(1-H)}.$$

The kernel

$$r_{\alpha,H}(u,v) := C(\alpha,H)\,\frac{\mathrm{e}^{-\alpha(1-H)(u-v)/H}}{|1-\mathrm{e}^{-\alpha(u-v)/H}|^{2(1-H)}} \tag{3.13}$$

is symmetric, i.e., $r_{\alpha,H}(u,v) = r_{\alpha,H}(v,u)$ *for all* $u,v\in\mathbf{R}$.

Proof. Recall the formula (see Gripenberg and Norros [7] Proposition 2.2)

$$\mathbf{E}\left(\int_{\mathbf{R}} f(s)dZ_s \int_{\mathbf{R}} g(t)dZ_t\right) = H(2H-1)\int_{\mathbf{R}}\int_{\mathbf{R}} f(s)g(t)|s-t|^{2H-2}\,dtds, \tag{3.14}$$

where $1/2 < H < 1$ and $f,g\in\mathbf{L}^2(\mathbf{R})\cap\mathbf{L}^1(\mathbf{R})$. Since

$$Y^{(\alpha)}_t := \int_0^t \mathrm{e}^{-\alpha s} dZ_{a_s} = \left(\frac{H}{\alpha}\right)^H \int_{a_0}^{a_t} s^{-H}\,dZ_s,$$

the claim follows by a straightforward application of (3.14). □

Remark 3.6. Notice that the kernel $r_{\alpha,H}$ is in $\mathbf{L}^2([0,T]\times[0,T])$ if and only if $H > 3/4$. Consequently, for $Y^{(1)}$ we have similar absolute continuity properties as for fractional Brownian motion (see Cheridito [2]). Namely, the measure induced by the process $\{B_t + Y^{(1)}_t : t\ge 0\}$, where $Y^{(1)}$ and the Brownian motion B are assumed to be independent, is absolutely continuous with respect to the Wiener measure.

For the next result, recall from Proposition 3.2 that the increments of $Y^{(\alpha)}$ are stationary.

Corollary 3.7. *The increments of* $Y^{(\alpha)}$ *are positively correlated. The increment process* $I_Y := \{Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n; n = 0,1,\dots\}$ *is stationary and short range dependent.*

Proof. From (3.12) it follows immediately that the increments are positively correlated. Of course, we may also deduce from (3.12) the stationarity of the increments of $Y^{(\alpha)}$. To show that I_Y is short range dependent consider

$$\begin{aligned}\mathbf{E}\left(Y^{(\alpha)}_1(Y^{(\alpha)}_{n+1} - Y^{(\alpha)}_n)\right) &= \int_n^{n+1} du \int_0^1 dv\ r_{\alpha,H}(u,v)\\ &= C(\alpha,H)\mathrm{e}^{-\alpha(1-H)n/H}\\ &\qquad\times\int_0^1 du\int_0^1 dv\,\mathrm{e}^{-\alpha(1-H)(u-v)/H}\,|1-\mathrm{e}^{-\alpha n/H}\,\mathrm{e}^{-\alpha(u-v)/H}|^{2(H-1)}.\end{aligned}$$

The integral term has a positive finite limit as $n \to \infty$. Indeed, Lebesgue's dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^1 du \int_0^1 dv\, \mathrm{e}^{-\alpha(1-H)(u-v)/H}\, |1 - \mathrm{e}^{-\alpha n/H}\, \mathrm{e}^{-\alpha(u-v)/H}|^{2(H-1)} = \int_0^1 du \int_0^1 dv\, \mathrm{e}^{-\alpha(1-H)(u-v)/H}.$$

Consequently,

$$\rho_{Y^{(\alpha)}}(n) := \mathbf{E}\left(Y_1^{(\alpha)}(Y_{n+1}^{(\alpha)} - Y_n^{(\alpha)})\right) = O\left(\mathrm{e}^{-\alpha(1-H)n/H}\right) \tag{3.15}$$

and, hence,

$$\lim_{N\to\infty} \mathbf{E}\left(Y_N^{(\alpha)} Y_1^{(\alpha)}\right) = \sum_{n=0}^{\infty} \rho_{Y^{(\alpha)}}(n) < +\infty \tag{3.16}$$

completing the proof. □

Next we study the asymptotic behaviour of the variance and covariance of $Y^{(\alpha)}$. For this, it is practical to rewrite the symmetric kernel $r_{\alpha,H}$ in (3.13) as

$$r_{\alpha,H}(t,s) = k_{\alpha,H}(t-s)$$

with

$$k_{\alpha,H}(x) := C(\alpha,H)\, \mathrm{e}^{-\alpha(1-H)x/H}\, |1 - \mathrm{e}^{-\alpha x/H}|^{2H-2}. \tag{3.17}$$

Proposition 3.8. *The following formulas hold:*

$$\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) = 2\int_0^{t-s} (t-s-x)\, k_{\alpha,H}(x)\, dx, \tag{3.18}$$

$$\mathbf{E}\left(Y_t^{(\alpha)} Y_s^{(\alpha)}\right) = \int_0^t (t-x)\, k_{\alpha,H}(x)\, dx + \int_0^s (s-x)\, k_{\alpha,H}(x)\, dx - \int_0^{t-s} (t-s-x) k_{\alpha,H}(x)\, dx. \tag{3.19}$$

Moreover,

$$\mathbf{E}\left((Y_t^{(\alpha)})^2\right) = O(t) \ \text{ as } t \to \infty, \tag{3.20}$$

and

$$\lim_{t\to\infty} \mathbf{E}(Y_t^{(\alpha)} Y_s^{(\alpha)}) = s\int_0^{\infty} k_{\alpha,H}(x) dx + \int_0^s (s-x)\, k_{\alpha,H}(x) dx. \tag{3.21}$$

Proof. We apply (3.12) to obtain (3.18):

$$\begin{aligned}\mathbf{E}\left((Y_t^{(\alpha)} - Y_s^{(\alpha)})^2\right) &= \int_s^t du \int_s^t dv\, r_{\alpha,H}(u,v) \\ &= 2\int_s^t du \int_s^u dv\, r_{\alpha,H}(u,v) \\ &= 2\int_0^{t-s} (t-s-x)\, k_{\alpha,H}(x)\, dx.\end{aligned}$$

Putting here $s = 0$ and using

$$\int_0^\infty k_{\alpha,H}(x)\,dx < \infty \quad \text{and} \quad \int_0^\infty x\,k_{\alpha,H}(x)\,dx < \infty$$

yield (3.20). Furthermore, straightforward computations produce formula (3.19) from (3.18). Likewise formula (3.21) is obtain fairly easily. We omit the details. □

Remark 3.9. The short range dependence property of $Y^{(\alpha)}$ also follows from (3.21) since (recall that $Y_0^{(\alpha)} = 0$)

$$\sum_{n=0}^{\infty} \rho_{Y^{(\alpha)}}(n) = \lim_{N\to\infty} \mathbf{E}\left(Y_N^{(\alpha)} Y_1^{(\alpha)}\right) < +\infty.$$

Proposition 3.10. *The covariance of* $U^{(D,\gamma)}$ *has the kernel representation*

$$\begin{aligned}&\mathbf{E}\left(U_t^{(D,\gamma)} U_s^{(D,\gamma)}\right)\\ &\quad = H(2H-1)H^{2H-2}\,\mathrm{e}^{-\gamma(t+s)} \int_{-\infty}^{t}\int_{-\infty}^{s} \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{u/H}-\mathrm{e}^{v/H}|^{2(1-H)}}\,du\,dv.\end{aligned}$$

Proof. As in the proof of Proposition 3.5, we use also here formula (3.14). However, now we need an extended version due to Pipiras and Taqqu [11] stating that (3.14) holds true for functions f and g satisfying

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |f(s)||g(t)||s-t|^{2H-2}\,dtds < \infty. \tag{3.22}$$

Consider

$$\begin{aligned} U_t^{(D,\gamma)} &= \mathrm{e}^{-\gamma t}\int_{-\infty}^{t} \mathrm{e}^{(\gamma-1)s} dZ_{a_s} \qquad (3.23)\\ &= H^{-(\gamma-1)H}\,\mathrm{e}^{-\gamma t}\int_0^{a_t} s^{(\gamma-1)H}\,dZ_s, \end{aligned}$$

where we have made the change of variable $s = H\mathrm{e}^{s/H}$.

To check that condition (3.22) is valid for $f(s) = g(s) = s^{(\gamma-1)H}\mathbf{1}_{(0,a_t)}(s)$ it is enough to consider

$$\begin{aligned}&\int_0^{a_t}\int_0^{a_t} (pr)^{(\gamma-1)H}|p-r|^{2H-2}\,dpdr\\ &\quad = \int_0^1\int_0^1 (a_t u a_t v)^{(\gamma-1)H}|a_t u - a_t v|^{2H-2}\,dudv\\ &\quad = 2a_t^{2\gamma H}\int_0^1 du\,u^{(\gamma-1)H}\int_0^u dv\,v^{(\gamma-1)H}(u-v)^{2H-2}\\ &\quad = 2a_t^{2\gamma H}\int_0^1 du\,u^{(\gamma-1)H}u^{(\gamma+1)H-1}\mathrm{Beta}(1+(\gamma-1)H, 2H-1)\\ &\quad = \frac{a_t^{2\gamma H}}{\gamma H}\,\mathrm{Beta}(1+(\gamma-1)H, 2H-1) < \infty,\end{aligned}$$

where Beta stands for the Beta function, i.e.

$$\mathrm{Beta}(a,b) = \int_0^1 x^{a-1}(1-x)^{b-1}dx,\ a>0,\ b>0.$$

To verify the claimed kernel representation is now a straightforward computation using formula (3.14). □

Recall from Corollary 3.7 that the increment process of $Y^{(1)}$ is short range dependent, and that if $Y^{(1)}$ is used as the driving process in the Langevin equation the solution is the process $U^{(D,\gamma)}$. In the next proposition we show that also $U^{(D,\gamma)}$ is short range dependent. Formula (3.24) can be compared with the corresponding formula (3.3) for $X^{(D,\alpha)}$. In fact, (3.3) with $\alpha = 1$ is (3.24) with $\gamma = 1$, as it should.

Proposition 3.11. *The rate of decay of the covariance of $U^{(D,\gamma)}$ is exponential. More precisely,*

$$\mathbf{E}\left(U_t^{(D,\gamma)}U_s^{(D,\gamma)}\right) = O\left(\exp\left(-\min\{\gamma,(1-H)/H\}\,t\right)\right), \quad \text{as } t\to\infty. \tag{3.24}$$

In particular, the stationary process $U^{(D,\gamma)}$ is short range dependent.

Proof. Without loss of generality, we may take $s=0$ and, hence, consider

$$\begin{aligned}\mathbf{E}&\left(U_t^{(D,\gamma)}U_0^{(D,\gamma)}\right)\\ &= H(2H-1)H^{2H-2}\,\mathrm{e}^{-\gamma t}\int_{-\infty}^t\int_{-\infty}^0 \frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{u/H}-\mathrm{e}^{v/H}|^{2(1-H)}}\,du\,dv\\ &= \Delta_1(t)+\Delta_2(t),\end{aligned}$$

where, for some fixed $T>0$,

$$\Delta_1(t) := H(2H-1)H^{2H-2}\,\mathrm{e}^{-\gamma t}\int_{-\infty}^T du\int_{-\infty}^0 dv\,\frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{u/H}-\mathrm{e}^{v/H}|^{2(1-H)}}$$

and

$$\Delta_2(t) := H(2H-1)H^{2H-2}\,\mathrm{e}^{-\gamma t}\int_{T}^t du\int_{-\infty}^0 dv\,\frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{u/H}-\mathrm{e}^{v/H}|^{2(1-H)}}.$$

Clearly,

$$\Delta_1(t) = O\left(\exp(-\gamma\, t)\right) \quad \text{as } t\to+\infty.$$

For the integral term in $\Delta_2(t)$ we have

$$\int_T^t du\int_{-\infty}^0 dv\,\frac{\mathrm{e}^{(\gamma-1+\frac{1}{H})(u+v)}}{|\mathrm{e}^{u/H}-\mathrm{e}^{v/H}|^{2(1-H)}} = \int_T^t du\int_{-\infty}^0 dv\,\frac{\mathrm{e}^{(\gamma+1-\frac{1}{H})\,u}\,\mathrm{e}^{(\gamma-1+\frac{1}{H})\,v}}{\left(1-\mathrm{e}^{(v-u)/H}\right)^{2(1-H)}}.$$

For $(u,v)\in(T,t)\times(-\infty,0)$

$$1 \le \left(1-\mathrm{e}^{(v-u)/H}\right)^{2(1-H)} \le \left(1-\mathrm{e}^{-T/H}\right)^{2(1-H)},$$

and, consequently, formula (3.24) holds. □

3.3. Weak convergence of $Y^{(1)}$ to Brownian motion. In Proposition 3.8 it is proved that the growth of the variance of $Y_t^{(1)}$ is asymptotically linear as $t \to +\infty$ (see (3.20)). This suggests that $Y^{(1)}$, when properly scaled, behaves asymptotically like a standard Brownian motion. We give the precise statement in the next proposition formulated for arbitrary $\alpha > 0$.

Proposition 3.12. *For $a > 0$ define*

$$Z_t^{(a,\alpha)} := \frac{1}{\sqrt{a}} Y_{at}^{(\alpha)}, \ t \geq 0,$$

and let $B = \{B_t : t \geq 0\}$ denote a standard Brownian motion started from 0. Then as $a \to +\infty$

$$\{Z_t^{(a,\alpha)} : t \geq 0\} \overset{\text{weakly}}{\Rightarrow} \{\sigma B_t : t \geq 0\},$$

where $\overset{\text{weakly}}{\Rightarrow}$ stands for weak convergence in the space of continuous functions and $\sigma = \sigma(\alpha, H)$ is a non-random quantity depending only on α and H (see (3.25)).

Proof. We show first that the finite dimensional distributions of $Z^{(a,\alpha)}$ converge to the finite dimensional distributions of σB. Since $Z^{(a,\alpha)}$ is a Gaussian process with mean zero it is enough to verify the convergence of the covariance function. From (3.19) in Proposition 3.8 we have for $t > s$

$$\begin{aligned}
\mathbf{E}\left(Z_t^{(a,\alpha)} Z_s^{(a,\alpha)}\right) &= \frac{1}{a}\mathbf{E}\left(Y_{at}^{(\alpha)} Y_{as}^{(\alpha)}\right) \\
&= \frac{1}{a}\Big(\int_0^{at} (at - x)\, k_{\alpha,H}(x)\, dx + \int_0^{as} (as - x)\, k_{\alpha,H}(x)\, dx \\
&\qquad - \int_0^{a(t-s)} (at - as - x) k_{\alpha,H}(x)\, dx \Big)
\end{aligned}$$

with $k_{\alpha,H}$ defined in (3.17). Letting here $a \to +\infty$ yields, after some simple computations,

$$\lim_{a\to\infty} \mathbf{E}\left(Z_t^{(a,\alpha)} Z_s^{(a,\alpha)}\right) = 2s \int_0^{\infty} k_{\alpha,H}(x)\, dx = \kappa(\alpha, H)\, s,$$

where

$$\kappa(\alpha, H) := 2\, C(\alpha, H)\, \frac{H}{\alpha}\, \mathrm{Beta}(1 - H, 2H - 1).$$

Since $\mathbf{E}(B_t B_s) = s$ for $t > s$ we have proved the convergence of finite dimensional distributions of $Z^{(a,\alpha)}$ of the finite dimensional distributions of σB with

$$\sigma = \sigma(\alpha, H) = \sqrt{\kappa(\alpha, H)}. \tag{3.25}$$

To prove tightness, it is enough to verify (see, e.g., Lamperti [8]) that there exists a constant C (which might depend on α and H) such that for all $a > 0$ and $t > s$

$$\Delta := \mathbf{E}\left(\left(Z_t^{(a,\alpha)} - Z_s^{(a,\alpha)}\right)^2\right) \leq C\,(t - s).$$

We have by formula (3.18) in Proposition 3.8

$$\Delta = \frac{1}{a}\,\mathbf{E}\left(\left(Y^{(\alpha)}_{at} - Y^{(\alpha)}_{as}\right)^2\right)$$

$$= 2\,\frac{1}{a}\int_0^{a(t-s)} (a(t-s)-x)\,k_{\alpha,H}(x)\,dx \le C\,(t-s)$$

with, e.g., $C = 2\int_0^\infty k_{\alpha,H}(x)\,dx$. This completes the proof. □

Acknowledgement. We thank Zhan Shi, Esko Valkeila and Marc Yor for discussions and comments on an early version of this paper. Thanks also to the anonymous referees for comments which improved the paper.

References

1. Beran, J.: *Statistics for Long Memory Processes*, Chapman & Hall, New York, 1994.
2. Cheridito, P.: Representations of Gaussian measures that are equivalent to Wiener measure. In Azéma J., Émery M., Ledoux M. and Yor M., editors *Séminaire de Probabilités XXXVII* number 1832 in Springer Lecture Notes in Mathematics, 81–89, Berlin, Heidelberg, New York, 2003.
3. Cheridito, P., Kawaguchi, H., and Maejima, M.: Fractional Ornstein-Uhlenbeck processes, *Electr. J. Prob.* **8** (2003) 1–14.
4. Decreusefond, L. and Üstünel, A.S.: Stochastic analysis of the fractional Brownian motion. *Potential Analysis* **10(2)** (1999) 177–214.
5. Doob, J. L.: The Brownian movement and stochastic equations, *Ann. Math.* **43(2)** (1942) 351–369.
6. Garrido-Atienza, M. J., Kloeden, P., and Neuenkirch, A.: Discretization of stationary solutions of stochastic systems driven by fractional Brownian motion, *Appl. Math. Optim.* **60** (2009) 151–172.
7. Gripenberg, G. and Norros, I.: On the prediction of fractional Brownian motion, *J. Appl. Probab.* **33** (1996) 400–410.
8. Lamperti, J.: On convergence of stochastic processes, *Transactions, Amer. Math. Soc.* **104** (1962) 430–435.
9. Lamperti, J.: Semi-stable Markov processes, *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **22** (1972) 205–225, University of California Press, Berkeley.
10. Maslowski, B. and Schmalfuss, B.: Random dynamical systems and stationary solutions of differential equations driven by the fractional Brownian motion, *Proc. Stoc. Anal. and Appl.* **22** (2004) 1577–1607.
11. Pipiras, V. and Taqqu, M.: Integration questions related to fractional Brownian motion, *Probab. Theory Relat. Fields* **118(2)** (2000) 251–291.
12. Zähle, M.: Integration with respect to fractal functions and stochastic calculus, I., *Probab. Theory Related Fields* **111(2)** (1998) 333–347.

Terhi Kaarakka: Department of Mathematics, Tampere University of Technology, FIN-33101 Tampere, Finland

E-mail address: terhi.kaarakka@tut.fi

Paavo Salminen: Mathematical Department, Åbo Akademi University, FIN-20500 Åbo, Finland

E-mail address: phsalmin@abo.fi