

CHARACTERIZATION OF MASS-STATIONARITY BY BERNOULLI AND COX TRANSPORTS

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ABSTRACT. Consider a random measure ξ on a locally compact Abelian group G acting on some random element X . Mass-stationarity – introduced in [10] – means informally that the origin is a typical location for (X, ξ) in the mass of ξ . It is an intrinsic characterization of Palm versions w.r.t. stationary random measures. In this paper we show that mass-stationarity w.r.t. discrete ξ is characterized by distributional invariance under shifts of the origin by certain mass-preserving transports involving a Bernoulli randomization of the group-identity and an allocation rule. We also show that mass-stationarity w.r.t. a general ξ is characterized by mass-stationarity w.r.t. a Cox process driven by ξ .

1. Introduction

Let ξ be a random measure on a locally compact Abelian group G , for instance $G = \mathbb{R}^d$. Let X be a random element in a space on which G acts, for instance a random field indexed by G . *Mass-stationarity* of (X, ξ) is a formalization of the intuitive idea that the origin is a typical location for (X, ξ) in the mass of ξ , just like stationarity of (X, ξ) means that the origin is a typical location for (X, ξ) in the space G . Stationarity is defined by distributional invariance under *deterministic* shifts of the origin, while mass-stationarity is defined by distributional invariance under certain *randomized* shifts. The formal definition is given in Section 2 below. If ξ is Haar measure and G acts continuously on the state space of X , then mass-stationarity of (X, ξ) boils down to ordinary stationarity of X , see Remark 2.1.

The word ‘typical’ needs some explanation. For a location $s \in G$ write $\theta_s \xi := \xi(\cdot - s)$ and also let $\theta_s(X, \xi)$ denote the pair (X, ξ) shifted by s . If ξ is finite and S is a random element in G with conditional distribution $\xi/\xi(G)$ given (X, ξ) then we say that S is a *typical* location for (X, ξ) in the mass of ξ , – and also that the *origin* is a typical location for $\theta_S(X, \xi)$ in the mass of $\theta_S \xi$. In this introduction we use the term ‘typical’ even for infinite ξ in order to explain informally the basic ideas of the paper.

Mass-stationarity was introduced in [10] as an extension to random measures of *point-stationarity*, which in turn was introduced in [12] for simple point processes in \mathbb{R}^d having a point at the origin. Point-stationarity formalizes the intuitive idea

Received 2010-3-15; Communicated by R. Roy.

2000 *Mathematics Subject Classification.* Primary 60G57, 60G55; Secondary 60G60.

Key words and phrases. Random measure, mass-stationarity, Palm measure, Bernoulli transport, Cox process, invariant transport kernel, allocation rule.

that the point at the origin is a typical point of the point process (think of the Poisson process on the line with an extra point added at the origin: shifting the origin to the n^{th} point on the right, – or to the n^{th} point on the left, – does not change the fact that the inter-point distances are i.i.d. exponential). The definition of point-stationarity in [12] involved an external randomization, but in [3] (and in [4] for the group case) it is shown that the definition of point-stationarity can be reduced to ‘distributional invariance under shifts of the origin by preserving allocation rules’: an *allocation rule* τ is a map taking each location $s \in G$ to another location $\tau(s) \in G$ depending on $\theta_s(X, \xi)$, and an allocation rule τ is *preserving* if the image of ξ under τ is ξ itself. In fact, [3] and [4] show that ‘matchings’ suffice for the definition: an allocation rule τ is a *matching* if τ is its own inverse.

In [12] it was shown that point-stationarity is an intrinsic characterization of Palm versions of stationary simple point processes, and [10] goes on to show that mass-stationarity is an intrinsic characterization of Palm versions of stationary random measures. In the present paper we derive further characterizations of mass-stationarity. Below we explain these results informally and sketch the plan of the paper.

After preliminaries in Section 2, the term ‘Bernoulli transport’ is introduced in Section 3. It refers to a randomized allocation rule that allows staying at a location s with a probability $p(s)$ depending on $\theta_s(X, \xi)$, and otherwise chooses another location according to a (non-randomized) allocation rule. This makes it possible to preserve discrete point-masses even if there are point-masses of different sizes. We show that mass-stationarity of (X, ξ) when ξ is *discrete* can be reduced to distributional invariance of (X, ξ) under shifts of the origin by preserving Bernoulli transports, Theorem 3.2. In the simple point process case mass-stationarity can even be reduced to distributional invariance under invariant matchings, see Proposition 3.8.

In Section 4 we introduce a Cox process ζ driven by (X, ξ) , that is, ζ is an integer-valued point process which conditionally on (X, ξ) is a Poisson process with intensity measure ξ . This we do in order to represent the mass of ξ by discrete points of unit mass, possibly several points at the same location in G if ξ is not diffuse. The Cox process ζ can be thought of as a collection of points placed independently at typical locations in the mass of ξ . Thus if the origin is also a typical location in the mass of ξ , – that is, if (X, ξ) is mass-stationary, – and we add an extra point at the origin to ζ to obtain $\zeta^0 := \zeta + \delta_0$, then the points of ζ^0 are all at typical locations in the mass of ξ . In fact, we might expect that the new point at the origin is a typical point of ζ^0 , in other words that $((X, \xi), \zeta^0)$ is also mass-stationary. We might even expect that (X, ξ) is mass-stationary *if and only if* $((X, \xi), \zeta^0)$ is mass-stationary. We will show that this is indeed the case in Theorem 4.1.

By a ‘Cox transport’ we mean an allocation rule preserving ζ^0 . Thus applying Cox transports reduces mass-stationarity with respect to a general random measure ξ to mass-stationarity with respect to the point process ζ^0 , Theorem 4.1. When ξ is diffuse then ζ^0 is a simple point process and mass-stationarity is reduced to point-stationarity. In particular, mass-stationarity for diffuse ξ is characterized

by applying matchings to the Cox process, Corollary 4.6, while for general ξ mass-stationarity is characterized by applying preserving Bernoulli transports to the Cox process, Corollary 4.3.

In Section 5 we discuss some potential applications of Bernoulli and Cox transports. In particular we suggest to study the costs associated with these transport.

2. Transports and Mass-stationarity

We consider a topological Abelian group G that is assumed to be a locally compact, second countable Hausdorff space with Borel σ -field \mathcal{G} and Haar measure λ . Let M denote the set of all locally finite measures on G equipped with the cylindrical σ -field \mathcal{M} . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a σ -finite measure space. Although \mathbb{P} need not be a probability measure, we still use a probabilistic language. A *random measure* is a random element ξ in M . We use the kernel notation $\xi(\omega, \cdot) := \xi(\omega)(\cdot)$, $\omega \in \Omega$. We equip (M, \mathcal{M}) with a *measurable flow* $\theta_s : M \rightarrow M$, $s \in G$, defined by $\theta_s \mu(B) := \mu(B + s)$, where $B \in \mathcal{G}$ and $B + s := \{t + s : t \in B\}$. Then $(\mu, s) \mapsto \theta_s \mu$ is a measurable mapping, θ_0 is the identity on M , and we have the flow property

$$\theta_s \circ \theta_t = \theta_{s+t}, \quad s, t \in G. \tag{2.1}$$

Here 0 denotes the neutral element in G and \circ denotes composition. Together with ξ we consider a random element X in a measurable space (W, \mathcal{W}) . We assume that this space is equipped with a measurable flow $\theta_s : W \rightarrow W$, $s \in G$, having the properties listed above. (Denoting this flow again by θ_s , $s \in G$, will cause no risk of ambiguity.)

Next we adapt some terminology from [10] to the setting established above. This makes some of the definitions more cumbersome. However, the present setting is closer to chapter 11 of [7] and chapter 9 of [13] and will allow for a more convenient formulation of our main results in Section 4. In the remainder of this paper we consider a pair (X, ξ) as introduced above such that $\mathbb{P}((X, \xi) \in \cdot)$ is σ -finite and $\mathbb{P}(\xi(G) = 0) = 0$. We call (X, ξ) *stationary* if $\mathbb{P}(\theta_s(X, \xi) \in \cdot) = \mathbb{P}((X, \xi) \in \cdot)$ for all $s \in G$. Here we define $\theta_s(w, \mu) := (\theta_s w, \theta_s \mu)$ for $s \in G$ and $(w, \mu) \in W \times M$. If (X, ξ) is stationary, then we also call $\mathbb{P}((X, \xi) \in \cdot)$ *invariant*. In this case the measure

$$\mathbb{P}_{X, \xi}(A) := \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(X(\omega), \xi(\omega))) \mathbf{1}_B(s) \xi(\omega, ds) \mathbb{P}(d\omega), \quad A \in \mathcal{W} \otimes \mathcal{M}, \tag{2.2}$$

is called the *Palm measure* of (X, ξ) (with respect to \mathbb{P}), see [11]. Here $B \in \mathcal{G}$ has $0 < \lambda(B) < \infty$. This measure is σ -finite. As the definition (2.2) is independent of B , we can use a monotone class argument to conclude the *refined Campbell theorem*

$$\iint f(\theta_s(X(\omega), \xi(\omega)), s) \xi(\omega, ds) \mathbb{P}(d\omega) = \iint f(w, \mu, s) ds \mathbb{P}_{X, \xi}(d(w, \mu))$$

for all measurable $f : W \times M \times G \rightarrow [0, \infty)$, where ds refers to integration with respect to the Haar measure λ . Using a standard convention in probability theory,

we write this as

$$\mathbb{E}_{\mathbb{P}} \left[\int f(\theta_s(X, \xi), s) \xi(ds) \right] = \iint f(w, \mu, s) ds \mathbb{P}_{X, \xi}(d(w, \mu)), \tag{2.3}$$

where \mathbb{E} denotes integration with respect to \mathbb{P} .

Next we define *mass-stationarity* of (X, ξ) . Let $C \in \mathcal{G}$ be a relatively compact set having $\lambda(C) > 0$ and $\lambda(\partial C) = 0$, where ∂C denotes the boundary of C . Let U, V be random elements in G , possibly obtained by extending $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that (X, ξ) and U are independent, U has the uniform distribution on C (w.r.t. Haar measure), and that the conditional distribution of V given (X, ξ, U) is uniform in the mass of ξ on $C - U$. Then (X, ξ) is called *mass-stationary* if

$$(\theta_V(X, \xi), U + V) \stackrel{d}{=} (X, \xi, U) \tag{2.4}$$

holds for all such C . In this case we call the distribution $\mathbb{P}((X, \xi) \in \cdot)$ mass-stationary. By Theorem 6.3 in [10] this is equivalent to the validity of the Mecke equation

$$\mathbb{E} \left[\int g(\theta_s(X, \xi), -s) \xi(ds) \right] = \mathbb{E} \left[\int g(X, \xi, s) \xi(ds) \right] \tag{2.5}$$

for all measurable $g : W \times M \times G \rightarrow [0, \infty)$. This equation identifies (X, ξ) as the Palm version of some stationary pair (X', ξ') , cf. [11] and the discussion of equation (2.7) in [10].

Remark 2.1. The random element X is *stationary* if $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$ for all $s \in G$ and if this measure is σ -finite. Mass-stationarity generalizes this concept. Indeed, assuming (2.5) for $\xi = \lambda$ we easily get that $\mathbb{P}(\theta_s X \in \cdot) = \mathbb{P}(X \in \cdot)$ for λ -a.e. $s \in G$. Assuming that W is a metric space with Borel σ -field \mathcal{W} and that $s \mapsto \theta_s X$ is \mathbb{P} -a.e. continuous, we obtain stationarity of X .

Remark 2.2. The definition implies that mass-stationarity of (X, ξ) is equivalent to mass-stationarity of $((X, \xi), \xi)$.

Remark 2.3. Recently the Mecke characterization (2.5) has been generalized to the much more general case of a random measure whose state space is subjected to a proper operation of a locally compact (not necessarily Abelian) group, see [2] and [8]. It appears worthwhile to define and to study mass-stationarity in such a general setting.

For the next definitions it is convenient to abbreviate $\Omega' := W \times M$ and $\mathcal{F}' := \mathcal{W} \otimes \mathcal{M}$. A *weighted transport kernel* is a kernel T from $\Omega' \times G$ to G such that $T(\omega', s, \cdot)$ is locally finite for all $(\omega', s) \in \Omega' \times G$. If T is Markovian, then it is called *transport kernel*. A weighted transport kernel is *invariant* if $T(\theta_s \omega', 0, B - s) = T(\omega', s, B)$ for all $(\omega', s) \in \Omega' \times G$ and $B \in \mathcal{G}$. An *allocation rule* is a measurable mapping $\tau : \Omega' \times G \rightarrow G$ which is *covariant*, i.e. which has $\tau(\theta_s \omega', 0) = \tau(\omega', s) - s$ for all ω', s . A weighted transport kernel T is *mass-preserving* if

$$\int T(w, \mu, s, \cdot) \mu(ds) = \mu(\cdot) \tag{2.6}$$

holds for all $(w, \mu) \in \Omega'$. An allocation rule is *mass-preserving* if

$$\int \mathbf{1}\{\tau(w, \mu, s) \in \cdot\} \mu(ds) = \mu(\cdot) \tag{2.7}$$

holds for all $(w, \mu) \in \Omega'$. If these relations hold almost everywhere w.r.t. some measure \mathbb{Q} on Ω' , then we say that T (resp. τ) is \mathbb{Q} -a.e. mass-preserving.

Remark 2.4. Let T be a locally finite kernel from $W \times M \times G$ to G . Assume that there is some $A \in \mathcal{W} \otimes \mathcal{M}$ such that

$$\int T(w, \mu, s, \cdot) \mu(ds) = \mu(\cdot), \tag{2.8}$$

holds for all $(w, \mu) \in A$. Then we can redefine T on $((W \times M) \setminus A) \times G$ by $T(w, \mu, s, \cdot) := \delta_s$, to obtain a kernel T satisfying (2.8) for all $(w, \mu) \in W \times M$. If A is *invariant* (i.e. $\theta_s A = A, s \in G$) and T is invariant, then the modified T is an invariant kernel too. A similar remark applies to allocation rules.

By Theorem 7.2 in [10] (X, ξ) is mass-stationary, iff

$$\mathbb{E} \left[\int \mathbf{1}\{\theta_t(X, \xi) \in A\} T(X, \xi, 0, dt) \right] = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{F}', \tag{2.9}$$

holds for all invariant mass-preserving weighted transport kernels T .

A measure $\mu \in M$ is *discrete* if

$$\mu = \sum_{s: \mu\{s\} > 0} \mu\{s\} \delta_s$$

and *diffuse* if $\mu\{s\} = 0$ for all $s \in G$. Lemma 1.39 in [7] shows that any $\mu \in M$ can be measurably and uniquely written as the sum of a discrete measure μ^d and a diffuse measure μ^c . The proof of this result shows that the mapping $\mu \mapsto (\mu^d, \mu^c)$ is covariant in the obvious sense. Therefore the characterization (2.5) of mass-stationarity together with $\xi = \xi^d + \xi^c$ implies the following result.

Proposition 2.5. *If $((X, \xi), \xi^d)$ and $((X, \xi), \xi^c)$ are both mass-stationary, then (X, ξ) is mass-stationary.*

3. Bernoulli Transports

A *Bernoulli transport kernel* is a transport kernel T of the form

$$T(w, \mu, s, \cdot) = p(w, \mu, s) \delta_s + (1 - p(w, \mu, s)) \delta_{\tau(w, \mu, s)}, \quad (w, \mu, s) \in W \times M \times G, \tag{3.1}$$

where $p : W \times M \times G \rightarrow [0, 1]$ is measurable and $\tau : W \times M \times G \rightarrow G$ is a measurable mapping. Invariance of Bernoulli transport kernels can easily be characterized as follows.

Lemma 3.1. *Let T be a Bernoulli transport kernel as in (3.1) such that for all $(w, \mu) \in W \times M$ it holds that $p(w, \mu, s) = 1$ iff $\tau(w, \mu, s) = s$. Then T is invariant iff τ is covariant and $p(w, \mu, s) = p(\theta_s(w, \mu), 0)$ for all $(w, \mu, s) \in W \times M \times G$.*

Recall that (X, ξ) is a random pair such that $\mathbb{P}((X, \xi) \in \cdot)$ is σ -finite and $\mathbb{P}(\xi(G) = 0) = 0$. We will show that the validity of (2.9) for all invariant Bernoulli transport kernels is sufficient for mass-stationarity of (X, ξ) . The *support* of a measure $\mu \in M$ is denoted by $\text{supp } \mu$. Here we need to make the weak assumption, that (W, \mathcal{W}) is a *Borel space*, i.e. Borel isomorphic to a Borel subset of $[0, 1]$, see e.g. Appendix A1 in [7].

Theorem 3.2. *Assume that (W, \mathcal{W}) is a Borel space, that $\mathbb{P}(0 \notin \text{supp } \xi) = 0$, and that $\mathbb{P}(\xi \neq \xi^d) = 0$. Assume also that (2.9) holds for all invariant mass-preserving Bernoulli transport kernels T . Then (X, ξ) is mass-stationary.*

Our proof of Theorem 3.2 requires the following generalization of a result in [4]. A proof can be found in [9]. A (partial) *matching* is an allocation rule τ such that the following holds for all $(w, \mu) \in W \times M$: $\tau(w, \mu, s) \in \text{supp } \mu$ and $\tau(w, \mu, \tau(w, \mu, s)) = s$ for all $s \in \text{supp } \mu$, and $\tau(w, \mu, s) = s$ for all $s \notin \text{supp } \mu$.

Lemma 3.3. *Assume that (W, \mathcal{W}) is a Borel space. Then there exist invariant matchings τ_k , $k \in \mathbb{N}$, such that for all $(w, \mu) \in W \times M$ with $\text{supp } \mu$ locally finite and $0 \in \text{supp } \mu$*

$$\{0\} \cup \{t \in \text{supp } \mu : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \tag{3.2}$$

For $n \in \mathbb{N}$ and $\mu \in M$ we define $\mu_n \in M$ by

$$\mu_n(B) := \int_B \mathbf{1}\{1/n \leq \mu\{s\} \leq n\} \mu(ds), \quad B \in \mathcal{G}.$$

Then $1/n \leq \mu_n\{s\} \leq n$, $s \in \text{supp } \mu_n$, and $\text{supp } \mu_n$ is locally finite. We will use the following version of Lemma 3.3.

Lemma 3.4. *Assume that (W, \mathcal{W}) is a Borel space and let $n \in \mathbb{N}$. Then there exist invariant matchings τ_k , $k \in \mathbb{N}$, such that for all $(w, \mu) \in W \times M$ with $0 \in \text{supp } \mu_n$*

$$\{0\} \cup \{t \in \text{supp } \mu_n : \theta_t(w, \mu) \neq (w, \mu)\} \subset \{\tau_k(w, \mu, 0) : k \in \mathbb{N}\}. \tag{3.3}$$

Furthermore, the τ_k can be chosen such that the following holds for all $(w, \mu) \in W \times M$. If $s \notin \text{supp } \mu_n$ then $\tau_k(w, \mu, s) = s$ and if $s \in \text{supp } \mu_n$ then $\tau_k(w, \mu, s) \in \text{supp } \mu_n$.

Proof. We apply Lemma 3.3 with W replaced by $W \times M$. This gives matchings τ'_k , $k \in \mathbb{N}$, such that for all $(w, \mu, \nu) \in W \times M \times M$ with $\text{supp } \nu$ locally finite and $0 \in \text{supp } \nu$

$$\{0\} \cup \{t \in \text{supp } \nu : \theta_t(w, \mu, \nu) \neq (w, \mu, \nu)\} \subset \{\tau'_k((w, \mu), \nu, 0) : k \in \mathbb{N}\}.$$

For any $k \in \mathbb{N}$ we define a mapping $\tau_k : W \times M \times G \rightarrow G$ by $\tau_k(w, \mu) := \tau'_k((w, \mu), \mu_n)$. Then (3.3) holds. (Note that $\theta_t(w, \mu, \mu_n) = (w, \mu, \mu_n)$ iff $\theta_t(w, \mu) = (w, \mu)$.) It is now easy to see that the τ_k are invariant matchings with the properties stated in the lemma. \square

Proof of Theorem 3.2. It is convenient (and no restriction of generality) to assume that $(\Omega, \mathcal{F}) = (W \times M, \mathcal{W} \otimes \mathcal{M})$, $\mathbb{P} = \mathbb{P}((X, \xi) \in \cdot)$, and that (X, ξ) is the identity on $W \times M$. We will prove the Mecke equation (2.5). Satz 2.5 in [11] (see also Section 2 in [10]) shows that \mathbb{P} is the Palm measure of (X, ξ) w.r.t. a σ -finite invariant

measure on Ω . By Theorem 7.3 in [10] this is equivalent to mass-stationarity of (X, ξ) .

In the sequel we fix $n \in \mathbb{N}$. Let τ be an invariant matching with the properties listed after (3.3). Define a Bernoulli transport kernel T by

$$T(w, \mu, s, \cdot) := \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_s + \frac{\mu\{\tau(s)\}}{\mu\{s\} + \mu\{\tau(s)\}} \delta_{\tau(s)} \quad (3.4)$$

if $s \in \text{supp } \mu_n$, and $T(w, \mu, s, \cdot) := \delta_s$, otherwise. Here and below we skip the argument (w, μ) whenever possible. This transport kernel is of the form (3.1) with

$$p(s) := \mathbf{1}\{\tau(s) \neq s\} \frac{\mu\{s\}}{\mu\{s\} + \mu\{\tau(s)\}} + \mathbf{1}\{\tau(s) = s\}, \quad (3.5)$$

where we recall that $\tau(s) = s$ for $s \notin \text{supp } \mu_n$. We have

$$p(\theta_s, 0) = \mathbf{1}\{\tau(\theta_s, 0) \neq 0\} \frac{\theta_s \mu\{0\}}{\theta_s \mu\{0\} + \theta_s \mu\{\tau(\theta_s, 0)\}} + \mathbf{1}\{\tau(\theta_s, 0) = 0\}.$$

Since $\tau(\theta_s, 0) = \tau(s) - s$ and $\theta_s \mu\{t\} = \mu\{t + s\}$, $t \in G$, we obtain that $p(\theta_s, 0) = p(s)$. Lemma 3.1 implies that T is invariant.

We next prove that T is mass-preserving, i.e.

$$\int T(w, \mu, s, \{t\}) \mu(ds) = \mu\{t\}, \quad t \in G, w \in W, \mu \in M. \quad (3.6)$$

Fix $w \in W$ and $\mu \in M$, and take $t \in G$. Assume first that $t \notin \text{supp } \mu_n$. Then $\tau(t) = t$ and $T(t, \{t\}) = 1$. Let $s \in G \setminus \{t\}$. If $s \notin \text{supp } \mu_n$ then $\tau(s) = s$ and $T(s, \{t\}) = 0$. If $s \in \text{supp } \mu_n$, then $T(s, \{t\}) > 0$ is only possible if $\tau(s) = t$, i.e. $\tau(t) = s$. As this would contradict $\tau(t) = t$, we again get $T(s, \{t\}) = 0$. Hence $T(s, \{t\}) = \mathbf{1}\{s = t\}$, implying (3.6) for $t \notin \text{supp } \mu_n$.

Assume now that $t \in \text{supp } \mu_n$. Then $T(s, \{t\}) = 0$ for $s \notin \text{supp } \mu_n$. (Otherwise we would obtain that $\tau(s) = t \neq s$.) For $s \in \text{supp } \mu_n$ we can have $T(s, \{t\}) > 0$ only if $s = t$ or $\tau(s) = t$. The latter equality implies $\tau(t) = s$. If $\tau(t) = t$ then $T(s, \{t\}) = 0$ for all $s \in \text{supp } \mu_n \setminus \{t\}$ and thus (3.6) holds. The only non-trivial case is $\tau(t) \neq t$. Then the left-hand side of (3.6) equals

$$\begin{aligned} & \mu\{t\} T(t, \{t\}) + \mu\{\tau(t)\} T(\tau(t), \{t\}) \\ &= \mu\{t\} \frac{\mu\{t\}}{\mu\{t\} + \mu\{\tau(t)\}} + \mu\{\tau(t)\} \frac{\mu\{t\}}{\mu\{t\} + \mu\{\tau(t)\}} = \mu\{t\}, \end{aligned}$$

where we have again used that $\tau(\tau(t)) = t$.

We have established that T is an invariant mass-preserving Bernoulli transport kernel and will now head towards (2.5). Let us define the *mass-shift* $\theta_\tau : \Omega \rightarrow \Omega$ by $\theta_\tau(\omega) := \theta_{\tau(\omega, 0)}(\omega)$. (We also define the random measure $\theta_\tau \xi$ by $\theta_\tau \xi(\omega) := \theta_{\tau(\omega, 0)} \xi(\omega)$; the random measure $\theta_\tau \xi_n = (\theta_\tau \xi)_n$ is defined in the same way.) A quick consequence of the matching property of τ is

$$\tau(\theta_\tau, 0) = -\tau(0). \quad (3.7)$$

In particular we have

$$\mathbf{1}_A(\theta_\tau) = \mathbf{1}_A, \quad (3.8)$$

where $A := \{\tau(0) \neq 0\}$. Note that $A \subset \{0 \in \text{supp } \xi_n, \tau(0) \in \text{supp } \xi_n\}$. Let $f : \Omega \rightarrow [0, \infty)$ be measurable with $\mathbb{E}[f] < \infty$. Let $B \in \mathcal{G}$ and define $g(\omega, s) := f(\omega)\mathbf{1}\{s \in B\}$. By assumption and the facts established above we can apply (2.9) for our specific T , to obtain

$$\begin{aligned} \mathbb{E}[\mathbf{1}_{Ag}(\theta_0, \tau(0))\xi\{\tau(0)\}] &= \mathbb{E}\left[\int \mathbf{1}_A(\theta_s)g(\theta_s, \tau(\theta_s, 0))\xi(\theta_s, \{\tau(\theta_s, 0)\})T(0, ds)\right] \\ &= \mathbb{E}[\mathbf{1}_{Ag}(\theta_0, \tau(0))\xi\{\tau(0)\}p(0)] + \mathbb{E}[\mathbf{1}_{Ag}(\theta_\tau, -\tau(0))\xi(\theta_\tau, \{-\tau(0)\})(1-p(0))], \end{aligned}$$

where we have used (3.8) and (3.7) for the second equality. (We suppress the dependence on (X, ξ) in the notation; for instance we use θ_s as a shorthand for $\theta_s(X, \xi)$.) Recalling the definition of p and using $\theta_\tau\xi\{-\tau(0)\} = \xi\{0\}$, we get

$$\begin{aligned} &\mathbb{E}[\mathbf{1}_{Ag}(\theta_0, \tau(0))\xi\{\tau(0)\}] \\ &= \mathbb{E}\left[\mathbf{1}_{Ag}(\theta_0, \tau(0))\frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}}\right] + \mathbb{E}\left[\mathbf{1}_{Ag}(\theta_\tau, -\tau(0))\frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}}\right]. \end{aligned}$$

Since for $0 \in \text{supp } \xi_n$ and $\tau(0) \in \text{supp } \xi_n$

$$g(\theta_0, \tau(0))\frac{\xi\{0\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}} \leq f\frac{n^3}{2}, \quad g(\theta_0, \tau(0))\xi\{\tau(0)\} \leq fn,$$

and $\mathbb{E}[f] < \infty$, we get by subtraction

$$\mathbb{E}\left[\mathbf{1}_{Ag}(\theta_0, \tau(0))\frac{\xi\{\tau(0)\}\xi\{\tau(0)\}}{\xi\{0\} + \xi\{\tau(0)\}}\right] = \mathbb{E}\left[\mathbf{1}_{Ag}(\theta_\tau, -\tau(0))\frac{\xi\{\tau(0)\}\xi\{0\}}{\xi\{0\} + \xi\{\tau(0)\}}\right]. \quad (3.9)$$

Consider the function $\tilde{g} : \Omega \times G \rightarrow [0, \infty)$ given by

$$\tilde{g}(s) := \mathbf{1}\{0 \in \text{supp } \xi_n, s \in \text{supp } \xi_n\} \frac{\xi\{0\} + \xi\{s\}}{\xi\{s\}}.$$

We have

$$\begin{aligned} \tilde{g}(\theta_\tau, -\tau(0)) &= \mathbf{1}\{0 \in \text{supp } \theta_\tau\xi_n, -\tau(0) \in \text{supp } \theta_\tau\xi_n\} \frac{\theta_\tau\xi\{0\} + \theta_\tau\xi\{-\tau(0)\}}{\theta_\tau\xi\{-\tau(0)\}} \\ &= \mathbf{1}\{\tau(0) \in \text{supp } \xi_n, 0 \in \text{supp } \xi_n\} \frac{\xi\{\tau(0)\} + \xi\{0\}}{\xi\{0\}}. \end{aligned}$$

Since $\tilde{g}(\theta_0, \tau(0)) \leq 2n^2$ and $\tilde{g}(\theta_\tau, -\tau(0)) \leq 2n^2$, we can apply (3.9) with $g \cdot \tilde{g}$ instead of g . Together with monotone convergence this gives for all measurable $g : \Omega \times G \rightarrow [0, \infty)$:

$$\mathbb{E}[\mathbf{1}\{\tau(0) \neq 0\}g(\theta_0, \tau(0))\xi_n\{\tau(0)\}] = \mathbb{E}[\mathbf{1}\{\tau(0) \neq 0\}g(\theta_\tau, -\tau(0))\xi_n\{\tau(0)\}]. \quad (3.10)$$

We now apply Lemma 3.4. If $0 \in \text{supp } \xi_n$, then (3.3) yields that

$$\int h(t)\mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\}\xi_n(dt) = \sum_{k \in \mathbb{N}} h_k(X, \xi, \tau_k(0))h(\tau_k(0))\xi_n\{\tau_k(0)\} \quad (3.11)$$

for all measurable $h : W \times G \rightarrow [0, \infty)$, where

$$h_k(t) := \mathbf{1}\{\theta_t(X, \xi) \neq (X, \xi)\}\mathbf{1}\{\tau_l(0) \neq t \text{ for } 1 \leq l \leq k-1\}.$$

We claim that

$$h_k(\theta_{\tau_k}(X, \xi), -\tau_k(0)) = h_k(X, \xi, \tau_k(0)), \quad k \in \mathbb{N}. \quad (3.12)$$

Indeed, for $k \geq 2$ and $l \leq k-1$ we have by covariance of τ_l that $\tau_l(\theta_{\tau_k}, 0) = -\tau_k(0)$ iff $\tau_l(\tau_k(0)) = 0$. By the matching property of τ_l this is in turn equivalent to $\tau_k(0) = \tau_l(0)$. From (3.11), (3.10) and (3.12) we obtain

$$\begin{aligned} & \mathbb{E} \left[\int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ &= \sum_{k \in \mathbb{N}} \mathbb{E} [h_k(X, \xi, \tau_k(0)) g(X, \xi, \tau_k(0)) \xi_n\{\tau_k(0)\}] \\ &= \sum_{k \in \mathbb{N}} \mathbb{E} [h_k(X, \xi, \tau_k(0)) g(\theta_{\tau(k)}(X, \xi), -\tau_k(0)) \xi_n\{\tau_k(0)\}]. \end{aligned}$$

Using (3.12) again we arrive at

$$\begin{aligned} & \mathbb{E} \left[\int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ &= \mathbb{E} \int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) \neq (X, \xi)\} g(\theta_t(X, \xi), -t) \xi_n(dt). \end{aligned} \quad (3.13)$$

Let $t \in \text{supp } \xi_n$ be such that $\theta_t(X, \xi) = (X, \xi)$. Then $\xi_n = \theta_{-t}\xi_n$ and

$$\xi_n\{t\} = \theta_t \xi_n\{0\} = \theta_{-t} \xi_n\{0\} = \xi\{-t\}.$$

Therefore,

$$\begin{aligned} & \mathbb{E} \left[\int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) = (X, \xi)\} g(X, \xi, t) \xi_n(dt) \right] \\ &= \mathbb{E} \left[\int \mathbf{1}\{0 \in \text{supp } \xi_n, \theta_t(X, \xi) = (X, \xi)\} g(\theta_t(X, \xi), -t) \xi_n(dt) \right]. \end{aligned}$$

Adding this to (3.13) and taking the limit as $n \rightarrow \infty$, yields (2.5) and hence the assertion of the theorem. \square

Remark 3.5. The last part of the preceding proof (starting with (3.12)) coincides with the second half of the proof of Theorem 1.1 in [4]. But it does also close a gap in the latter proof in that it is using Lemma 3.3 instead of the (slightly) weaker Theorem 3.6 in [4]. This theorem is not sufficient for the conclusion made in [4].

The definitions of the previous section apply in particular in the case where W is a singleton. In this case we can identify $W \times M$ with M and abbreviate the set of all mass-preserving invariant weighted transport kernels as \mathbf{T} and the set of all mass-preserving allocation rules as \mathbf{A} . Moreover, the set of all mass-preserving invariant Bernoulli transport kernels (a subset of \mathbf{T}) is denoted by \mathbf{T}_b , while the set of all invariant matchings (a subset of \mathbf{A}) is denoted by \mathbf{A}_m .

The proof of Theorem 3.2 yields the following result without a Borel assumption on the space W .

Proposition 3.6. *Assume that $\mathbb{P}(0 \notin \text{supp } \xi) = 0$ and $\mathbb{P}(\xi \neq \xi^d) = 0$. Assume further that*

$$\mathbb{E} \left[\int \mathbf{1}\{\theta_t(X, \xi) \in A\} T(\xi, 0, dt) \right] = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (3.14)$$

holds for all $T \in \mathbf{T}_b$. Then, for all measurable $g : W \times M \times G \rightarrow [0, \infty)$,

$$\mathbb{E} \left[\int \mathbf{1}\{\theta_t \xi \neq \xi\} g(\theta_t(X, \xi), -t) \xi(dt) \right] = \mathbb{E} \left[\int \mathbf{1}\{\theta_t \xi \neq \xi\} g(X, \xi, t) \xi(dt) \right]. \quad (3.15)$$

Proof. Let $n \in \mathbb{N}$. We apply Lemma 3.4 in the case where W is a singleton. We can then proceed as in the proof of Theorem 3.2, to obtain as at (3.13)

$$\mathbb{E} \left[\int \mathbf{1}\{\theta_t \xi \neq \xi\} g(X, \xi, t) \xi'_n(dt) \right] = \mathbb{E} \left[\int \mathbf{1}\{\theta_t \xi \neq \xi\} g(\theta_t(X, \xi), -t) \xi'_n(dt) \right].$$

where $\xi'_n(dt) = \mathbf{1}\{0 \in \text{supp } \xi_n\} \xi_n(dt)$. Letting $n \rightarrow \infty$ gives the assertion. \square

Let $N \subset M$ be the set of all discrete measures μ on G having $\mu\{s\} \in \{0, 1\}$ for all $s \in G$. Strengthening the assumptions of Proposition 3.6, we can use a simplified version of the proof of Theorem 3.2 to get the following result. We refer here also to Theorem 1.1 in [4].

Proposition 3.7. *Assume $\mathbb{P}(0 \notin \text{supp } \xi) = 0$, $\mathbb{P}(\xi \notin N) = 0$, and that*

$$\mathbb{P}(\theta_\tau(X, \xi) \in A) = \mathbb{P}((X, \xi) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}, \quad (3.16)$$

holds for all $\tau \in \mathbf{A}_m$, where $\theta_\tau : W \times M \rightarrow \Omega$ is defined by $\theta_\tau(w, \mu) := \theta_{\tau(\mu, 0)}(w, \mu)$. Then (3.15) holds for all measurable $g : W \times M \times G \rightarrow [0, \infty)$.

A measure $\mu \in M$ is called *periodic* if $\theta_t \mu = \mu$ for some $t \neq 0$. A measure \mathbb{Q} on M is called *aperiodic* if it is supported by the set of all measures $\mu \in M$ that are not periodic. Since the Mecke equation (2.5) implies mass-stationarity, Propositions 3.6 and 3.7 give the following result.

Proposition 3.8. *Assume that $\mathbb{P}(0 \notin \text{supp } \xi) = 0$ and $\mathbb{P}(\xi \neq \xi^d) = 0$. Assume further that $\mathbb{P}(\xi \in \cdot)$ is aperiodic. If either (3.14) holds for all $T \in \mathbf{T}_b$ or $\mathbb{P}(\xi \notin N) = 0$ and (3.16) holds for all $\tau \in \mathbf{A}_m$, then (X, ξ) is mass-stationary.*

Remark 3.9. Assume that $\mathbb{P}(0 \notin \text{supp } \xi) = 0$ and $\mathbb{P}(\xi \neq \xi^d) = 0$. If (3.14) holds for all $T \in \mathbf{T}_b$ we conjecture that (X, ξ) is mass-stationary without the additional aperiodicity assumption. Of course, if X is constant, then this is implied by Theorem 3.2.

Remark 3.10. Let \mathbb{P} satisfy the assumptions of Proposition 3.8 and assume in addition that $\mathbb{P}(\xi \notin N) = 0$. If $\mathbb{P}(\xi \in \cdot)$ is not aperiodic, then Proposition 3.8 does not apply. However, we might assume that (3.16) holds for all $\tau \in \mathbf{A}$. We believe that this implies mass-stationarity of (X, ξ) . In case $G = \mathbb{R}^d$ this was established in Theorem 4.1 in [3].

Remark 3.11. Let the assumptions of Proposition 3.7 be satisfied. Example 7.1 in [10] shows that invariance of $\mathbb{P}((X, \xi) \in \cdot)$ under mass-preserving allocation rules (in the sense of (3.16)) is not enough to imply mass-stationarity of (X, ξ) . Therefore Theorem 3.2 does not only solve Problem 7.3 in [10] for discrete random measures (up to the fact that in case of periodicities we have to allow the weighted transport kernels to depend on X) but is also the natural (and minimal) extension of Theorem 1.1 in [4] to discrete random measures.

4. Cox Transports

For any $\alpha \in M$ we let Π_α denote the distribution of a Poisson process with intensity measure α . It is convenient to consider Π_α as a probability measure on M . It is concentrated on those $\mu \in M$ having locally finite support and $\mu\{s\} \in \mathbb{N}_0$, $s \in G$. We consider a *Cox process* (see e.g. [7]) driven by (X, ξ) , i.e. a random measure ζ on G satisfying

$$\mathbb{P}((X, \xi, \zeta) \in \cdot) = \mathbb{E} \left[\int \mathbf{1}\{(X, \xi, \mu) \in \cdot\} \Pi_\xi(d\mu) \right]. \tag{4.1}$$

Possibly extending $(\Omega, \mathcal{F}, \mathbb{P})$, the existence of ζ can be assumed without loss of generality. Let $\zeta^0 := \zeta + \delta_0$ and define $\Pi_\alpha^0 := \int \mathbf{1}\{\mu + \delta_0 \in \cdot\} \Pi_\alpha(d\mu)$, $\alpha \in M$.

Theorem 4.1. *Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite. Then (X, ξ) is mass-stationary iff (X, ζ^0) is mass-stationary. In this case even $((X, \xi), \zeta^0)$ is mass-stationary.*

We will prove this theorem later in this section.

Remark 4.2. Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite and that (X, ξ) is mass-stationary. Then Theorem 4.1 and (2.9) imply

$$\mathbb{E} \left[\int \int \mathbf{1}_A(\theta_s X, \theta_s \xi, \theta_s \mu) T(X, \xi, \mu, 0, ds) \Pi_\xi^0(d\mu) \right] = \mathbb{E} \left[\int \mathbf{1}_A(X, \xi, \mu) \Pi_\xi^0(d\mu) \right] \tag{4.2}$$

for all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ and all mass-preserving invariant weighted transport kernels T from $(W \times M) \times M \times G$ to G .

Combining Theorem 4.1 with Theorem 3.2 gives the following characterization of mass-stationarity via Bernoulli transport kernels.

Corollary 4.3. *Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite. Then (X, ξ) is mass-stationary iff (4.2) holds for all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ and all invariant mass-preserving Bernoulli transport kernels T .*

The Bernoulli transport kernels T used in Corollary 4.3 are allowed to depend on X . Under an additional assumption on ξ (or on G) we can improve this result. To do so we introduce G_0 as the set of all group elements of finite order. Hence $s \in G_0$ if $ks = 0$ for some $k \in \mathbb{N}_0$, where ks is defined inductively, by $0s := 0$ and $(k + 1)s := ks + s$. Obviously, G_0 is a measurable set. In the following result we assume that the discrete part ξ^d of ξ does not charge G_0 . In the important special cases $G = \mathbb{R}^d$ and $G = \mathbb{Z}^d$ this is no restriction of generality. Recall the definitions of the sets \mathbf{T} , \mathbf{T}_b , \mathbf{A} , and \mathbf{A}_m given before Remark 2.4.

Corollary 4.4. *Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite and that $\mathbb{P}(\xi^d(G_0) > 0) = 0$. Then (X, ξ) is mass-stationary iff*

$$\mathbb{E} \left[\iint \mathbf{1}_A(\theta_s X, \theta_s \mu) T(\mu, 0, ds) \Pi_\xi^0(d\mu) \right] = \mathbb{E} \left[\int \mathbf{1}_A(X, \mu) \Pi_\xi^0(d\mu) \right] \quad (4.3)$$

holds for all $A \in \mathcal{W} \otimes \mathcal{M}$ and all $T \in \mathbf{T}_b$.

For the proof of Corollary 4.4 we need the following lemma.

Lemma 4.5. *Let η be a Poisson process with intensity measure $\alpha \in M$ and define $\eta^0 := \eta + \delta_0$. Assume that $\alpha^d(G_0) = 0$. Then*

$$\mathbb{P}(\theta_s \eta^0 = \eta^0 \text{ for some } s \in \text{supp } \eta \setminus \{0\}) = 0. \quad (4.4)$$

Proof. By assumption on α we have that $\mathbb{P}(\eta(G_0) > 0) = 0$. Hence we have to show that the event

$$A := \{\theta_s \eta^0 = \eta^0 \text{ for some } s \in (G \setminus G_0) \cap \text{supp } \eta\}$$

has probability 0. (We assume here that \mathbb{P} is a probability measure.) We introduce the measurable sets $B_c := G_0^c \cap (G \setminus \text{supp } \alpha^d)$ and $B_d := G_0^c \cap \text{supp } \alpha^d$ and define the events

$$A_c := \{\theta_s \eta^0 = \eta^0 \text{ for some } s \in B_c \cap \text{supp } \eta\},$$

$$A_d := \{\text{there is some } s \in B_d \cap \text{supp } \eta \text{ s. t. } \theta_{ks} \eta^0 = \eta^0 \text{ and } ks \in B_d \text{ for all } k \in \mathbb{N}\}.$$

Assume that $\theta_s \eta^0 = \eta^0$ for some $s \in G_0^c \cap \text{supp } \eta$, implying that $\theta_{ks} \eta^0 = \eta^0$ for all $k \in \mathbb{N}$. Then either $ks \in B_d$ for all $k \in \mathbb{N}$, that is A_d holds, or $ks \in B_c$ for some $k \in \mathbb{N}$, that is A_c holds. Hence $A = A_c \cup A_d$ and we have to show that $\mathbb{P}(A_c) = \mathbb{P}(A_d) = 0$.

Let C_i , $i \in \mathbb{N}$, be a partition of B_c such that $\alpha(C_i) < \infty$. Then the restriction of η to B_c can be represented as $\hat{\eta} = \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} \delta_{X_{ij}}$ where the N_i , X_{ij} , $i, j \in \mathbb{N}$, are independent random variables, N_i Poisson with parameter $\alpha(C_i)$, and X_{ij} with distribution $\alpha(\cdot | C_i)$. If there is an $s \neq 0$ such that $\theta_s \hat{\eta}^0 = \hat{\eta}^0$ then there are n, i, j such that $N_n \geq 1$ and $X_{ij} = 2X_{n1}$. Since the X_{ij} are independent with diffuse distributions, this happens with probability 0. Hence $\mathbb{P}(A_c) = 0$.

It remains to show that $\mathbb{P}(A_d) = 0$. Since $\text{supp } \alpha^d$ is countable, it is enough to show that $\theta_s \eta^0 = \eta^0$ occurs with zero probability for any fixed $s \in \text{supp } \alpha^d$ such that $s \notin G_0$ and $ks \in \text{supp } \alpha^d$ for all $k \in \mathbb{N}$. Put $X_k := \eta\{ks\}$, $k \in \mathbb{N}$. Since $s \notin G_0$, the points ks are all different and the random variables X_k are independent Poisson random variables with means $\alpha_k := \alpha\{sk\}$. Fix an integer $m > 0$. Let α_{n_k} be a subsequence tending to a limit $\alpha \in [0, \infty]$. If $\alpha = 0$ or $\alpha = \infty$ then $e^{-\alpha_{n_k}} \alpha_{n_k}^m / m! \rightarrow 0$. If $0 < \alpha < \infty$ then $e^{-\alpha_{n_k}} \alpha_{n_k}^m / m! \rightarrow e^{-\alpha} \alpha^m / m! < 1$. In both cases the product $\mathbb{P}(X_1 = m) \mathbb{P}(X_2 = m) \dots$ is zero. Sum over $m > 0$ and use the independence to obtain $\mathbb{P}(0 < X_1 = X_2 = \dots) = 0$. Noting that $\theta_s \eta^0 = \eta^0$ implies that $0 < X_1 = X_2 = \dots$ completes the proof. \square

Proof of Corollary 4.4: If (X, ξ) is mass-stationary then (4.3) follows as a special case of (4.2). Conversely, assume that (4.3) holds and that $\mathbb{P}(\xi^d(G_0) > 0) = 0$.

By Lemma 4.5, $\mathbb{P}(\zeta^0 \in \cdot)$ is aperiodic. Hence we obtain from Proposition 3.8 that (X, ζ^0) is mass-stationary. Theorem 4.1 yields mass-stationarity of (X, ξ) . \square

For diffuse random measures the condition (4.3) can be simplified as follows.

Corollary 4.6. *Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite and that $\mathbb{P}(\xi \neq \xi^c) = 0$. Then (X, ξ) is mass-stationary iff*

$$\mathbb{E} \left[\int \mathbf{1}_A(\theta_{\tau(\mu,0)}X, \theta_{\tau(\mu,0)}\mu) \Pi_\xi^0(d\mu) \right] = \mathbb{E} \left[\int \mathbf{1}_A(X, \mu) \Pi_\xi^0(d\mu) \right], \quad A \in \mathcal{W} \otimes \mathcal{M}, \tag{4.5}$$

holds for all $\tau \in \mathbf{A}_m$.

Proof. Using the second part of Proposition 3.8, the result can be proved as Corollary 4.4. \square

Remark 4.7. Equation (4.3) can be written as

$$\mathbb{E} \left[\int \mathbf{1}_A(\theta_s X, \theta_s \zeta^0) T(\zeta^0, 0, ds) \right] = \mathbb{P}((X, \zeta^0) \in A), \quad A \in \mathcal{W} \otimes \mathcal{M}. \tag{4.6}$$

The point here is that the random measure ξ is not entering this equation explicitly, but only implicitly, as random intensity measure of ζ .

Remark 4.8. Assume that $\mathbb{P}(X \in \cdot)$ is σ -finite. Let T be a mass-preserving and σ -finite transport kernel. Define another transport kernel T' by

$$T'(w, \alpha, s, \cdot) := \int T(w, \mu + \delta_s, s, \cdot) \Pi_\alpha(d\mu). \tag{4.7}$$

Then (4.8) below implies invariance of T' , while (4.9) easily implies that T' is mass-preserving. If (X, ξ) is mass-stationary, then Remark 4.2 yields

$$\mathbb{E} \left[\int \mathbf{1}\{\theta_s(X, \xi) \in \cdot\} T'(X, \xi, 0, ds) \right] = \mathbb{P}((X, \xi) \in \cdot).$$

We do not know whether the validity of this equation for all such *Cox transport kernels* T' is enough to imply mass-stationarity of (X, ξ) . We refer here also to Problem 7.3 in [10].

Remark 4.9. Take $\tau \in \mathbf{A}$, and let $V := \tau(\zeta^0, 0)$. Then (4.5) can be written as

$$(\theta_V X, \theta_V \zeta^0) \stackrel{d}{=} (X, \zeta^0).$$

Remark 4.10. Assuming that $\mathbb{P}(\xi \in \cdot)$ is σ -finite is stronger than only assuming that $\mathbb{P}((X, \xi) \in \cdot)$ is σ -finite. If, for instance, X is a constant, $\mathbb{P}(X \in \cdot)$ can only be σ -finite, if \mathbb{P} is a finite measure. We do not know, whether the results of this section remain true in the more general case, where only $\mathbb{P}((X, \xi) \in \cdot)$ is σ -finite.

Proof of Theorem 4.1: First we recall that

$$\int \mathbf{1}\{\mu \in \cdot\} \Pi_{\theta_s \alpha}(d\mu) = \int \mathbf{1}\{\theta_s \mu \in \cdot\} \Pi_\alpha(d\mu), \quad \alpha \in M, s \in G, \tag{4.8}$$

and

$$\iint \mathbf{1}\{(\mu, s) \in \cdot\} \mu(ds) \Pi_\alpha(d\mu) = \iint \mathbf{1}\{(\mu + \delta_s, s) \in \cdot\} \alpha(ds) \Pi_\alpha(d\mu), \quad \alpha \in M. \tag{4.9}$$

The first equation comes directly from the definition of Π_α , while the second is from [11].

Assume now that (X, ξ) is mass-stationary. By Theorem 6.3 in [10] there is a stationary σ -finite measure \mathbb{Q} on $W \times M$ such that

$$\mathbb{P}((X, \xi) \in \cdot) = \lambda(B)^{-1} \iint \mathbf{1}_A(\theta_s(w, \mu)) \mathbf{1}_B(s) \mu(ds) \mathbb{Q}(d(w, \mu)), \quad A \in \mathcal{W} \otimes \mathcal{M}, \tag{4.10}$$

where $0 < \lambda(B) < \infty$. This means that $\mathbb{P}((X, \xi) \in \cdot)$ is the Palm measure of the projection from $W \times M$ onto M with respect to \mathbb{Q} , cf. (2.2).

Consider the measurable space $(\Omega^*, \mathcal{F}^*) := (\Omega \times M \times M, \mathcal{F} \otimes \mathcal{M} \otimes \mathcal{M})$ equipped with the measurable flow $\theta_s^*(w, \alpha, \mu) := (\theta_s w, \theta_s \alpha, \theta_s \mu)$. Define a measure \mathbb{Q}^* on $(\Omega^*, \mathcal{F}^*)$ by

$$\mathbb{Q}^* := \iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\} \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)). \tag{4.11}$$

Since \mathbb{Q} is σ -finite, so is \mathbb{Q}^* . Using (4.8), we get for any measurable $f : \Omega^* \rightarrow [0, \infty)$

$$\begin{aligned} \int f(\theta_s^*(w, \alpha, \mu)) \mathbb{Q}^*(d(w, \alpha, \mu)) &= \iint f(\theta_s w, \theta_s \alpha, \mu) \Pi_{\theta_s \alpha}(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iint f(w, \alpha, \mu) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)), \end{aligned}$$

where the second equality comes from stationarity of \mathbb{Q} . Hence \mathbb{Q}^* is invariant under the flow $\{\theta_s^* : s \in G\}$.

Denote by (X^*, ξ^*, ζ^*) the identity on Ω^* . Our next aim is to compute the Palm measure of $((X^*, \xi^*), \zeta^*)$ w.r.t. \mathbb{Q}^* . Using (4.8) and (4.9), we obtain for all measurable $f : \Omega^* \times G \rightarrow [0, \infty)$ that

$$\begin{aligned} \iint f(\theta_s(w, \alpha), \theta_s \mu, s) \mu(ds) \mathbb{Q}^*(d(w, \alpha, \mu)) &= \iiint f(\theta_s(w, \alpha), \theta_s \mu, s) \mu(ds) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \theta_s(\mu + \delta_s), s) \alpha(ds) \Pi_\alpha(d\mu) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s) \Pi_{\theta_s \alpha}(d\mu) \alpha(ds) \mathbb{Q}(d(w, \alpha)) \\ &= \iiint f(\theta_s(w, \alpha), \mu + \delta_0, s) \Pi_\alpha(d\mu) ds \mathbb{P}((X, \xi) \in d(w, \alpha)), \end{aligned}$$

where the final equality is due to (4.10) and the refined Campbell theorem (2.3) for the pair $(\mathbb{Q}, \mathbb{P}((X, \xi) \in \cdot))$. Therefore

$$\iint \mathbf{1}\{(w, \alpha, \mu) \in \cdot\} \Pi_\alpha^0(d\mu) \mathbb{P}((X, \xi) \in d(w, \alpha)) = \mathbb{P}((X, \xi, \zeta^0) \in \cdot) \tag{4.12}$$

is the Palm measure of $((X^*, \xi^*), \zeta^*)$ w.r.t. \mathbb{Q}^* . Theorem 6.3 in [10] implies that $((X, \xi), \zeta^0)$ is mass-stationary and that

$$\mathbb{E} \left[\int g(\theta_s(X, \xi), \theta_s \zeta^0, -s) \zeta^0(ds) \right] = \mathbb{E} \left[\int g(X, \xi, \zeta^0, s) \zeta^0(ds) \right] \quad (4.13)$$

for any measurable $g : W \times M \times M \times G \rightarrow [0, \infty)$. In particular we have

$$\mathbb{E} \left[\int g(\theta_s(X, \zeta^0), -s) \zeta^0(ds) \right] = \mathbb{E} \left[\int g(X, \zeta^0, s) \zeta^0(ds) \right] \quad (4.14)$$

for any measurable $g : W \times M \times G \rightarrow [0, \infty)$. As σ -finiteness of $\mathbb{P}(X \in \cdot)$ entails the same property of $\mathbb{P}((X, \zeta^0) \in \cdot)$, we conclude that (X, ζ^0) is mass-stationary.

To prove the other implication, we assume that (X, ζ^0) is mass-stationary. Since mass-stationarity is equivalent to the Mecke equation (4.14), we have

$$\begin{aligned} \mathbb{E} \left[\iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s) (\mu + \delta_0)(ds) \Pi_\xi(d\mu) \right] \\ = \mathbb{E} \left[\iint f(X, \mu + \delta_0, s) (\mu + \delta_0)(ds) \Pi_\xi(d\mu) \right] \end{aligned}$$

for all measurable $f : W \times M \times G \rightarrow [0, \infty)$. If $\mathbb{E}[\int f(X, \mu + \delta_0, 0) \Pi_\xi(d\mu)] < \infty$, we obtain

$$\begin{aligned} \mathbb{E} \left[\iint f(\theta_s X, \theta_s \mu + \delta_{-s}, -s) \mu(ds) \Pi_\xi(d\mu) \right] \\ = \mathbb{E} \left[\iint f(X, \mu + \delta_0, s) \mu(ds) \Pi_\xi(d\mu) \right]. \end{aligned}$$

Since $\mathbb{P}(X \in \cdot)$ is σ -finite, this remains true for any measurable $f : W \times M \times G \rightarrow [0, \infty)$. Using (4.9) and then (4.8) we get

$$\begin{aligned} \mathbb{E} \left[\iint f(\theta_s X, \mu + \delta_{-s} + \delta_0, -s) \Pi_{\theta_s \xi}(d\mu) \xi(ds) \right] \\ = \mathbb{E} \left[\iint f(X, \mu + \delta_s + \delta_0, s) \Pi_\xi(d\mu) \xi(ds) \right]. \end{aligned}$$

We apply this with $f(w, \mu, s) := \mathbf{1}\{\mu\{s\} \geq 1, \mu\{0\} \geq 1\} f_1(w, \mu - \delta_s - \delta_0, s)$ for a measurable function $f_1 : W \times M \times G \rightarrow [0, \infty)$. It follows that

$$\mathbb{E} \left[\iint f_1(\theta_s X, \mu, -s) \Pi_{\theta_s \xi}(d\mu) \xi(ds) \right] = \mathbb{E} \left[\iint f_1(X, \mu, s) \Pi_\xi(d\mu) \xi(ds) \right]. \quad (4.15)$$

Take $B \in \mathcal{G}$ and measurable functions $h_1 : W \rightarrow \mathbb{R}$ and $h : M \rightarrow \mathbb{R}$. Equation (4.15) implies

$$\mathbb{E} \left[\int h_1(\theta_s X) h^*(\theta_s \xi) \mathbf{1}_B(-s) \xi(ds) \right] = \mathbb{E}[h_1(X) \xi(B) h^*(\xi)], \quad (4.16)$$

where the measurable function $h^* : M \rightarrow [0, \infty]$ is defined by

$$h^*(\alpha) := \int h(\mu) \Pi_\alpha(d\mu). \quad (4.17)$$

Our next aim is to show that the class of measurable functions defined by (4.17) is rich enough, to conclude from (4.16) that

$$\mathbb{E} \left[\int h_1(\theta_s X) g(\theta_s \xi) \mathbf{1}_B(-s) \xi(ds) \right] = \mathbb{E}[h_1(X) \xi(B) g(\xi)] \tag{4.18}$$

holds for all measurable $g : M \rightarrow \mathbb{R}$. For $n \in \mathbb{N}$ and $\mu \in M$ we define a measure $\mu^{(n)}$ on G^n by

$$\mu^{(n)}(C) := \int \cdots \int \mathbf{1}_C(s_1, \dots, s_n) \mu_{s_1, \dots, s_{n-1}}(ds_n) \cdots \mu_{s_1}(ds_2) \mu(ds_1),$$

where, for $1 \leq k \leq n - 1$, the measure μ_{s_1, \dots, s_k} on G is defined by

$$\mu_{s_1, \dots, s_k} := \mathbf{1}\{\mu - \delta_{s_1} - \dots - \delta_{s_k}(\{s_1, \dots, s_k\}) \geq 0\}(\mu - \delta_{s_1} - \dots - \delta_{s_k}).$$

A well-known property of a Poisson process (following from (4.9) and induction) is

$$\int \mu^{(n)}(C) \Pi_\alpha(d\mu) = \alpha^n(C), \quad C \in \mathcal{G}^{\otimes n}, \alpha \in M.$$

For $k, i_1, \dots, i_k \in \mathbb{N}$ and relatively compact sets $B_1, \dots, B_k \in \mathcal{G}$ this gives

$$\int \mu^{(i_1 + \dots + i_k)}(B_1^{i_1} \times \dots \times B_k^{i_k}) \Pi_\alpha(d\mu) = \alpha(B_1)^{i_1} \cdots \alpha(B_k)^{i_k}. \tag{4.19}$$

Now we consider the measurable function

$$h(\mu) := c_0 + \sum_{i_1, \dots, i_k \in \mathbb{N}} c_{i_1, \dots, i_k} \mu^{(i_1 + \dots + i_k)}(B_1^{i_1} \times \dots \times B_k^{i_k}), \tag{4.20}$$

where $c_0 \in \mathbb{R}$ and the numbers $c_{i_1, \dots, i_k} \in \mathbb{R}$ satisfy

$$\sum_{i_1, \dots, i_k \in \mathbb{N}} |c_{i_1, \dots, i_k}| x_1^{i_1} \cdots x_k^{i_k} < \infty$$

for all $x_1, \dots, x_k \geq 0$. Let the entire function $f : \mathbb{R}^k \rightarrow \mathbb{R}$ be given by

$$f(x_1, \dots, x_k) := c_0 + \sum_{i_1, \dots, i_k \in \mathbb{N}} c_{i_1, \dots, i_k} x_1^{i_1} \cdots x_k^{i_k}.$$

Then (4.19) and dominated convergence implies that

$$h^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k)), \quad \alpha \in M, \tag{4.21}$$

where we recall the definition (4.17) of h^* . Let $B \in \mathcal{G}$ be relatively compact and $c > 0$. Consider the function $f(x_1, \dots, x_{k+1}) := f(x_1, \dots, x_k) e^{-cx_{k+1}}$, $x_1, \dots, x_{k+1} \in \mathbb{R}$, where f is as in (4.21). Define \tilde{h} as in (4.20) with (B_1, \dots, B_k) replaced by (B_1, \dots, B_k, B) and with the appropriate coefficients $c_{i_1, \dots, i_{k+1}} \in \mathbb{R}$. Then $\tilde{h}^*(\alpha) = f(\alpha(B_1), \dots, \alpha(B_k)) e^{-c\alpha(B)}$ and we get from (4.16) that

$$\mathbb{E} \left[\int h_1(\theta_s X) h(\theta_s \xi) \mathbf{1}_B(-s) e^{-c\xi(B+s)} \xi(ds) \right] = \mathbb{E}[h_1(X) h(\xi) \xi(B) e^{-c\xi(B)}] \tag{4.22}$$

holds for all $c > 0$ and all functions h in the class \mathcal{H} of bounded measurable functions of the form (4.21). Assume that $\mathbb{E}[|h_1(X)|] < \infty$. Applying (4.22) with $h \equiv 1$ and h_1 replaced with $|h_1|$, yields

$$\mathbb{E} \left[\int |h_1(\theta_s X)| \mathbf{1}_B(-s) e^{-c\xi(B+s)} \xi(ds) \right] = \mathbb{E}[|h_1(X)| \xi(B) e^{-c\xi(B)}] < \infty.$$

Therefore the class of all bounded measurable functions h satisfying (4.22) is a vector space containing the constant functions and being closed under monotone bounded convergence. Since \mathcal{H} is stable under multiplication and generates the σ -field \mathcal{M} , we can apply a well-known functional version of the monotone class theorem to obtain that (4.22) holds for any bounded measurable function h . Assume that $h \geq 0$. Since $\mathbb{P}(X \in \cdot)$ is σ -finite, (4.22) remains true for any measurable $h_1 : W \rightarrow [0, \infty)$. Moreover, for $c \rightarrow 0$ we get from monotone convergence the desired equation (4.18), and in particular

$$\mathbb{E} \left[\int \mathbf{1}_A(\theta_s X, \theta_s \xi, -s) \xi(ds) \right] = \mathbb{E} \left[\int \mathbf{1}_A(X, \xi, s) \xi(ds) \right], \quad (4.23)$$

for all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$ that are of product form. The measure on the right-hand side of (4.23) is finite on product sets of the form $C \times \{\alpha \in M : \alpha(B) \leq k\} \times B$, where $\mathbb{Q}(X \in C) < \infty$, $B \in \mathcal{G}$ is compact, and $k \in \mathbb{N}$. Since $W \times M \times G$ is the monotone union of countably many such sets, (4.23) extends to all $A \in \mathcal{W} \otimes \mathcal{M} \otimes \mathcal{M}$. This is equivalent to the Mecke equation (2.5) and hence to mass-stationarity of (X, ξ) . \square

5. Comments on Bernoulli and Cox Transports

In this paper we have been concerned with using Bernoulli and Cox transports to characterize mass-stationarity. However, we believe that such transports could be used also for other purposes. In this section we briefly indicate some possible directions assuming that $G = \mathbb{R}^d$. Assume given the general setting of Section 2 with \mathbb{P} a probability measure. Consider a stationary and ergodic random pair (X, ξ) and assume that ξ has intensity $\mathbb{E}[\xi([0, 1]^d)] = 1$. Then the Palm measure $\mathbb{P}_{X, \xi}$ defined by (2.2) is a probability measure. Let (X^0, ξ^0) be a random pair with distribution $\mathbb{P}_{X, \xi}$.

Let T be transport kernel and consider a *cost function* $g : [0, \infty) \rightarrow [0, \infty)$. Define the g -cost of T by

$$c(g, T) := \mathbb{E} \left[\int g(|s|) T(X^0, \xi^0, 0, ds) \right]. \quad (5.1)$$

This number can be interpreted as the mean unit cost of transporting mass from a typical location in the mass of ξ .

5.1. Bernoulli transports. Let T be a Bernoulli transport kernel as defined at (3.1). Let us briefly consider the case $p \equiv 1$, that is $T = \delta_\tau$. Then (5.1) takes the form

$$c(g, T) = \mathbb{E}[g(\tau(X^0, \xi^0, 0))].$$

For instance, take $g(u) := \mathbf{1}\{u \geq r\}$ for $r \geq 0$. If ξ is a simple point process and τ is a matching, then $c(r, \tau) := c(g, T)$ is the probability that a typical point of ξ is matched to a point at distance at least r . In the Poisson case and for certain (*one-color*) matchings τ the asymptotic behaviour of $c(r, \tau)$ (as $r \rightarrow \infty$) has recently been studied in [5] and [14].

Let us now return to the general Bernoulli transport kernel (3.1), i.e. p need not be identically 1. Assume that τ is a matching and that X^0 is constant. Consider

the cost $c(r, T) := c(g, T)$ (with g as above) under the condition that

$$\mathbb{E}[T(\xi^0, 0, \{0\})] \leq c_0$$

for some $c_0 \geq 0$. This amounts to accepting a rate c_0 of points of ξ that are not being matched. It would be interesting to check whether the Bernoulli randomization can reduce the costs of such (partial) matchings.

A similar problem for *two-color* matchings [5, 14] can be formulated as follows. Let $\eta := X$ and ξ be simple point processes. Consider an invariant Bernoulli transport kernel T (in case $W = M$) satisfying

$$\int T(\eta, \xi, s, \cdot)\eta(ds) = \xi, \quad \int T(\eta, \xi, s, \cdot)\xi(ds) = \eta \quad \mathbb{P}\text{-a.s.} \tag{5.2}$$

Again one could then consider the cost $c(r, T)$ under the condition that

$$\mathbb{E}[T(\eta^0, \xi^0, 0, \{0\})] \leq c_0.$$

It is easy to see that (5.2) implies that

$$\int T(\nu, \mu, 0, \{0\})\mathbb{P}_{\eta, \xi}(d(\nu, \mu)) = \int T(\nu, \mu, 0, \{0\})\mathbb{P}_{\xi, \eta}(d(\nu, \mu)).$$

5.2. Cox transports. Assume that ξ is a diffuse random measure and (for simplicity) that X is constant. Let ζ be a Cox process on \mathbb{R}^d driven by ξ . Since ξ is assumed ergodic, it is easy to see that ζ is ergodic as well. Let τ be an allocation rule balancing ζ and Lebesgue measure λ^d , that is,

$$\int \mathbf{1}\{\tau(\zeta, s) \in \cdot\} ds = \zeta \quad \mathbb{P}\text{-a.s.} \tag{5.3}$$

The existence of such τ is guaranteed by the results in [6] and [1]. We now define an invariant transport kernel T by

$$T(\alpha, s, B) = \int \mathbf{1}\{\tau(\mu, s) \in B\} \Pi_\alpha(d\mu) = \mathbb{P}(\tau(\zeta, s) \in B | \xi = \alpha). \tag{5.4}$$

From (5.3) and $\mathbb{E}[\zeta(\cdot) | \xi] = \xi$ it easily follows that T is \mathbb{P} -a.s. (λ^d, ξ) -balancing, that is

$$\int T(\xi, s, \cdot) ds = \xi \quad \mathbb{P}\text{-a.s.} \tag{5.5}$$

In particular we obtain from Theorem 4.1 in [10] that

$$\mathbb{P}_\xi = \mathbb{E} \left[\int \mathbf{1}\{\theta_s \xi \in \cdot\} T(\xi, 0, ds) \right] = \mathbb{P}(\theta_{\tau(\zeta, 0)} \in \cdot). \tag{5.6}$$

Equation (5.5) yields a more explicit version of Theorem 5.1 in [10]. It would be interesting to study the cost (5.1) associated with this transport. A similar task can be formulated for the transport kernels mentioned in Remark 4.8.

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