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A GENERAL THEOREM FOR PORTFOLIO GENERATING FUNCTIONS

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Abstract. In the present work, we show that dynamic equity portfolios can be generated by positive continuously differentiable functions of both the original and the ranked capitalization weights of an equity market. The return on these portfolios relative to the market is given by a stochastic differential equation and is expressed in terms of the logarithmic change in the value of the generating function, and a drift process that is of bounded variation.

1. Introduction

Functionally generated equity portfolios first appeared in [10] with the entropy-weighted portfolio and constitute one of the basics tools of stochastic portfolio theory. Entropy appeared in the stochastic portfolio theory as a measurement of the degree of diversity in the market and was used to derive conditions under which market diversity is consistent with capital market equilibrium in the sense of Sharpe (see [22]). Functionally generated equity portfolios are a natural generalization of entropy weighted portfolio. In [11], Fernholz showed that a broad class of functions can be used to generate portfolio: More precisely, he proved that dynamic equity portfolios can be generated by positive twice continuously differentiable functions of market weights in an equity market. Moreover, such a function is a measure of diversity if it is symmetric and concave.

In stochastic portfolio theory, the distribution of capital is of crucial importance as are functionally generated portfolios. In connections with this distribution of capital, it is better to associate the stocks with their rank rather than their name. Although functionally generated portfolios had useful theoretical properties, the construction was not sufficiently general to allow for the study of portfolios composed of stocks selected by their rank, as occurs in many equity indices. In fact rank functions are not differentiable, therefore, the results in [10, 11] cannot directly be applicable. In [12], functions of the ranked market weights were considered, and it was shown that under appropriate conditions they also generate portfolios.

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This allowed the author to examine portfolios composed exclusively of large stocks, which are identified by ranked market weights.

The Gini coefficient is another important function used by economists to measure the diversity of the distribution of wealth. Despite the fact that this function fails to be C^2, it was shown in [13] that it can generate portfolios. The natural questions which arise are: Is there a bigger class of functions that can generate a portfolio? In other words, as asked by Fernholz in Problem 4.2.3 of [13], is there a general theorem for portfolio generating functions that includes at least Theorem 3.1.5, Theorem 4.2.1, and Example 4.2.2 in [13]?

In the present paper, we give an answer to these questions when the price processes are given by reversible continuous semimartingales. We derive general theorems for functionally generated portfolios for the original and the ranked capitalization weights of an equity market (Theorem 4.1 and Theorem 4.6.) In order to obtain the results for ranked capitalization weight, a semimartingale decomposition of ranked processes is needed (see Theorem 3.3 or [15, Theorem 2.3].) The main results are given for a broad class of functions, namely C^1 functions, and, extend previous works by Ferhnolz in [13]. The proof of the main results relies on Corollary 3.7 and on the generalized Itô formula (see [4, 5, 6, 7, 8]). We also extend the result to the time-dependent generating functions. Let us mention that, the generalized Itô formula for continuous semimartingales (not necessarily reversible) in n-dimensions or even 2-dimensions is in general difficult and sparsely covered by the present literature. See [9] and references therein. In [9], the authors use the stochastic Lebesgue-Stieltjes integrals of two parameters to derive under quite general conditions on the function, a generalized Itô formula in 2-dimensions. The extension of the latter result in n-dimensions is still open. Such an extension is of higher interest in stochastic portfolio theory, since, it can be used to find portfolio generated by some exotic options on stocks.

The paper is organized as follows. In Section 2, we give some basic definitions and some preliminary results concerning equity portfolios. Section 3 is devoted to definitions and results on local time and ranked processes. In Section 4, we prove our main results.

2. Equity Portfolios

In this Section, we briefly recall the concepts of stochastic portfolio theory. The material in this Section is from [13]. We shall work in a market $\mathcal{M}$ consisting of n stocks represented by their price processes $X_1,\dots,X_n$. The price processes evolve according to the equations

$$d\log X_i(t)=\gamma_i(t)dt+\sum_{\nu=1}^{n}\xi_{i\nu}(t)dW_\nu(t),\ \ t\in[0,T]\,,$$

for, $i=1,\dots,n$. Here $(W_1,\dots,W_n)$ is a standard n-dimensional Brownian motion defined on a complete filtered probability space $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ satisfying the usual conditions. The growth rate processes $\gamma_i=\{\gamma_i(t),\ t\in[0,T]\}\,,\ i=1,\dots,n$ are measurable, adapted and satisfy the growth condition $\int_0^T|\gamma_i(t)|\,dt<\infty$ for all $T>0$ a.s. For $i,\nu=1,\dots,n$, the volatility processes $\xi_{i\nu}=\{\xi_{i\nu}(t),t\in[0,T]\}$ are measurable, adapted and satisfy $\int_0^T \xi_{i\nu}^2(t)dt < \infty$, a.s., with $\xi_{i1}^2(t) + \ldots + \xi_{in}^2(t) > 0,\ t \in [0,T]$, a.s.

Consider the matrix-valued process ξ defined by $\xi(t) = (\xi_{i\nu}(t))_{1\le i,\nu\le n}$, and define the covariance process σ where $\sigma(t) = \xi(t)\xi^T(t)$. Then for all $t \in [0,T]$, $\sigma_{ij}(t)dt = d\left\langle \log X_i, \log X_j \right\rangle_t$, a.s. The conditions on the volatility processes ensure that the $\{\sigma_{ij}(\cdot)\}$ are a.s., L^1 functions.

Definition 2.1. A *portfolio* of the stocks $X_1,\ldots,X_n$ in $\mathcal{M}$ is a measurable, adapted process π that is bounded on $[0,T]\times\Omega$ and satisfies $\pi_1(t)+\ldots+\pi_n(t) = 1$, for $t \in [0,T]$, a.s.

For each i, the process π_i represents the proportion, or weight, of X_i in the portfolio. A negative value for $\pi_i(t)$ indicates a short sale.

Suppose $Z_\pi(t)$ represents the value of an investment in π at time t. Then the process $Z_\pi(t)$ is called the portfolio value process for π and it satisfies

$$dZ_\pi(t) = Z_\pi(t)\sum_{i=1}^n \pi(t)\frac{dX_i(t)}{X_i(t)},\ \text{for } t \in [0,T],\ \text{and } Z_\pi(0) > 0. \tag{2.1}$$

By Itô's formula, Equation (2.1) becomes

$$d\big(\log Z_\pi(t)\big) = \sum_{i=1}^n \pi_i(t)d\big(\log X_i(t)\big) + \gamma_\pi^*(t)dt \tag{2.2}$$

where

$$\gamma_\pi^*(t) = \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)\sigma_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\sigma_{ij}(t)\right),\ \ t\in[0,T] \tag{2.3}$$

is called the *excess growth rate* of π.

Definition 2.2. The portfolio μ with weights $\mu_1,\ldots,\mu_n$ defined by

$$\mu_i(t) = \frac{X_i(t)}{X_1(t)+\ldots+X_n(t)},\ \ t\in[0,T] \tag{2.4}$$

for $i = 1,\ldots,n$, is called the *market portfolio*, and the weights μ_i are called the *market weights.*

If we let $Z_\mu(t) = X_1(t)+\ldots+X_n(t),\ \ t \in [0,T]$, then $Z_\mu(t)$ satisfies Equation (2.1) with proportion $\mu_i(t)$ given by Equation (2.4).

The instantaneous relative return of X_i with respect to the market at time t is given by

$$d\left(\log(X_i(t)/Z_\mu(t))\right),\ \ t\in[0,T]$$

for $i = 1,\ldots,n$. Since $\mu_i = X_i/Z_\mu$, the relative return process $\log(X_i/Z_\mu)$ can be represented by $\log\mu_i$. The cross-variation processes for the relative returns of the stocks in the market generate the (matrix-valued) relative covariance process $\tau(t) = (\tau_{ij}(t))_{1\le i,j\le n}$, which is defined for all $t\in[0,T]$ a.s., by

$$\tau_{ij}(t)dt = d\left\langle \log\mu_i, \log\mu_j\right\rangle_t \tag{2.5}$$

$$= \sigma_{ij}(t) - \sum_{k=1}^n \mu_k\sigma_{ik}(t) - \sum_{k=1}^n \mu_k\sigma_{kj}(t) + \sum_{k,l=1}^n \mu_k\mu_l\sigma_{kl}(t). \tag{2.6}$$

Combining (2.6) with (2.3), we get

$$\gamma_\pi^*(t) = \frac{1}{2}\bigg(\sum_{i=1}^n \pi_i(t)\tau_{ii}(t) - \sum_{i,j=1}^n \pi_i(t)\pi_j(t)\tau_{ij}(t)\bigg). \tag{2.7}$$

In the following, we define the notion of *portfolio generating function*. This idea was introduced by Fernholz in [11] and is used theoretically to study market diversity and arbitrage, and, practically to explain the size effect which means the historical observed movement of smaller companies to have bigger return than larger companies. (See [13] and the references therein.)

Definition 2.3. Let O be an open neighborhood in $\mathbb{R}^n$ of the open simplex

$$\Delta^n = \left\{ x \in \mathbb{R}^n \, : \, \sum_{i=1}^n x_i = 1, \; 0 < x_i < 1, \; i = 1, \dots, n \right\},$$

and let $\mathbf{S}$ be a positive function defined in O. Then $\mathbf{S}$ *generates the portfolio* π if there exists a measurable, adapted process of bounded variation Θ such that

$$\log\left(Z_\pi(t)/Z_\mu(t)\right) = \log \mathbf{S}(\mu(t)) + \Theta(t), \;\; t \in [0,T], a.s. \tag{2.8}$$

The next section is consecrated to some definitions and important results on local time and ranked processes that will be used to proved the main theorems.

3. Results on Local Time and Ranked Processes

We consider a complete filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ which satisfies the usual conditions. In our study, any given semimartingale X is supposed to satisfy the following condition (A):

$$\sum_{0<s\leq t} |\Delta X_s| < \infty \quad \text{a.s.} \quad \text{for all} \quad t > 0\,, \tag{A}$$

where $\Delta X_s = X_s - X_{s-}$.

We begin by giving the definition of the local time of a semimartingale X.

Definition 3.1. Let $X = (X_t)_{t\geq 0}$ be a semimartingale and $a \in \mathbb{R}$. The *local time* $L_t^a(X)$ of X at a is defined by the following Tanaka-Meyer formula

$$\begin{aligned} |X_t - a| = |X_0 - a| &+ \int_0^t \operatorname{sgn}(X_{s-} - a)\, dX_s + L_t^a(X) \\ &+ \sum_{s\leq t} \big(|X_s - a| - |X_{s-} - a| - \operatorname{sgn}(X_{s-} - a)\Delta X_s\big), \end{aligned}$$

where $\operatorname{sgn}(x) = -1_{(-\infty,0]}(x) + 1_{(0,\infty)}(x)$.

As has been proved by Yor [24], under the condition (A), a measurable version of $(a,t,\omega) \mapsto L_t^a(X)(\omega)$ exists which is continuous in t and right continuous with left limits (i.e. *càdlàg*) in a. We will only deal with this version and in the remaining part of the paper, we will focus our attention on continuous-time processes.

With the purpose of giving the results for portfolios that are generated by functions of ranked market weight, let us recall the definition of the k-th rank process of a family of n semimartingales.

Definition 3.2. Let $X_1,\ldots,X_n$ be semimartingales. For $1 \le k \le n$, the *k-th rank process* $X_{(k)}$ of $X_1,\ldots,X_n$ is defined by

$$X_{(k)} = \max_{1\le i_1<\ldots<i_k\le n} \min(X_{i_1},\ldots,X_{i_k})\,. \tag{3.1}$$

Note that, according to Definition 3.2, it can be seen that $X^{(k)} \in \{X_1,\ldots,X_n\}$ and for $t \in \mathbb{R}^+$,

$$\max_{1\le i\le n} X_i(t) = X_{(1)}(t) \ge X_{(2)}(t) \ge \ldots \ge X_{(n)}(t) = \min_{1\le i\le n} X_i(t), \tag{3.2}$$

so that at any given time, the values of the ranked processes represent the values of the original processes arranged in descending order (i.e. the (reverse) order statistics).

The following theorem proved in [15] for semimartingales not necessarily continuous shows that the ranked processes can be expressed in terms of the stochastic integrals with respect to the original processes adjusted by local times. We will require the following definitions

$$S_t(k) = \{i : X_i(t) = X_{(k)}(t)\} \quad \text{and} \quad N_t(k) = |S_t(k)|\,. \tag{3.3}$$

Then, for $t>0$, $N_t(k)$ is the number of subscripts $\{i\}$ such that $X_i(t) = X_{(k)}(t)$. It is a predictable process and we have the following explicit decomposition.

Theorem 3.3. *Let $X_1,\ldots,X_n$ be continuous semimartingales. Then the k-th ranked processes $X_{(k)}$, $k \in \{1,2,\ldots,n\}$ are semimartingales and the following equality holds:*

$$\begin{aligned} dX_{(k)}(t) \;=\; & \sum_{i=1}^{n} \frac{1}{N_t(k)} 1_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t) + \sum_{i=k+1}^{n} \frac{1}{N_t(k)}\, d\,L_t^0(X_{(k)} - X_{(i)}) \\ & - \sum_{i=1}^{k-1} \frac{1}{N_t(k)}\, d\,L_t^0(X_{(i)} - X_{(k)}) \text{ for all } t>0 \text{ a.s.} \end{aligned} \tag{3.4}$$

Moreover,

$$\sum_{i=1}^{n} 1_{\{X_{(k)}(t)=X_{(i)}(t)\}}\, dX_{(i)}(t) = \sum_{i=1}^{n} 1_{\{X_{(k)}(t)=X_i(t)\}}\, dX_i(t), \text{ for all } t>0 \text{ a.s.} \tag{3.5}$$

Define

$$\begin{aligned} U = \{&p_\cdot(\cdot) : [0,\infty)\times\{1,\ldots,n\} \to \{1,\ldots,n\} \mid p_\cdot(k) \text{ is predictable and} \\ & X_{(k)}(t) = X_{p_t(k)}(t) \text{ for all } t>0,\ k=1,\ldots,n\big\}\,. \end{aligned} \tag{3.6}$$

For any $t \in [0,T]$, we can define p_t as a random permutation of $\{1,\ldots,n\}$ (see [1]), which in some cases coincides with the random permutation defined in (4.1.26) of [13]. In these cases, $p_t(k)$ represents the name (index) of the stock that has the k-th rank in terms of relative capitalization at time t.

From now on, we consider the collection of semimartingales $X_1,\ldots,X_n$ with decompositions $X_i(\cdot) = X_i(0) + M_i(\cdot) + V_i(\cdot)$; here $M_1,\ldots,M_n$ are continuous local martingales and $V_1,\ldots,V_n$ are of locally bounded variation. We make the following two assumptions:

$$dV_1(\cdot),\ldots,dV_n(\cdot) << \text{Leb}, a.s. \tag{3.7}$$

$$\{t\,:\,X_i(t)=X_j(t)\}\ \text{is Lebesgue-null a.s., for all } i\neq j. \tag{3.8}$$

We will also need the subsequent result from [1].

Theorem 3.4. *If $X_1,...,X_n$ are continuous semimartingales satisfying (3.7) and (3.8) above, then*

$$\begin{aligned} dX_{(k)}(t) \;=\; & \sum_{i=1}^{n} 1_{\{p_t(k)=i\}}\; dX_i(t) + \sum_{i=k+1}^{n} \frac{1}{N_t(k)}\, d\,L_t^0(X_{(k)}-X_{(i)}) \\ & - \sum_{i=1}^{k-1} \frac{1}{N_t(k)}\, d\,L_t^0(X_{(i)}-X_{(k)}) \end{aligned} \tag{3.9}$$

for all $t>0$ a.s., for any p in the set U of (3.6), where $N_t(k)$ is as in (3.3).

We also get

Lemma 3.5. *Suppose that (3.7) and (3.8) are satisfied. Suppose, moreover, that for all $i<j<k$ we have $\{t\,:\,X_i(t)=X_j(t)=X_k(t)\}=\varnothing$, a.s. Then for any k,*

$$\sum_{i=1}^{n}\int_0^t 1_{\{p_s(k)=(i)\}}\, dX_{(i)}(s) \tag{3.10}$$

is independent of the choice of $p\in U$ for all $t>0$, a.s.

Proof. We wish to show that

$$\sum_{i=1}^{n}\int_0^t 1_{\{p_s(k)=(i)\}}\, dX_{(i)}(s) - \sum_{i=1}^{n}\int_0^t 1_{\{q_s(k)=(j)\}}\, dX_{(j)}(s) = 0$$

for any $p,q\in U$. Rewrite the left hand side as

$$\begin{aligned} & \sum_{i,j=1}^{n}\int_0^t 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\; 1_{\left\{X_{(i)}(s)-X_{(j)}(s)=0\right\}} d\left(X_{(i)}(s)-X_{(j)}(s)\right) \\ &= \sum_{i,j=1}^{n}\int_0^t 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\; 1_{\left\{X_{(i)}(s)-X_{(j)}(s)=0\right\}} \\ &\quad \left(\sum_{l=1}^{n} 1_{\{u_t(i)=l\}}\; dX_l(t) - \sum_{r=1}^{n} 1_{\{v_t(j)=r\}}\; dX_r(t)\right) \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\; 1_{\left\{X_{(i)}(s)-X_{(j)}(s)=0\right\}} \\ &\quad \left(dL_t\left(X_{(i)}-X_{(i+1)}\right) - dL_t\left(X_{(i-1)}-X_{(i)}\right)\right) \\ &\quad - \frac{1}{2}\sum_{i,j=1}^{n}\int_0^t 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\; 1_{\left\{X_{(i)}(s)-X_{(j)}(s)=0\right\}} \\ &\quad \left(dL_t\left(X_{(j)}-X_{(j+1)}\right) - dL_t\left(X_{(j-1)}-X_{(j)}\right)\right) \end{aligned} \tag{3.11}$$

where the equality follows from the decomposition of the rank process $X_{(k)}$ (see Corollary 2.6 in [1] or Proposition 4.1.11 in [13]). The first term of the right side of (3.11) is identically zero a.s., (see Proposition 2.4 in [1]). Let study the two other terms.

(1) Case $i = j$: In this case it is clear that the second and the third terms sum to zero.
(2) Case $i \neq j$: We claim that each term of both sums is zero.

Since $\{q_s(k) = (j)\} \subset \left\{X_{(k)}(s) = X_{(j)}(s)\right\}$, we have

$$\begin{aligned}
&1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\, 1_{\left\{X_{(i)}(s)-X_{(j)}(s)=0\right\}} \\
&= 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\, 1_{\left\{X_{(i)}(s)=X_{(j)}(s)\right\}} 1_{\left\{X_{(k)}(s)=X_{(i)}(s)\right\}} 1_{\left\{X_{(k)}(s)=X_{(j)}(s)\right\}} \\
&= 1_{\{q_s(k)=(j)\}}\, 1_{\{p_s(k)=(i)\}}\, 1_{\left\{X_{(i)}(s)=X_{(j)}(s)=X_{(k)}(s)\right\}} \\
&= 0.
\end{aligned}$$

Where the last equality follows by assumption. □

Theorem 3.6. *Let $X_1, \ldots, X_n$ be continuous semimartingales. Assume the conditions of Lemma 3.5. Then the k-th ranked processes $X_{(k)}$, $k \in \{1, 2, \ldots, n\}$ are semimartingales and we have*

$$\begin{aligned}
dX_{(k)}(t) \;=\; &\sum_{i=1}^{n} 1_{\{p_t(k)=(i)\}}\, dX_{(i)}(t) + \frac{1}{2}\, d\, L_t^0(X_{(k)} - X_{(k+1)}) \\
&- \frac{1}{2}\, d\, L_t^0(X_{(k-1)} - X_{(k)})
\end{aligned} \tag{3.12}$$

for all $t > 0$ a.s., for any p in the set U of (3.6).

Proof. Using Lemma 3.5, the result follows from the proofs of Theorem 2.5 and Theorem 2.3 in [1] □

Combining Theorem 3.4 with Theorem 3.6 we get

Corollary 3.7. *Let $X_1, \ldots, X_n$ be continuous semimartingales. Suppose that (3.7) and (3.8) are satisfied. Then, for $k \in \{1, 2, \ldots, n\}$, the following equality holds:*

$$\sum_{i=1}^{n} 1_{\{p_t(k)=i\}}\, dX_i(t) = \sum_{i=1}^{n} 1_{\{p_t(k)=(i)\}}\, dX_{(i)}(t) \tag{3.13}$$

for all $t > 0$ a.s. for any p in the set U.

Proof. It follows from Equations (3.9) and (3.12). □

4. Main Results

In this Section, we intend to use definitions of Section 2 and results from Section 3 to give an answer to Problem 4.2.3 of [13]. Note first that the incidence $X_i(t) = X_j(t)$ corresponds to the incidence $\mu_i(t) = \mu_j(t)$ and that the rankings of μ_i is equivalent to the rankings of X_i. Under this condition, the ranked market weights are denoted $\mu_{(k)}(\cdot)$, for $k=1,\dots,n$. Let us now fix $p\in U$. The *ranked covariance process relative to the market* is defined by

$$\tau_{(ij)}(t) := \tau_{p_t(i)p_t(j)}(t)$$

for all $t>0$ and $i,j=1,\dots,n$.

4.1. Time independent case. In this Section, the result is presented for time independent functions.

Theorem 4.1. *Let $\mathcal{M}$ be a market of stocks $X_1,\dots,X_n$ that are pathwise mutually nondegenerate, let p_t be the random permutation defined by (3.6), and let $\boldsymbol{S}$ be a function defined on a neighborhood O of Δ^n. Assume that $\mu_{\alpha(k)}(t)$ is a reversible semimartingale. Moreover, suppose that there exists a positive C^1 function S defined on O such that for $(x_1,\dots,x_n)\in U$,*

$$\boldsymbol{S}(x_1,\cdots,x_n) = S\left(x_{(1)},\cdots,x_{(n)}\right), \tag{4.1}$$

and for $i=1,\dots,n$, $x_iD_i\log S(x)$ is bounded for $x\in\Delta^n$. Then $\boldsymbol{S}$ generates portfolio π such that for $k=1,\dots,n$

$$\pi_{\alpha(k)}(t) = \left(D_k\log S(\mu_{\alpha(\cdot)}(t)) + 1 - \sum_{j=1}^{n}\mu_{(j)}D_j\log S(\mu_{\alpha(\cdot)}(t))\right)\mu_{\alpha(k)}(t) \tag{4.2}$$

for all $t\in[0,T]$, a.s., with the drift process Θ that satisfies

$$d\Theta(t) = -\frac{1}{S(\mu_{(\cdot)}(t))}\sum_{k=1}^{n}\int_{\mathbb{R}} D_kS(\mu_{(\cdot)}(t))d_xL_t^x(\mu_{(k)}) \tag{4.3}$$

for all $t\in[0,T]$, a.s., where $\pi_{\alpha(k)}=\pi_k$ or $\pi_{(k)}$.

Remark 4.2. Note that when $\mu_{\alpha(k)}=\mu_k$, the proof is similar to the proof of Theorem 3.1 in [11]. Conditions under which $\mu_{\alpha(k)}$ is a reversible semimartingale can be found in [17], for instance if $\mu_{\alpha(k)}(t)$ has a density $\nu_t(x)$ for all $t\in[0,T]$ a.s.

Proof. Let show that the portfolio π defined by (4.2) and the drift Θ defined by (4.3) satisfies (2.8). In order to achieve this, we examine the generating function term $\log S(\mu(t))$ in (2.8) and the relative return process $\log\left(Z_\pi(t)/Z_\mu(t)\right)$, and show that the difference of these two terms satisfies (4.3).

Theorem 2.3 in [15] states that the ranked weight processes $\mu_{(k)}$, for $k=1,\dots,n$, satisfy

$$\begin{aligned} d\log\mu_{(k)}(t) \;=\; & \sum_{i=1}^{n}\frac{1}{N_t(k)}1_{\{\log\mu_{(k)}(t)=\log\mu_{(i)}(t)\}}\,d\log\mu_{(i)}(t) \\ & + \sum_{i=k+1}^{n}\frac{1}{N_t(k)}\,dL_t^0(\log\mu_{(k)}-\log\mu_{(i)}) \\ & - \sum_{i=1}^{k-1}\frac{1}{N_t(k)}\,dL_t^0(\log\mu_{(i)}-\log\mu_{(k)}) \end{aligned} \tag{4.4}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \frac{1}{N_t(k)} 1_{\{\log \mu_{(k)}(t) = \log \mu_i(t)\}}\, d \log \mu_i(t) \\
&\quad + \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\, L_t^0((\log \mu_{(k)} - \log \mu_i)^+) \\
&\quad - \sum_{i=1}^{n} \frac{1}{N_t(k)}\, d\, L_t^0((\log \mu_{(k)} - \log \mu_i)^-)
\end{aligned} \tag{4.5}
$$

for $t \in [0,T]$, a.s. Combining (4.5) and (2.5), we have for $i, j = 1, \ldots, n$,

$$
d \left\langle \log \mu_{(i)}, \log \mu_{(j)} \right\rangle_t = \tau_{(ij)}(t)dt,\ t \in [0,T],\ a.s.
$$

Applying Itô's formula to $\mu_{(i)}(t) = \exp(\log \mu_{(i)}(t))$, a.s., for all $t \in [0,T]$,

$$
d\mu_{(i)}(t) = \mu_{(i)}(t) d \log \mu_{(i)}(t) + \frac{1}{2} \mu_{(i)}(t) \tau_{(ij)}(t) dt, \tag{4.6}
$$

we have

$$
d \left\langle \mu_{(i)}, \mu_{(j)} \right\rangle_t = \mu_{(i)}(t)\mu_{(j)}(t)\tau_{(ij)}(t)dt,\ t \in [0,T],\ a.s. \tag{4.7}
$$

Let note that for all $t \in [0,T]$, $\sum_{i=1}^{n} \mu_{(i)}(t) = 1$, and hence

$$
\sum_{i=1}^{n} d\mu_{(i)}(t) = 0.
$$

Consider now the generating function component of the relative return, $\log \mathbf{S}(\mu(t))$. Applying a generalized Itô's formula for reversible semimartingales (see [6, 7, 8]) together with (4.7), we get a.s., for $t \in [0,T]$,

$$
\begin{aligned}
d \log \mathbf{S}(\mu(t)) &= d \log S(\mu_{(\cdot)}(t)) \\
&= \frac{dS(\mu_{(\cdot)}(t))}{S(\mu_{(\cdot)}(t))} - \frac{d \left\langle S(\mu_{(\cdot)}) \right\rangle_t}{2S^2(\mu_{(\cdot)}(t))} \\
&= \frac{1}{S(\mu_{(\cdot)}(t))} \left[\sum_{i=1}^{n} D_i S(\mu_{(\cdot)}(t)) d\mu_{(i)}(t) + \sum_{i=1}^{n} d \left\langle D_i S(\mu_{(\cdot)}), \mu_{(i)} \right\rangle_t \right] \\
&\quad - \frac{1}{2S^2(\mu_{(\cdot)}(t))} \sum_{i=1}^{n} \sum_{i=1}^{n} D_i S(\mu_{(\cdot)}(t)) D_j S(\mu_{(\cdot)}(t)) d \left\langle \mu_{(i)}, \mu_{(j)} \right\rangle_t \\
&= \frac{1}{S(\mu_{(\cdot)}(t))} \left[\sum_{i=1}^{n} D_i S(\mu_{(\cdot)}(t)) d\mu_{(i)}(t) \right. \\
&\quad \left. + \sum_{i=1}^{n} \int_{\mathbb{R}} D_i S(\mu_{(\cdot)}(t)) \Big|_{\mu_{(i)(t)} = x} dL_t^x(\mu_{(i)}) \right] \\
&\quad - \frac{1}{2S^2(\mu_{(\cdot)}(t))} \sum_{i=1}^{n} \sum_{i=1}^{n} D_i S(\mu_{(\cdot)}(t)) D_j S(\mu_{(\cdot)}(t)) \mu_{(i)}(t) \mu_{(j)}(t) \tau_{(ij)}(t) dt.
\end{aligned} \tag{4.8}
$$

Now let consider the relative return process $\log(Z_\pi(t)/Z_\mu(t))$. From Equation (2.2) and Corollary 3.7, we have a.s., for $t \in [0,T]$,

$$d\log(Z_\pi(t)/Z_\mu(t)) = \sum_{i=1}^{n}\pi_i(t)d\log\mu_i(t) \;+\; \gamma_\pi^*(t)dt \tag{4.9}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}1_{\{p_t(k)=i\}}\pi_{p_t(k)}(t)d\log\mu_i(t) \;+\; \gamma_\pi^*(t)dt$$

$$= \sum_{k=1}^{n}\pi_{p_t(k)}(t)\sum_{i=1}^{n}1_{\{p_t(k)=i\}}d\log\mu_i(t) \;+\; \gamma_\pi^*(t)dt$$

$$= \sum_{k=1}^{n}\pi_{p_t(k)}(t)\sum_{i=1}^{n}1_{\{p_t(k)=(i)\}}d\log\mu_{(i)}(t) \;+\; \gamma_\pi^*(t)dt \tag{4.10}$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n}\pi_{p_t(k)}(t)1_{\{p_t(k)=(i)\}}d\log\mu_{(i)}(t) \;+\; \gamma_\pi^*(t)dt$$

$$= \sum_{i=1}^{n}\pi_{(i)}(t)d\log\mu_{(i)}(t) \;+\; \gamma_\pi^*(t)dt, \tag{4.11}$$

where (4.10) follows from (3.5). Substituting (2.7) and (4.6) into (4.11), we get

$$d\log(Z_\pi(t)/Z_\mu(t)) = \sum_{i=1}^{n}\frac{\pi_{(i)}(t)}{\mu_{(i)}(t)}d\mu_{(i)}(t) - \frac{1}{2}\sum_{i,j=1}^{n}\pi_{(i)}(t)\pi_{(j)}(t)\tau_{(ij)}(t)dt. \tag{4.12}$$

Let us simplify the first term on the right-hand side of (4.12). If the weights $\pi_i,\;\; i=1,\dots,n$ satisfy (4.2), then

$$\pi_{(k)}(t) = \big(D_k\log S(\mu_{(\cdot)}(t)) + \varphi(t)\big)\,\mu_{(k)}(t)$$

for $k=1,\dots,n$, where

$$\varphi(t) = 1 - \sum_{j=1}^{n}\mu_{(j)}D_j\log S(\mu_{(\cdot)}(t)) \;\; t\in[0,T]\,.$$

In this case

$$\sum_{i=1}^{n}\frac{\pi_{(i)}(t)}{\mu_{(i)}(t)}d\mu_{(i)}(t) = \sum_{i=1}^{n}\mu_{(i)}(t)D_i\log S(\mu_{(\cdot)}(t)) + \varphi(t)\sum_{i=1}^{n}d\mu_{(i)}(t)$$

$$= \sum_{i=1}^{n}\mu_{(i)}(t)D_i\log S(\mu_{(\cdot)}(t)), \tag{4.13}$$

since $\sum_{i=1}^{n}d\mu_{(i)}(t) = 0$.

Now consider the last term in (4.12). It follows from Lemma 1.2.2 in [13] and the definition of $\pi_{(k)}$ that a.s., for $t\in[0,T]$,

$$\sum_{i,j=1}^{n}\pi_{(i)}(t)\pi_{(j)}(t)\tau_{(ij)}(t)dt = \frac{1}{S^2(\mu_{(\cdot)}(t))}\sum_{i,j=1}^{n}D_iS(\mu_{(\cdot)}(t))D_jS(\mu_{(\cdot)}(t))\tau_{(ij)}(t)dt. \tag{4.14}$$

Equations (4.12), (4.13) and (4.14), imply that a.s., for $t \in [0,T]$,

$$d\log\left(Z_\pi(t)/Z_\mu(t)\right) = \sum_{i=1}^{n} \mu_{(i)} D_i \log S(\mu_{(\cdot)}(t)) + \frac{1}{2}\frac{1}{S^2(\mu_{(\cdot)}(t))}\sum_{i,j=1}^{n} D_i S(\mu_{(\cdot)}(t)) D_j S(\mu_{(\cdot)}(t))\tau_{(ij)}(t)dt. \tag{4.15}$$

This equation and (4.8) imply (4.3). □

As a consequence, we have the following Corollary which corresponds to Theorem 3.1.5 in [13].

Corollary 4.3. *Let S be a positive C^2 function defined on a neighborhood O of Δ^n such that for all i, $x_i D_i \log S(x)$ is bounded Δ^n. Then S generates portfolio π with weights*

$$\pi_k(t) = \left(D_k \log S(\mu(t)) + 1 - \sum_{j=1}^{n} \mu_j D_j \log S(\mu(t))\right)\mu_k(t) \tag{4.16}$$

for $t \in [0,T]$ and $i = 1,\dots,n$ and with the drift process Θ such that a.s., for $t \in [0,T]$,

$$d\Theta(t) = -\frac{1}{S(\mu(t)}\sum_{i,j=1}^{n} D_{ij} S(\mu(t))\mu_i(t)\mu_j(t)\tau_{ij}(t)dt. \tag{4.17}$$

Proof. It follows from Theorem 1 in [14]. □

We also have

Corollary 4.4. *Let $\boldsymbol{S}$ be the Gini function defined as in Example 4.2.2 in* [13] *on a neighborhood O of Δ^n. Then $\boldsymbol{S}$ generates portfolio π with weights*

$$\pi_k(t) = \left(\frac{\operatorname{sgn}(n^{-1} - \mu_i(t))}{2\boldsymbol{S}(\mu(t))} + 1 - \sum_{j=1}^{n} \frac{\mu_j(t)\operatorname{sgn}(n^{-1} - \mu_j(t))}{2\boldsymbol{S}(\mu(t))}\right)\mu_k(t), \tag{4.18}$$

for $i = 1,\dots,n$ and a drift process that satisfies

$$d\Theta(t) = \frac{1}{\boldsymbol{S}(\mu(t))}\sum_{i=1}^{n} dL_t^0(\mu_i - n^{-1}),\ \text{for } t \in [0,T],\ a.s. \tag{4.19}$$

4.2. Time dependent case. In this Section, we generalize the definition of generating functions to include time-dependent generating functions. We also give a Theorem similar to Theorem 4.1. The next definition is from [13].

Definition 4.5. Let $\mathbf{S}$ be a positive continuous function defined on $\Delta^n \times [0,T]$, $0 < T < \infty$, and π be a portfolio. Then $\mathbf{S}$ generates π if there exists a measurable, adapted process of bounded variation Θ such that

$$d\log\left(Z_\pi(t)/Z_\mu(t)\right) = d\log \mathbf{S}(\mu(t),t) - D_t \log \mathbf{S}(\mu(t),t) + d\Theta(t),\ \text{for } t \in [0,T], a.s. \tag{4.20}$$

The process Θ is called the *drift process* corresponding to $\mathbf{S}$.

Theorem 4.1 can be extended to the time-dependent generating functions. Let $\mathbf{S}$ be a $C^{1,1}$ function, let D_t represent the partial derivative with respect to the last variable.

Theorem 4.6. *Let $\mathcal{M}$ be a market of stocks $X_1, \ldots, X_n$ that are pathwise mutually nondegenerate, let p_t be the random permutation defined by (3.6), and let $\boldsymbol{S}$ be a function defined on a neighborhood O of Δ^n. Assume that $\mu_{\alpha(k)}(t)$ is a reversible semimartingale. Moreover, suppose that there exists a positive $C^{1,1}$ function S defined on $O \times [0,T]$ such that for $(x_1, \ldots, x_n) \in U$,*

$$\boldsymbol{S}(x_1, \cdots, x_n, t) = S\left(x_{(1)}, \cdots, x_{(n)}, t\right), \tag{4.21}$$

and for $i = 1, \ldots, n$, $x_i D_i \log S(x,t)$ is bounded for $x \in \Delta^n \times [0,T]$. Then $\boldsymbol{S}$ generates portfolio π such that for $k = 1, \ldots, n$

$$\pi_{\alpha(k)}(t) = \left(D_k \log S(\mu_{\alpha(\cdot)}(t), t) + 1 - \sum_{j=1}^{n} \mu_{(j)} D_j \log S(\mu_{\alpha(\cdot)}(t), t) \right) \mu_{\alpha(k)}(t) \tag{4.22}$$

for all $t \in [0,T]$, a.s., with the drift process Θ that satisfies

$$\begin{aligned} d\Theta(t) = &-\frac{1}{S(\mu_{(\cdot)}(t), t)} \sum_{k=1}^{n} \int_0^T \int_{\mathbb{R}} D_k S(\mu_{(\cdot)}(t)) dL_t^x(\mu_{(k)}) \\ &- D_t \log S(\mu_{\alpha(\cdot)}(t), t) dt, \quad t \in [0,T], \quad a.s., \end{aligned} \tag{4.23}$$

where $\pi_{\alpha(k)} = \pi_k$ or $\pi_{(k)}$.

Proof. The fact that the weights $\pi_{\alpha(k)}$ sum to 1 and the conditions on S ensure that π is a portfolio. From its expression, it is clear that Θ is of bounded variation. The rest of the proof follows from the generalized Itô formula and the proof of Theorem 4.1. □

Remark 4.7. The time reversibility of the processes $\mu_{\alpha(k)}$ is essential in the proof of the main results. A natural question will then be: What happens if the processes fail to be time reversible? An answer to this question could be the generalization to the n-dimensional case of the result obtain by Feng and Zhao in [9] and this is an object of a forthcoming paper [16].

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