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THE LONG-RANGE DEPENDENCE OF LINEAR LOG-FRACTIONAL STABLE MOTION

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ABSTRACT. We study the dependence structure of the α-stable random process *linear log-fractional stable motion* (Ulog-FSM). It is defined for $\alpha \in (1, 2]$ and real numbers $(a, b) \neq (0, 0)$. Ulog-FSM is actually a collection of processes parametrized by α, a, and b. All of its moments of order $p \geq \alpha$ are infinite, including the variance. It also has stationary increments. In the "well-balanced" case $a = b$, it reduces to log-fractional stable motion (log FSM), a *self-similar* process. Unlike log-FSM, it is not self-similar in the "unbalanced" case $a \neq b$. Since the covariance does not exist, other measures are necessary to analyze the dependence structure of Ulog-FSM. We use the *codifference* and the *covariation*. Ulog-FSM exhibits *long-range dependence* because over long lags of time, the codifference and the covariation decay "slowly" to zero.

1. Introduction

We examine here the dependence structure of the process *linear log-fractional stable motion* (Ulog-FSM). Our principal objective is to determine the precise nature of its asymptotic dependence over long lags of time, by studying its increments, which are stationary and may be called linear log-stable noise.

1.1. The Ulog-FSM process. Denoted by $U(a, b) := \{U(a, b; t), t \in \mathbb{R}\}$, where $a, b \in \mathbb{R}$ satisfy $|a| + |b| > 0$, linear log-fractional stable motion is defined as

$$U(a, b; t) =$$
$$\int_{\mathbb{R}} \left(a \left[\ln_0 (t - x)_+ - \ln_0 (-x)_+\right] + b \left[\ln_0 (t - x)_- - \ln_0 (-x)_-\right]\right) M_\alpha(\mathrm{d}x)$$
$$=: \int_{\mathbb{R}} u(a, b; t, x) \, M_\alpha(\mathrm{d}x). \tag{1.1}$$

The function $\ln_0(x)$ equals $\ln x$ if $x > 0$ and equals 0 if $x \leq 0$, and we write $x_+ = \max(x, 0)$ and $x_- = -\min(x, 0)$. The random measure M_α is assumed to be symmetric α-stable ($S\alpha S$) with $1 < \alpha \leq 2$ and to have Lebesgue control measure

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(see also Section 2). As a consequence, the process $U(a,b)$ is $S\alpha S$. $U(a,b)$ is said to be "well-balanced" if $a=b$ and "unbalanced" otherwise.

For fixed t, the characteristic function of $U(a,b)$ is given by

$$\mathrm{E}\exp\{i\theta U(a,b;t)\} = \exp\{-|\theta|^\alpha \|U(a,b;t)\|_\alpha^\alpha\}$$

where

$$\|U(a,b;t)\|_\alpha := \left(\int_E |u(a,b;t,x)|^\alpha \mathrm{d}x\right)^{1/\alpha} < \infty.$$

serves as a scale parameter. Let d be a positive integer and $\phi, \theta_1, \ldots, \theta_d$, and $t_1, \ldots, t_d$ be any real numbers. The finite-dimensional distributions of $U(a,b;t)$ are

$$\begin{aligned}\mathrm{E}\exp\left\{i\phi\sum_{j=1}^d \theta_j U(a,b;t_j)\right\} &= \exp\left\{-\int_E \left|\phi\sum_{j=1}^d \theta_j u(a,b;t_j,x)\right|^\alpha dx\right\} \\ &= \exp\left\{-|\phi|^\alpha \left\|\sum_{j=1}^d \theta_j U(a,b;t_j)\right\|_\alpha^\alpha\right\}.\end{aligned}$$

When $\alpha = 2$, $U(a,b)$ is Gaussian and $\|U(a,b;t)\|_2$ equals its standard deviation divided by $\sqrt{2}$. If $\alpha < 2$, then $U(a,b)$ is a non-Gaussian stable process. The probability tails of $\sum_{j=1}^d \theta_j U(a,b;t_j)$ behave like $c|x|^{-\alpha}, c>0$ as $|x| \to \infty$.

In (1.1), the index of stability, α, is restricted to $(1,2]$. This is due to the relation

$$\|U(a,b;t)\|_\alpha^\alpha \le (|a| \vee |b|)^\alpha \int_{\mathbb{R}} |\ln|t-x| - \ln|x||^\alpha \,\mathrm{d}x,$$

which is finite since when $x \sim 0, \int_0^\delta (\ln|x|)^\alpha \,\mathrm{d}x < \infty$, while if $|x| \sim \infty$, then $\ln|t-x| - \ln|x| \sim -t/x$ and hence for $A>0$, $\int_A^\infty x^{-\alpha}\mathrm{d}x < \infty$ if and only if $\alpha > 1$. Thus, Ulog-FSM is $S\alpha S$, provided $1 < \alpha < 2$. It therefore has finite moments of order p only when $p < \alpha$, but infinite moments if $p \ge \alpha$. Since the variance, in particular, does not exist, the covariance is evidently unsuitable for calibrating the dependence structure of $U(a,b)$.

1.2. The log-FSM process. Ulog-FSM appears in [6, p. 355] as a generalization of the "well-balanced" case $a=b$.[1] (The "U" in "Ulog" stands for "unbalanced.") When $a=b=1$, one gets the well-balanced case which is the process called *log-fractional stable motion* (log-FSM),

$$U(1,1;t) = \int_{\mathbb{R}} (\ln_0|t-x| - \ln_0|-x|)\, M_\alpha(\mathrm{d}x) = \int_{\mathbb{R}} (\ln|t-x| - \ln|-x|)\, M_\alpha(\mathrm{d}x).$$

Log-FSM has the property of H-self-similarity with $H = 1/\alpha$. Recall that a process $\{X(t), t\in\mathbb{R}\}$, is H-self-similar (H-ss) if there exists $H>0$ such that for every $c>0$

$$\{X(ct), t\in\mathbb{R}\} \stackrel{d}{=} \{c^H X(t), t\in\mathbb{R}\}$$

[1]In that paper neither the underlying α-stable random measure nor the process is necessarily symmetric.

where $\stackrel{d}{=}$ denotes equality of the finite-dimensional distributions. An H-ss process with stationary increments is refereed to as H-sssi. Thus, a self-similar process involves a rescaling between the time domain and the spatial domain. We refer to the monograph [2] and [6, Chapter 7] about self-similar processes in general and to [6, Section 7.6] about log-FSM in particular.

Log-FSM can be viewed as an extension to the "boundary" case $H = 1/\alpha$ $(1 < \alpha < 2)$ of the H-ss process *linear fractional stable motion* (LFSM). LFSM is defined for $0 < \alpha < 2, 0 < H < 1, H \neq 1/\alpha$ and $(a, b) \neq (0, 0)$ by $L_{\alpha,H}(a, b) := \{L_{\alpha,H}(a, b; t), t \in \mathbb{R}\}$, where

$$L_{\alpha,H}(a,b;t) = \int_{\mathbb{R}} l_{\alpha,H}(a,b;t) M_\alpha(\mathrm{d}x) \tag{1.2}$$

with

$$l_{\alpha,H}(a,b;t) := \\ a\left[\left((t-x)_+\right)^{H-1/\alpha} - \left((-x)_+\right)^{H-1/\alpha}\right] + b\left[\left((t-x)_-\right)^{H-1/\alpha} - \left((-x)_-\right)^{H-1/\alpha}\right].$$

and M_α is as above. See [6, Section 7.4] for further details.

However, unlike log-FSM or LFSM, Ulog-FSM is not H-ss if $a \neq b$. (See [6, p. 355].)

1.3. The ΔUlog-FSM process. We will now focus on the increment processes of Ulog-FSM, which are defined for any $h \in \mathbb{R}$, as $\Delta U(a, b; h) := \{\Delta U(a, b; h, t), t \in \mathbb{R}\}$, where

$$\begin{aligned} \Delta U(a,b;h,t) &= U(a,b;t+h) - U(a,b;h) \\ &= \int_{\mathbb{R}} \left(a\left[\ln_0(t+h-x)_+ - \ln_0(h-x)_+\right]\right. \\ &\quad \left. + b\left[\ln_0(t+h-x)_- - \ln_0(h-x)_-\right]\right) M_\alpha(\mathrm{d}x) \\ &=: \int_{\mathbb{R}} \tilde{u}(a,b;h,t,x) M_\alpha(\mathrm{d}x). \end{aligned}$$

$\Delta U(a, b; h)$ is *strictly stationary* because

$$\begin{aligned} &\mathrm{E}\exp\left\{i\phi\sum_{j=1}^d \theta_j \Delta U(a,b;h,t_j)\right\} = \exp\left\{-\int_E \left|\phi\sum_{j=1}^d \theta_j \tilde{u}(a,b;h,t_j,x)\right|^\alpha dx\right\} \\ &= \exp\left\{-\int_E \left|\phi\sum_{j=1}^d \theta_j \tilde{u}(a,b;0,t_j,x)\right|^\alpha dx\right\} = \mathrm{E}\exp\left\{i\phi\sum_{j=1}^d \theta_j \Delta U(a,b;0,t_j)\right\}. \end{aligned}$$

Thus,

$$\{\Delta U(a,b;h,t), t \in \mathbb{R}\} \stackrel{d}{=} \{\Delta U(a,b;0,t), t \in \mathbb{R}\}, \quad h \in \mathbb{R}.$$

Our goal is to analyze the asymptotic dependence of the *one-step* increment process, $\Delta U(a,b)$. We call it ΔUlog-FSM and denote it for $t \in \mathbb{R}$, as

$$\begin{aligned}\Delta U(a,b;t) &= U(a,b;t+1) - U(a,b;t) \\ &= \int_{\mathbb{R}} \Big(a\left[\ln_0(t+1-x)_+ - \ln_0(t-x)_+\right] \\ &\qquad + b\left[\ln_0(t+1-x)_- - \ln_0(t-x)_-\right]\Big) M_\alpha(\mathrm{d}x) \\ &=: \int_{\mathbb{R}} \tilde{u}(a,b;t,x)\, M_\alpha(\mathrm{d}x). \end{aligned} \tag{1.3}$$

We will analyze the dependence structure as $t \to \infty$ of the stationary process $\Delta U(a,b)$. This, in turn, will identify the corresponding structure for $U(a,b)$, since the dependence of an H*-self-similar stationary increment* process is determined by the dependence of its increments, which often are referred to as "noise." The asymptotic dependence as $t \to \infty$ of log-FSM, which is $1/\alpha$-sssi, has been studied in [1], [5], and [7], but can also be derived from the results of this paper (see Theorem 3.2 and Theorem 3.3 below). Another $1/\alpha$-sssi process is α*-stable motion* and it, too, lies on the $H = 1/\alpha$-boundary of LFSM. It has *independent* increments, hence, its dependence structure is trivial.

The paper is structured as follows. Section 2 reviews the two measures of dependence which will be applied to ΔUlog-FSM. They are the *codifference* and the *covariation*. They have been used in previous studies to obtain the asymptotic dependence for other processes, for example, LFSM. In Section 3 the principal theorems about the asymptotic codifference and the asymptotic covariation of ΔUlog-FSM appear. The ensuing work generally considers all $(a,b) \neq (0,0)$, including the well-balanced case $a = b$. The proofs for the asymptotic codifference and the asymptotic covariation of ΔUlog-FSM are established in Section 4 and Section 5. Facts about the measures are proved in the Appendix.

2. Two Measures of Dependence

To study the dependence of ΔUlog-FSM we rely on two "standard" measures, the *codifference* and the *covariation*. They replace the covariance when the variance does not exist.

2.1. Codifference and generalized codifference. Consider, first, an evaluation of the codependence of random variables $Y(t)$ and $Y(0)$ that is given by

$$\begin{aligned} r(t) &= \mathrm{E}\exp\{i(\theta_1 Y(t) + \theta_2 Y(0))\} - \mathrm{E}\exp\{i\theta_1 Y(t)\}\,\mathrm{E}\exp\{i\theta_2 Y(0)\} \\ &= e^{-A(\theta_1,\theta_2)}\left(e^{-I_Y(\theta_1,\theta_2;t)} - 1\right), \end{aligned} \tag{2.1}$$

where $\theta_1, \theta_2 \in \mathbb{R}$ and

$$e^{-A(\theta_1,\theta_2)} = \mathrm{E}\exp\{i\theta_1 Y(t)\}\,\mathrm{E}\exp\{i\theta_2 Y(0)\}.$$

If $\theta_1 = 0$ or $\theta_2 = 0$, then $r(t) \equiv 0$, so we will exclude the trivial case $\operatorname{sign}(\theta_1\theta_2) = 0$ in the sequel. The quantity

$$\begin{aligned} I_Y(\theta_1, \theta_2; t) &= -\ln \mathrm{E} \exp\{i(\theta_1 Y(t) + \theta_2 Y(0))\} + \ln \mathrm{E} \exp\{i\theta_1 Y(t)\} \\ &\quad + \ln \mathrm{E} \exp\{i\theta_2 Y(0)\} \\ &= \|\theta_1 Y(t) + \theta_2 Y(0)\|_\alpha^\alpha - \|\theta_1 Y(t)\|_\alpha^\alpha - \|\theta_2 Y(0)\|_\alpha^\alpha \end{aligned} \tag{2.2}$$

is called the *generalized codifference*. Taking $(\theta_1, \theta_2) = (1, -1)$ obtains the *codifference*,

$$\begin{aligned} \tau_Y(t) : &= -I_Y(1, -1; t) \\ &= \ln \mathrm{E} \exp\{i(Y(t) - Y(0))\} - \ln \mathrm{E} \exp\{iY(t)\} - \ln \mathrm{E} \exp\{-iY(0)\} \\ &= -\|Y(t) - Y(0)\|_\alpha^\alpha + \|Y(t)\|_\alpha^\alpha + \|Y(0)\|_\alpha^\alpha . \end{aligned} \tag{2.3}$$

Suppose that $Y := \{Y(t), t \in \mathbb{R}\}$ is a strictly stationary $S\alpha S$ with $0 < \alpha \leq 2$ and can be represented by

$$Y(t) = \int_E g(t, x) M(\mathrm{d}x) \tag{2.4}$$

where $g : \mathbb{R} \times \mathbb{E} \to \mathbb{R}$ is deterministic and M is a $S\alpha S$ random measure on a measure space (the "control" space) $(\mathbb{E}, \mathcal{E}, m)$.[2] For fixed t the random variable $Y(t)$ is characterized by

$$\mathrm{E} \exp\{i\theta Y(t)\} = \exp\{-|\theta|^\alpha \|Y(t)\|_\alpha^\alpha\} \tag{2.5}$$

with scale parameter

$$\|Y(t)\|_\alpha := \left(\int_E |g(t, x)|^\alpha m(\mathrm{d}x) \right)^{1/\alpha} < \infty. \tag{2.6}$$

By using (2.2), (2.5), and (2.6) the generalized codifference of Y then can be expressed as

$$I_Y(\theta_1, \theta_2; t) = \int_E \left(|\theta_1 g(t, x) + \theta_2 g(0, x)|^\alpha - |\theta_1 g(t, x)|^\alpha - |\theta_2 g(0, x)|^\alpha \right) m(\mathrm{d}x), \tag{2.7}$$

and by (2.3), the *codifference* equals

$$\tau_Y(t) = 2 \int_E |g(0, x)|^\alpha m(\mathrm{d}x) - \int_E |g(t, x) - g(0, x)|^\alpha m(\mathrm{d}x). \tag{2.8}$$

When $\alpha = 2$, Y is stationary Gaussian, and the codifference and covariance coincide. Thus, one can regard the codifference as "extending" the covariance in the $S\alpha S$ case, $\alpha < 2$.

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[2] m is the control measure for M. This means that if $A \in \mathcal{E}$ with $m(A) < \infty$, then $M(A)$ is a $S\alpha S$ random variable with characteristic function $\mathrm{E} \exp\{i\theta M(A)\} = \exp\{-|\theta|^\alpha m(A)\}$, $\theta \in \mathbb{R}$. In addition, suppose $\{A_n\}$, $n = 1, 2, \ldots$, is a pairwise disjoint sequence satisfying $m(A_n) < \infty$. Then (i) any finite subcollection of $\{M(A_n)\}$ are independent random variables and (ii) if $m\left(\cup_{n=1}^\infty A_n\right) < \infty$, $M\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty M(A_n)$ almost surely. (i) and (ii) identify M as being *independently scattered* and *σ-additive*. Refer to [6, chapter 3.3].

We can now specify the generalized codifference of ΔUlog-FSM from (1.3) and (2.7), taking the control measure m to be Lebesgue measure, as

$$\begin{aligned} I_{\Delta U(a,b)}(\theta_1,\theta_2;t) &= \int_{\mathbb{R}} \left(\left|\theta_1 \tilde{u}\,(a,b;t,x)+\theta_2 \tilde{u}\,(a,b;0,x)\right|^{\alpha} \right. \\ &\quad \left. -\left|\theta_1 \tilde{u}\,(a,b;t,x)\right|^{\alpha}-\left|\theta_2 \tilde{u}\,(a,b;0,x)\right|^{\alpha}\right) \mathrm{d}x \\ &=: \Xi_{-\infty}^{\infty}\left(\theta_1 \tilde{u}\,(a,b;t,x)\,, \theta_2 \tilde{u}\,(a,b;0,x)\right). \end{aligned} \tag{2.9}$$

The codifference, by (2.8), is

$$\tau_{\Delta U(a,b)}(t)=2\int_{\mathbb{R}}\left|\tilde{u}\,(a,b;0,x)\right|^{\alpha} \mathrm{d}x-\int_{\mathbb{R}}\left|\tilde{u}\,(a,b;t,x)-\tilde{u}\,(a,b;0,x)\right|^{\alpha} \mathrm{d}x. \tag{2.10}$$

An interesting fact is that the generalized codifference is symmetric with respect to both (a,b) and (θ_1,θ_2). (The proof is in the Appendix.)

Proposition 2.1. $I_{\Delta U(a,b)}(\theta_1,\theta_2;t)=I_{\Delta U(b,a)}(\theta_2,\theta_1;t)$.

The codifference $\tau_{\Delta U(a,b)}(t)$ is symmetric with respect to (a,b). We have here (see the Appendix for the proof),

Corollary 2.2. $\tau_{\Delta U(a,b)}(t)=\tau_{\Delta U(b,a)}(t)$.

2.2. Covariation. We turn to the second measure of dependence, the *covariation*. It is restricted to $S\alpha S$ processes Y (not necessarily stationary) for which $1<\alpha\leq 2$ and have the representation (2.4). For any two components $Y_1(t)$ and $Y(t_2)$ the covariation is defined by

$$[Y(t_1),Y(t_2)]_{\alpha}:=\int_E g(t_1,x)g(t_2,x)^{\langle\alpha-1\rangle}m(\mathrm{d}x)$$

where $y^{\langle p\rangle}:=|y|^p\mathrm{sign}(y)$ is the signed p'th power. Usually,

$$[Y(t_1),Y(t_2)]_{\alpha}\neq[Y(t_2),Y(t_1)]_{\alpha}$$

when $\alpha<2$, so the covariation is not symmetric in its arguments ([6, Corollary 2.7.10, p. 91]). In the Gaussian case $\alpha=2$, the covariation is symmetric in its arguments and equals one-half the covariance.

If Y is also stationary then

$$[Y(t_1),Y(t_2)]_{\alpha}=[Y(t_1-t_2),Y(0)]_{\alpha}.$$

One can therefore define the covariation of a stationary $S\alpha S$ process by

$$[Y(t),Y(0)]_{\alpha}:=\int_E g(t,x)g(0,x)^{\langle\alpha-1\rangle}m(\mathrm{d}x).$$

The covariation of ΔUlog-FSM is

$$[\Delta U(a,b;t),\Delta U(a,b;0)]_{\alpha}=\int_{\mathbb{R}}\tilde{u}\,(a,b;t,x)\,\tilde{u}\,(a,b;0,x)^{\langle\alpha-1\rangle}\,\mathrm{d}x. \tag{2.11}$$

Unlike the codifference, the covariation of $\Delta U(a,b)$ is not symmetric in a and b. Interchanging a and b, however, obtains the covariation of $\Delta U(b,a)$, which in turn equals the *reverse covariation* of $\Delta U(a,b)$, namely $[\Delta U(a,b;0),\Delta U(a,b;t)]_{\alpha}$. Indeed (the proof is in the Appendix),

Proposition 2.3. $[\Delta U(a,b;0),\Delta U(a,b;t))]_{\alpha}=[\Delta U(b,a;t),\Delta U(b,a;0)]_{\alpha}$.

2.3. Rates of decay. The ΔUlog-FSM process $\Delta U(a,b)$ is a *stationary moving average* since by (1.3) it is representable as $\int_{\mathbb{R}} g(t-x)M(\mathrm{d}x)$. As a consequence, both its codifference and covariation converge to zero.[3] Our goal is to determine their rates of decay. They are useful, in particular, for evaluating the convergence or divergence of the series

$$\sum_{t=1}^{\infty} |\tau_{\Delta U(a,b)}(t)| \quad \text{and} \quad \sum_{t=1}^{\infty} |[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha|,$$

where $\tau_{\Delta U(a,b)}(t)$ and $[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha$ are defined by (2.10) and (2.11). If these series diverge, the process $\{\Delta U(a,b;t), t \in \mathbb{R}\}$ is said to be *long-range dependent* by analogy to the finite-variance case. If they converge, the process is said to be *short-range dependent* (see [6]). One applies, by extension, the same terminology to $\{U(a,b;t), t \in \mathbb{R}\}$, namely, to the process Ulog-FSM.

It turns out (Theorem 3.4) that the codifference $\tau_{\Delta U(a,b)}(t)$ behaves as $t \to \infty$ like

$$c_1 t^{1-\alpha} \to 0$$

for all $(a,b) \neq (0,0)$ where $c_1 > 0$. The exponent $1-\alpha$ is called the *intensity* of the convergence. Since $\alpha < 2$, the series $\sum_{t=1}^{\infty} |\tau_{\Delta U(a,b)}(t)|$ diverges.

We will also show (Theorem 3.5) that, in fact, the covariation of $\Delta U(a,b)$ converges to zero like

$$c_2 t^{1-\alpha} \ln t \quad \text{if } 0 \neq b \neq a \text{ with } c_2 \neq 0.$$

On the other hand, if either $b = 0$ or $a = b$, then it behaves like $c_3 t^{1-\alpha}$ and $c_3 > 0$. In all cases $(a,b) \neq (0,0)$, therefore, $\sum_{t=1}^{\infty} [|\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha|$ diverges.

As a consequence, $\Delta U(a,b)$ displays long-range dependence for $(a,b) \neq (0,0)$ when either measure is applied. The codifference, however, converges faster than the covariation when $0 \neq b \neq a$. The process has a *stronger* long-range dependence from the covariation since it it converges more slowly to zero as $t \to \infty$.

3. Main Results

We examine first the asymptotic behavior of its generalized codifference (2.9).

Theorem 3.1. *As $t \to \infty$, the generalized codifference of the $S\alpha S$ ΔUlog-FSM satisfies*

$$I_{\Delta U(a,b)}(\theta_1, \theta_2; t) \sim G_u t^{1-\alpha} \tag{3.1}$$

where

$$\begin{aligned} G_u := G_u(a,b,\theta_1,\theta_2) \quad &= \quad |a|^\alpha\, \Xi_{-\infty}^{0}\left(\frac{\theta_1}{1-x}, \frac{\theta_2}{-x}\right) + \Xi_0^1\left(\frac{a\theta_1}{1-x}, \frac{-b\theta_2}{x}\right) \\ &\quad + |b|^\alpha\, \Xi_0^\infty\left(\frac{\theta_2}{1+x}, \frac{\theta_1}{x}\right) \end{aligned} \tag{3.2}$$

and Ξ is defined by (2.9).

[3]See, for example, [6, Theorem 4.7.3, p. 212] regarding the codifference converging to zero. A similar proof also holds for the covariation.

A more explicit formulation for the constant G_u (the index "u" stands for unbalanced) is

$$\begin{aligned} G_u &= |a|^\alpha \int_{-\infty}^0 \left(\left| \frac{\theta_1}{1-x} + \frac{\theta_2}{-x} \right|^\alpha - \left| \frac{\theta_1}{1-x} \right|^\alpha - \left| \frac{\theta_2}{-x} \right|^\alpha \right) \mathrm{d}x \\ &\quad + \int_0^1 \left(\left| \frac{a\theta_1}{1-x} - \frac{b\theta_2}{x} \right|^\alpha - \left| \frac{a\theta_1}{1-x} \right|^\alpha - \left| \frac{b\theta_2}{x} \right|^\alpha \right) \mathrm{d}x \\ &\quad + |b|^\alpha \int_0^\infty \left(\left| \frac{\theta_2}{1+x} + \frac{\theta_1}{x} \right|^\alpha - \left| \frac{\theta_2}{1+x} \right|^\alpha - \left| \frac{\theta_1}{x} \right|^\alpha \right) \mathrm{d}x. \qquad (3.3) \end{aligned}$$

The sign of G_u is of interest and in particular, whether G_u is zero or not, as this affects the asymptotic behavior of (3.1). As mentioned earlier, the case $\theta_1\theta_2 = 0$ is ignored, since then $r(t) \equiv 0 \equiv I_{\Delta U(a,b)}(\theta_1, \theta_2; t)$ in (2.1).

Proposition 3.2. *Suppose* $\operatorname{sign}(ab) \neq 1$.

(1) *If* $\operatorname{sign}(\theta_1\theta_2) = 1$*, then* $G_u > 0$*.*
(2) *If* $\operatorname{sign}(\theta_1\theta_2) = -1$*, then* $G_u < 0$*.*

Proof. In (3.3) the first and third integrals are identical. If $\operatorname{sign}(\theta_1\theta_2) = -1$, they are negative because $|c-d|^\alpha \leq |c|^\alpha + |d|^\alpha$. They are positive if $\operatorname{sign}(\theta_1\theta_2) = 1$ since $\alpha > 1$ and due to the inequality $|c+d|^\alpha > |c|^\alpha + |d|^\alpha$ for $\operatorname{sign}(cd) = 1$. When $\operatorname{sign}(ab) = 0$, the second integral and either the first or the third integral vanish, hence, $\operatorname{sign}(G_u) = \operatorname{sign}(\theta_1\theta_2)$. The second integral is positive if $\operatorname{sign}(ab\theta_1\theta_2) = -1$; however, it is negative if $\operatorname{sign}(ab\theta_1\theta_2) = 1$. It follows that when $\operatorname{sign}(ab) = -1$ and $\operatorname{sign}(\theta_1\theta_2) = -1$, all three integrals are negative, but if $\operatorname{sign}(ab) = -1$ and $\operatorname{sign}(\theta_1\theta_2) = 1$ then they are all positive. □

If $\operatorname{sign}(ab) = 1$ and $\operatorname{sign}(\theta_1\theta_2) \neq 0$, then the first and third integrals have the opposite sign of the second integral. In this case $\operatorname{sign}(G_u)$ cannot be determined so easily and probably varies with respect to these four parameters. However, for the codifference where $\theta_1 = 1$ and $\theta_2 = -1$, the signs can be determined explicitly (see Theorem 3.4).

When $a = b$, ΔUlog-FSM becomes log-FSN (up to the multiplicative constant a). The generalized codifference is obtained by factoring a from each of the integrals in (3.3). The integrands of $\int_{-\infty}^0$ and $\int_0^1$ are now the same, so $\int_{-\infty}^0 + \int_0^1 = \int_{-\infty}^1$. We get

Corollary 3.3. *The generalized codifference of SαS log-FSN satisfies (3.1) with*

$$\begin{aligned} G_u(a, a, \theta_1, \theta_2) &= |a|^\alpha \left[\int_{-\infty}^1 \left(\left| \frac{\theta_1}{1-x} - \frac{\theta_2}{x} \right|^\alpha - \left| \frac{\theta_1}{1-x} \right|^\alpha - \left| \frac{\theta_2}{x} \right|^\alpha \right) \mathrm{d}x \right. \\ &\quad \left. + \int_0^\infty \left(\left| \frac{\theta_2}{1+x} + \frac{\theta_1}{x} \right|^\alpha - \left| \frac{\theta_2}{1+x} \right|^\alpha - \left| \frac{\theta_1}{x} \right|^\alpha \right) \mathrm{d}x \right]. \qquad (3.4) \end{aligned}$$

The codifference of ΔUlog-FSM,

$$\tau_{\Delta U}(t) = -I_{\Delta U(a,b)}(1, -1; t),$$

is formulated in (2.10). From (3.1) its rate of convergence has intensity $1-\alpha$. The coefficient of asymptoticity is obtained by setting $\theta_1 = 1$ and $\theta_2 = -1$ in (3.2) and premultiplying by -1; it equals

$$H_u := H_u(a,b) = -G_u(a,b,1,-1). \tag{3.5}$$

As shown in the next theorem, the rate is exact, since H_u does not vanish for any $(a,b) \neq (0,0)$.

Theorem 3.4. *The codifference of $S\alpha S$ ΔUlog-FSM satisfies as $t \to \infty$*

$$\tau_{\Delta U(a,b)}(t) \sim H_u t^{1-\alpha}$$

where

$$\begin{aligned} H_u &= (|a|^\alpha + |b|^\alpha) \int_0^\infty \left[\left(\frac{1}{1+x}\right)^\alpha + \left(\frac{1}{x}\right)^\alpha - \left(\frac{1}{x} - \frac{1}{1+x}\right)^\alpha \right] \mathrm{d}x \\ &\quad + \int_0^1 \left[\left|\frac{a}{1-x}\right|^\alpha + \left|\frac{b}{x}\right|^\alpha - \left|\frac{a}{1-x} + \frac{b}{x}\right|^\alpha \right] \mathrm{d}x \end{aligned} \tag{3.6}$$

is positive.

ΔUlog-FSM exhibits long-range dependence since $\sum |\tau_{\Delta U}(t)|$ diverges.

To get the rate in the "well-balanced" case, $a = b$, either substitute $a = b$ in (3.6) and perform basic algebra, or apply (3.4) with this substitution and set $\theta_1 = 1, \theta_2 = -1$ in (3.5). The latter approach leads immediately to

$$\begin{aligned} H_u &= |a|^\alpha \int_{-\infty}^1 \left[\left|\frac{1}{1-x}\right|^\alpha + \left|\frac{1}{x}\right|^\alpha - \left|\frac{1}{1-x} + \frac{1}{x}\right|^\alpha \right] \mathrm{d}x \\ &\quad + \int_0^\infty \left[\left(\frac{1}{1+x}\right)^\alpha + \left(\frac{1}{x}\right)^\alpha - \left(\frac{1}{x} - \frac{1}{1+x}\right)^\alpha \right] \mathrm{d}x \\ &=: |a|^\alpha P. \end{aligned} \tag{3.7}$$

When $a = 1$ the codifference of ΔUlog-FSM reduces to the codifference of log-FSN obtained previously in [5, Theorem 3.1, p. 6], for which P in (3.7) is the coefficient of asymptoticity.

We turn to the covariation of ΔUlog-FSM. It is expressed by (2.11). Its asymptotic behavior is stated in the next result. Recall that the beta function is defined for $p > 0$ and $q > 0$ by

$$\beta(p,q) = \int_0^1 x^{p-1} (1-x)^{q-1} \, \mathrm{d}x. \tag{3.8}$$

Theorem 3.5. *As $t \to \infty$,*

$$[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha \sim b^{\langle \alpha-1 \rangle} (b-a)\, t^{1-\alpha} \ln t + Q_0 t^{1-\alpha} \tag{3.9}$$

where

$$Q_0 = |a|^\alpha \beta(2-\alpha, \alpha-1) + b^{\langle \alpha-1 \rangle} [bQ_2 + aQ_3 + (b-a)\,Q_4], \tag{3.10}$$

with

$$Q_2 = \int_1^\infty x^{-1} (1+x)^{1-\alpha} \, \mathrm{d}x, \quad Q_3 = \int_0^1 x^{-1} \left[(1+x)^{1-\alpha} - (1-x)^{1-\alpha} \right] \mathrm{d}x, \tag{3.11}$$

and

$$Q_4 \;=\; \int_0^1 x^{-1}\left[(1+x)^{1-\alpha} - 1\right] \mathrm{d}x.$$

Corollary 3.6. *As $t \to \infty$,*

$$[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha \;\sim\; b^{\langle \alpha-1\rangle}\,(b-a)\,t^{1-\alpha}\ln t \begin{cases} >0 & \textit{iff} \;\; b > a_+ \;\; \textit{or} \;\; b < -a_- \\ <0 & \textit{iff} \;\; a < b < 0 \;\; \textit{or} \;\; 0 < b < a. \end{cases} \tag{3.12}$$

In particular,

$$[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha \sim$$

$$\begin{cases} |b|^\alpha\, t^{1-\alpha}\ln t & \textit{iff} \;\; a = 0, b \neq 0 \\ |a|^\alpha\, \beta(2-\alpha, \alpha-1)\, t^{1-\alpha} & \textit{iff} \;\; a \neq 0, b = 0 \;\; \textit{or} \;\; a = b \neq 0, \alpha = 3/2 \\ |a|^\alpha\, Q t^{1-\alpha} & \textit{iff} \;\;\; a = b \neq 0, \alpha \neq 3/2 \end{cases} \tag{3.13}$$

where

$$\begin{aligned} Q \;&=\; \beta(2-\alpha,\alpha-1) \;+\; Q_2 \;+\; Q_3 \\ &=\; \int_0^1 \left[(1+x)^{1-\alpha}\left(x^{-1}+x^{\alpha-2}\right) - (1-x)^{1-\alpha}\left(x^{-1}-x^{\alpha-2}\right)\right] \mathrm{d}x \;>\; 0. \end{aligned}$$

Observe that ΔUlog-FSM also displays long-range dependence since $\alpha > 1$. Comparing Theorems 3.1 and 3.5, we get that the codifference and covariation are not asymptotically proportional if $b \neq 0$ and $a \neq b$ because as $t \to \infty$

$$\tau_{\Delta U(a,b)}(t)/[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha \;\to\; 0.$$

Specific results for the one-sided processes $a \neq 0, b = 0$ and $a = 0, b \neq 0$ are given by (3.13). When $(a,b) = (0,1)$ the covariation decays more slowly than when $a = b \neq 0$ or $(a,b) = (1,0)$. The well-balanced ΔUlog-FSM has the asymptotic covariation $|a|^\alpha\, Qt^{1-\alpha}$ obtained by setting $a = b$ in (3.10). When $a = b = 1$, the covariation of log-FSN is recovered. Its rate $Qt^{1-\alpha}$ and the positivity of Q (its integrand is positive) have been established in [5, Theorem 3.2, p. 6]. If a and b have different signs and $b \neq 0$, then the rate is the positive $b^{\langle\alpha-1\rangle}\,(b-a)\,t^{1-\alpha}\ln t$. It turns out that the restriction $\alpha = 1.5$ in the second part of (3.13) is equivalent to $Q_2 + Q_3 = 0$ (see Section 5).

Previous work on the asymptotic codifference and covariation also have been carried out for other processes. A particular example is the work of Kokoskza and Taqqu who study the fractional autoregressive moving average, or FARIMA. This is the stationary increment process - it is in fact a collection of processes indexed by $0 < \alpha < 2$ and $H \in (0,1)$ - whose partial sums appropriately renormalized are asymptotically H-self-similar. The authors apply both measures to FARIMA. They prove that the codifference ($0 < \alpha \leq 2$) and covariation ($1 < \alpha \leq 2$) are always asymptotically proportional for all α and H ([3, Theorem 4.1, pp. 35-36 and Theorem 5.1, p. 43]).

Another result involves *linear fractional stable noise* (LFSN), $\Delta L_{\alpha,H}(a,b) = \{\Delta L_{\alpha,H}(a,b;t), t \in \mathbb{R}\}$. This is the one-step increment process of LFSM. It follows from (1.2) that

$$\Delta L_{\alpha,H}(a,b;t) = L_{\alpha,H}(a,b;t+1) - L_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} \tilde{l}_{\alpha,H}(a,b;t,x) M_\alpha(\mathrm{d}x)$$

with

$$\tilde{l}_{\alpha,H}(a,b;t,x) := \\ a\left[(t+1-x)_+^{H-\frac{1}{\alpha}} - (t-x)_+^{H-\frac{1}{\alpha}}\right] + b\left[(t+1-x)_-^{H-\frac{1}{\alpha}} - (t-x)_-^{H-\frac{1}{\alpha}}\right].$$

LFSN is clearly a stationary process, hence, LFSM is H-sssi. We mention this process because in the studies of its codependence, it also displays a special case in which the codifference and covariation are not asymptotically proportional as they converge to 0. Indeed, in the special circumstance

$$1 < \alpha < 2,\ 0 < H < 1 - 1/[\alpha(\alpha-1)] \quad \text{and} \quad (a,b) = (0,1),$$

the codifference $\tau_{\Delta L_{\alpha,H}(0,1)}(t)$ of $S\alpha S$ LFSN satisfies, as $t \to \infty$,

$$\tau_{\Delta L_{\alpha,H}(0,1)}(t) \sim A_\tau t^{H-\frac{1}{\alpha}-1} \quad \text{where} \quad A_\tau > 0$$

([4, Theorem 2.3.1, p. 9]). On the other hand, for the same α and H, the covariation $[\Delta L_{\alpha,H}(0,1;t), \Delta L_{\alpha,H}(0,1;0)]_\alpha$ satisfies, as $t \to \infty$,

$$[\Delta L_{\alpha,H}(0,1;t), \Delta L_{\alpha,H}(0,1;0)]_\alpha \sim C t^{\alpha H - \alpha} \quad \text{where} \quad C < 0$$

([4, Theorem 3.2, p13]). The constants A_τ and C have *opposite* signs. In contrast, also, to ΔUlog-FSM, the covariation of LFSN converges more rapidly than its codifference since it has a *smaller* intensity:

$$0 < H < 1 - 1/[\alpha(\alpha-1)] \quad \text{and} \quad \alpha > 1$$

imply

$$\alpha H - \alpha < H - 1/\alpha - 1.$$

Both $\sum_{t=1}^\infty \tau_{\Delta L_{\alpha,H}(a,b)}(t)$ and $\sum_{t=1}^\infty |[\Delta L_{\alpha,H}(0,1;t), \Delta L_{\alpha,H}(0,1;0)]_\alpha|$ nevertheless converge, hence display, *short-range dependence*.[4]

4. Proofs for the Generalized Codifference and Codifference

4.1. Proof of Theorem 3.1. The proof uses results from [1]. We take $t > 1$ and write from (2.9),

$$I_{\Delta U(a,b)}(\theta_1,\theta_2;t) = \sum_{j=1}^{5} \tilde{I}_j(t), \quad \tilde{I}_j(t) \equiv \tilde{I}_j(a,b,\theta_1,\theta_2;t) \tag{4.1}$$

with

$$\tilde{I}_1(t) = \Xi_{-\infty}^0, \quad \tilde{I}_2(t) = \Xi_0^1, \quad \tilde{I}_3(t) = \Xi_1^t, \quad \tilde{I}_4(t) = \Xi_t^{t+1}, \quad \tilde{I}_5(t) = \Xi_{t+1}^\infty.$$

[4] See [4, Sections 2 and 3] about more explicit asymptotic results for the codifference and covariation of $S\alpha S$ $\Delta L_{\alpha,H}(0,1;t)$ and $\Delta L_{\alpha,H}(1,0;t)$. Corresponding results about the asymptotic codifference for any $(a,b) \neq (0,0)$ can be found in [6, Section 7.10].

More explicitly, using (1.3) one has,

$$
\begin{aligned}
\tilde{I}_1(t) &= \Xi_{-\infty}^{0}\left(\theta_1 a\left[\ln(t+1-x)-\ln(t-x)\right],\, \theta_2 a\left[\ln(1-x)-\ln(-x)\right]\right),\\
\tilde{I}_2(t) &= \Xi_{0}^{1}\left(\theta_1 a\left[\ln(t+1-x)-\ln(t-x)\right],\, \theta_2\left[a\ln(1-x)-b\ln x\right]\right),\\
\tilde{I}_3(t) &= \Xi_{1}^{t}\left(\theta_1 a\left[\ln(t+1-x)-\ln(t-x)\right],\, \theta_2 b\left[\ln(x-1)-\ln x\right]\right),\\
\tilde{I}_4(t) &= \Xi_{t}^{t+1}\left(\theta_1\left[a\ln(t+1-x)-b\ln(x-t)\right],\, \theta_2 b\left[\ln(x-1)-\ln x\right]\right),
\end{aligned}
$$

and

$$
\tilde{I}_5(t) = \Xi_{t+1}^{\infty}\left(\theta_1 b\left[\ln(x-1-t)-\ln(x-t)\right],\, \theta_2 b\left[\ln(x-1)-\ln x\right]\right).
$$

The change of variables $x \mapsto t+1-x$ applied to $\tilde{I}_4$ obtains

$$
\begin{aligned}
\tilde{I}_4(t) &= \Xi_{0}^{1}\left(\theta_1\left[a\ln x-b\ln(1-x)\right],\, \theta_2 b\left[\ln(t-x)-\ln(t+1-x)\right]\right)\\
&= \Xi_{0}^{1}\left(\theta_2 b\left[\ln(t+1-x)-\ln(t-x)\right],\, \theta_1\left[b\ln(1-x)-a\ln x\right]\right)\\
&= \tilde{I}_2\left(b,a,\theta_2,\theta_1;t\right) \qquad (4.2)
\end{aligned}
$$

since $\Xi_l^h(u,v) = \Xi_l^h(-u,-v) = \Xi_l^h(v,u)$. The same change of variables also gets

$$
\begin{aligned}
\tilde{I}_5(t) &= \Xi_{-\infty}^{0}\left(\theta_1 b\left[\ln(-x)-\ln(1-x)\right],\, \theta_2 b\left[\ln(t+1-x)-\ln(t-x)\right]\right)\\
&= \Xi_{-\infty}^{0}\left(\theta_2 b\left[\ln(t+1-x)-\ln(t-x)\right],\, \theta_1 b\left[\ln(1-x)-\ln(-x)\right]\right)\\
&= \tilde{I}_1\left(b,a,\theta_2,\theta_1;t\right). \qquad (4.3)
\end{aligned}
$$

The analysis of $\lim_{t\to\infty}\tilde{I}(t)$ now is obtained from $\lim_{t\to\infty}\tilde{I}_j(t), j = 1,2,3$. The relevant results are stated below.

Proposition 4.1. *As* $t \to \infty$,

$$
\tilde{I}_1(t) \sim |a|^{\alpha}\,\Xi_{-\infty}^{0}\left(\frac{\theta_1}{1-x}, \frac{\theta_2}{-x}\right) t^{1-\alpha}.
$$

Proposition 4.2. *As* $t \to \infty$,

$$
\tilde{I}_2(t) = \mathrm{O}\big(t^{-1}\big) = \mathrm{o}\big(t^{1-\alpha}\big).
$$

Proposition 4.3. *As* $t \to \infty$,

$$
\tilde{I}_3(t) \sim \Xi_{0}^{1}\left(\frac{a\theta_1}{1-x}, \frac{-b\theta_2}{x}\right) t^{1-\alpha}.
$$

Although $(a,b) \neq 0$ is arbitrary, the proofs of Propositions 4.1-4.3 follow closely the respective proofs for $\lim_{t\to\infty} I_j(t), j = 1,2,3$, carried out in [1, Propositions 6.1-6.3, pp. 26-27] when $a = b = 1$.

We conclude from (4.2) that

$$
\tilde{I}_4(t) = \mathrm{O}\big(t^{-1}\big) = \mathrm{o}\big(t^{1-\alpha}\big) \quad \text{as } t \to \infty,
$$

and from (4.3), that

$$
\tilde{I}_5(t) \sim |b|^{\alpha}\,\Xi_{-\infty}^{0}\left(\frac{\theta_2}{1-x}, \frac{\theta_1}{-x}\right) t^{1-\alpha} = |b|^{\alpha}\,\Xi_{0}^{\infty}\left(\frac{\theta_2}{1+x}, \frac{\theta_1}{x}\right) t^{1-\alpha} \quad \text{as } t \to \infty.
$$

$\tilde{I}_1(t), \tilde{I}_3(t)$, and $\tilde{I}_5(t)$ make the principal contributions to $\lim_{t\to\infty}$. Combining all five relations for $\lim_{t\to\infty}\tilde{I}_j(t)$ with (4.1) now proves (3.1) and obtains the coefficient G_u in (3.2). □

4.2. Proof of Theorem 3.4. The asymptotic expression for $\tau_{\Delta U(a,b)}(t)$ has been noted previously with a constant of asymptoticity H_u given by (3.5), from which a more definite form (3.6) is obtained.

By Proposition 3.2, $G_u < 0$ if $\operatorname{sign}(ab) \neq 1$ and $\operatorname{sign}(\theta_1\theta_2) = -1$, in which case $H_u > 0$. To complete the proof one must show that $H_u > 0$ if $\operatorname{sign}(ab) = 1$.

We begin by writing $b = \lambda a$ with $\lambda > 0$. (3.6) becomes

$$\begin{aligned} H_u &= |a|^\alpha \left\{ (1+\lambda^\alpha) \int_0^\infty \left[\left(\frac{1}{1+x}\right)^\alpha + \left(\frac{1}{x}\right)^\alpha - \left(\frac{1}{x} - \frac{1}{x+1}\right)^\alpha \right] \mathrm{d}x \right. \\ &\quad \left. + \int_0^1 \left[\left(\frac{1}{1-x}\right)^\alpha + \left(\frac{\lambda}{x}\right)^\alpha - \left(\frac{1}{1-x} + \frac{\lambda}{x}\right)^\alpha \right] \mathrm{d}x \right\} \\ &=: |a|^\alpha J(\lambda). \end{aligned}$$

(Note: Since $1 < \alpha < 2$, $\int_0^\infty$ is positive but $\int_0^1$ is negative.) The proof will involve the useful relation

$$J(\lambda^{-1}) = \lambda^{-\alpha} J(\lambda), \tag{4.4}$$

which is easily verified. On exchanging variables $x \mapsto x/(1+x)$ in the first integral,

$$\int_0^\infty = \int_0^\infty \frac{x^\alpha + (1+x)^\alpha - 1}{x^\alpha (1+x)^\alpha} \mathrm{d}x = \int_0^1 \frac{x^\alpha + 1 - (1-x)^\alpha}{x^\alpha (1-x)^{2-\alpha}} \mathrm{d}x.$$

Let

$$N(\alpha, \lambda; x) := (1+\lambda^\alpha)(1-x)^{2\alpha-2} \left[x^\alpha + 1 - (1-x)^\alpha\right] + x^\alpha + \lambda^\alpha (1-x)^\alpha - \left[x + \lambda(1-x)\right]^\alpha.$$

Then

$$\begin{aligned} J(\lambda) &= (1+\lambda^\alpha) \int_0^1 \frac{x^\alpha + 1 - (1-x)^\alpha}{x^\alpha (1-x)^{2-\alpha}} \mathrm{d}x \\ &\quad + \int_0^1 \frac{x^\alpha + \lambda^\alpha (1-x)^\alpha - \left[x + \lambda(1-x)\right]^\alpha}{(1-x)^\alpha x^\alpha} \mathrm{d}x \\ &= \int_0^1 \frac{N(\alpha, \lambda; x)}{x^\alpha (1-x)^\alpha} \mathrm{d}x \\ &= \int_0^{1/2} + \int_{1/2}^1 = \int_0^{1/2} \frac{N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x)}{x^\alpha (1-x)^\alpha} \mathrm{d}x. \end{aligned} \tag{4.5}$$

In the last integral,

$$\begin{aligned} N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x) &= (1+\lambda^\alpha)\left[g(\alpha, x) + g(\alpha, 1-x)\right] \\ &\quad - \left[x + \lambda(1-x)\right]^\alpha - (1 - x + \lambda x)^\alpha, \end{aligned} \tag{4.6}$$

with

$$0 < x \leq 1/2 \implies g(\alpha, x) := (1-x)^{2\alpha-2}\left[x^\alpha + 1 - (1-x)^\alpha\right] + x^\alpha > 0 \quad \text{and} \quad g(\alpha, 1-x) > 0.$$

It can be proved that $N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x)$ is positive for $\lambda > 0$ and $0 < x \leq 1/2$. This will obtain $J_\lambda > 0$, hence $H_u > 0$. However, we will consider proving the first fact only for the two cases $\lambda = 1$ and $\lambda > 1$, and then apply

(4.4) to validate $J_\lambda > 0$ for $0 < \lambda < 1$. (Alternatively, one can prove $N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x) > 0$ for $\lambda < 1$ and then use (4.4) to get $J_\lambda > 0$ for $\lambda > 1$.) Although already by [1, Theorem 3.1, p. 6] $H_u > 0$ if $\lambda = 1$, its proof does not involve exactly $N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x)$. We therefore reprove this case below by analyzing $N(\alpha, 1; x) + N(\alpha, 1; 1-x)$ for purpose of completeness.

Proposition 4.4. *If* $\lambda = 1$ *and* $0 < x \le 1/2$*, then* $N(\alpha, 1; x) + N(\alpha, 1; 1-x) > 0$.

Proof. Let $x \in (0, 1/2]$. Substituting $\lambda = 1$ into (4.6) we get

$$\begin{aligned} N(\alpha, 1; x) + N(\alpha, 1; 1-x) &= 2\left[g(\alpha, x) + g(\alpha, 1-x)\right] - 2 \\ &> 0 \iff \left[g(\alpha, x) + g(\alpha, 1-x)\right] > 1. \quad (4.7) \end{aligned}$$

It is straightforward to check the facts $\lim_{\alpha\to 1}\left[g(\alpha, x) + g(\alpha, 1-x)\right] = 3$ and $\lim_{\alpha\to 2}\left[g(\alpha, x) + g(\alpha, 1-x)\right] = 1$. Showing now that $\left[g(\alpha, x) + g(\alpha, 1-x)\right]$ is strictly decreasing over $\alpha \in (1, 2)$ will prove $\left[g(\alpha, x) + g(\alpha, 1-x)\right] > 1$. Certainly, its partial derivative with respect to α satisfies

$$\begin{aligned} \frac{\partial}{\partial\alpha}\left[g(\alpha, x) + g(\alpha, 1-x)\right] &= 2(1-x)^{2\alpha-2}\ln(1-x)\left[x^\alpha + 1 - (1-x)^\alpha\right] \\ &\quad + (1-x)^{2\alpha-2}\left[x^\alpha \ln x - (1-x)^\alpha \ln(1-x)\right] \\ &\quad + x^\alpha \ln x + 2x^{2\alpha-2}\ln x\left[(1-x)^\alpha + 1 - x^\alpha\right] \\ &\quad + x^{2\alpha-2}\left[(1-x)^\alpha \ln(1-x) - x^\alpha \ln x\right] \\ &\quad + (1-x)^\alpha \ln(1-x). \end{aligned}$$

It is obvious that the first, the third, the fourth, and the sixth terms on the right side of the preceding expression are negative. Combining the second and fifth terms obtains the product $[(1-x)^{2\alpha-2} - x^{2\alpha-2}][x^\alpha \ln x - (1-x)^\alpha \ln(1-x)]$. If $x = 1/2$ the product equals zero. If $0 < x < 1/2$, since $\alpha > 1$ the first factor is positive but the second factor is negative due to

$$x^\alpha \ln x - (1-x)^\alpha \ln(1-x) < 0 \quad \Leftrightarrow \quad \left(\frac{x}{1-x}\right)^\alpha < \frac{x}{1-x} < 1 < \frac{\ln(1-x)}{\ln x}.$$

Thus, $\partial[g(\alpha, x) + g(\alpha, 1-x)]/\partial\alpha < 0$ for $0 < x \le 1/2$. This implies $g(\alpha, x) + g(\alpha, 1-x)$ is strictly decreasing over $\alpha \in (1, 2)$. It follows that $\left[g(\alpha, x) + g(\alpha, 1-x)\right] > 1$ so, by (4.7), $N(\alpha, 1; x) + N(\alpha, 1; 1-x) > 0$. □

Proposition 4.5. *If* $\lambda > 1$ *and* $0 < x \le 1/2$*, then*

$$N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x) > 0.$$

Proof. In this case, we differentiate both sides of (4.6) with respect to λ:

$$\begin{aligned} \frac{\partial\left[N(\alpha, \lambda; x) + N(\alpha, \lambda; 1-x)\right]}{\partial\lambda} &= \alpha\lambda^{\alpha-1}\left[g(\alpha, x) + g(\alpha, 1-x)\right] \\ &\quad - \alpha(1-x)\left[x + \lambda(1-x)\right]^{\alpha-1} \\ &\quad - \alpha x(1 - x + \lambda x)^{\alpha-1}. \end{aligned}$$

Again, since $g(\alpha, x) + g(\alpha, 1-x) > 1$, then

$$\begin{aligned}\alpha^{-1}\,\frac{\partial\,[N(\alpha,\lambda;x)+N(\alpha,\lambda;1-x)]}{\partial\lambda} &> \lambda^{\alpha-1}-(1-x)\,[x+\lambda\,(1-x)]^{\alpha-1}\\ &\quad -x\,(1-x+\lambda x)^{\alpha-1}\\ &> \lambda^{\alpha-1}-(1-x)\,\lambda^{\alpha-1}-x\lambda^{\alpha-1} \;=\; 0.\end{aligned}$$

The second inequality follows from the fact that $\alpha > 1, \lambda > 1$, and $0 < x < 1$ imply $[x+\lambda(1-x)]^{\alpha-1} < \lambda^{\alpha-1}$ and $[1-x+\lambda x]^{\alpha-1} < \lambda^{\alpha-1}$. Thus, $N(\alpha,\lambda;x)+N(\alpha,\lambda;1-x) > 0$ is strictly increasing over $\lambda \in [1,\infty)$. Since $N(\alpha,1;x)+N(\alpha,1;1-x) > 0$ by Proposition 4.4, then also $N(\alpha,\lambda;x)+N(\alpha,\lambda;1-x) > 0$ for $\lambda > 1$. □

If we combine Propositions 4.4 and 4.5, then from (4.5) we get that $J_\lambda > 0$ when $\lambda > 1$. By (4.4) $J_\lambda > 0$ for all $\lambda > 0$; in turn, $H_u > 0$ when $\mathrm{sign}(ab) = 1$. This completes the proof of Theorem 3.4. □

5. Proof of Theorem 3.5

We begin here the proof of (3.9) by following the structure of the proof of [5, Theorem 3.2, p. 6]. In fact, the second-leading term $Q_0 t^{1-\alpha}$ is derived similarly to the leading term $Qt^{1-\alpha}$ in that result. We introduce, as in that former proof, constants δ and t_0 so that

$$1/2 < \delta < 1, \quad t_0 := 2(1+\delta). \tag{5.1}$$

The first-order relation

$$f(x+h) - f(x) = h\int_0^1 f'(x+\theta h)\,\mathrm{d}\theta$$

is valid for any f continuous on $[c,d]$ and differentiable on (c,d) with $c < x < x+h < d$. Applying it to $f(x) = \ln x$ and $x > 0$,

$$\ln(x+1) - \ln x < \frac{1}{x}. \tag{5.2}$$

When f has a second derivative,

$$f(x+h) - f(x) = hf^{'}(x) + h^2\int_0^1 (1-\theta)f^{''}(x+\theta h)\,\mathrm{d}\theta.$$

From this one gets

$$\ln(x+1) - \ln x = \frac{1}{x} + J(x), \quad -\frac{1}{2x^2} \le J(x) < 0. \tag{5.3}$$

These two mean-value relations for $\ln(\cdot)$ will be used in the sequel.

We now choose $t > t_0$ and decompose the covariation of ΔUlog-FSM in (2.11):

$$[\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha = \int_{-\infty}^0 + \int_0^1 + \int_t^{t+1} + \left(\int_1^t + \int_{t+1}^\infty\right) =: \sum_{j=1}^4 \tilde{L}_j(t). \tag{5.4}$$

The terms $\tilde{L}_j$ correspond to the terms L_j in the former proof and the behavior of $\lim_{t\to\infty} \tilde{L}_j(t)$ will be resolved by separate propositions for $j = 1,2,3,4$.

Proposition 5.1. *As* $t \to \infty$,

$$\tilde{L}_1(t) \sim |a|^\alpha \beta(2-\alpha, \alpha-1) t^{1-\alpha}.$$

Proof. When $-\infty < x \le 0$, we get from (5.4)

$$\begin{aligned} \tilde{L}_1(t) &= \int_{-\infty}^{0} a\left[\ln_0(t+1-x) - \ln_0(t-x)\right] \times \\ &\qquad \times \left\{a\left[\ln_0(1-x) - \ln_0(-x)\right]\right\}^{\langle\alpha-1\rangle} \mathrm{d}x \\ &= |a|^\alpha \int_{-\infty}^{0} \left[\ln(t+1-x) - \ln(t-x)\right]\left[\ln(1-x) - \ln(-x)\right]^{\langle\alpha-1\rangle} \mathrm{d}x \\ &=: |a|^\alpha L_1(t) \sim |a|^\alpha \beta(2-\alpha, \alpha-1) t^{1-\alpha} \end{aligned}$$

by [5, Proposition 5.1, p. 9], where $\beta(\cdot,\cdot)$ is the beta function from (3.8). □

Proposition 5.2. *As* $t \to \infty$,

$$\tilde{L}_2(t) = \mathrm{O}\big(t^{-1}\big) = \mathrm{o}\big(t^{1-\alpha}\big). \tag{5.5}$$

Proof. We will use dominated convergence to prove this. Note, first, that

$$\begin{aligned} \tilde{L}_2(t) &= \int_0^1 a\left[\ln_0(t+1-x) - \ln_0(t-x)\right]\left[a\ln_0(1-x) - b\ln_0 x\right]^{\langle\alpha-1\rangle} \mathrm{d}x \\ &= a\int_0^1 \left[\ln(t+1-x) - \ln(t-x)\right]\left[a\ln(1-x) - b\ln x\right]^{\langle\alpha-1\rangle} \mathrm{d}x. \end{aligned}$$

For fixed $x \in [0,1]$, (5.3) leads to

$$t[\ln(t+1-x) - \ln(t-x)] \sim t/(t-x) \to 1$$

as $t \to \infty$, which gets

$$t\left[\ln(t+1-x) - \ln(t-x)\right]\left[a\ln(1-x) - b\ln x\right]^{\langle\alpha-1\rangle} \sim \left[a\ln(1-x) - b\ln x\right]^{\langle\alpha-1\rangle}.$$

On the other hand, since $t > t_0$, then by (5.1) and by (5.2) (with $t-x$ instead of x),

$$t\left|\ln(t+1-x) - \ln(t-x)\right| \le t\left(\frac{1}{t-x}\right) \le \frac{t}{t-1} = 1 + \frac{1}{t-1} < 1 + \frac{1}{1+2\delta}.$$

In turn,

$$\begin{aligned} &\sup_{t>t_0} t\left|\ln(t+1-x) - \ln(t-x)\right|\left|a\ln(1-x) - b\ln x\right|^{\alpha-1} \\ &\qquad\qquad \le \left(1 + \frac{1}{1+2\delta}\right)\left[\left|a\ln(1-x)\right|^{\alpha-1} + \left|b\ln x\right|^{\alpha-1}\right] \\ &\qquad\qquad \in \mathrm{L}^1[0,1]. \end{aligned}$$

We have used the inequality $|c+d|^p \le |c|^p + |d|^p$ with $0 < p < 1$ and the fact $(\ln x)^{\alpha-1} \in \mathrm{L}^1[0,1]$. As a consequence, from dominated convergence

$$\lim_{t\to\infty} t\tilde{L}_2(t) = a\int_0^1 \left[a\ln(1-x) - b\ln x\right]^{\langle\alpha-1\rangle} \mathrm{d}x.$$

(5.5) now follows. □

When $a = b$, $\int_0^1 [a \ln(1-x) - b \ln x]$ vanishes, so the faster rate $\tilde{L}_2(t) = aL_2(t) = \mathrm{o}(t^{-1})$ prevails (see also [5, Proposition 5.2, p. 11]).

Proposition 5.3. *As $t \to \infty$,*

$$\tilde{L}_3(t) \sim b^{\langle \alpha-1 \rangle} (a-b) \, t^{1-\alpha}. \tag{5.6}$$

Proof.

$$\begin{aligned}
\tilde{L}_3(t) &= \int_t^{t+1} [a \ln_0(t+1-x) - b \ln_0(x-t)] \{b [\ln_0(1-x) - \ln_0 x]\}^{\langle \alpha-1 \rangle} \, \mathrm{d}x \\
&= -b^{\langle \alpha-1 \rangle} \int_t^{t+1} [a \ln(t+1-x) - b \ln(x-t)] [\ln x - \ln(x-1)]^{\langle \alpha-1 \rangle} \, \mathrm{d}x \\
&= -b^{\langle \alpha-1 \rangle} \int_0^1 [a \ln(1-x) - b \ln x] [\ln(x+t) - \ln(x+t-1)]^{\alpha-1} \, \mathrm{d}x
\end{aligned}$$

by exchanging variables, $x \mapsto x - t$, and by factoring -1. Note that

$$\lim_{t \to \infty} \{t[\ln(x+t) - \ln(x+t-1)]\} = \lim_{t \to \infty} [t/(x+t-1)] = 1$$

by (5.3); hence,

$$\lim_{t \to \infty} [a \ln(1-x) - b \ln x] [\ln(x+t) - \ln(x+t-1)]^{\alpha-1} = a \ln(1-x) - b \ln x.$$

Moreover,

$$t \, |\ln(x+t) - \ln(x+t-1)| \, \frac{t}{x+t-1} < 1 + \frac{1}{1+2\delta}$$

for $t > t_0$. It follows that

$$\begin{aligned}
&\sup_{t > t_0} |a \ln(1-x) - b \ln x| \, [t \, |\ln(x+t) - \ln(x+t-1)|]^{\alpha-1} \\
&\qquad \le [|a \ln(1-x)| + |b \ln x|] \left(1 + \frac{1}{1+2\delta}\right)^{\alpha-1} \in \mathrm{L}^1[0,1].
\end{aligned}$$

One now has

$$\lim_{t \to \infty} t^{\alpha-1} \tilde{L}_3 = -b^{\langle \alpha-1 \rangle} \int_0^1 [a \ln(1-x) - b \ln x] \, \mathrm{d}x = b^{\langle \alpha-1 \rangle} (a-b)$$

since $\int_0^1 \ln x \, \mathrm{d}x = -1$. This indeed proves (5.6). □

If $a = b$, then

$$\tilde{L}_3 = -|b|^{\alpha} L_3(t) = \mathrm{o}(t^{1-\alpha}),$$

which agrees with [5, Proposition 5.3, p. 11].

For "most" values a and b, the leading term in $\lim_{t \to \infty} [\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha$ is $t^{1-\alpha} \ln t$. It is realized next from analyzing $\lim_{t \to \infty} \tilde{L}_4(t)$.

Proposition 5.4. *As $t \to \infty$,*

$$\tilde{L}_4(t) \sim b^{\langle \alpha-1 \rangle} (b-a) \, t^{1-\alpha} \ln t + b^{\langle \alpha-1 \rangle} [bQ_2 + aQ_3 + (b-a)(Q_4+1)] \, t^{1-\alpha}, \tag{5.7}$$

where

$$Q_2 = \int_1^\infty x^{-1} (1+x)^{1-\alpha} \, \mathrm{d}x, \tag{5.8}$$

$$Q_3 = \int_0^1 x^{-1}\left[(1+x)^{1-\alpha} - (1-x)^{1-\alpha}\right] \mathrm{d}x, \tag{5.9}$$

and

$$Q_4 = \int_0^1 x^{-1}\left[(1+x)^{1-\alpha} - 1\right] \mathrm{d}x. \tag{5.10}$$

Proof. We will generalize the proof of [5, Proposition 5.4, p. 12] which supposes $a = b \neq 0$. We will get a different asymptotic behavior. Unlike that proof, we must deal with the universal instance $a \neq b$ instead of $a = b = 1$.

From (5.4) we get

$$\begin{aligned}\tilde{L}_4(t) \;\; = \;\; & \int_1^t a\left[\ln_0(t+1-x) - \ln_0(t-x)\right]\{b\left[\ln_0(x-1) - \ln_0 x\right]\}^{\langle\alpha-1\rangle}\,\mathrm{d}x \\ & + \int_{t+1}^{\infty} b\left[\ln_0(x-t-1) - \ln_0(x-t)\right]\{\left[\ln_0(x-1) - \ln_0 x\right]\}^{\langle\alpha-1\rangle}\,\mathrm{d}x.\end{aligned}$$

The first integral equals

$$\begin{aligned}& -ab^{\langle\alpha-1\rangle}\int_1^t \left[\ln(t+1-x) - \ln(t-x)\right]\left[\ln x - \ln(x-1)\right]^{\alpha-1}\,\mathrm{d}x \\ & \quad = -\,ab^{\langle\alpha-1\rangle}\int_0^{t-1} \left[\ln(x+1) - \ln x\right]\left[\ln(t-x) - \ln(t-x-1)\right]^{\alpha-1}\,\mathrm{d}x\end{aligned}$$

on changing variables $x \mapsto t - x$. The second integral satisfies

$$\begin{aligned}& |b|^{\alpha}\int_{t+1}^{\infty} \left[\ln(x-t) - \ln(x-t-1)\right]\left[\ln x - \ln(x-1)\right]^{\langle\alpha-1\rangle}\,\mathrm{d}x \\ & \quad = |b|^{\alpha}\int_0^{\infty} \left[\ln(x+1) - \ln x\right]\left[\ln(t+x+1) - \ln(t+x)\right]^{\alpha-1}\,\mathrm{d}x,\end{aligned}$$

via $x \mapsto x - t - 1$. Expressing the last integral as $\int_0^{t-1} + \int_{t-1}^{\infty}$, we get

$$\tilde{L}_4(t) = \tilde{K}(t) + \tilde{M}(t), \tag{5.11}$$

$$\begin{aligned}\tilde{K}(t) : \;\; = \;\; & \int_0^{t-1} \Big\{-ab^{\langle\alpha-1\rangle}\left[\ln(x+1) - \ln x\right]\left[\ln(t-x) - \ln(t-x-1)\right]^{\alpha-1} \\ & + |b|^{\alpha}\left[\ln(x+1) - \ln x\right]\left[\ln(t+x+1) - \ln(t+x)\right]^{\alpha-1}\Big\}\,\mathrm{d}x,\end{aligned}$$

and

$$\tilde{M}(t) := |b|^{\alpha}\int_{t-1}^{\infty} \left[\ln(x+1) - \ln x\right]\left[\ln(t+x+1) - \ln(t+x)\right]^{\alpha-1}\,\mathrm{d}x.$$

The expression $|b|^{-\alpha}\tilde{M}(t) = M(t)$ has been estimated by [5, relation (5.17), p. 13], and hence,

$$\tilde{M}(t) \sim |b|^{\alpha} Q_2 t^{1-\alpha}, \quad t \to \infty \tag{5.12}$$

with Q_2 given by (5.8).

We will analyze $\tilde{K}(t)$ similarly to $K(t)$ in [5]. We factor $b^{\langle\alpha-1\rangle}$ from the integrand and use the fact that $|b|^\alpha/b^{\langle\alpha-1\rangle} = b$ to express

$$\begin{aligned}\tilde{K}(t) &= b^{\langle\alpha-1\rangle}\int_0^{t-1}[\ln(x+1)-\ln x]\Big\{b\,[\ln(t+x+1)-\ln(t+x)]^{\alpha-1} \\ &\quad -a\,[\ln(t-x)-\ln(t-x-1)]^{\alpha-1}\Big\}\,\mathrm{d}x \\ &= b^{\langle\alpha-1\rangle}\left(\int_0^{\rho_1(t)}+\int_{\rho_1(t)}^{t-1}\right) =: b^{\langle\alpha-1\rangle}\left(\tilde{K}_1(t)+\tilde{K}_2(t)\right). \qquad (5.13)\end{aligned}$$

Proceeding as in [5], we stipulate

$$\rho_1(t) := t(1-\rho(t)) \qquad (5.14)$$

where

$$\frac{1+\delta}{t} < \rho(t) < \frac{1}{2}, \qquad (5.15)$$

$$\lim_{t\to\infty}\rho(t) = 0, \qquad (5.16)$$

and

$$\lim_{t\to\infty}\frac{\left(\frac{\ln t}{t}\right)^{\frac{1}{\alpha}}}{\rho(t)} = 0. \qquad (5.17)$$

Indeed, $t > t_0 = 2(1+\delta)$ with $1/2 < \delta < 1$, hence $(1+\delta)/t < 1/2$ and so ρ can be chosen appropriately in (5.15)-(5.17). Note also that $t\rho(t) > 1$ implies $\rho_1(t) < t-1$, hence, $\tilde{K}_2$ is nontrivial.

It is relatively easy to estimate $\lim_{t\to\infty}\tilde{K}_2(t)$.

$$\begin{aligned}\left|\tilde{K}_2(t)\right| &\le \int_{\rho_1(t)}^{t-1}[\ln(x+1)-\ln x]\,\Big|b\,[\ln(t+x+1)-\ln(t+x)]^{\alpha-1} \\ &\quad -a\,[\ln(t-x)-\ln(t-x-1)]^{\alpha-1}\Big|\,\mathrm{d}x \\ &\le \int_{\rho_1(t)}^{t-1}[\ln(x+1)-\ln x]\,(|b|+|a|)\,[\ln(t-x)-\ln(t-x-1)]^{\alpha-1}\,\mathrm{d}x \\ &= \mathrm{o}\big(\mathrm{t}^{1-\alpha}\big), \quad t\to\infty. \qquad (5.18)\end{aligned}$$

The second inequality follows because $\ln(y+1)-\ln y$ is a decreasing function of y. This implies that

$$\ln(t+x+1)-\ln(t+x) < \ln(t-x-1)-\ln(t-x).$$

The equality is obtained by [5, relation (5.23), p. 13]. In particular, the rate is a consequence of the definition of $\rho_1(t)$, (5.16), and $\lim_{t\to\infty} t\rho(t) = \infty$ due to (5.17).

The estimation of $\tilde{K}_1(t)$ is more delicate. Begin by applying the equation in (5.3) to the expressions $\ln(t+x+1)-\ln(t+x)$ and $\ln(t-x)-\ln(t-x-1)$.

$$\begin{aligned}
&b\,[\ln(t+x+1)-\ln(t+x)]^{\alpha-1}-a\,[\ln(t-x)-\ln(t-x-1)]^{\alpha-1}\\
&=b\left(\frac{1}{t+x}\right)^{\alpha-1}-a\left(\frac{1}{t-x-1}\right)^{\alpha-1}\\
&\quad+b\left(\frac{1}{t+x}\right)^{\alpha-1}\left([1+(t+x)J(t+x)]^{\alpha-1}-1\right)\\
&\quad-a\left(\frac{1}{t-x-1}\right)^{\alpha-1}\left([1+(t-x-1)J(t-1-x)]^{\alpha-1}-1\right).
\end{aligned}$$

This relation is valid since $|(t+x)J(t+x)|<1/(2t)<1$ for $t>t_0$ and, since if $0<x\le\rho_1(t)$, then

$$\begin{aligned}
|(t-x-1)J_1(t-1-x)| &\le 1/[2(t-1-x)]\le\\
\le 1/[2(t-1-\rho_1(t))] &= 1/[2(-1+t\rho(t)]<1/(2\delta)<1
\end{aligned}$$

by (5.14) and the first inequality in (5.15). We next write

$$\tilde{K}_1(t)=\tilde{K}_{11}(t)+\tilde{K}_{12}(t)+\tilde{K}_{13}(t) \tag{5.19}$$

with

$$\tilde{K}_{11}(t):=\int_0^{\rho_1(t)}[\ln(x+1)-\ln x]\left[b\left(\frac{1}{t+x}\right)^{\alpha-1}-a\left(\frac{1}{t-x-1}\right)^{\alpha-1}\right]\mathrm{d}x,$$

$$\begin{aligned}
&\tilde{K}_{12}(t):=\\
&\quad\int_0^{\rho_1(t)}[\ln(x+1)-\ln x]\,b\left(\frac{1}{t+x}\right)^{\alpha-1}\left([1+(t+x)J(t+x)]^{\alpha-1}-1\right)\mathrm{d}x,
\end{aligned}$$

and

$$\begin{aligned}
\tilde{K}_{13}(t): &= -\int_0^{\rho_1(t)}[\ln(x+1)-\ln x]\,a\left(\frac{1}{t-x-1}\right)^{\alpha-1}\times\\
&\qquad\times\left([1+(t-x-1)J(t-x-1)]^{\alpha-1}-1\right)\mathrm{d}x.
\end{aligned}$$

Both $\tilde{K}_{12}(t)$ and $\tilde{K}_{13}(t)$ are negligible. More precisely,

$$\tilde{K}_{12}(t)=bK_{12}(t)=\mathrm{O}\left(\mathrm{t}^{-\alpha}\ln\rho_1(\mathrm{t})\right)=\mathrm{o}\left(\mathrm{t}^{1-\alpha}\right),\quad t\to\infty, \tag{5.20}$$

by [5, (5.28), p. 16]. The last equality follows since (5.14) and (5.16) imply $(\rho_1(t)\sim t)$. Similarly,

$$\tilde{K}_{13}(t)=aK_{13}(t)=\mathrm{O}\left(\frac{\ln\mathrm{t}}{(\mathrm{t}\rho(\mathrm{t}))^{\alpha}}\right)=\mathrm{o}\left(\mathrm{t}^{1-\alpha}\right),\quad t\to\infty, \tag{5.21}$$

follows from [5, (5.29), p. 16], where the final rate is a consequence of (5.17).

$\tilde{K}_{11}(t)$ provides the leading asymptotic rate $t^{1-\alpha}\ln t$. In order to obtain this rate, we begin by factoring $t^{1-\alpha}$ from the integrand and write

$$\begin{aligned}
\tilde{K}_{11}(t) &= t^{1-\alpha}\int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\left[b\left(1+\frac{x}{t}\right)^{1-\alpha} - a\left(1-\frac{x+1}{t}\right)^{1-\alpha}\right]\mathrm{d}x \\
&= t^{1-\alpha}\left\{\int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\,(b-a)\left(1+\frac{x}{t}\right)^{1-\alpha}\mathrm{d}x\right. \\
&\quad \left. + \int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\,a\left[\left(1+\frac{x}{t}\right)^{1-\alpha} - \left(1-\frac{x+1}{t}\right)^{1-\alpha}\right]\mathrm{d}x\right\} \\
&=: t^{1-\alpha}\left(N_1(t)+N_2(t)\right). \qquad (5.22)
\end{aligned}$$

Observe that

$$\begin{aligned}
N_1(t) &= (b-a)\left\{\int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\left[\left(1+\frac{x}{t}\right)^{1-\alpha}-1\right]\mathrm{d}x\right. \\
&\quad \left. + \int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\,\mathrm{d}x\right\} \\
&=: (b-a)\left(N_{11}(t)+N_{12}(t)\right).
\end{aligned}$$

Make the variable exchange $x \mapsto x/t$ in $N_{11}(t)$ to get

$$N_{11}(t) = \int_0^1 [\ln(xt+1)-\ln xt]\left[(1+x)^{1-\alpha}-1\right] 1_{[0,\rho_1(t)/t]}(x)\,t\mathrm{d}x.$$

By (5.3), $\lim_{t\to\infty} t[\ln(xt+1)-\ln(xt)] = \lim_{t\to\infty} t(1/(xt)) = 1/x$, hence,

$$\begin{aligned}
&\lim_{t\to\infty} [\ln(xt+1)-\ln xt]\left[(1+x)^{1-\alpha}-1\right] 1_{[0,\rho_1(t)/t]}(x)\,t \\
&\qquad = x^{-1}\left[(1+x)^{1-\alpha}-1\right] 1_{[0,1]}(x)
\end{aligned}$$

since $\rho_1(t)/t \to 1$. Moreover, $\rho_1(t) < t$, so $[0,\rho_1(t)/t] \subset [0,1]$ and

$$\begin{aligned}
&\sup_{t>t_0}\left|[\ln(xt+1)-\ln xt]\left[(1+x)^{1-\alpha}-1\right] 1_{[0,\rho_1(t)/t]}(x)\,t\right| \\
&\leq \sup_{t>t_0}\frac{1}{xt}\left[(1+x)^{1-\alpha}-1\right] = x^{-1}\left[(1+x)^{1-\alpha}-1\right] \in \mathrm{L}^1[0,1].
\end{aligned}$$

As a consequence, dominated convergence implies

$$\lim_{t\to\infty} N_{11}(t) = \int_0^1 x^{-1}\left[(1+x)^{1-\alpha}-1\right]\mathrm{d}x =: Q_4,$$

the constant in (5.10). On the other hand, direct evaluation of $N_{12}(t)$ obtains, as $t\to\infty$,

$$\int_0^{\rho_1(t)} [\ln(x+1)-\ln x]\,\mathrm{d}x = \rho_1(t)\ln\left(1+\frac{1}{\rho_1(t)}\right) + \ln\left(\rho_1(t)+1\right) \sim 1+\ln t.$$

Combining the two estimations for $N_{11}(t)$ and $N_{12}(t)$, we have as $t\to\infty$

$$N_1(t) \sim (b-a)\left(Q_4+1+\ln t\right).$$

The second term in (5.22) satisfies $t^{1-\alpha}a^{-1}N_2(t) = K_{11}(t) \sim Q_3 t^{1-\alpha}$ by [5, (5.27) p. 16], where Q_3 is defined by (5.9). Thus,

$$N_2(t) \sim aQ_3.$$

This estimate and the preceding one for $N_1(t)$ together get

$$\tilde{K}_{11}(t) \sim (b-a)\, t^{1-\alpha} \ln t + [aQ_3 + (b-a)(Q_4+1)]\, t^{1-\alpha}, \quad t \to \infty. \tag{5.23}$$

We now substitute relations (5.23), (5.20), (5.21) back into (5.19). This achieves along with (5.18), and from (5.13), as $t \to \infty$,

$$\begin{aligned} \tilde{K}(t) &\sim b^{\langle \alpha-1 \rangle} \tilde{K}_1(t) \sim b^{\langle \alpha-1 \rangle} \tilde{K}_{11}(t) \\ &\sim b^{\langle \alpha-1 \rangle}(b-a)\, t^{1-\alpha} \ln t \\ &\quad + b^{\langle \alpha-1 \rangle}[aQ_3 + (b-a)(Q_4+1)]\, t^{1-\alpha}. \end{aligned} \tag{5.24}$$

Combining (5.11), (5.12), and (5.24) now obtains, as $t \to \infty$

$$\begin{aligned} \tilde{L}_4(t) &\sim b^{\langle \alpha-1 \rangle}(b-a)\, t^{1-\alpha} \ln t \\ &\quad + \left\{ |b|^\alpha Q_2 + b^{\langle \alpha-1 \rangle}[aQ_3 + (b-a)(Q_4+1)] \right\} t^{1-\alpha} \\ &= b^{\langle \alpha-1 \rangle}(b-a)\, t^{1-\alpha} \ln t + b^{\langle \alpha-1 \rangle}[bQ_2 + aQ_3 + (b-a)(Q_4+1)]\, t^{1-\alpha}. \end{aligned}$$

This proves (5.7). □

Propositions 5.1-5.4 and (5.4) accomplish that as $t \to \infty$,

$$\begin{aligned} [\Delta U(a,b;t), \Delta U(a,b;0)]_\alpha &\sim |a|^\alpha \beta(2-\alpha, \alpha-1) t^{1-\alpha} \\ &\quad + b^{\langle \alpha-1 \rangle}(a-b)\, t^{1-\alpha} + b^{\langle \alpha-1 \rangle}(b-a)\, t^{1-\alpha} \ln t \\ &\quad + b^{\langle \alpha-1 \rangle}[bQ_2 + aQ_3 + (b-a)(Q_4+1)]\, t^{1-\alpha} \\ &= b^{\langle \alpha-1 \rangle}(b-a)\, t^{1-\alpha} \ln t + Q_0 t^{1-\alpha}, \end{aligned}$$

on simplifying the coefficient of $t^{1-\alpha}$ and defining Q_0 by relation (3.10). (3.9) is now proved. This concludes the proof of Theorem 3.5. □

We turn to Corollary 3.6. Its proof requires the following.

Proposition 5.5. *Let Q_2 and Q_3 be defined as in (3.11). Then Q_2+Q_3 is strictly decreasing in $\alpha \in (1,2)$ from ∞ to $-\infty$. Its unique zero occurs at $\alpha = 3/2$.*

Proof. We have

$$Q_2 + Q_3 = \int_1^\infty x^{-1}(1+x)^{1-\alpha}\, dx + \int_0^1 x^{-1}\left[(1+x)^{1-\alpha} - (1-x)^{1-\alpha}\right] dx.$$

Since both integrands are strictly decreasing in $\alpha \in (1,2)$, then so is Q_2+Q_3, which therefore has a unique zero. It occurs at $\alpha = 3/2$ because, by direct integration, at this value

$$\begin{aligned} \int_1^\infty x^{-1}(1+x)^{-1/2}\, dx &= \ln\left(3 + 2\sqrt{2}\right) \\ &= \int_0^1 x^{-1}\left[(1-x)^{-1/2} - (1+x)^{-1/2}\right] dx. \end{aligned}$$

$Q_2 + Q_3$ must be positive for $1 < \alpha < 3/2$ and negative for $3/2 < \alpha < 2$.

Note that Q_2 increases to ∞ as $\alpha \to 1$, by monotone convergence. Q_3, which is negative, increases to 0 as $\alpha \to 1$. Thus, $\lim_{\alpha\to 1}(Q_2 + Q_3) = \infty$. On the other hand, $\lim_{\alpha\to 2} Q_2 = \int_0^1 x^{-1}(1+x)^{-1}\mathrm{d}x = \ln 2$ and $\lim_{\alpha\to 2} Q_3 = -\infty$, hence $\lim_{\alpha\to 2}(Q_2 + Q_3) = -\infty$. □

To complete the proof of Corollary 3.6, note that (3.12) follows from (3.9). Also, the first part of the asymptotic relation in (3.13) follows from (3.9) and (3.10). The leading term for $\lim_{t\to\infty}[\Delta U_t, \Delta U_0]_\alpha$ vanishes if and only if $b = 0$ or $a = b \neq 0$, hence, the second and third parts of the relation follow from Proposition 5.5 as well as from (3.9) and (3.10). Finally, the positivity of the constant Q is a consequence of [5, Theorem 3.2, p. 6]. □

6. Appendix

The proofs of Proposition 2.1, Corollary 2.2, and Proposition 2.3 are included.

6.1. Proof of Proposition 2.1. We have from (2.9)

$$\begin{aligned} I_{\Delta U(a,b)}(\theta_1, \theta_2; t) &= \Xi_{-\infty}^{\infty}\left(\theta_1 \tilde{u}\left(a, b; t, x\right), \theta_2 \tilde{u}\left(a, b; 0, x\right)\right) \\ &= \Xi_{-\infty}^{\infty}\left(\theta_1 \tilde{u}\left(a, b; t, t+1-x\right), \theta_2 \tilde{u}\left(a, b; 0, t+1-x\right)\right) \end{aligned} \tag{6.1}$$

on changing variables $x \mapsto t + 1 - x$. From (1.3) and the fact that $y_+ = (-y)_-$,

$$\begin{aligned} \tilde{u}(a, b; t, t+1-x) &= a\left[\ln_0(x)_+ - \ln_0(x-1)_+\right] + b\left[\ln_0(-x)_- - \ln_0(x-1)_-\right] \\ &= a\left[\ln_0(-x)_- - \ln_0(1-x)_-\right] \\ &\quad + b\left[\ln_0(-x)_+ - \ln_0(1-x)_+\right] \\ &= -\tilde{u}(b, a; 0, x). \end{aligned} \tag{6.2}$$

Similarly,

$$\begin{aligned} \tilde{u}(a, b; 0, t+1-x) &= a\left[\ln_0(x-t)_+ - \ln_0(x-t-1)_+\right] \\ &\quad + b\left[\ln_0(x-t)_- - \ln_0(x-t-1)_-\right] \\ &= a\left[\ln_0(t-x)_- - \ln_0(t+1-x)_-\right] \\ &\quad + b\left[\ln_0(t-x)_+ - \ln_0(t+1-x)_+\right] \\ &= -\tilde{u}(b, a; t, x). \end{aligned} \tag{6.3}$$

Substituting (6.2) and (6.3) into (6.1),

$$\begin{aligned} I_{\Delta U(a,b)}(\theta_1, \theta_2; t) &= \Xi_{-\infty}^{\infty}\left(-\theta_1 \tilde{u}\left(b, a; 0, x\right), -\theta_2 \tilde{u}\left(b, a; t, x\right)\right) \\ &= \Xi_{-\infty}^{\infty}\left(\theta_2 \tilde{u}\left(b, a; t, x\right), \theta_1 \tilde{u}\left(b, a; 0, x\right)\right) = I_{\Delta U(b,a)}(\theta_2, \theta_1; t), \end{aligned}$$

where the second equality follows from the identities

$$\Xi_{-\infty}^{\infty}\left(-v(x), -u(x)\right) = \Xi_{-\infty}^{\infty}\left(v(x), u(x)\right) = \Xi_{-\infty}^{\infty}\left(u(x), v(x)\right).$$

This completes the proof. □

6.2. Proof of Corollary 2.2. If $\theta_1 = \theta = -\theta_2$, then

$$\begin{aligned} I_{\Delta U(a,b)}(\theta,-\theta;t) &= \Xi_{-\infty}^{\infty}\left(\theta\tilde{u}\left(a,b;t,x\right),-\theta\tilde{u}\left(a,b;0,x\right)\right) \\ &= \Xi_{-\infty}^{\infty}\left(-\theta\tilde{u}\left(a,b;t,x\right),\theta\tilde{u}\left(a,b;0,x\right)\right) = I_{\Delta U(a,b)}(-\theta,\theta;t) \\ &= I_{\Delta U(b,a)}(\theta,-\theta;t) \end{aligned}$$

by Proposition 2.1. Taking $\theta = 1$ and premultiplying by -1,

$$\tau_{\Delta U(a,b)}(t) = -I_{\Delta U(a,b)}(1,-1;t) = -I_{\Delta U(b,a)}(1,-1;t) = \tau_{\Delta U(b,a)}(t),$$

thus completing the proof. □

6.3. Proof of Proposition 2.3. Switch t and 0 in (2.11). We get that

$$\begin{aligned} \left[\Delta U(a,b;0),\Delta U(a,b;t)\right]_{\alpha} &= \int_{\mathbb{R}} \tilde{u}\left(a,b;0,x\right)\tilde{u}\left(a,b;t,x\right)^{\langle\alpha-1\rangle}\,\mathrm{d}x \\ &= \int_{\mathbb{R}} \tilde{u}\left(a,b;0,t+1-x\right)\tilde{u}\left(a,b;t,t+1-x\right)^{\langle\alpha-1\rangle}\,\mathrm{d}x \qquad (\text{via } x\mapsto t+1-x) \\ &= \int_{\mathbb{R}} -\tilde{u}\left(b,a;t,x\right)\left(-\tilde{u}\left(b,a;0,x\right)\right)^{\langle\alpha-1\rangle}\,\mathrm{d}x \qquad (\text{by (6.3) and (6.2)}) \\ &= \left[\Delta U(b,a;t),\Delta U(b,a;0)\right]_{\alpha} \end{aligned}$$

on using the fact $-u(-v)^{\langle\alpha-1\rangle} = uv^{\langle\alpha-1\rangle}$. □

References

1. Astrauskas, A., Levy, J. B., and Taqqu, M. S.: The asymptotic dependence structure of the linear fractional Lévy motion, *Lietuvos Matematikos Rinkinys (Lithuanian Mathematical Journal)* **31** (1991), no. 1, 1–28.
2. Embrechts, P. and Maejima, M.: *Selfsimilar Processes*, Princeton University Press, Princeton, NJ, 2002.
3. Kokoskza, P. S. and Taqqu, M. S.: Fractional ARIMA with stable innovations, *Stochastic Processes and Their Applications* **60** (1995), 19–47.
4. Levy, J. B., Taqqu, and M. S.: The asymptotic dependence of one-sided linear fractional stable noise, Preprint (2005).
5. Levy, J. B. and Taqqu, M. S.: The asymptotic codifference and covariation of log-fractional stable noise, Preprint (2006).
6. Samorodnitsky, G. and Taqqu, M. S.: *Stable non-Gaussian random processes: Models with infinite variance*, Chapman and Hall, New York, 1994.
7. Taqqu, M. S. and Levy, J. B.: The dependence structure of log-fractional stable noise with analogy to fractional Gaussian noise, *Rendiconti di Matematica, Serie VII* **28** (2008), no. 1, 97–115.

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