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ITÔ'S FORMULA FOR A SUB-FRACTIONAL BROWNIAN MOTION*

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Abstract. In this paper we consider stochastic calculus connected with sub-fractional Brownian motion S^H with $H \in (\frac{1}{2}, 1)$ and narrow the focus to obtain various versions of Itô's formula. We introduce the integral of deterministic functions f with respect to the local time $\mathcal{L}^H(x,t)$ of S^H and the *weighted quadratic covariation* $[f(S^H), S^H]^{(W)}$. We establish the Bouleau-Yor identity

$$(2-2^{2H-1})\left[f(S^H), S^H\right]^{(W)}_t = -\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t), \quad t \geq 0,$$

and show that Itô's formula admits the following form:

$$F(S^H_t) = F(0) + \int_0^t f(S^H_s)dS^H_s + \frac{1}{2}(2-2^{2H-1})\left[f(S^H), S^H\right]^{(W)}_t,$$

provided f of bounded p-variation with $1 \leq p < \frac{2H}{1-H}$, where F is an absolutely continuous function with derivative f and the stochastic integral is the Skorokhod integral.

1. Introduction

Recently, the long-range dependence property has become an important aspect of stochastic models in various scientific areas including hydrology, telecommunication, turbulence, image processing and finance. The best known and most widely used process that exhibits the long-range dependence property is *fractional Brownian motion* (fBm in short). The fBm is a suitable generalization of the standard Brownian motion, but exhibits long-range dependence, self-similarity and it has stationary increments. It is impossible to list here all the contributors in previous topics. Some surveys and complete literatures could be found in Biagini *et al.* [2], Hu [17], Mishura [19], Nualart [20]. On the other hand, many authors have proposed to use more general self-similar Gaussian processes and random fields as stochastic models. Such applications have raised many interesting theoretical questions about self-similar Gaussian processes and fields in general. Therefore, some generalizations of the fBm has been introduced. However, contrast to the extensive studies on fractional Brownian motion, there has been little systematic investigation on other self-similar Gaussian processes. The main reason for this is

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the complexity of dependence structures for self-similar Gaussian processes which do not have stationary increments. Therefore, it seems interesting to study sub-fractional Brownian motion with $\frac{1}{2} < H < 1$.

As an extension of Brownian motion, recently, Bojdecki *et al.* [3] introduced and studied a rather special class of self-similar Gaussian processes which preserves many properties of the fBm. This process arises from occupation time fluctuations of branching particle systems with Poisson initial condition. This process is called the *sub-fractional Brownian motion*. The so-called sub-fractional Brownian motion (sub-fBm in short) with index $H \in (0,1)$ is a mean zero Gaussian process $S^H = \{S_t^H, t \geq 0\}$ with $S_0^H = 0$ and the covariance

$$R_H(t,s) \equiv E\left[S_t^H S_s^H\right] = s^{2H} + t^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right] \tag{1.1}$$

for all $s,t \geq 0$. For $H = 1/2$, S^H coincides with the standard Brownian motion B. S^H is neither a semimartingale nor a Markov process unless $H = 1/2$, so many of the powerful techniques from stochastic analysis are not available when dealing with S^H. As a Gaussian process, it is possible to construct a stochastic calculus of variations with respect to S^H (see, for example, Alós *et al.* [1]). The sub-fBm has properties analogous to those of fBm (self-similarity, long-range dependence, Hölder paths), and satisfies the following estimates:

$$[(2-2^{2H-1}) \wedge 1](t-s)^{2H} \leq E\left[\left(S_t^H - S_s^H\right)^2\right] \leq [(2-2^{2H-1}) \vee 1](t-s)^{2H}. \tag{1.2}$$

Thus, Kolmogorov's continuity criterion implies that sub-fractional Brownian motion is Hölder continuous of order γ for any $\gamma < H$. But its increments are not stationary. More works for sub-fractional Brownian motion can be found in Bojdecki *et al.* [4, 5, 6], Shen-Yan [24], Tudor [25, 26, 28, 27] and Yan-Shen [32].

In this paper, throughout we assume that $\frac{1}{2} < H < 1$. Our purpose is to prove various versions of Itô's formula for sub-fBm. The case $0 < H < \frac{1}{2}$ will be discussed in a forthcoming paper. It is noteworthiness that sub-fBm S^H is also of finite quadratic variation provided $\frac{1}{2} < H < 1$. Some surveys and a complete literature could be found in Russo and Vallois [23] (see also Russo and Vallois [22]) for finite quadratic variation processes. On the other hand, as a Gaussian process, Alós *et al.* [1] have obtained stochastic calculus and the Itô formula for C^2-functions. However, our method and technique used here are different from theirs. Moreover, by fully focusing on the sub-fBm case with $\frac{1}{2} < H < 1$, our study allows to weaken the hypotheses of some formulas, and provides various original versions of Itô's formula.

This paper is organized as follows. In Section 2 we present some preliminaries for sub-fBm. In Section 3 we derive an Itô formula. We start with the one dimensional sub-fBm S^H. We mention that the Itô formula has been already proved by Alós *et al.* [1] but here we propose an alternative proof based on the Taylor expansion and the hypothesis on f is weakened, which appears to be also useful in the multidimensional settings. In Section 4 the Tanaka formula is obtained from the Itô formula by a limit argument and it involves the so-called weighted local time

$$\mathcal{L}^H(t,x) = 2H(2-2^{2H-1})\int_0^t \delta(S_s^H - x)s^{2H-1}ds.$$

In Section 5, we consider the integral of the form

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(t,dx),$$

and show that it is well-defined provided f is of bounded p-variation with $1 \le p < \frac{2H}{1-H}$. As an application we show that Bouleau-Yor's formula

$$F(S^H_t) = F(0) + \int_0^t f(S^H_s)dS^H_s - \frac{1}{2}\int_{\mathbb{R}} f(x)\mathcal{L}^H(t,dx) \tag{1.3}$$

holds for all absolutely continuous function

$$F(x) = F(0) + \int_0^x f(y)dy,$$

where the derivative f is of bounded p-variation with $1 \le p < \frac{2H}{1-H}$. Clearly, the difference of two convex functions is an absolutely continuous function with derivative of bounded variation. Thus, as a direct consequence we obtain the following Itô-Tanaka formula

$$F(S^H_t) = F(0) + \int_0^t F'_-(S^H_s)dS^H_s + \frac{1}{2}\int_{\mathbb{R}} \mathcal{L}^H(t,x)F''{}_-(dx), \tag{1.4}$$

holds, where F is the difference of two convex functions. In Section 6, inspired by Yan *et al.* [30, 31] we study the *weighted quadratic covariation* $[f(S^H), S^H]^{(W)}$ of $f(S^H)$ and S^H defined by

$$[f(S^H), S^H]^{(W)}_t = \lim_{\varepsilon\downarrow 0} \frac{1}{\varepsilon^{2H}} \int_0^t 2Hs^{2H-1}\left\{f(S^H_{s+\varepsilon}) - f(S^H_s)\right\}(S^H_{s+\varepsilon} - S^H_s)ds, \tag{1.5}$$

for $t \ge 0$, where the limit is uniform in probability and $x \mapsto f(x)$ is a deterministic function. In particular, we have

$$[S^H, S^H]^{(W)}_t = t^{2H}.$$

Clearly, this is the usual quadratic covariation $[f(S^H), S^H]_t$ provided the expression (1.5) does not involve the *weight* $2Hs^{2H-1}$ and the coefficient of the integral in expression (1.5) is ε^{-1}, and moreover, $[S^H, S^H]_t = 0$ for $\frac{1}{2} < H < 1$. We show that the *weighted quadratic covariation* $[f(S^H), S^H]^{(W)}$ exists and

$$(2 - 2^{2H-1})[f(S^H), S^H]^{(W)}_t = -\int_{\mathbb{R}} f(x)\mathcal{L}^H(t,dx) \tag{1.6}$$

if f is of bounded p-variation with $1 \le p < \frac{2H}{1-H}$.

2. Preliminaries

Let $(S^H_t, t \in [0,T])$ be a sub-fBm with $\frac{1}{2} < H < 1$, defined on the complete probability space $(\Omega, \mathcal{F}, P)$. S^H can be written as a Volterra process. It is possible to construct a stochastic calculus of variations with respect to the Gaussian process S^H, which will be related to the Malliavin calculus. Some surveys and complete literatures could be found in Alòs *et al* [1], Nualart [20] and Tudor [27]. We recall here the basic definitions and results of this calculus. The crucial ingredient is the canonical Hilbert space $\mathcal{H}$ (is also said to be reproducing kernel Hilbert space) associated to the sub-fBm which is defined as the closure of the linear space $\mathcal{E}$

generated by the indicator functions $\{1_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product

$$\langle 1_{[0,t]}, 1_{[0,s]} \rangle_{\mathcal{H}} = R_H(t,s) = t^{2H} + s^{2H} - \frac{1}{2}\left[(s+t)^{2H} + |t-s|^{2H}\right].$$

The mapping $1_{[0,s]} \to S_s^H$ can be extended to a linear isometry between $\mathcal{H}$ and the Gaussian space associated with S^H. We will denote the isometry by $\varphi \to S^H(\varphi)$. For $\frac{1}{2} < H < 1$ we denote by $\mathcal{S}$ the set of smooth functionals of the form

$$F = f(S^H(\varphi_1), \dots, S^H(\varphi_n)),$$

where $f \in C_b^\infty(\mathbb{R}^n)$ and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of a functional F as above is given by

$$D^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}(S^H(\varphi_1), \dots, S^H(\varphi_n))\varphi_i$$

and this operator can be extended to the closure $\mathbb{D}^{m,2}(m \geq 1)$ of $\mathcal{S}$ with respect to the norm

$$\|F\|_{m,2}^2 \equiv E|F|^2 + E\|D^H F\|_{\mathcal{H}}^2 + \dots + E\|D^{H,m} F\|_{\mathcal{H}^{\widehat{\otimes} m}}^2$$

where $\mathcal{H}^{\widehat{\otimes} m}$ denotes the m fold symmetric tensor product and the mth derivative $D^{H,m}$ is defined by iteration. The divergence integral δ^H is the adjoint operator of D^H. Concretely, a random variable $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of the divergence operator δ^H (in symbol $\mathrm{Dom}(\delta^H)$) if

$$E|\langle D^H F, u\rangle_{\mathcal{H}}| \leq c\|F\|_{L^2}$$

for every $F \in \mathcal{S}$. In this case $\delta^H(u)$ is given by the duality relationship

$$E(F\delta^H(u)) = E\langle D^H F, u\rangle_{\mathcal{H}}$$

for any $F \in \mathbb{D}^{1,2}$, and we have the following integration by parts:

$$F\delta^H(u) = \delta^H(Fu) + \langle D^H F, u\rangle_{\mathcal{H}} \tag{2.1}$$

for any $u \in \mathrm{Dom}(\delta^H)$, $F \in \mathbb{D}^{1,2}$ such that $Fu \in L^2(\Omega, \mathcal{H})$. It follows that

$$E\left[\delta^H(u)^2\right] = E\|u\|_{\mathcal{H}}^2 + E\langle D^H u, (D^H u)^*\rangle_{\mathcal{H}\otimes\mathcal{H}}, \tag{2.2}$$

where $(D^H u)^*$ is the adjoint of $D^H u$ in the Hilbert space $\mathcal{H} \otimes \mathcal{H}$, and

$$\|u\|_{\mathcal{H}}^2 = \int_0^T \int_0^T u_s u_r \phi_H(s,r) ds dr, \tag{2.3}$$

where

$$\phi_H(s,r) = \frac{\partial^2 R_H}{\partial s \partial r}(s,r) = H(2H-1)\left(|s-r|^{2H-2} - |s+r|^{2H-2}\right) \geq 0,$$

and for $\varphi : [0,T]^2 \to \mathbb{R}$, we have

$$\|\varphi\|_{\mathcal{H}\otimes\mathcal{H}}^2 = \int_{[0,T]^4} \varphi(t,s)\varphi(t',s')\phi_H(t,t')\phi_H(s,s') dt ds dt' ds'.$$

We denote by $|\mathcal{H}|$ the subspace of $\mathcal{H}$, which is defined as the set of measurable function f on $[0,T]$ with

$$\|f\|^2_{|\mathcal{H}|} := \int_0^T \int_0^T |f(s)||f(r)|\phi_H(s,r)dsdr < \infty. \tag{2.4}$$

We can show that the space $|\mathcal{H}| \subset \mathcal{H}$ is a Banach space for the norm $\|\cdot\|_{|\mathcal{H}|}$ and

$$E(\delta^H(u))^2 \le E\|u\|^2_{|\mathcal{H}|} + E\|D^H(u)\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} \tag{2.5}$$

where, if $\varphi : [0,T]^2 \to \mathbb{R}$ then

$$\|\varphi\|^2_{|\mathcal{H}|\otimes|\mathcal{H}|} = \int_{[0,T]^4} |\varphi(t,s)||\varphi(t^{'},s^{'})|\phi_H(t,t^{'})\phi_H(s,s^{'})dtdsdt^{'}ds^{'}.$$

We also will use the notation

$$\delta^H(u) = \int_0^T u_s dS^H_s$$

to express the Skorokhod integral of an adapted process u. It is also possible to introduce multiple integrals $I_n(f_n)$, $f_n \in \mathcal{H}^{\otimes n}$ with respect to S^H. For the divergence integral we have the following convergence: if $\{u_n\}$ is a sequence of elements in $\mathrm{Dom}(\delta^H)$ such that $u_n \to u$ in $L^2(\Omega;\mathcal{H})$, and $\delta^H(u_n) \to G$ in L^2, then we have

$$u \in \mathrm{Dom}(\delta^H) \text{ and } \delta^H(u) = G. \tag{2.6}$$

3. Itô's Formula for C^2-functions

In this section, we use Taylor expansion to obtain Itô formula for one dimensional sub-fBm. Note that the Itô formula has been already proved in Alós *et al.* [1]. However, our method is different, and we propose an alternative proof based on the Taylor expansion and the hypothesis on f is weakened, which appears to be also useful in the multidimensional settings. But, this method beseems only to the case $\frac{1}{2} < H < 1$. For simplicity we let C stand for a positive constant and its value may be different in different appearance.

Lemma 3.1. *Consider the function on* $[1,\infty)$

$$h(y) = y^{2H} + (y-1)^{2H} - \frac{1}{2^{2H-1}}(2y-1)^{2H},$$

where $0 < H < 1$. *Then* $h(y) \to 0$, *as* y *tends to infinity. Moreover if* $2H = 1$, $h(y) = 0$.

Theorem 3.2. *Let* $f \in C^2(\mathbb{R})$ *and* $\frac{1}{2} < H < 1$. *Then*

$$f(S^H_t) = f(0) + \int_0^t f^{'}(S^H_s)dS^H_s + H(2-2^{2H-1})\int_0^t f^{''}(S^H_s)s^{2H-1}ds. \tag{3.1}$$

Proof. Let us fix $t > 0$ and let $\pi \equiv \{t_j = \frac{jt}{n}; j = 0,1,...,n\}$ be a partition of $[0,t]$. By the localization argument and the fact that the process S^H is continuous, we can assume that f has compact support, and so $f, f^{'}, f^{''}$ are bounded. Using Taylor expansion, we have

$$f(S_t^H) = f(0) + \sum_{j=1}^{n} f^{'}(S_{t_{j-1}}^H)(S_{t_j}^H - S_{t_{j-1}}^H) + \sum_{j=1}^{n} \frac{1}{2} f^{''}(S_j^H(\theta_j))(S_{t_j}^H - S_{t_{j-1}}^H)^2$$
$$\equiv f(0) + I^n + J^n \tag{3.2}$$

where $S_j^H(\theta_j) = S_{t_{j-1}}^H + \theta_j(S_{t_j}^H - S_{t_{j-1}}^H)$ with $\theta_j \in (0,1)$ being a random variable. A straightforward calculation shows that

$$E|J^n|^2 \le \frac{nK}{4} \sum_{j=1}^{n} E(S_{t_j}^H - S_{t_{j-1}}^H)^4 \le \frac{K}{4} \frac{t^{4H}}{n^{4H-2}} \longrightarrow 0,$$

by (1.2), as n tends to infinity, where K is a constant depends on $f^{''}$, which yields $J^n \to 0$ in $L^2(\Omega)$, as n tends to infinity. By (2.1) we have

$$\begin{aligned} I^n &= \sum_{j=1}^{n} f^{'}(S_{t_{j-1}}^H)(\delta^H(1_{(t_{j-1},t_j]})) \\ &= \delta^H(\sum_{j=1}^{n} f^{'}(S_{t_{j-1}}^H) 1_{(t_{j-1},t_j]}(\cdot)) + \sum_{j=1}^{n} f^{''}(S_{t_{j-1}}^H) \langle 1_{(0,t_{j-1}]}, 1_{(t_{j-1},t_j]} \rangle_{\mathcal{H}} \\ &\equiv I_1^n + I_2^n. \end{aligned}$$

Clearly, we have

$$I_2^n \to H(2 - 2^{2H-1}) \int_0^t f^{''}(S_s^H) s^{2H-1} ds \tag{3.3}$$

in L^2 as n tends to infinity. On the other hand, we have

$$\begin{aligned} & E\Big\| \sum_{j=1}^{n} (f^{'}(S_{t_{j-1}}^H) - f^{'}(S_{\cdot}^H)) 1_{(t_{j-1},t_j]} \Big\|_{|\mathcal{H}|}^2 \\ &= E \sum_{j,l=1}^{n} \int_{t_{j-1}}^{t_j} \int_{t_{l-1}}^{t_l} |f^{'}(S_{t_{j-1}}^H) - f^{'}(S_u^H)| |f^{'}(S_{t_{l-1}}^H) - f^{'}(S_v^H)| \phi_H(u,v) du dv \\ &\le C_H (\sup_{x \in \mathbb{R}} |f^{''}(x)|)^2 \frac{t^{2H}}{n^{2H}} \int_0^t \int_0^t \phi_H(u,v) du dv \longrightarrow 0, \end{aligned}$$

as n tends to infinity, which implies

$$\sum_{j=1}^{n} f^{'}(S_{t_{j-1}}^H) 1_{(t_{j-1},t_j]}(\cdot) \longrightarrow f^{'}(S_{\cdot}^H) 1_{(0,t]}(\cdot)$$

in $L^2(\Omega; |\mathcal{H}|)$, as n tends to infinity.

This implies that I_1^n converges to $\delta^H(f^{'}(S_{\cdot}^H 1_{(0,t]}(.))$ in $L^2(\Omega)$. Therefore (3.1) is established due to (2.6). □

Now, we extend Itô's formula (3.1) to the multidimensional case. Let $\widetilde{H} = (H_1, \ldots, H_d)$ be a multiindex with $0 < H_i < 1, i = 1, 2, \ldots, d$ and let $S^{\widetilde{H}} =$

$(S^{H_1}, S^{H_2}, \dots, S^{H_d})$ be a d-dimensional sub-fBm such that $S_0^{H_i} = 0$, $ES_t^{H_i} = 0$ $(t \geq 0)$ and

$$E\left[S_t^{H_i} S_s^{H_j}\right] = \left\{s^{2H_i} + t^{2H_i} - \frac{1}{2}\left[(s+t)^{2H_i} + |t-s|^{2H_i}\right]\right\}\delta_{ij} \tag{3.4}$$

for all $i, j = 1, 2, \dots, d$, where

$$\delta_{i,j} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

This means that we are considering d independent sub-fBms with Hurst parameters $H_1, H_2, \dots, H_d$, respectively.

Theorem 3.3. *Let $S_t^{\widetilde{H}} = (S_t^{H_1}, S_t^{H_2}, \dots, S_t^{H_d})$ be a d-dimensional sub-fBm and let f be a function of class $C^2(\mathbb{R}^d)$. Suppose that $\frac{1}{2} < H_i < 1$ for all $i = 1, 2, \dots, d$, then*

$$f(S_t^{\widetilde{H}}) = f(0) + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(S_s^{\widetilde{H}}) dS_s^{H_i} + \sum_{i=1}^d H_i(2 - 2^{2H_i - 1}) \int_0^t \frac{\partial^2 f}{\partial x_i^2}(S_s^{\widetilde{H}}) s^{2H_i - 1} ds. \tag{3.5}$$

Proof. Similar to the proof of Theorem 3.2 we can prove the theorem. □

One can easily generalize the above theorem to the time-dependent case.

Theorem 3.4. *Let $S_t^{\widetilde{H}} = (S_t^{H_1}, S_t^{H_2}, \dots, S_t^{H_d})$ be a d-dimensional sub-fBm and let f be a function of class $C^{1,2}(\mathbb{R}_+ \times \mathbb{R}^d)$. Suppose that $\frac{1}{2} < H_i < 1$ for all $i = 1, 2, \dots, d$, then*

$$\begin{aligned} f(t, S_t^{\widetilde{H}}) = f(0,0) &+ \int_0^t \frac{\partial f}{\partial s}(s, S_s^{\widetilde{H}}) ds + \sum_{i=1}^d \int_0^t \frac{\partial f}{\partial x_i}(s, S_s^{\widetilde{H}}) dS_s^{H_i} \\ &+ \sum_{i=1}^d H_i(2 - 2^{2H_i - 1}) \int_0^t \frac{\partial^2 f}{\partial x_i^2}(s, S_s^{\widetilde{H}}) s^{2H_i - 1} ds. \end{aligned} \tag{3.6}$$

4. Local Times and Tanaka Formula

In this section, we consider the local times of sub-fBm and find their Wiener chaos expansions. By a limit argument and the Itô formula (3.1) we obtain a Tanaka formula involving the so-called weighted local time.

From Geman-Horowitz [16] one can find that sub-fBm S^H has a local time $L^H(x,t)$ continuous in $(x,t) \in \mathbb{R} \times [0, \infty)$, with compact support, which satisfies the occupation formula

$$\int_0^t \phi(S_s^H, s) ds = \int_{\mathbb{R}} dx \int_0^t \phi(x, s) L^H(x, ds) \tag{4.1}$$

for every continuous and bounded function $\phi(x,t) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, and such that

$$L^H(x,t) = \int_0^t \delta(S_s^H - x) ds = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \lambda\big(s \in [0,t], |S_s^H - x| < \epsilon\big),$$

where λ denotes Lebesgue measure and $\delta(x)$ is the Dirac delta function. Define the so-called *weighted local time* $\mathcal{L}^H(x,t)$ of S^H at x as follows

$$\mathcal{L}^H(x,t) = 2H(2-2^{2H-1})\int_0^t s^{2H-1}L^H(x,ds)$$
$$\equiv 2H(2-2^{2H-1})\int_0^t \delta(S_s^H - x)s^{2H-1}ds.$$

Then the occupation formula (4.1) can be rewritten as

$$2H(2-2^{2H-1})\int_0^t \phi(S_s^H, s)s^{2H-1}ds = \int_{\mathbb{R}} dx \int_0^t \phi(x,s)\mathcal{L}^H(x,ds). \tag{4.2}$$

Let

$$p_\varepsilon(x) = \frac{1}{\sqrt{2\pi\varepsilon}}e^{-\frac{x^2}{2\varepsilon}} \qquad (x \in \mathbb{R})$$

be the heat kernel with variance $\varepsilon > 0$. We denote by H_n the nth Hermite polynomial defined for $n \geq 1$ by

$$H_n(x) = (-1)^n \frac{1}{n!}e^{\frac{x^2}{2}}\frac{d^n}{dx^n}e^{-\frac{x^2}{2}}$$

and $H_0(x) = 1$. There is a standard method to compute the Wiener-Itô chaos expansion of weighted local time. It consists in approaching the Dirac function by mean-zero Gaussian kernels p_ε of variance ε and to take the limit in L^2 as $\varepsilon \to 0$.

Lemma 4.1. *Let $0 \leq s < t \leq T$. We have*

$$0 \leq R_H(t,s) \leq \frac{1}{2}(2-2^{2H-1})\left[t^{2H} + s^{2H} - (t-s)^{2H}\right] \tag{4.3}$$

for $\frac{1}{2} < H < 1$.

Proof. This is a calculus exercise. □

Lemma 4.2 ([11])**.** *Let $0 < H < 1$. We then have*

$$\int_0^1 \frac{\left[1 + x^{2H} - (1-x)^{2H}\right]^n}{2^n x^{H(n+1)}}dx \leq \frac{C_H}{n^{\frac{1}{2H}}} \tag{4.4}$$

for all integers $n \geq 1$, where $K_H > 0$ is a constant depending only on H.

Denote

$$\mu([0,u]\times[0,v]) := R_H(u,v) + R_H(0,0) - R_H(u,0) - R_H(0,v) = R_H(u,v)$$

for $u, v \geq 0$. It follows from Lemma 4.1 that

$$\int_0^t\int_0^t \frac{|\mu([0,u]\times[0,v])|^n}{(|\mu([0,u]\times[0,u])||\mu([0,v]\times[0,v])|)^{\frac{n+1}{2}}}dvdu$$
$$= \frac{2}{(2-2^{2H-1})^{n+1}}\int_0^t\int_0^u \frac{R_H(u,v)^n}{(u^{2H}v^{2H})^{\frac{n+1}{2}}}dvdu$$
$$\leq \frac{C_H t^{2-2H}}{(1-H)(2-2^{2H-1})}n^{-\frac{1}{2H}}$$

for $\frac{1}{2} < H < 1$, which implies that

$$\sum_{n\geq 1} n^{-\frac{1}{2}} \int_0^t \int_0^t \frac{|\mu([0,u]\times[0,v])|^n}{(|\mu([0,u]\times[0,u])||\mu([0,v]\times[0,v])|)^{\frac{n+1}{2}}} dvdu < \infty \tag{4.5}$$

for all $t \geq 0$. Thus, similar to Eddahbi *et al.* [11] (see also Kruk-Russo-Tudor [18] and Coutin *et al* [8]) one can prove the following.

Proposition 4.3. *Let $\frac{1}{2} < H < 1$. For each $x \in \mathbb{R}$, and $t \in [0,T]$, the random variables*

$$\int_0^t p_\varepsilon(S_s^H - x)ds$$

converges to $L^H(x,t)$ in L^2, as ε tends to zero. Moreover the local time $L^H(x,t)$ has the following Wiener chaos expansion:

$$L^H(x,t) = \sum_{n=0}^{\infty} \int_0^t \frac{p_{(2-2^{2H-1})s^{2H}}(x)}{((2-2^{2H-1})s^{2H})^{\frac{n}{2}}} H_n\left(\frac{x}{\sqrt{(2-2^{2H-1})s^{2H}}}\right) I_n(1_{[0,s]}^{\otimes n})ds \tag{4.6}$$

for all $t \in [0,T], x \in \mathbb{R}$, and the L^2 norm of $L^H(x,t)$ is finite, where I_n stands for multiple Wiener integrals with respect to S^H.

Let us consider now the Tanaka formula for the sub-fBm S^H.

Theorem 4.4. *Let $S^H = \{S_t^H, 0 \leq t \leq T\}$ be a sub-fBm with $\frac{1}{2} < H < 1$. Then the following Tanaka formula holds:*

$$|S_t^H - x| = |x| + \int_0^t \text{sign}(S_s^H - x)dS_s^H + \mathcal{L}^H(x,t). \tag{4.7}$$

Proof. Consider the heat kernel $p_\varepsilon(x)$ and define

$$F_\varepsilon'(z) = 2\int_{-\infty}^z p_\varepsilon(y)dy - 1, \quad F_\varepsilon(z) = \int_0^z F_\varepsilon'(y)dy.$$

Then, for all $x \in \mathbb{R}$

$$F_\varepsilon'(x) \longrightarrow \text{sign}(x), \qquad F_\varepsilon(x) \longrightarrow |x|,$$

as ε tends to zero, and by Lebesgue's dominated convergence theorem we have, as $\varepsilon \to 0$

$$F_\varepsilon(S_t^H - x) \to |S_t^H - x|$$

in L^2. Moreover $F_\varepsilon \in C^2(\mathbb{R})$ and the Theorem 3.2 implies that, for any $x \in \mathbb{R}$

$$F_\varepsilon(S_t^H - x) = F_\varepsilon(-x) + \int_0^t F_\varepsilon'(S_s^H - x)dS_s^H + 2H\beta_H \int_0^t p_\varepsilon(S_s^H - x)s^{2H-1}ds, \tag{4.8}$$

where $\beta_H = (2 - 2^{2H-1})$. By a slight adaptation of Proposition 4.3, we can obtain that

$$\mathcal{L}^H(x,t) = \lim_{\varepsilon\to 0} 2H\beta_H \int_0^t p_\varepsilon(S_s^H - x)s^{2H-1}ds \tag{4.9}$$

in L^2, and $\mathcal{L}^H(x,t)$ admits the following chaotic representation:

$$\mathcal{L}^H(x,t) = 2H\beta_H \sum_{n=0}^{\infty} \int_0^t \frac{p_{\beta_H s^{2H}}(x)}{(\beta_H s^{2H})^{\frac{n}{2}}} s^{2H-1} H_n\left(\frac{x}{\sqrt{\beta_H s^{2H}}}\right) I_n(1_{[0,s]}^{\otimes n})ds.$$

Combining this with (4.8), we get

$$\int_0^t F_\varepsilon^{'}(S_s^H - x)dS_s^H = F_\varepsilon(S_t^H - x) - F_\varepsilon(-x) - 2H\beta_H \int_0^t p_\varepsilon(S_s^H - x)s^{2H-1}ds$$
$$\longrightarrow |S_t^H - x| - |x| - \mathcal{L}^H(x,t)$$

in L^2, as $\varepsilon \to 0$. On the other hand, we have

$$F_\varepsilon^{'}(S^H - x) \to \text{sign}(S^H - x)$$

in $L^2(\Omega; |\mathcal{H}|)$, as $\varepsilon \to 0$, and the theorem follows since δ^H is closed in L^2. □

5. An Itô Formula for Absolutely Continuous Functions

We assume that in this section $\frac{1}{2} < H < 1$. Tanaka formula for the sub-fBm can be applied to derive a generalization of Itô formula to a convex function F (similar to Proposition 7 of Coutin *et al* [8]). We recall that if a function F is convex, its second derivative F'' in the sense of distributions is a positive measure.

Proposition 5.1. *Suppose that F is the difference of two convex functions with the derivative $F' = f$. Then we have*

$$F(S_t^H) = F(0) + \int_0^t f(S_s^H)dS_s^H + \frac{1}{2}\int_{\mathbb{R}} \mathcal{L}^H(x,t)F''(dx). \tag{5.1}$$

It is well-known that if F is the difference of two convex functions, then F is an absolutely continuous function with derivative of bounded variation. Thus, $F' = f$ is of bounded variation and (5.1) can be rewritten as

$$F(S_t^H) = F(0) + \int_0^t f(S_s^H)dS_s^H + \frac{1}{2}\int_{\mathbb{R}} \mathcal{L}^H(x,t)df(x)$$
$$\equiv F(0) + \int_0^t f(S_s^H)dS_s^H - \frac{1}{2}\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)$$

provided the final integral in the second expression

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t) \tag{5.2}$$

is well-defined. This encourages us to consider the integral (5.2), and to prove a more general result, an analogue of Bouleau-Yor's formula, such that Proposition 5.1 is a straightforward corollary.

We first define the integral (5.2) as a Riemann-Stieltjes integral, and show that it is well-defined provided f is of bounded p-variation for $1 \le p < \frac{2H}{1-H}$. To this end we need three auxiliary statements.

Lemma 5.2 (Yan-Shen [32])**.** *For all $s, r \in [0,T], r < s$ and $0 < H < 1$ we have*

$$C_{1,H}(s-r)^{2H}r^{2H} \le \beta_H^2 s^{2H}r^{2H} - \mu^2 \le C_{2,H}(s-r)^{2H}r^{2H}, \tag{5.3}$$

where $\beta_H = 2 - 2^{2H-1}$ and $\mu = E(S_s^H S_r^H)$.

Define the function $(x,y)\mapsto\psi(x,y)$ by

$$\psi_{s,r}(x,y)=\frac{1}{2\pi}e^{-\frac{1}{2\rho^2}\left(\beta_H r^{2H}x^2-2\mu xy+\beta_H s^{2H}y^2\right)}$$

for $s>r>0$, where $\rho^2=\beta_H^2 s^{2H}r^{2H}-\mu^2$. Then $\varphi_{s,r}(x,y)=\frac{1}{\rho}\psi_{s,r}(x,y)$ is the density function of (S_s^H,S_r^H). Denote

$$\Delta_{s,r}(x,y):=\psi_{s,r}(y,y)-\psi_{s,r}(y,x)-\psi_{s,r}(x,y)+\psi_{s,r}(x,x)$$

for all $x,y\in\mathbb{R}$ and $x\le y$. For $0<x\le y$ we have

$$\begin{aligned}\Delta^1_{s,r}(x,y)&:=e^{-\frac{\beta_H r^{2H}-2\mu+\beta_H s^{2H}}{2\rho^2}x^2}-e^{-\frac{1}{2\rho^2}\left(\beta_H r^{2H}y^2-2\mu xy+\beta_H s^{2H}x^2\right)}\\&=e^{-\frac{\beta_H r^{2H}-2\mu+\beta_H s^{2H}}{2\rho^2}x^2}\left\{1-e^{-\frac{1}{2\rho^2}\left(-2\mu x(y-x)+\beta_H r^{2H}(y^2-x^2)\right)}\right\}\\&\le 1-e^{-\frac{y-x}{2\rho^2}\left(-2\mu x+\beta_H r^{2H}(y+x)\right)}.\end{aligned}$$

An elementary calculus can show that

$$2+2a^{2H}-(1+a)^{2H}-(1-a)^{2H}-(4-2^{2H})a^{2H}\ge 0$$

for all $a\in[0,1]$ and $H\ge\frac{1}{2}$. We get

$$\begin{aligned}-2\mu x+\beta_H r^{2H}(y+x)&=\beta_H r^{2H}(y-x)-2\left(\mu-\beta_H r^{2H}\right)x\\&=\beta_H r^{2H}(y-x)\\&\quad-\left[2s^{2H}+2r^{2H}-(s+r)^{2H}-(s-r)^{2H}-(4-2^{2H})r^{2H}\right]x\\&\le\beta_H r^{2H}(y-x)\end{aligned}$$

for $0<x<y$, which implies

$$\Delta^1_{s,r}(x,y)\le 1-e^{-\frac{\beta_H r^{2H}}{2\rho^2}(y-x)^2}\le C_H\frac{r^{2H}(y-x)^2/\rho^2}{1+r^{2H}(y-x)^2/\rho^2}$$

for all $0<x<y$. Similarly, we have also

$$\begin{aligned}\Delta^2_{s,r}(x,y):&=e^{-\frac{\beta_H r^{2H}-2\mu+\beta_H s^{2H}}{2\rho^2}y^2}-e^{-\frac{1}{2\rho^2}\left(\beta_H r^{2H}x^2-2\mu xy+\beta_H s^{2H}y^2\right)}\\&\le C_H\frac{s^{2H}(y-x)^2/\rho^2}{1+s^{2H}(y-x)^2/\rho^2}\end{aligned}$$

for all $0<x<y$. By symmetry we get

$$\Delta^i_{s,r}(x,y)\le\frac{C_H(y-x)^2}{\rho^2/s^{2H}+(y-x)^2},\quad i=1,2$$

for all $x<y$, $xy\ge 0$.

Lemma 5.3. *For $a\le b$, $ab\ge 0$ and $0<r<s$ we have*

$$\int_{\mathbb{T}}\frac{1}{\rho}\Delta_{s,r}(a,b)\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta\le C_{H,t}(b-a)^{1+\alpha}\tag{5.4}$$

for all $\alpha\in[0,1)$ and $\frac{1}{2}<H<1$, where $\mathbb{T}=\{0<\xi<s<t,0<\eta<r<t\}$.

Proof. Without loss of generality we may assume that $b \geq a \geq 0$. Denote

$$f(a,b) = \int_{\mathbb{T}} \frac{1}{\rho} \Delta_{s,r}(a,b)\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta$$

for $b > a > 0$. Then $f(a,b) \geq 0$, and

$$\begin{aligned} f(a,b) &= \int_{\mathbb{T}} \frac{1}{\rho} \Delta^1_{s,r}(a,b)\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta \\ &\quad + \int_{\mathbb{T}} \Delta^2_{s,r}(a,b)\phi(\xi,r)\phi(\eta,s)\frac{1}{\rho}dsdrd\xi d\eta \\ &\leq C_H \int_{\mathbb{T}} \frac{(b-a)^2}{\rho^2/s^{2H} + (b-a)^2}\phi(\xi,r)\phi(\eta,s)\frac{1}{\rho}dsdrd\xi d\eta. \end{aligned}$$

Notice that the function

$$x \mapsto g(x) = \frac{x^{1-\alpha}}{\kappa^2 + x^2}, \qquad x \in \mathbb{R}, \quad \kappa > 0$$

is increasing for $|x| \leq \sqrt{\frac{1-\alpha}{1+\alpha}}\kappa$, and decreasing for $|x| \geq \sqrt{\frac{1-\alpha}{1+\alpha}}\kappa$. We get

$$\frac{f(a,b)}{(b-a)^{1+\alpha}} \leq \int_{\mathbb{T}} \frac{C_H(b-a)^{1-\alpha}}{\rho^2/s^{2H} + (b-a)^2}\frac{1}{\rho}\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta \longrightarrow 0, \tag{5.5}$$

as $b - a \to 0$. On the other hand, for all $\alpha \in [0,1)$ and $b - a \geq 1$ we have

$$\begin{aligned} \frac{f(a,b)}{(b-a)^{1+\alpha}} &\leq \int_{\mathbb{T}} \frac{1}{\rho}\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta \\ &\leq C_H \int_{\mathbb{T}} \frac{1}{(r \wedge s)^H |s-r|^H}|\xi - r|^{2H-2}|\eta - s|^{2H-2}dsdrd\xi d\eta \leq C_H t^{2H} \end{aligned}$$

by the inequality (5.3). Combining this with (5.5) and the continuity of $f(a,b)$, we obtain

$$f(a,b) \leq C_{H,t}(b-a)^{1+\alpha}$$

for all $\alpha \in [0,1)$, and the lemma follows. □

Proposition 5.4. *For $t \geq 0, x \in \mathbb{R}$ we set $\widetilde{S}^H_t(x) := \int_0^t 1_{(S^H_s > x)}dS^H_s$. Then the estimate*

$$E\left[\left(\widetilde{S}^H_t(b) - \widetilde{S}^H_t(a)\right)^2\right] \leq C_{H,t}(b-a)^{1+\alpha} \tag{5.6}$$

holds for all $\frac{1}{2} < H < 1$, $0 < \alpha < \frac{2H-1}{H}$ and $a, b \in \mathbb{R}, a < b$.

Proof. Using the formula for the expectation of the product of two divergence integrals we obtain

$$\begin{aligned} E\left|\int_0^t 1_{(a,b]}(S^H_s)dS^H_s\right|^2 &= \int_0^t \int_0^t E\left[1_{(a,b]}(S^H_s)1_{(a,b]}(S^H_r)\right]\phi(s,r)dsdr + \\ \int_{\mathbb{T}} E\left\{\delta(S^H_s - a) - \delta(S^H_s - b)\right\}&\left\{\delta(S^H_r - a) - \delta(S^H_r - b)\right\}\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta \\ &\equiv I + II \end{aligned}$$

where $\mathbb{T}=\{0<\xi<s<t, 0<\eta<r<t\}$ and δ denotes the Dirac delta function. This formula can be proved by approximating the function $1_{(a,b]}(x)$ by smooth functions and then taking the limit in L^2.

On the other hand, we get

$$\begin{aligned}
E1_{(a,b]^2}(S^H_s,S^H_r) &= \int_a^b\int_a^b \frac{1}{\rho}\psi_{s,r}(x,y)dxdy \\
&= \frac{1}{2\pi\sqrt{\beta_H}s^H}\int_a^b e^{-\frac{x^2}{2\beta_H s^{2H}}}dx\int_{\frac{\sqrt{\beta_H}s^H}{\rho}(a-\frac{\mu}{\beta_H s^{2H}}x)}^{\frac{\sqrt{\beta_H}s^H}{\rho}(b-\frac{\mu}{\beta_H s^{2H}}x)} e^{-\frac{1}{2}y^2}dy \\
\le \frac{1}{\sqrt{2\pi\beta_H}s^H}\int_a^b e^{-\frac{x^2}{2\beta_H s^{2H}}}dx &\left(\frac{1}{\sqrt{2\pi}}\int_{\frac{\sqrt{\beta_H}s^H}{\rho}(a-\frac{\mu}{\beta_H s^{2H}}x)}^{\frac{\sqrt{\beta_H}s^H}{\rho}(b-\frac{\mu}{\beta_H s^{2H}}x)} e^{-\frac{1}{2}y^2}dy\right)^\alpha \\
&\le (\beta_H)^{\frac{1}{2}(\alpha-1)}\left(\frac{s^H(b-a)}{\rho}\right)^\alpha \frac{1}{s^H}\int_a^b e^{-\frac{x^2}{2\beta_H s^{2H}}}dx \\
&\le (\beta_H)^{\frac{1}{2}(\alpha-1)}\frac{s^{(\alpha-1)H}}{\rho^\alpha}(b-a)^{1+\alpha} \\
&\le C_{H,\alpha}r^{-\alpha H}s^{(\alpha-1)H}(s-r)^{-\alpha H}(b-a)^{1+\alpha}
\end{aligned}$$

for all $s>r>0$ and all $\alpha\in[0,1]$ by Lemma 5.2, which gives

$$\begin{aligned}
I &= \int_0^t\int_0^t E1_{(a,b]\times(a,b]}(S^H_s,S^H_r)\phi(s,r)dsdr \\
&\le C_{H,\alpha}\int_0^t\int_0^s E1_{(a,b]\times(a,b]}(S^H_s,S^H_r)|s-r|^{2H-2}dsdr \\
&\le C_{H,\alpha}t^{H(1-\alpha)}(b-a)^{1+\alpha}
\end{aligned}$$

if $0<\alpha<\frac{2H-1}{H}$.

Now, let us estimate II for $a\le b$, $ab\ge 0$. From Lemma 5.3 we have

$$II=\int_{\mathbb{T}}\frac{1}{\rho}\Delta_{s,r}(a,b)\phi(\xi,r)\phi(\eta,s)dsdrd\xi d\eta \le C_{H,t}(b-a)^{1+\alpha}$$

for all $\alpha\in[0,1)$. Thus, we get the desired estimate

$$E\left|\int_0^t 1_{(a,b]}(S^H_s)dS^H_s\right|^2 = I+II\le C_{H,t}(b-a)^{1+\alpha},\quad a\le b,\ ab\ge 0$$

for all $0<\alpha<\frac{2H-1}{H}$. It follows that

$$\begin{aligned}
E\left[\left(\widetilde{S}^H_t(b)-\widetilde{S}^H_t(a)\right)^2\right] &\le 2E\left[\left(\widetilde{S}^H_t(b)-\widetilde{S}^H_t(0)\right)^2\right]+2E\left[\left(\widetilde{S}^H_t(a)-\widetilde{S}^H_t(0)\right)^2\right] \\
&\le C_{H,t}\left(b^{1+\alpha}+(-a)^{1+\alpha}\right)\le C_{H,t}(b-a)^{1+\alpha}
\end{aligned}$$

for $b>0>a$ and $0<\alpha<\frac{2H-1}{H}$, and the proposition follows. □

Recall that a measurable function $f : [a,b] \mapsto \mathbb{R}$ is said to be of bounded p-variation ($p \geq 1$) if

$$v_p(f) := \sup_{\triangle_n} \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)|^p < \infty,$$

where the supremum is taken over all partition $\triangle_n = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a,b]$. The following theorem indicates that the function $x \mapsto \mathcal{L}^H(x,t)$ is of bounded p-variation for every t.

Theorem 5.5. *Let $\frac{1}{2} < H < 1$. Then the weighted local time $\mathcal{L}^H(x,t)$ of sub-fBm S^H is of bounded p-variation in x for any $0 \leq t \leq T$, for all $p > \frac{2H}{3H-1}$, almost surely.*

Proof. Keep the notation in Proposition 5.4. By Tanaka formula (4.7) we can write

$$\mathcal{L}^H(t,x) = 2\big(\phi_t(x) - \widetilde{S}_t^H(x)\big),$$

where

$$\phi_t(x) = (S_t^H - x)^+ - (-x)^+. \tag{5.7}$$

Let $[-N,N]$ contain the support of $\mathcal{L}^H(t,x)$ in x and let

$$D := \{-N = a_0 < a_1 < \cdots < a_n = N\}$$

be a partition of $[-N,N]$. Noting that the function $\phi_t(x)$ is Lipschitz continuous in x with Lipschitz constant 2, we get

$$\sup_D \sum_i |\phi_t(a_{i+1}) - \phi_t(a_i)|^p \leq \sup_D 2^p \sum_i (a_{i+1} - a_i)^p \leq 2^p (2N)^p < \infty.$$

On the other hand, Proposition 5.4 yields

$$\left|\widetilde{S}_t^H(x) - \widetilde{S}_t^H(y)\right| \leq G_{H,T}\left(|x-y|^{\frac{1+m}{2}}\right), \quad \text{a.s.}$$

for all $x, y \in \mathbb{R}$ by Garsia-Rodemich-Rumsey Lemma, where $0 < m < \frac{2H-1}{H}$, and $G_{H,T}$ is a nonnegative random variable such that $E(G_{H,T}^q) < \infty$ for all $q \geq 1$. It follows that for all $p \geq \frac{2}{1+m} > \frac{2H}{3H-1}$

$$\begin{aligned}\sup_D \sum_i (\widetilde{S}_t^H(a_{i+1}) - \widetilde{S}_t^H(a_i))^p &\leq G_{H,T}^p \sup_D \sum_i \left[(a_{i+1} - a_i)^{\frac{p(1+m)}{2}}\right] \\ &\leq G_{H,T}^p \left[(2N)^{\frac{p(1+m)}{2}}\right] < \infty, \quad \text{a.s.}\end{aligned}$$

This completes the proof. □

We now can establish one parameter integral of the weighted local time of sub-fBm. Denote by $\mathcal{W}_p([a,b])$ ($p \geq 1$) the set of all measurable functions f on $[a,b]$ such that $v_p(f) < \infty$. For $1 \leq p < \infty$ define

$$\|f\|_{(p)} := v_p(f)^{1/p}.$$

Then $\|\cdot\|_{(p)}$ is a seminorm on $\mathcal{W}_p([a,b])$, which is called the p-variation seminorm. For $1 \leq p \leq \infty$ define

$$\|f\|_{[p]} := \|f\|_{(p)} + \|f\|_\infty,$$

where $\|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$. Then $\|\cdot\|_{[p]}$ is a norm on $\mathcal{W}_p([a,b])$, which is called the p-variation norm. The space $(\mathcal{W}_p([a,b]), \|\cdot\|_{[p]})$ is a Banach space for $p \geq 1$, and $f \in \mathcal{W}_p$ means that $f \in \mathcal{W}_p([a,b])$ for any $a, b \in \mathbb{R}$, and moreover $f \in \mathcal{W}_p$ is locally bounded. For these, see Dudley-Norvaiša [9] and Young [33].

Lemma 5.6 (Dudley-Norvaiša [9])**.** *Let $f \in \mathcal{W}_p([a,b])$ and $g \in \mathcal{W}_q([a,b])$, where $p, q \geq 1$ and $\frac{1}{p} + \frac{1}{q} > 1$. If f and g have no common discontinuities, then the Young integral*

$$\int_a^b f(x)dg(x) := \lim_{|\triangle_n|\to 0} \sum_{j=1}^n f(\xi_j)(g(x_j) - g(x_{j-1}))$$

exists, where $\xi_j \in [x_{j-1}, x_j]$ $(j = 1, 2, \ldots)$, $|\triangle_n| = \max_{1\leq j\leq n} |x_j - x_{j-1}|$ and the Love-Young inequality

$$\left| \int_a^b f(x)dg(x) \right| \leq C_{p,q} \|f\|_{[p]} \|g\|_{(p)}$$

holds.

Proposition 5.7. *Let $1\leq p < \frac{2H}{1-H}$. If $f \in \mathcal{W}_p$, then the Young integral*

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t)$$

exists for any $0 \leq t \leq T$, almost surely, and

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t) = -\int_{\mathbb{R}} \mathcal{L}^H(x, t)df(x).$$

Now, we consider the properties and characterization of one parameter integral. We first have

$$\begin{aligned}\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t) &= \lim_{\Delta_n\to 0} \sum_{j=1}^n f(x_{j-1}) \left(\mathcal{L}^H(x_j, t) - \mathcal{L}^H(x_{j-1}, t)\right) \\ &= \lim_{\Delta_n\to 0} \left[\sum_{j=1}^n f(x_{j-1})\mathcal{L}^H(x_j, t) - \sum_{j=0}^{n-1} f(x_j)\mathcal{L}^H(x_j, t)\right] \\ &= -\lim_{\Delta_n\to 0} \sum_{j=1}^n \left(f(x_j) - f(x_{j-1})\right) \mathcal{L}^H(x_j, t)\end{aligned}$$

by adding some points in the partition Δ_n to make

$$\mathcal{L}^H(x_1, t) = 0, \qquad \mathcal{L}^H(x_n, t) = 0,$$

which yields the following

Corollary 5.8. *For $f \in C^1(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx, t) = -\int_{\mathbb{R}} \mathcal{L}^H(x, t)f'(x)dx. \tag{5.8}$$

Define the mollifier θ by

$$\theta(x) = \begin{cases} c\exp(\frac{1}{(x-1)^2-1}), & x \in (0,2), \\ 0, & x \notin (0,2), \end{cases} \tag{5.9}$$

where c is a normalizing constant such that $\int_{\mathbb{R}} \theta(x)dx = 1$. Set $\theta_n(x) = n\theta(nx)$. For a locally integrable function $g(x)$ we define

$$g_n(x) = \int_{\mathbb{R}} \theta_n(x-y)g(y)dy = \int_0^2 \theta(z)g(x-\frac{z}{n})dz, \quad n \geq 1.$$

Then $g_n \in C^\infty(\mathbb{R})$.

Lemma 5.9. *Let $g \in \mathcal{W}_p$ with $1 \leq p < \frac{2H}{1-H}$. Suppose that g_n is defined as above, then $g_n \in \mathcal{W}_p$ for every $n \geq 1$, and moreover, the convergence*

$$\lim_{n\to\infty} \int_{\mathbb{R}} g_n(x)\mathcal{L}^H(dx,t) = \int_{\mathbb{R}} g(x)\mathcal{L}^H(dx,t)$$

holds in L^2.

Recall that a measurable function F is called absolutely continuous with measurable derivative f, if

$$F(x) = F(0) + \int_0^x f(y)dy.$$

By using the above lemmas, we immediately get an extension of Itô formula stated as follows, which is an analogue of Bouleau-Yor's formula.

Theorem 5.10. *Let $1 \leq p < \frac{2H}{1-H}$ and let $f \in \mathcal{W}_p$ be a left continuous function. If F is an absolutely continuous function with derivative $F' = f$, then we have*

$$F(S_t^H) = F(0) + \int_0^t f(S_s^H)dS_s^H - \frac{1}{2}\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t). \tag{5.10}$$

For $H = \frac{1}{2}$, the process S^H is classical Brownian motion B and the above theorem has been studied by Bouleau–Yor [7], Eisenbaum [10], Föllmer *et al.* [13], Feng–Zhao [12], Peskir [21]. Moreover, Yan *et al* [29, 30, 31] considered the case of fractional Brownian motion.

Proof of Theorem 5.10. It is sufficient to prove (5.10) for all uniformly bounded function $f \in \mathcal{W}_p$ by means of a localization argument. In fact, for any $k \geq 0$ we may consider the set

$$\Omega_k = \left\{ \sup_{0\leq t\leq T} |S_t^H| < k \right\}$$

and let $f^{[k]}$ be a measurable function such that $f^{[k]} = f$ on $[-k,k]$, and such that $f^{[k]}$ vanishes outside. Notice that $f \in \mathcal{W}_p$ is locally bounded. Then $f^{[k]}$ is a uniformly bounded function. Set $\frac{d}{dx}F^{[k]} = f^{[k]}$ and $F^{[k]} = F$ on $[-k,k]$. If the theorem is true for all uniformly bounded functions on $\mathcal{W}_p$, then we get the desired formula

$$F^{[k]}(S_t^H) = F^{[k]}(0) + \int_0^t f^{[k]}(S_s^H)dS_s^H - \frac{1}{2}\int_{\mathbb{R}} f^{[k]}(x)\mathcal{L}^H(dx,t)$$

on the set Ω_k. Letting k tend to infinity we deduce the Itô formula (5.10) for all $f \in \mathcal{W}_p$.

Let now $F' = f \in \mathcal{W}_p$ be uniformly bounded and left continuous. For $n \geq 1$ we set

$$F_n(x) := \int_{\mathbb{R}} \theta_n(x-y)F(y)dy,$$

where θ_n is the mollifier defined in (5.9). Then, $F_n \in C^\infty(\mathbb{R})$ and the Itô formula yields

$$F_n(S_t^H) = F_n(0) + \int_0^t F_n'(S_s^H)dS_s^H + H(2-2^{2H-1})\int_0^t s^{2H-1}F_n''(S_s^H)ds \quad (5.11)$$

for all $n \geq 1$. Notice that $F_n' = f_n \in C^\infty(\mathbb{R}) \cap \mathcal{W}_p$,

$$f_n(x) = \int_{\mathbb{R}} \theta_n(x-y)f(y)dy = \int_0^2 \theta(z)f(x-\frac{z}{n})dz$$

for every $n \geq 1$ and

$$|f_n(x)| \leq \lambda := \sup_x |f(x)| < \infty, \qquad n = 1, 2, \ldots, \ x \in \mathbb{R}.$$

On the other hand, using Lebesgue's dominated convergence theorem, one can prove that as $m \to \infty$, for each x,

$$\lim_{n\to\infty} F_n(x) = F(x), \quad \lim_{n\to\infty} F_n'(x) = f(x)$$

as $n \to \infty$, which implies that $F_n(S_t^H)$ and $F_n'(S_t^H)$ converge to $F(S_t^H)$ and $f(S_t^H)$ in L^2, as n tends to infinity, respectively. Moreover, we have

$$\begin{aligned} 2H(2-2^{2H-1})\int_0^t s^{2H-1}F_n''(S_s^H)ds &= \int_{\mathbb{R}} \mathcal{L}^H(x,t)F_n''(x)dx \\ &= -\int_{\mathbb{R}} F_n'(x)\mathcal{L}^H(dx,t) \longrightarrow -\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t) \end{aligned}$$

in L^2, as n tends to infinity, by occupation formula (4.2) and Lemma 5.9. This yields

$$\begin{aligned} \int_0^t F_n'(S_s^H)dS_s^H &= F_n(S_t^H) - F_n(0) - H(2-2^{2H-1})\int_0^t s^{2H-1}F_n''(S_s^H)ds \\ &\longrightarrow F(S_t^H) - F(0) + \frac{1}{2}\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t) \end{aligned}$$

in L^2, as $n \to \infty$, and the theorem follows since δ^H is closed in L^2. □

6. The Weighted Quadratic Covariation

In this section, we use one parameter integral

$$\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)$$

to study the *weighted quadratic covariation* of $f(S^H)$ and S^H. Denote

$$J_\varepsilon(f) := \frac{2H}{\varepsilon^{2H}}\int_0^t s^{2H-1}\left\{f(S_{s+\varepsilon}^H) - f(S_s^H)\right\}(S_{s+\varepsilon}^H - S_s^H)ds, \qquad t \geq 0$$

for $\varepsilon > 0$.

Recall that for a stochastic process X, the quadratic covariation $[f(X), X]$ of $f(X)$ and X is defined by

$$[f(X), X]_t := \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t \{f(X_{s+\varepsilon}) - f(X_s)\} (X_{s+\varepsilon} - X_s) ds$$

provided the limit exists uniformly in probability. More works for finite quadratic variation processes could be found in Russo and Vallois [23] (see also Russo and Vallois [22]). For a standard Brownian motion W, if f is locally square integrable, then the quadratic covariation $[f(W), W]$ of $f(W)$ and W exists in L^1 and

$$[f(W), W]_t = -\int_{\mathbb{R}} f(x) \mathcal{L}^W(dx, t),$$

where $\mathcal{L}^W(x,t)$ is the local time of W. For this see Föllmer *et al.* [13] and Eisenbaum [10]. However, this is not true for sub-fBm S^H, i.e. in general

$$\left[f(S^H), S^H\right]_t \neq -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t), \qquad 0 \leq t \leq T,$$

because $\left[f(S^H), S^H\right]_t = 0$ for $1/2 < H < 1$. Thus, we need to find a substitution tool of the quadratic variation.

Definition 6.1. Let $0 < H < 1$ and let f be a measurable function on $\mathbb{R}$. The limit

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon(f)$$

is said to be the *weighted quadratic covariation* of $f(S^H)$ and S^H, denoted by $[f(S^H), S^H]_t^{(W)}$, provided the limit exists uniformly in probability.

Our main object of this section is to explain and prove the following theorem.

Theorem 6.2. *Let $\frac{1}{2} < H < 1$ and let $f \in \mathcal{W}_p$ with $1 \leq p < \frac{2H}{1-H}$. Then the weighted quadratic covariation $\left[f(S^H), S^H\right]_t^{(W)}$ exists, and Bouleau-Yor's identity takes the form*

$$(2 - 2^{2H-1}) \left[f(S^H), S^H\right]_t^{(W)} = -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t), \quad t \geq 0. \tag{6.1}$$

In order to prove the theorem we need some preliminaries. Recently, Gradinaru-Nourdin [14] introduced the following very useful result:

Lemma 6.3. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying*

$$|g(x) - g(y)| \leq C|x - y|^a (1 + x^2 + y^2)^b, \quad (C > 0, 0 < a \leq 1, b > 0), \tag{6.2}$$

for all $x, y \in \mathbb{R}$ and let X be a locally Hölder continuous paths process with index $\gamma \in (0,1)$. Assume that V is a bounded variation continuous paths process. Set

$$X_\varepsilon^g(t) = \int_0^t g(\frac{X_{s+\varepsilon} - X_s}{\varepsilon^\gamma}) ds$$

for $t \geq 0$, $\varepsilon > 0$. If for each $t \geq 0$, as $\varepsilon \to 0$,

$$\|X_\varepsilon^g(t) - V_t\|_{L^2}^2 = O(\varepsilon^\alpha) \tag{6.3}$$

with $\alpha > 0$, then, for any $t \geq 0$, $\lim_{\varepsilon\to 0} X_\varepsilon^g(t) = V_t$ almost surely, and if g is non-negative, for any continuous stochastic process $\{Y_t:\ t\geq 0\}$,

$$\lim_{\varepsilon\to 0}\int_0^t Y_s g(\frac{X_{s+\varepsilon}-X_s}{\varepsilon^\gamma})ds \longrightarrow \int_0^t Y_s dV_s, \tag{6.4}$$

almost surely, uniformly in t on each compact interval.

Consider the function $x\mapsto h_u(x)$ on $(0,1]$ defined by

$$h_u(x) = (2u+x)^{2H} - 2^{2H-1}\left((u+x)^{2H}+u^{2H}\right)$$

with $u>0$. Then, the proof of Lemma 3.1 implies that

$$\begin{aligned} h_u(x) &= -2^{2H-1}(u+x)^{2H}\left\{1+(1-\frac{x}{u+x})^{2H} - \frac{1}{2^{2H-1}}\left(2-\frac{x}{u+x}\right)^{2H}\right\} \\ &= -2^{2H-1}\frac{1}{(u+x)^{2-2H}}\left[\frac{1}{2}H(2H-1)x^2+o(x^2)\right] \qquad (x\to 0). \end{aligned}$$

Proposition 6.4. *Let $f\in C(\mathbb{R})$ and $0<H<1$. Then the convergence*

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon^{2H}}\int_0^t f(S_s^H)(S_{s+\varepsilon}^H - S_s^H)^2 s^{2H-1}ds \longrightarrow \int_0^t f(S_s^H)s^{2H-1}ds, \tag{6.5}$$

holds almost surely, uniformly in t on each compact interval.

Proof. Let $\frac{1}{2}<H<1$. Similarly, one can prove the proposition for $0<H<\frac{1}{2}$. Denote

$$X_\varepsilon(t) = \frac{1}{\varepsilon^{2H}}\int_0^t (S_{s+\varepsilon}^H - S_s^H)^2 ds$$

for $\varepsilon>0$. We need to estimate

$$E\left|X_\varepsilon(t)-t\right|^2 = \frac{1}{\varepsilon^{4H}}\int_0^t\int_0^t A_\varepsilon(s,r)dsdr$$

for $t\geq 0$ and $\varepsilon>0$, where

$$\begin{aligned} A_\varepsilon(s,r) :&= E\left((S_{s+\varepsilon}^H - S_s^H)^2-\varepsilon^{2H}\right)\left((S_{r+\varepsilon}^H - S_r^H)^2-\varepsilon^{2H}\right) \\ &= E(S_{s+\varepsilon}^H - S_s^H)^2(S_{r+\varepsilon}^H - S_r^H)^2 + \varepsilon^{4H} \\ &\quad - \varepsilon^{2H}E\left((S_{r+\varepsilon}^H - S_r^H)^2 + (S_{s+\varepsilon}^H - S_s^H)^2\right). \end{aligned}$$

Clearly, we have

$$E(S_{s+\varepsilon}^H - S_s^H)^2 = h_s(\varepsilon)+\varepsilon^{2H}$$

for all $s\geq 0, \varepsilon>0$. An elementary calculus can show that

$$\begin{aligned} E[(S_{s+\varepsilon}^H - S_s^H)^2(S_{r+\varepsilon}^H - S_r^H)^2] &= E\left[(S_{s+\varepsilon}^H - S_s^H)^2\right]E\left[(S_{r+\varepsilon}^H - S_r^H)^2\right] \\ &\quad + 2\left(E\left[(S_{s+\varepsilon}^H - S_s^H)(S_{r+\varepsilon}^H - S_r^H)\right]\right)^2 \\ &= \left[h_s(\varepsilon)+\varepsilon^{2H}\right]\left[h_r(\varepsilon)+\varepsilon^{2H}\right] + 2(\mu_{s,r})^2 \end{aligned}$$

for all $r,s\geq 0$ and $\varepsilon>0$, where $\mu_{s,r} := E\left[(S_{s+\varepsilon}^H - S_s^H)(S_{r+\varepsilon}^H - S_r^H)\right]$, which yields

$$A_\varepsilon(s,r) = h_s(\varepsilon)h_r(\varepsilon) + 2(\mu_{s,r})^2.$$

We now estimate the expression

$$\mu_{s,r} = E\left[(S^H_{s+\varepsilon} - S^H_s)(S^H_{r+\varepsilon} - S^H_r)\right]$$

for $0 \le r < s \le T$ and $\varepsilon > 0$. Denote $s' = s+\varepsilon$, $r' = r+\varepsilon$ and $s > r+\varepsilon$. Define the function $x \mapsto g(x)$ by

$$g(x) = (x+r)^{2H} - (x+r')^{2H} + (x-r)^{2H} - (x-r')^{2H}, \qquad x \ge r'.$$

Then, by mean value theorem we have

$$\begin{aligned} g'(x) &= 2H\left[(x+r)^{2H-1} + (x-r)^{2H-1} - (x+r')^{2H-1} - (x-r')^{2H-1}\right] \\ &= 2H(2H-1)(r'-r)\left[(x-\xi)^{2H-2} - (x+\xi)^{2H-2}\right] \ge 0 \end{aligned}$$

and

$$\begin{aligned} \mu_{s,r} &= E\left[(S^H_{s+\varepsilon} - S^H_s)(S^H_{r+\varepsilon} - S^H_r)\right] \\ &= E\left[(S^H_{s'} - S^H_s)(S^H_{r'} - S^H_r)\right] = \frac{1}{2}(g(s') - g(s)) = \frac{1}{2}(s'-s)g'(\eta) \ge 0. \end{aligned}$$

for some $\xi \in (r, r')$ and $\eta \in (s, s')$. Notice that

$$\begin{aligned} 0 \le g'(\eta) &= 2H\left[(\eta+r)^{2H-1} + (\eta-r)^{2H-1} - (\eta+r')^{2H-1} - (\eta-r')^{2H-1}\right] \\ &\le 2H\left[(\eta-r)^{2H-1} - (\eta-r')^{2H-1}\right] \\ &\le 2H\frac{(r'-r)^H}{(\eta-r)^{1-H}} \le 2H\frac{(r'-r)^H}{(s-r)^{1-H}} \end{aligned}$$

by $1 - x^{2H-1} \le 1 - x \le (1-x)^H$. We get the desired estimate

$$|\mu_{s,r}| \le H\frac{\varepsilon^{1+H}}{(s-r)^{1-H}}$$

for $s > r > 0$ and $\varepsilon > 0$. Combining this with

$$h_s(\varepsilon)h_r(\varepsilon) \le C_H\frac{\varepsilon^4}{(s+\varepsilon)^{2-2H}(r+\varepsilon)^{2-2H}} \le C_H\frac{\varepsilon^{2+2H}}{(sr)^{1-H}}$$

for all $s, r > 0$ and $\varepsilon > 0$ small enough, we see that we have

$$A_\varepsilon(s,r) = h_s(\varepsilon)h_r(\varepsilon) + 2(\mu_{s,r})^2 \le C_H\frac{\varepsilon^{2+2H}}{(sr)^{1-H}} + H^2\frac{\varepsilon^{2+2H}}{(s-r)^{2-2H}}$$

for all $s > r > 0$ and $\varepsilon > 0$ small enough, which yields

$$E|X_\varepsilon(t) - t|^2 \le C_H t^{2H}\varepsilon^{2-2H}$$

for all $t \ge 0$ and $\varepsilon > 0$ small enough. Thus, we get the desired estimate

$$\|X_\varepsilon(t) - t\|^2_{L^2} = O\left(\varepsilon^\beta\right) \qquad (\varepsilon \to 0)$$

for each $t \ge 0$ and some $\beta > 0$. Notice that $g(x) = x^2$ satisfies the condition (6.2). We obtain the proposition by taking $Y_s = f(S^H_s)s^{2H-1}$ for $s \ge 0$. ◻

Corollary 6.5. *For all $t \in [0,T]$ we have*

$$\left[S^H, S^H\right]^{(W)}_t = t^{2H}$$

for all $0 < H < 1$.

Thanks to the above proposition, as a direct consequence, for $f \in C^1(\mathbb{R})$ we have

$$\left[f(S^H), S^H\right]_t^{(W)} = 2H \int_0^t f'(S_s^H) s^{2H-1} ds \tag{6.6}$$

for all $0 < H < 1$. In fact, by the Hölder continuity of sub-fBm S^H we have

$$U_\varepsilon(t) := \frac{2H}{\varepsilon^{2H}} \int_0^t o(S_{s+\varepsilon}^H - S_s^H)(S_{s+\varepsilon}^H - S_s^H) s^{2H-1} ds \longrightarrow 0$$

in L^1, as $\varepsilon \to 0$. Together this and proposition 6.4 lead to

$$J_\varepsilon(f) = \frac{2H}{\varepsilon^{2H}} \int_0^t f'(S_s^H)(S_{s+\varepsilon}^H - S_s^H)^2 s^{2H-1} ds + U_\varepsilon(t) \longrightarrow 2H \int_0^t f'(S_s^H) s^{2H-1} ds$$

uniformly in probability, as ε tends to zero. Thus, the next corollary follows from the occupation formula (4.1).

Corollary 6.6. *For $f \in C^1(\mathbb{R})$ we have*

$$(2 - 2^{2H-1}) \left[f(S^H), S^H\right]_t^{(W)} = -\int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t) \tag{6.7}$$

for all $t \geq 0$.

We can now show our main result in this section.

Proof of Theorem 6.2. By a localization argument we may assume that the function f is uniformly bounded.

If $f \in C^1(\mathbb{R})$, then the identity (6.1) follows from (6.7). Let now $f \notin C^1(\mathbb{R})$ and $f \in \mathcal{W}_p$ and let θ be a the mollifier defined in (5.9). Define the functions $f_n, n = 1, 2, \ldots$ by

$$f_n(x) = \int_{\mathbb{R}} \theta_n(x - y) f(y) dy$$

for all $x \in \mathbb{R}$, where $\theta_n(x) = n\theta(nx)$. Then $f_n \in C^\infty(\mathbb{R})$ for all $n \geq 0$, and $f_n \to f$ in $\mathcal{W}_p$, as n tends to infinity. So, for all $n \geq 0$ and $t \geq 0$

$$\lim_{\varepsilon \downarrow 0} J_\varepsilon(f_n) = \left[f_n(S^H), S^H\right]_t^{(W)} = -\frac{1}{2 - 2^{2H-1}} \int_{\mathbb{R}} f_n(x) \mathcal{L}^H(dx, t) \tag{6.8}$$

for all $t \in [0, T]$, almost surely, and

$$\lim_{n \to \infty} \int_{\mathbb{R}} f_n(x) \mathcal{L}^H(dx, t) = \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t) \tag{6.9}$$

in L^2 by Lemma 5.9. For $f \in \mathcal{W}_p$ we denote

$$\|f\| := E \left| \int_{\mathbb{R}} f(x) \mathcal{L}^H(dx, t) \right|.$$

Then

$$\|f\| \leq C_{p,p'} \|f\|_{[p]} E(\|\mathcal{L}^H(\cdot, t)\|_{(p')}) \leq C_{H,T,p,p'} \|f\|_{[p]}$$

for $p' > \frac{2H}{3H-1}$, by the Love-Young inequality.

On the other hand, we have

$$|f_n(x)| \leq \lambda := \sup_x |f(x)| < \infty$$

and

$$|f_n'(x)| \le n\int_0^2 |\theta'(y)f(x-\frac{y}{n})|dy \le n\lambda\int_0^2 |\theta'(y)|dy < \infty$$

for every $n = 1, 2, \ldots$ and $x \in \mathbb{R}$. Combining this with $E|U_\varepsilon(t)| \le C_H t^{2H} < \infty$, we get

$$\begin{aligned} E|J_\varepsilon(f_n)| &\le \frac{2H}{\varepsilon^{2H}}\int_0^t E|f_n'(S^H_{s+\varepsilon})|(S^H_{s+\varepsilon}-S^H_s)^2 s^{2H-1}ds + E|U_\varepsilon(t)| \\ &\le C_H t^{2H}\left(n\lambda\int_0^2 |\theta'(y)|dy\right) + C_H t^{2H} \end{aligned}$$

for all $\varepsilon > 0$ and $n \ge 1$, which yields $\sup_{\varepsilon\ge 0} E|J_\varepsilon(f_n)| < \infty$ for any $n \ge 1$. Thus, for all $n \ge 1$ the convergence (6.8) holds in L^q with $0 < q < 1$, and

$$\lim_{\varepsilon\downarrow 0} E\left(|J_\varepsilon(f_n)|^q\right) = E\left(\left|\left[f_n(S^H), S^H\right]_t^{(W)}\right|^q\right) \le C_{H,q}\|f_n\|^q.$$

In particular, for n large enough we have

$$\lim_{\varepsilon\downarrow 0} E\left(|J_\varepsilon(f_n)|^q\right) \le C_{H,q}(\|f\|^q + \frac{1}{n})$$

because f_n converges to f in $\mathcal{W}_p$, as n tends to infinity. This shows that, for ε small enough

$$E|J_\varepsilon(f)|^q \le C_{H,q}\|f\|^q.$$

It follows from

$$\lim_{\varepsilon\downarrow 0} P\left(|\beta_H J_\varepsilon(f_n) + \int_{\mathbb{R}} f_n(x)\mathcal{L}^H(dx,t)| \ge \delta\right) = 0$$

for all $n \ge 1$ and $\beta_H = 2 - 2^{2H-1}$ that

$$\begin{aligned} &\lim_{\varepsilon\downarrow 0} P\left(|\beta_H J_\varepsilon(f) + \int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)| \ge \delta\right) \\ &\quad \le (\frac{3}{\delta})^q C_{H,q}\|f-f_n\|^q + P\left(|\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t) - \int_{\mathbb{R}} f_n(x)\mathcal{L}^H(dx,t)| \ge \frac{\delta}{3}\right) \end{aligned}$$

for n large enough and every $\delta > 0$, which gives the desired convergence

$$\lim_{\varepsilon\downarrow 0}(2-2^{2H-1})J_\varepsilon(f) = -\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)$$

in probability. This completes the proof. □

According to Theorem 5.10, we get an Itô's formula.

Corollary 6.7. *Let $\frac{1}{2} < H < 1$ and let $f \in \mathcal{W}_p$ be a left continuous function with $1 \le p < \frac{2H}{1-H}$. If F is an absolutely continuous function with the derivative $F' = f$, then the Itô type formula*

$$F(S^H_t) = F(0) + \int_0^t f(S^H_s)dS^H_s + \frac{1}{2}(2-2^{2H-1})\left[f(S^H), S^H\right]_t^{(W)} \tag{6.10}$$

holds for all $t \in [0, T]$.

Clearly, this is an analogue of Föllmer-Protter-Shiryayev's formula (see Föllmer *et al.* [13]). It is an improvement in terms of the hypothesis on f and it is also quite interesting itself. Recall that a process X is called a finite quadratic variation process if its quadratic variation $[X,X]$ exists. Sub-fBm with the index $H \geq \frac{1}{2}$ is of finite quadratic variation. In fact, we have $[S^H, S^H]_t = 0$ for $H > \frac{1}{2}$. We refer to Russo and Vallois [23] for a complete description of stochastic calculus with respect to finite quadratic variation process. If X is a finite quadratic variation process and if $F \in C^2(\mathbb{R})$, then the following Itô's formula holds:

$$F(X_t) = F(0) + \int_0^t f(X_s)d^- X_s + \frac{1}{2}\left[f(X), X\right]_t, \tag{6.11}$$

where $f = F'$ and the integral $\int_0^t f(X_s)d^- X_s$ is the forward (pathwise) integral defined by

$$\int_0^t f(X_s)d^- X_s = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_0^t f(X_s)(X_{s+\varepsilon} - X_s)ds$$

and $[f(X), X]_t = \int_0^t f'(X_s)d[X,X]_s$. If X is a continuous semimartingale, then the integral

$$\int_0^t f(X_s)d^- X_s = \int_0^t f(X_s)dX_s$$

is classical Itô's integral. For sub-fBm S^H with $\frac{1}{2} < H < 1$ we have

$$\int_0^t f(S^H_s)d^- S^H_s = \int_0^t f(S^H_s)dS^H_s + H(2 - 2^{2H-1}) \int_0^t f'(S^H_s)s^{2H-1}ds.$$

More generally, we have

Corollary 6.8. *Let $\frac{1}{2} < H < 1$ and let $f \in \mathcal{W}_p$ be a left continuous function with $1 \leq p < \frac{2H}{1-H}$. Then we have*

$$\int_0^t f(S^H_s)d^- S^H_s = \int_0^t f(S^H_s)dS^H_s + \frac{1}{2}(2 - 2^{2H-1}) \left[f(S^H), S^H\right]^{(W)}_t.$$

Remark 6.9. By a proof similar to Lemma 3.1 in Gradinaru–Nourdin [14], one can obtain the following convergence (see also Gradinaru–Nourdin [15]):

$$\lim_{n\to\infty} \sum_{j=1}^n j^{2H-1} g(S^H_{t_j})(S^H_{t_j} - S^H_{t_{j-1}})^2 = \int_0^t g(S^H_s)s^{2H-1}ds \tag{6.12}$$

almost surely, where $t_j = tj/n$ and $g \in C(\mathbb{R})$. Thus, similar to the proof of Theorem 5.10 we can show that the convergence

$$2H\beta_H \lim_{n\to\infty} \sum_{j=1}^n j^{2H-1}\{f(S^H_{t_j}) - f(S^H_{t_{j-1}})\}(S^H_{t_j} - S^H_{t_{j-1}}) = -\int_{\mathbb{R}} f(x)\mathcal{L}^H(dx,t)$$

holds with $\beta_H = 2 - 2^{2H-1}$, which yields

$$2H \lim_{n\to\infty} \sum_{j=1}^n j^{2H-1}\{f(S^H_{t_j}) - f(S^H_{t_{j-1}})\}(S^H_{t_j} - S^H_{t_{j-1}}) = [f(S^H), S^H]^{(W)},$$

where $f \in \mathcal{W}_p$ with $1 \le p < \frac{2H}{1-H}$ and the limits are uniform in probability. Thus, we can redefine the *weighted quadratic covariation* as follows:

The limit

$$2H \lim_{\|\pi_n\|\to 0} \sum_{t_j \in \pi_n} (\Lambda_j)^{2H-1} \{f(S^H_{t_j}) - f(S^H_{t_{j-1}})\}(S^H_{t_j} - S^H_{t_{j-1}}) \tag{6.13}$$

is called the *weighted quadratic covariation* of $f(S^H)$ and S^H, provided the limit exists uniformly in probability, where $\pi_n = \{0 = t_0 < t_1 < \cdots < t_n = t\}$ denotes an arbitrary partition of the interval $[0,t]$ with $\|\pi_n\| = \sup_j(t_j - t_{j-1})$ tends to zero, and $\Lambda_j = \frac{t_j}{t_j - t_{j-1}}$, $j = 1, 2, \ldots, n$.

Remark 6.10. By using some ideas from Feng-Zhao [12] one can define the integral over the plane

$$\int_{\mathbb{R}} \int_0^t f(x,s)\mathcal{L}^H(dx,ds)$$

and study the existence of *weighted quadratic covariation* $\left[f(S^H,\cdot), S^H\right]^{(W)}$, where $f:\ \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}$ is a measurable function.

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