

ROBUSTNESS OF OPTION PRICES AND THEIR DELTAS IN MARKETS MODELLED BY JUMP-DIFFUSIONS

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ABSTRACT. We study the robustness of option prices to model variation within a jump-diffusion framework. In particular we consider models in which the small variations in price dynamics are modeled with a Poisson random measure with infinite activity and models in which these small variations are modeled with a Brownian motion. We show that option prices are robust. Moreover we study the computation of the deltas in this framework with two approaches, the Malliavin method and the Fourier method. We show robustness of the deltas to the model variation.

1. Introduction

The delta of an option is defined as the sensitivity of the option price with respect to the state of the underlying asset. In mathematical terms, this is given as the derivative of $\mathbb{E}[f(X(t))]$ with respect to $X(0) = x$, where $X(t)$ is the price dynamics of the underlying asset. In complete markets, the delta is known to be the number of assets $X(t)$ to hold in a self-financing portfolio exactly replicating the option $f(X(t))$. This is known as the delta-hedge. This is important also in incomplete markets for the construction of partial hedges (see for instance Cont and Tankov [4] for more on incomplete markets and partial hedging). Moreover, the delta being a sensitivity evaluation of the option price to variations in the underlying, it gives important information of the risk associated to an investment in the option both in complete and incomplete markets.

In general it is not possible to obtain analytical expressions for deltas. Thus, numerical approaches are called for, and we refer to Glasserman [11] for an overview of such methods. In this paper we apply the *Malliavin approach* proposed in Fournié *et al.* [10], which is a technique yielding expectation functionals suitable for Monte Carlo simulation. The approach has the advantage of not differentiating the payoff function f of the option. Indeed, options like digitals have non-differentiable payoff functions. However, different from Fournié *et al.* [10], we consider a jump-diffusion framework.

The Malliavin approach is well-developed for the Brownian motion case, but for jump-diffusion models it is not straightforwardly generalized due to the lack of a classical chain rule. Davis and Johansson [5] propose to use the Malliavin

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approach *only* on the Wiener term in the jump-diffusion dynamics where the jump part is driven by a Poisson process. We extend this idea to substantially more general jump-diffusion processes. Our results are based on the Malliavin calculus for jump processes developed by Solé, Utzet, and Vives [20] and Di Nunno [6] (see also Di Nunno, Øksendal, and Proske [7]). We demonstrate that one may use the Malliavin approach also in cases where there are no continuous martingale components in the jump-diffusion dynamics. In this situation, one can *approximate* the small jumps by a continuous martingale with appropriately scaled variance (see Proposition 3.3) and it turns out that the derived delta based on this approximation is close to the true one (see Theorem 4.1). This idea was first initiated by Rydelberg [18], and Asmussen and Rosinski [1] who studied the approximation of small jumps in a Lévy process with an approximately scaled Brownian motion. This opens up for applying Monte Carlo methods to compute deltas for a rich class of models. Our results also show that the deltas in jump-diffusion models are robust towards small changes in the underlying dynamics. Hence, the Malliavin approach can be used to derive approximative deltas in the case when we face a jump-diffusion model without any continuous martingale part present in the dynamics. This is an important consideration also from the modeling point of view, in fact it is very hard from the point of view of statistics, if at all possible, to decide which model for price dynamics is best between one where the small variations in the asset dynamics come from a jump process with infinite activity or from a continuous martingale. Our results show that, for what option pricing is concerned, the difference is for practical purposes negligible. We remark that there are different ways of applying the same Malliavin method, with the result that there are several equivalent expressions of the same delta.

Besides the Malliavin approach, this paper deals also with another method for computing the deltas, this is the *Fourier approach*. This method, in fact, has the advantage that it can be directly applied to models with or without continuous martingale part. However, it is actually difficult to implement since it requires an explicit solution of the stochastic differential equation describing the first variation process (see (4.17)). Within this methodology we again study the expressions for the deltas and prove robustness. Some examples are also detailed.

The paper is organized as follows. In Section 2 we introduce the notation and give a short introduction to the Malliavin calculus for mixtures of Gaussian and compensated Poisson random measures. Section 3 is dedicated to jump-diffusions and results about the robustness of the models and the option prices. Section 4 deals directly with the computation of the deltas and the related analysis of robustness to the model. Here, both the Malliavin and the Fourier approaches are introduced.

2. Some Mathematical Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space equipped with a filtration

$$\{\mathcal{F}_t\}_{t \in [0, T]}, \quad (T > 0)$$

satisfying the *usual conditions* (see Karatzas and Shreve [14]). We introduce the generic notation $L(t)$ for a Lévy process on the given probability space and denote

by $B(t)$ a Brownian motion independent of $L(t)$, with $t \in [0, T]$ and $L(0) = B(0) = 0$ by convention. We work with the RCLL¹ version of the Lévy process and let $\Delta L(t) := L(t) - L(t-)$. Denote the Lévy measure of $L(t)$ by $\ell(dz)$. Recall that $\ell(dz)$, $z \in \mathbb{R}_0$, is a σ -finite Borel measure on $\mathbb{R}_0 := \mathbb{R} - \{0\}$.

We also recall the Lévy-Itô decomposition of a Lévy process (see Sato [19]):

Theorem 2.1. *For $t \geq 0$, let $L(t)$ be a Lévy process on \mathbb{R} and ℓ its Lévy measure. Then we have:*

- ℓ verifies

$$\int_{\mathbb{R}_0} \min(1, z^2) \ell(dz) < \infty.$$

- The jump measure of $L(t)$, denoted by $N(dt, dz)$, is a Poisson random measure on $[0, \infty[\times \mathbb{R}_0$ with intensity measure $\ell(dz) dt$.
- There exists a Brownian motion $W(t)$ and two constants $a, b \in \mathbb{R}$ such that

$$L(t) = at + bW(t) + Z(t) + \lim_{\varepsilon \downarrow 0} \tilde{Z}_\varepsilon(t), \tag{2.1}$$

where

$$Z(t) := \sum_{s \in [0, t]} \Delta L(s) \mathbf{1}_{\{|\Delta L(s)| \geq 1\}} = \int_0^t \int_{|z| \geq 1} z N(ds, dz)$$

and

$$\tilde{Z}_\varepsilon(t) := \sum_{s \in [0, t]} \Delta L(s) \mathbf{1}_{\{\varepsilon \leq |\Delta L(s)| < 1\}} - t \int_{\varepsilon \leq |z| < 1} z \ell(dz) = \int_0^t \int_{\varepsilon \leq |z| < 1} z \tilde{N}(ds, dz),$$

where \tilde{N} is the compensated Poisson random measure of $L(t)$. The convergence of $\tilde{Z}_\varepsilon(t)$ in (2.1) is almost sure and uniform on $t \in [0, T]$. The components W , Z and \tilde{Z}_ε are independent.

In various applications involving statistical and numerical methods, it is often useful to approximate the small jumps by a scaled Brownian motion. This approximation was advocated in Rydberg [18] as a way to simulate the path of a Lévy process with NIG distributed increments, and later studied in detail by Asmussen and Rosinski [1]. We shall make use of it to study robustness of option prices and their deltas based on jump-diffusion models.

We introduce the following notation for the variation of the Lévy process $L(t)$ close to the origin:

$$\sigma^2(\varepsilon) := \int_{|z| < \varepsilon} z^2 \ell(dz), \quad 0 < \varepsilon \leq 1. \tag{2.2}$$

Since every Lévy measure $\ell(dz)$ integrates z^2 in an open interval around zero, we have that $\sigma^2(\varepsilon)$ is finite for any $\varepsilon > 0$. Note that the $\sigma^2(\varepsilon)$ is the variance of the jumps smaller than ε of $L(t)$ in the case it is symmetric and has mean zero. By dominated convergence $\sigma^2(\varepsilon)$ converges to zero when $\varepsilon \downarrow 0$.

¹Right-continuous with left limits, also called *càdlàg*.

Recall the Lévy-Itô decomposition of a Lévy process $L(t)$ and introduce now an approximative Lévy process (in law)

$$L_\varepsilon(t) := at + bW(t) + \sigma(\varepsilon)B(t) + Z(t) + \tilde{Z}_\varepsilon(t), \quad (2.3)$$

with $\sigma^2(\varepsilon)$ as in (2.2), and $B(t)$ being a Brownian motion independent of $L(t)$ (which in particular means independent of $W(t)$). From the definition of \tilde{Z}_ε , we see that we have substituted the small jumps (compensated by their expectation) in $L(t)$ by a Brownian motion scaled with $\sigma(\varepsilon)$, the standard deviation of the compensated small jumps. We have the following result taken from Benth, Di Nunno, and Khedher [2]. We include here the proof for the convenience of the reader:

Proposition 2.2. *Let the process $L(t)$ respectively $L_\varepsilon(t)$ be defined as in equation (2.1), respectively (2.3). Then, for every $t \geq 0$,*

$$\lim_{\varepsilon \rightarrow 0} L_\varepsilon(t) = L(t) \quad \mathbb{P} - a.s.$$

In fact, the limit above also holds in $L^1(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\mathbb{E} [|L_\varepsilon(t) - L(t)|] \leq 2\sigma(\varepsilon)\sqrt{t}.$$

Proof. The \mathbb{P} -a.s. convergence follows from the proof of the Lévy-Kintchine formula (See Thm. 19.2 in Sato [19]). Concerning the L^1 -convergence, we argue as follows. The combined application of the triangle and Cauchy-Schwarz inequalities give

$$\begin{aligned} \mathbb{E} [|L_\varepsilon(t) - L(t)|] &= \mathbb{E} \left[\left| \sigma(\varepsilon)B(t) - \int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz) \right| \right] \\ &\leq \sigma(\varepsilon)\mathbb{E} [|B(t)|] + \mathbb{E} \left[\left| \int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz) \right| \right] \\ &\leq \sigma(\varepsilon)\mathbb{E} [B^2(t)]^{1/2} + \mathbb{E} \left[\left(\int_0^t \int_{0 < |z| < \varepsilon} z \tilde{N}(ds, dz) \right)^2 \right]^{1/2} \\ &\leq 2\sigma(\varepsilon)\sqrt{t}. \end{aligned}$$

This proves the proposition. \square

We shall make use of the approximation and its convergence properties in our analysis.

2.1. Chaotic representation for Lévy processes and Malliavin derivative.

In Itô [13], multiple stochastic integrals with respect to a Poisson random measure are defined (see Di Nunno [6] for an extension to general random measures with independent values). We recall the construction, which follows the same steps as in the Wiener case (see Kuo [15]).

Here and in the sequel we assume that the Lévy measure satisfies

$$\sigma^2(\infty) := \int_{\mathbb{R}_0} z^2 \ell(dz) < \infty. \quad (2.4)$$

Consider a Lévy process L having a representation as in (2.1) with $b = 1$. Introduce the measure M on the Borel σ -algebra $\mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ such that for $E \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$,

$$M(E) = \int_{E(0)} dt + \int_{E'} z^2 dt \ell(dz),$$

where $E(0) = \{t \in \mathbb{R}_+; (t, 0) \in E\}$ and $E' = E - \{(t, 0) \in E\}$. Define

$$\mu(E) = \int_{E(0)} dW(t) + \lim_{n \rightarrow \infty} \int_{\{(t,z) \in E; \frac{1}{n} < |z| < n\}} z \tilde{N}(dt, dz),$$

where the convergence is in $L^2(\Omega)$. We denote by $L^2(\Omega) := L^2(\Omega, \mathcal{F}, \mathbb{P})$ the Hilbert space of square-integrable random variables, equipped with the norm $\|F\|_2 = (E[F^2])^{1/2} < \infty$. The set function μ is a centered random measure such that for $E_1, E_2 \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ with $M(E_1) < \infty$ and $M(E_2) < \infty$,

$$\mathbb{E}[\mu(E_1)\mu(E_2)] = M(E_1 \cap E_2).$$

Denote by $L_n^2 = L^2((\mathbb{R}_+ \times \mathbb{R})^n, \mathcal{B}((\mathbb{R}_+ \times \mathbb{R})^n), M^{\otimes n})$, with the standard norm $|\cdot|_n$. Let

$$f = \mathbf{1}_{E_1 \times \dots \times E_n},$$

where the sets $E_1, \dots, E_n \in \mathcal{B}(\mathbb{R}_+ \times \mathbb{R})$ are pairwise disjoint and

$$M(E_1) < \infty, \dots, M(E_n) < \infty.$$

The multiple stochastic integral of the elementary function f is an element in $L^2(\Omega)$ defined as follows

$$I_n(f) := \int_{(\mathbb{R}_+ \times \mathbb{R})^n} f \mu^{\otimes n} := \mu(E_1) \cdots \mu(E_n).$$

By standard arguments, I_n can be extended to the symmetric function in L_n^2 by appealing to linearity and continuity. Moreover, for any symmetric functions $f \in L_n^2$ and $g \in L_m^2$ we have

$$\mathbb{E}[I_n(f)I_m(g)] = \delta_{n,m} n! \int_{(\mathbb{R}_+ \times \mathbb{R})^n} \tilde{f} \tilde{g} d\mu^{\otimes n},$$

where $\delta_{n,m} = 1$, if $n = m$ and 0 otherwise. Itô [13] proves the following chaos expansion for elements of $L^2(\Omega)$:

Theorem 2.3. *For any $F \in L^2(\Omega)$ there exists a unique sequence $(f_n)_{n=0}^\infty$ of symmetric functions $f_n \in L_n^2$ such that*

$$F = \sum_{n=0}^\infty I_n(f_n),$$

(with convergence in $L^2(\Omega)$). Moreover, it holds

$$\|F\|_2^2 = \sum_{n=0}^\infty n! |f_n|_n^2.$$

Note that, among all the stochastic measures with independent values in $L^2(\Omega)$ it is only in the case of mixtures of Gaussian and Poisson measures that it is possible to achieve chaos representation type of results. This is proved in Theorem 2.2 in Di Nunno [6].

In Solé, Utzet and Vives [20] (see also Di Nunno [6] for random measures with independent values) a stochastic derivative is defined on a subspace of $L^2(\Omega)$. The idea is to exploit chaos expansion representations much in the same manner as done for the Malliavin derivative in the Wiener space (see Nualart [16]). Suppose $F \in L^2(\Omega)$ has a chaotic representation $F = \sum_{n=0}^{\infty} I_n(f_n)$ such that

$$\sum_{n=1}^{\infty} nn! |f_n|_n^2 < \infty. \tag{2.5}$$

Then, the *Malliavin derivative* $DF : \mathbb{R}_+ \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ of F is the random field defined as

$$D_{\zeta}F := \sum_{n=1}^{\infty} nI_{n-1}(f_n(\zeta, \cdot)), \quad \zeta \in \mathbb{R}_+ \times \mathbb{R}, \tag{2.6}$$

with convergence in $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, M \otimes \mathbb{P})$. Note that the Malliavin derivative can be viewed as an annihilation operator, shifting the chaos expansion of F by one to the left.

Denote by $\text{Dom}D$ the set of functionals $F \in L^2(\Omega)$ that satisfy (2.5). This becomes a Hilbert space equipped with the scalar product

$$\langle F, G \rangle = \mathbb{E}[FG] + \mathbb{E}\left[\int_{\mathbb{R}_+ \times \mathbb{R}} D_{\zeta}F D_{\zeta}G M(d\zeta)\right],$$

on which D is a closed operator from $\text{Dom}D$ to $L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, M \otimes \mathbb{P})$. Furthermore, let $\text{Dom}D^0$ be the set of random variables $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{\mathbb{R}_+ \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f_n^2((t, 0), \zeta_1, \dots, \zeta_{n-1}) dt dM^{\otimes(n-1)}(\zeta_1, \dots, \zeta_{n-1}) < \infty,$$

For $F \in \text{Dom}D^0$ we define the square integrable stochastic process

$$D_{t,0}F := \sum_{n=1}^{\infty} nI_{n-1}(f_n((t, 0), \cdot)),$$

where the convergence is in $L^2(\mathbb{R}_+ \times \Omega, dt \otimes \mathbb{P})$. Analogously, for $\ell(dz) \neq 0$, let $\text{Dom}D^J$ be the set of $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(\Omega)$ such that

$$\sum_{n=1}^{\infty} nn! \int_{(\mathbb{R}_+ \times \mathbb{R}_0) \times (\mathbb{R}_+ \times \mathbb{R})^{n-1}} f_n^2((t, z), \zeta_1, \dots, \zeta_{n-1}) dM^{\otimes(n)}(\zeta_1, \dots, \zeta_{n-1}) < \infty.$$

For $F \in \text{Dom}D^J$, define the random field $D_{t,z}^J F : \mathbb{R}_+ \times \mathbb{R} \times \Omega \mapsto \mathbb{R}$ such that

$$D_{t,z}F := \sum_{n=1}^{\infty} nI_{n-1}(f_n((t, z), \cdot)),$$

where the convergence is in $L^2(\mathbb{R}_+ \times \mathbb{R}_0 \times \Omega, z^2 dt d\ell(x) \otimes \mathbb{P})$. We remark that the derivative $D_{t,0}$ is essentially a derivative with respect to the Brownian part of L ,

and in many situations the usual rules of classical Malliavin calculus on Wiener space apply.

Let $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ and $(\Omega_J, \mathcal{F}_J, \mathbb{P}_J)$ be the canonical spaces for the Brownian motion and pure jump Lévy process, resp. We can interpret

$$\Omega = \Omega_W \times \Omega_J, \quad \mathcal{F} = \mathcal{F}_W \otimes \mathcal{F}_J, \quad \mathbb{P} = \mathbb{P}_W \otimes \mathbb{P}_J.$$

The following chain rule for $D_{t,0}$ is proved by Solé, Utzet and Vives [20].

Proposition 2.4. *Assume $F = f(Z, Z') \in L^2(\Omega_W \times \Omega_J)$, with $Z \in \text{Dom}D^W$, $Z' \in L^2(\Omega_J)$, and $f(x, y)$ being a continuously differentiable function with bounded partial derivative in the first variable. Then $F \in \text{Dom}D^0$, and*

$$D_{t,0}F = \frac{\partial f}{\partial x}(Z, Z')D_t^W Z,$$

where D^W is the Malliavin derivative in $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$ and $\text{Dom}D^W$ its domain.

In Solé, Utzet and Vives [20] the Skorohod integral with respect to a mixture of Gaussian and Poisson random measures is also defined (see Di Nunno [6] and Di Nunno and Rozanov [8] for the treatment with respect to general stochastic measures in $L^2(\Omega)$). Let us consider

$$G(\zeta) = \sum_{n=0}^{\infty} I_n(\widehat{f}_n(\zeta, \cdot)), \quad \zeta \in \mathbb{R}_+ \times \mathbb{R},$$

where $f_n \in L^2_{n+1}$ is symmetric in the last n variables. We denote \widehat{f}_n the symmetrization of f_n in all $n + 1$ variables. If

$$\sum_{n=0}^{\infty} (n + 1)! |\widehat{f}_n|_{n+1}^2 < \infty, \tag{2.7}$$

the Skorohod integral of $G(\zeta)$, $\zeta \in \mathbb{R}_+ \times \mathbb{R}$, is defined by

$$\delta(G) := \sum_{n=0}^{\infty} I_{n+1}(\widehat{f}_n),$$

where the convergence of the series on the right-hand side is in $L^2(\Omega)$. Denote by $\text{Dom}\delta$ the set of random fields $G(\zeta)$ satisfying (2.7). The following is a duality formula proven by Solé, Utzet and Vives [20]:

Proposition 2.5. *Let $G \in L^2(\mathbb{R}_+ \times \mathbb{R} \times \Omega, \mu \otimes \mathbb{P})$. The random field G belongs to $\text{Dom}\delta$ if and only if there is a constant C such that for all $F \in \text{Dom}D$,*

$$|\mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} G(\zeta) D_\zeta F M(d\zeta)]| \leq C \|F\|_2.$$

If $G \in \text{Dom}\delta$, then $\delta(G)$ is the element of $L^2(\Omega)$ characterized by

$$\mathbb{E}[\delta(G)F] = \mathbb{E}[\int_{\mathbb{R}_+ \times \mathbb{R}} G(\zeta) D_\zeta F M(d\zeta)],$$

for any $F \in \text{Dom}D$.

The Malliavin derivatives introduced above will become useful when we analyze the delta of option prices based on jump-diffusion models, see Section 4.

3. Robustness of Jump-diffusions and Option Prices

In this section we consider the robustness of jump-diffusions given by the solution of stochastic differential equations of the form

$$\begin{aligned} X(t) &= x + \int_0^t \alpha(X(s-)) ds + \int_0^t \beta(X(s-)) dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \gamma(X(s-), z) \tilde{N}(ds, dz). \end{aligned} \quad (3.1)$$

We assume that the coefficient functions $\alpha(x)$ and $\beta(x)$ have linear growth and are Lipschitz continuous and that γ is of the form $\gamma(x, z) = \gamma_1(x)g(z)$, $x \in \mathbb{R}$, $z \in \mathbb{R}_0$, where the (stochastic) factor $\gamma_1(x)$ has linear growth and is Lipschitz continuous and the (deterministic) factor $g(z)$ satisfies

$$G^2(\infty) = \int_{\mathbb{R}_0} g^2(z) \ell(dz) < \infty,$$

which will ensure that $X(t)$ has finite variance. We also define

$$G^2(\varepsilon) = \int_{|z| < \varepsilon} g^2(z) \ell(dz),$$

for later use. Notice that $G^2(\varepsilon)$ converges to zero when $\varepsilon \downarrow 0$. A jump-diffusion of type (3.1) is, e.g., considered in Example 4.2.

Note that we consider a stochastic differential equation with the roles of W and \tilde{N} separated, that is, we do not consider an equation using L as the integrator, but rather split the roles of the continuous martingale and the pure-jump parts. This is more in line with common formulations of such stochastic differential equations (see for example Davis and Johansson [5]). Introduce the approximative jump-diffusion dynamics where the small jumps part in (3.1) has been substituted by a Brownian motion B independent of W and appropriately scaled, namely

$$\begin{aligned} X_\varepsilon(t) &= x + \int_0^t \alpha(X_\varepsilon(s-)) ds + \int_0^t \beta(X_\varepsilon(s-)) dW(s) \\ &\quad + \int_0^t \left(\int_{|z| < \varepsilon} (\gamma^2(X_\varepsilon(s-), z) \ell(dz))^{1/2} dB(s) \right. \\ &\quad \left. + \int_0^t \int_{|z| \geq \varepsilon} \gamma(X_\varepsilon(s-), z) \tilde{N}(ds, dz) \right) \\ &= x + \int_0^t \alpha(X_\varepsilon(s-)) ds + \int_0^t \beta(X_\varepsilon(s-)) dW(s) \\ &\quad + \int_0^t G(\varepsilon) \gamma_1(X_\varepsilon(s-)) dB(s) \\ &\quad + \int_0^t \int_{|z| \geq \varepsilon} \gamma(X_\varepsilon(s-), z) \tilde{N}(ds, dz). \end{aligned} \quad (3.2)$$

The existence and uniqueness of the solutions $X(t)$ and $X_\varepsilon(t)$ are ensured by the following theorem collected from Ikeda and Watanabe [12] (Thm 9.1. Chap IV):

Theorem 3.1. *Let U be an open set in \mathbb{R}_0 , α and β be two measurable functions $\mathbb{R} \rightarrow \mathbb{R}$ and γ be a measurable function $\mathbb{R} \times U \rightarrow \mathbb{R}$ such that, for some positive constant K ,*

$$\begin{aligned} & |\alpha(x)|^2 + |\beta(x)|^2 + \int_U |\gamma(x, z)|^2 \ell(dz) \\ & \leq K(1 + |x|^2), \quad x \in \mathbb{R}, \end{aligned} \tag{3.3}$$

$$\begin{aligned} & |\alpha(x) - \alpha(y)|^2 + |\beta(x) - \beta(y)|^2 + \int_U |\gamma(x, z) - \gamma(y, z)|^2 \ell(dz) \\ & \leq K|x - y|^2, \quad x, y \in \mathbb{R}. \end{aligned} \tag{3.4}$$

Then there exists a unique \mathcal{F}_t -adapted right-continuous process $X(t)$ with left-hand limits which satisfies the following stochastic differential equation

$$\begin{aligned} X(t) = & x + \int_0^t \alpha(X(s-)) ds + \int_0^t \beta(X(s-)) dW(s) \\ & + \int_0^t \int_U \gamma(X(s-), z) \tilde{N}(ds, dz). \end{aligned} \tag{3.5}$$

Before proving that $X_\varepsilon(t)$ converges to $X(t)$ in $L^2(\Omega)$, we need a lemma which shows the boundedness of X in $L^2([0, T] \times \Omega)$ for $T < \infty$.

Lemma 3.2. *Let $X(t)$ and $X_\varepsilon(t)$, $t \in [0, T]$, be the unique solutions of (3.1) and (3.2), respectively. For every $0 \leq t \leq T < \infty$, we have the following type of estimate for the respective norms*

$$\|X(t)\|_2^2, \|X_\varepsilon(t)\|_2^2 \leq ae^{bt},$$

where a and b are positive constants depending on T but independent of ε in the case of X_ε .

Proof. By the Cauchy-Schwartz inequality and the application of the Itô isometry, we find that

$$\begin{aligned} \|X(t)\|_2^2 \leq & C|x|^2 + C\mathbb{E} \left[\int_0^t \alpha^2(X(s)) ds \right] + C\mathbb{E} \left[\int_0^t \beta^2(X(s)) ds \right] \\ & + CG^2(\infty)\mathbb{E} \left[\int_0^t \gamma_1^2(X(s)) ds \right], \end{aligned}$$

for some positive constant C . By linear growth, it follows that

$$|\alpha(x)|^2 \leq K(1 + |x|^2)$$

for some positive constant K . Hence, by using the same property for β and γ_1 , it follows that

$$\|X(t)\|_2^2 \leq C_1 + C_2 \int_0^t \|X(s)\|_2^2 ds,$$

for two positive constants C_1, C_2 , which depend only on $K, T, G^2(\infty)$ and x . By Gronwall's inequality, the lemma follows for $X(t)$.

Concerning the estimate for $X_\varepsilon(t)$, we proceed in the way as for $X(t)$. In this case, however, we get an additional contribution from the term

$$\int_0^t G(\varepsilon)\gamma_1(X_\varepsilon(s)) dB(s),$$

whereas the jump-term is including only jumps in absolute value greater than ε . However, after applying the Itô isometry, we can merge the contributions from these two terms into $G^2(\infty)\mathbb{E}[\int_0^t \gamma_1^2(X_\varepsilon(s)) ds]$. Hence, we are back to the same estimation type as for $X(t)$. This completes the proof. \square

We use the lemma to prove the following robustness result:

Proposition 3.3. *For every $0 \leq t \leq T < \infty$, we have*

$$\|X(t) - X_\varepsilon(t)\|_2^2 \leq CG^2(\varepsilon),$$

where X and X_ε are solutions of (3.1) and (3.2), respectively and C is a positive constant depending on T , but independent of ε .

Proof. We have

$$\begin{aligned} X(t) - X_\varepsilon(t) &= \int_0^t (\alpha(X(s-)) - \alpha(X_\varepsilon(s-))) ds \\ &\quad + \int_0^t (\beta(X(s-)) - \beta(X_\varepsilon(s-))) dW(s) \\ &\quad + \int_0^t \int_{0 < |z| < \varepsilon} \gamma(X(s-), z) \tilde{N}(ds, dz) \\ &\quad - \int_0^t G(\varepsilon)\gamma_1(X(s-)) dB(s) \\ &\quad + \int_0^t G(\varepsilon)(\gamma_1(X(s-)) - \gamma_1(X_\varepsilon(s-))) dB(s) \\ &\quad + \int_0^t \int_{|z| \geq \varepsilon} (\gamma(X(s-), z) - \gamma(X_\varepsilon(s-), z)) \tilde{N}(ds, dz). \end{aligned}$$

Therefore, using the Hölder inequality and the Itô isometry, we get

$$\begin{aligned} \|X(t) - X_\varepsilon(t)\|_2^2 &\leq T\mathbb{E} \left[\int_0^t (\alpha(X(s)) - \alpha(X_\varepsilon(s)))^2 ds \right] \\ &\quad + \mathbb{E} \left[\int_0^t (\beta(X(s)) - \beta(X_\varepsilon(s)))^2 ds \right] \\ &\quad + 2G^2(\varepsilon)\mathbb{E} \left[\int_0^t \gamma_1^2(X(s)) ds \right] \\ &\quad + G^2(\varepsilon)\mathbb{E} \left[\int_0^t (\gamma_1(X(s)) - \gamma_1(X_\varepsilon(s)))^2 ds \right] \\ &\quad + (G^2(\infty) - G^2(\varepsilon))\mathbb{E} \left[\int_0^t (\gamma_1(X(s)) - \gamma_1(X_\varepsilon(s)))^2 ds \right]. \end{aligned}$$

Hence, by the Lipschitz continuity of the three coefficient functions and the triangle inequality, we find

$$\begin{aligned} \|X(t) - X_\varepsilon(t)\|_2^2 &\leq K(T + 1 + G^2(\infty)) \int_0^t \|X(s) - X_\varepsilon(s)\|_2^2 ds \\ &\quad + 2G^2(\varepsilon)K \int_0^t (1 + \|X(s)\|_2^2) ds. \end{aligned}$$

Applying Gronwall’s inequality and Lemma 3.2, we prove the Proposition. \square

This result has various applications, one of which is the numerical simulations of the solution of (3.1). First, we observe that the speed of convergence is explicitly given by $G(\varepsilon)$, which in many situations will be a rate of ε . See e.g. Asmussen and Rosinski [1] for examples in the case $g(z) = z$. In practice, it may be difficult to simulate from a Lévy process $L(t)$ directly. One may in such circumstances approximate the small jumps by an appropriate scaled Brownian motion and observe that the remaining process is a compound Poisson process. Brownian motion and compound Poisson processes are simple to simulate on a computer, and the approximative dynamics may next be discretized for instance, by an Euler scheme. Our result in Prop. 3.3 provides the mathematical foundation for such a procedure, ensuring for instance that expectation functionals of the type $\mathbb{E}[f(X_\varepsilon(t))]$ converge to $\mathbb{E}[f(X(t))]$ under mild assumptions on f . We have the following corollary:

Corollary 3.4. *Suppose f is a Lipschitz continuous function and X and X_ε solve (3.1) and (3.2), resp. Then, for every $0 \leq t \leq T < \infty$, there exists a positive constant C depending on T but independent of ε such that*

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq CG^2(\varepsilon).$$

Proof. Letting K be the Lipschitz constant of f , we have from the Jensen inequality,

$$|\mathbb{E}[f(X_\varepsilon(t))] - \mathbb{E}[f(X(t))]| \leq K\mathbb{E}[|X_\varepsilon(t) - X(t)|].$$

Hence, from the Cauchy-Schwarz inequality and Prop. 3.3 the result follows. \square

This result has an immediate interpretation in terms of robustness of option prices. If we assume that $X(t)$ represents the dynamics of some asset on which there is written an option with payoff $f(X(t))$ at an exercise time t , then the discounted risk-neutral expected value of $f(X(t))$ is the option price. Supposing that we model $X(t)$ directly under the risk-neutral probability (i.e., assuming \mathbb{P} is the risk-neutral probability), the discounted asset dynamics must be a martingale, that is, $\alpha(x) = rx$, with r being the risk-free interest rate. But the approximative dynamics X_ε is also a martingale after discounting when $\alpha(x) = rx$, and henceforth, we obtain from the Corollary above that option prices are stable with respect to perturbation in the underlying dynamics when we substitute small jumps with an appropriate continuous martingale. In practical terms, we may interpret this as having two competing models, one where we suppose that small variations in the asset dynamics come from a jump process of infinite activity, and another where we model this by continuous martingale. It is very hard, if possible, to decide

which model is better from a statistical point of view. However, the result above shows that the effect on option prices is very small. From a different perspective, if we want to perform a numerical evaluation of the option price, we may apply the above result in order to quantify the error if we approximate small jumps by a Brownian motion dynamics. The error is explicit in terms of $G(\varepsilon)$, the volatility of the jumps smaller than ε .

4. Computation of the Delta Using the Malliavin Method and Robustness

In this section we present the Malliavin approach to computing the delta for option prices based on a jump-diffusion market model. Our approach extends the method proposed in Davis and Johansson [5]. We apply the results to study robustness of the delta to small-jump approximations in the underlying jump-diffusion model. These results explain to us that we may use the Malliavin approach to approximate the delta in cases when there are no continuous martingale part in the jump-diffusion dynamics.

Let $\mathcal{F}_t^{\tilde{N}} = \sigma\{\int_0^s \int_A \tilde{N}(du, dz); \quad s \leq t, \quad A \in \mathcal{B}(R_0)\}$. Assume that α, β and γ are continuously differentiable functions with bounded derivatives and consider Markov jump diffusions, X of the form (3.1), for which we have a continuously differentiable function h with bounded derivative in the first argument such that

$$X(t) = h(X^c(t), X^d(t)), \quad X(0) = x. \tag{4.1}$$

Here X^c satisfies a stochastic differential equation

$$\begin{aligned} dX^c(t) &= \alpha_c(X^c(t))dt + \beta_c(X^c(t))dW(t), \\ X^c(0) &= x = h(X^c(0), X^d(0)), \end{aligned} \tag{4.2}$$

with continuously differentiable coefficients α_c, β_c , while X^d is adapted to the natural filtration $\mathcal{F}^{\tilde{N}}$ of the compensated compound Poisson process \tilde{N} . In particular, X^d does not depend on x . The jump-diffusion process of type (4.1) is called *separable*.

We associate to the process X^c , a process V given by

$$V(t) = 1 + \int_0^t \alpha'_c(X^c(s))V(s)ds + \int_0^t \beta'_c(X^c(s))V(s)dW(s), \tag{4.3}$$

The process V is called *the first variation process* for X^c and we have

$$V(t) = \frac{\partial X^c(t)}{\partial x}.$$

Theorem 4.1. *Let X be a diffusion of the form (3.1). We assume that it is separable. Define*

$$\Gamma = \left\{ a \in L^2[0, T] \mid \int_0^T a(t)dt = 1 \right\}.$$

Then for $a \in \Gamma$ and $f(X(T)) \in L^2(\Omega)$,

$$\Delta = \mathbb{E}\left[f(X(T)) \int_0^T a(t)\beta_c^{-1}(X^c(t))V(t)dW(t) \right], \tag{4.4}$$

where V is given by (4.3).

Proof. Assume that $f \in C_K^\infty(\mathbb{R})$, the set of infinitely differentiable functions with compact support. Then

$$\frac{\partial}{\partial x} \mathbb{E}[f(X(T))] = \mathbb{E}\left[f'(X(T)) \frac{\partial X(T)}{\partial x}\right] = \mathbb{E}\left[f'(X(T)) \frac{\partial X(T)}{\partial X^c(T)} V(T)\right], \quad (4.5)$$

where V is the first variation process for X^c . By the chain rule (Proposition 2.4), we have

$$D_{t,0}X(T) = \frac{\partial X(T)}{\partial X^c(T)} D_t^W X^c(T) = \frac{\partial X(T)}{\partial X^c(T)} V(T)(V(t))^{-1} \beta_c(X^c(t)).$$

See Proposition 5.1 in the Appendix for more details. Therefore,

$$\frac{\partial X(T)}{\partial X^c(T)} V(T) = D_{t,0}X(T) V(t) \beta_c^{-1}(X^c(t)).$$

Multiply by $a(t)$ and integrate,

$$\frac{\partial X(T)}{\partial X^c(T)} V(T) = \int_0^T D_{t,0}X(T) a(t) \beta_c^{-1}(X^c(t)) V(t) dt. \quad (4.6)$$

Inserting (4.6) in (4.5), the chain rule (Proposition 2.4) and the Duality formula (Proposition 2.5) yield

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E}[f(X(T))] &= \mathbb{E}\left[\int_0^T f'(X(T)) D_{t,0}X(T) a(t) \beta_c^{-1}(X^c(t)) V(t) dt\right] \\ &= \mathbb{E}\left[\int_0^T D_{t,0}f(X(T)) a(t) \beta_c^{-1}(X^c(t)) V(t) dt\right] \\ &= \mathbb{E}\left[f(X(T)) \int_0^T a(t) \beta_c^{-1}(X^c(t)) V(t) dW(t)\right]. \end{aligned}$$

Then we can extend this formula to $f(X(T)) \in L^2(\Omega)$ following the Proposition 5.2 in the Appendix. \square

We provide an example of a jump-diffusion dynamics satisfying our assumptions and at the same time illustrating the result (4.4).

Example 4.2. Consider a jump-diffusion of the form

$$dX(t) = \alpha X(t-)dt + \beta X(t-)dW(t) + \int_{\mathbb{R}_0} (e^z - 1)X(t-) \tilde{N}(dt, dz), \quad (4.7)$$

where α and β are constants. We introduce the process $X^c(t)$ defined by

$$\begin{aligned} dX^c(t) &= \left\{ \alpha + \int_{\mathbb{R}_0} (1 + z - e^z) \ell(dz) \right\} X^c(t) dt + \beta X^c(t) dW(t), \\ X(0) &= x. \end{aligned}$$

Then by applying the Itô formula to $\hat{X}(t) = e^{\tilde{Z}(t)} X^c(t)$, where

$$\tilde{Z}(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(dt, dz),$$

we get,

$$\begin{aligned} d\widehat{X}(t) &= e^{\widetilde{Z}(t-)}dX^c(t) + \int_{\mathbb{R}_0} (e^{\widetilde{Z}(t-)+z}X^c(t) - e^{\widetilde{Z}(t-)}X^c(t))\widetilde{N}(dt, dz) \\ &+ X^c(t)e^{\widetilde{Z}(t-)} \int_{\mathbb{R}_0} (-1 - z + e^z)\ell(dz)dt \\ &= \alpha\widehat{X}(t-)dt + \beta\widehat{X}(t-)dW(t) + \int_{\mathbb{R}_0} (e^z - 1)\widehat{X}(t-)\widetilde{N}(dt, dz), \end{aligned} \tag{4.8}$$

Therefore, $\widehat{X}(t) = X(t)$, a.e. and we see that the process X given by equation (4.7) is a separable process. Now, to illustrate the result in Theorem 4.1, we consider a differentiable claim $f(X(T)) = X^2(T)$, where X is given by (4.7) with $\alpha = 0$ and $\beta = 1$. In this case, an explicit solution of X is given by $X(t) = \text{exp}\{W(t) - \frac{t}{2} + \int_0^t \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)\}$ and the first variation process is $V(t) = \frac{X^c(t)}{x}$. We can apply the formula (4.4) with $a(t) = \frac{1}{T}$ and easily see that

$$\Delta = \frac{1}{xT}\mathbb{E}[X^2(T)W(T)] = \frac{1}{xT}\mathbb{E}\left[W(T)e^{2W(T)}e^{2\int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)}\right].$$

Put $Y(T) = e^{2\int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)}$. Since the two random variables $W(T)$ and $Y(T)$ are independent, we have

$$\Delta = \frac{xe^{-T}}{T}\mathbb{E}[W(T)e^{2W(T)}]\mathbb{E}[Y(T)] = 2xe^T\mathbb{E}[Y(T)].$$

On the other hand side note that in this example the delta can be computed directly by simple differentiation, this gives

$$\begin{aligned} \Delta &= 2xe^{-T}\mathbb{E}[e^{2W(T)}e^{2\int_0^T \int_{\mathbb{R}_0} z\widetilde{N}(ds, dz)}] \\ &= 2xe^{-T}\mathbb{E}[e^{2W(T)}]\mathbb{E}[Y(T)] = 2xe^T\mathbb{E}[Y(T)]. \end{aligned}$$

This confirms the result found before.

Let X_ε be a jump diffusion of the form (3.2). We assume that it is separable. Then the process X_ε^c is given by

$$X_\varepsilon^c(t) = x + \int_0^t \alpha_c(X_\varepsilon^c(s))ds + \int_0^t \beta_c(X_\varepsilon^c(s))dW(s) + \int_0^t G(\varepsilon)\gamma_{1,c}(X_\varepsilon^c(s))dB(s)$$

and the first variation process V_ε of X_ε^c is given by

$$\begin{aligned} V_\varepsilon(t) &= x + \int_0^t \alpha'_c(X_\varepsilon^c(s))V_\varepsilon(s)ds + \int_0^t \beta'_c(X_\varepsilon^c(s))V_\varepsilon(s)dW(s) \\ &+ \int_0^t G(\varepsilon)\gamma'_{1,c}(X_\varepsilon^c(s))V_\varepsilon(s)dB(s). \end{aligned}$$

We are now ready to study the delta related to the approximating model. We propose four ways of applying the Malliavin approach with related assumptions. The first two (4.9) and (4.10) are completely equivalent in the sense that the computations can be carried out either with respect to the original Brownian component W or with respect to the additional one B . The expression (4.11) derived from

the fact that the evaluation of the delta depends on the distribution and we consider a Brownian motion $\widetilde{W}_\varepsilon$ that merges W and B . In the last case, (4.12), the delta is computed starting from an approximating model created by modifying the coefficients of the original Brownian component W instead of considering a new independent Brownian motion B .

Theorem 4.3. *Let X_ε be a diffusion of the form (3.2) and assume that it is separable. Let $a \in \Gamma$, V_ε the first variation process of X_ε^c and $f(X_\varepsilon(T)) \in L^2(\Omega)$. Then*

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \beta_c^{-1}(X_\varepsilon^c(t)) V_\varepsilon(t) dW(t) \right], \quad (4.9)$$

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \gamma_{1,c}^{-1}(X_\varepsilon^c(t)) \frac{V_\varepsilon(t)}{G(\varepsilon)} dB(t) \right]. \quad (4.10)$$

We assume $\beta(x) = \gamma_1(x)$. Then

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \gamma_{1,c}^{-1}(X_\varepsilon^c(t)) \frac{V_\varepsilon(t)}{\sqrt{G^2(\varepsilon) + 1}} d\widetilde{W}_\varepsilon(t) \right], \quad (4.11)$$

where

$$\widetilde{W}_\varepsilon(t) = \frac{1}{\sqrt{G^2(\varepsilon) + 1}} W(t) + \frac{G(\varepsilon)}{\sqrt{G^2(\varepsilon) + 1}} B(t).$$

If we approximate the small jumps of $X(t)$ (equation (3.1)) by $X_\varepsilon(t)$, where $B(t) = W(t)$, then

$$\Delta_\varepsilon = \mathbb{E} \left[f(X_\varepsilon(T)) \int_0^T a(t) \{G(\varepsilon) \gamma_{1,c}(X_\varepsilon^c(t)) + \beta_c(X_\varepsilon^c(t))\}^{-1} V_\varepsilon(t) dW(t) \right]. \quad (4.12)$$

Proof. By the chain rule (Proposition 2.4), we have

$$D_{t,0} X_\varepsilon(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} D_t^W X_\varepsilon^c(T).$$

Here, D^W is the Malliavin derivative with respect to the Brownian motion W . By Thm 2.2.1 in Nualart [16],

$$\begin{aligned} D_t^W X_\varepsilon^c(T) &= \beta_c(X_\varepsilon^c(t)) + \int_t^T \alpha'_c(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) ds \\ &+ \int_t^T \beta'_c(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) dW(s) \\ &+ \int_t^T G(\varepsilon) \gamma'_{1,c}(X_\varepsilon^c(s)) D_t^W X_\varepsilon^c(s) dB(s). \end{aligned}$$

Then

$$D_t^W X_\varepsilon^c(T) = V_\varepsilon(T) (V_\varepsilon(t))^{-1} \beta_c(X_\varepsilon^c(t)).$$

However, we find the expression (4.9) for the Δ_ε following the same steps of the Thm 4.1.

We can apply the chain rule again with differentiation taken with respect to B (Proposition 2.4), then we get

$$D_{t,0}X_\varepsilon(T) = \frac{\partial X_\varepsilon(T)}{\partial X_\varepsilon^c(T)} D_t^B X_\varepsilon^c(T),$$

where D^B is the Malliavin derivative with respect to the Brownian motion B . Then, following the same steps as above we obtain the expression (4.10) for the Δ_ε .

We assume now that we are in the case of the approximation (3.2), with $\beta(x) = \gamma_1(x)$. Then the process X_ε^c is given by

$$X_\varepsilon^c(t) = x + \int_0^t \alpha_c(X_\varepsilon^c(s)) ds + \int_0^t \gamma_{1,c}(X_\varepsilon^c(s)) \sqrt{G^2(\varepsilon) + 1} d\widetilde{W}_\varepsilon(t).$$

By Thm 4.1, expression (4.11) follows. The last case (4.12) also follows by application of Thm 4.1. \square

Note that, if $\varepsilon = 0$, we are in the case of no-approximation and we have the same method as proposed in Davis and Johansson [5], except for more general jump parts. This shows us how to use the Malliavin approach for these jump diffusions of general type. Next, in the case of jump-diffusions with no continuous component, i.e. $\beta = 0$, we have an expression which can be used as the approximation for the delta.

We next address the question of robustness of the delta with respect to approximations of the small jumps by an appropriately scaled continuous martingale. It turns out that this question can be efficiently answered by means of *Fourier transform*. The methods of Fourier transform will translate the question of convergence of the delta to a question of convergence of the derivative of the characteristic function of the approximating dynamics.

One may ask why we do not study the expression derived above for the delta directly. The reason is that in the singular case of $\beta = 0$, the expressions inside the expectation for the delta in Thm 4.3 will involve singular weights which in general are hard to study in the limit (see Benth, Di Nunno and Khedher [2] for simple examples of such singular weights). The Fourier approach avoids this problem.

The approach we choose can be used also for efficient computations of the delta, however, only for those cases where the characteristic function is easily computable which is in general not the case for stochastic differential equations like (3.1) and (3.2). We also note that the application of the Fourier transform requires also the explicit solution of the first variation process dynamics (4.17).

Assume that $f \in L^1(\mathbb{R})$, the space of integrable functions on the real line. The Fourier transform of f is defined by

$$\widehat{f}(u) = \int_{\mathbb{R}} f(y) e^{iuy} dy. \quad (4.13)$$

Suppose in addition that $\widehat{f} \in L^1(\mathbb{R})$. Then the inverse Fourier transform is well-defined, and we have

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iuy} \widehat{f}(u) du. \quad (4.14)$$

We refer to Folland [9] for definitions and results on the Fourier transform. Following Carr and Madan [3], we calculate,

$$\begin{aligned} \mathbb{E}[f(X_\varepsilon^x(t))] &= \int_{\mathbb{R}} \left\{ \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iyu} \widehat{f}(u) du \right\} P_{X_\varepsilon^x(t)}(dy) \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} e^{-iyu} P_{X_\varepsilon^x(t)}(dy) \right\} \widehat{f}(u) du \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) \mathbb{E} \left[e^{-iuX_\varepsilon^x(t)} \right] du, \end{aligned} \tag{4.15}$$

where $P_{X_\varepsilon^x(t)}(dy)$ is the distribution of $X_\varepsilon(t) = X_\varepsilon^x(t)$, the solution of (3.2) with $X_\varepsilon(0) = X_\varepsilon^x(0) = x$. Fubini-Tonelli's Theorem (see Folland [9]) is applied to commute the integrations. Similarly, we get for $X(t) = X^x(t)$ being the solution of (3.1) with $X(0) = X^x(0) = x$,

$$\mathbb{E}[f(X^x(t))] = \frac{1}{2\pi} \int_{\mathbb{R}} \widehat{f}(u) \mathbb{E} \left[e^{-iuX^x(t)} \right] du. \tag{4.16}$$

Thus, in order to study the delta, we need to be able to move differentiation inside the inverse Fourier transform. But, furthermore, we must have accessible the derivative of $X_\varepsilon^x(t)$ and $X^x(t)$ with respect to x . Before moving on with the robustness of deltas, we study this.

Introduce the stochastic differential equation

$$\begin{aligned} Y^y(t) &= y + \int_0^t \alpha'(X^x(s-)) Y^y(s-) ds + \int_0^t \beta'(X^x(s-)) Y^y(s-) dW(s) \\ &\quad + \int_0^t \int_{\mathbb{R}_0} \gamma'(X^x(s-), z) Y^y(s-) \widetilde{N}(ds, dz). \end{aligned} \tag{4.17}$$

Since the derivatives of α, β and γ are assumed to be bounded, it follows from Thm. 3.1 that there exists a unique solution $Y^y(t)$ of (4.17). From Thm 40 in Chapter V of Protter [17], it follows that $X^x(t)$ is differentiable with respect to x , and that

$$\frac{\partial X^x(t)}{\partial x} = Y^1(t) \quad (\text{i.e. } y = 1). \tag{4.18}$$

By the same considerations, $X_\varepsilon^x(t)$ is differentiable with respect to x , and

$$\frac{\partial X_\varepsilon^x(t)}{\partial x} = Y_\varepsilon^1(t), \tag{4.19}$$

with $Y_\varepsilon^y(t)$ being the unique solution of the stochastic differential equation

$$\begin{aligned} Y_\varepsilon^y(t) &= y + \int_0^t \alpha'(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) ds + \int_0^t \beta'(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) dW(s) \\ &\quad + \int_0^t G(\varepsilon) \gamma'_1(X_\varepsilon^x(s-)) Y_\varepsilon^y(s-) dB(s) \end{aligned} \tag{4.20}$$

$$+ \int_0^t \int_{|z| \geq \varepsilon} \gamma'(X_\varepsilon^x(s-), z) Y_\varepsilon^y(s-) \widetilde{N}(ds, dz). \tag{4.21}$$

We have the following regularity of Y and Y_ε :

Proposition 4.4. *Let $Y(t)$ and $Y_\varepsilon(t)$ be the solutions of (4.17) and (4.20), resp. For $0 \leq t \leq T < \infty$ it holds that*

$$\|Y^y(t)\|_2^2, \|Y_\varepsilon^y(t)\|_2^2 < ae^{bt},$$

for positive constants a and b depending on T but independent of ε in the case of Y_ε . Moreover,

$$\|Y^y(t) - Y_\varepsilon^y(t)\|_2^2 \leq CG^2(\varepsilon),$$

for a positive constant C independent of ε .

Proof. The proof follows the same lines as the arguments for Lemma 3.2 and Prop. 3.3. The only modification is that we use the boundedness of the derivatives $\alpha'(x), \beta'(x)$ and $\gamma'(x)$ rather than the Lipschitz continuity of α, β and γ . \square

In the next Proposition we derive the expressions for the delta based on X and X_ε using the Fourier method.

Proposition 4.5. *Let $X^x(t)$ and $Y^y(t)$ be solutions of (3.1) and (4.17), resp., and $X_\varepsilon^x(t)$ and $Y_\varepsilon^y(t)$ of (3.2) and (4.20), resp. Let $u\hat{f}(u) \in L^1(\mathbb{R})$. Then, for $0 \leq t \leq T$,*

$$\begin{aligned} \frac{\partial}{\partial x} \mathbb{E}[f(X^x(t))] &= \frac{1}{2\pi} \int_{\mathbb{R}} (-iu)\hat{f}(u) \mathbb{E}\left[Y^1(t)e^{-iuX^x(t)}\right] du \\ \frac{\partial}{\partial x} \mathbb{E}[f(X_\varepsilon^x(t))] &= \frac{1}{2\pi} \int_{\mathbb{R}} (-iu)\hat{f}(u) \mathbb{E}\left[Y_\varepsilon^1(t)e^{-iuX_\varepsilon^x(t)}\right] du. \end{aligned}$$

Proof. By the dominated convergence theorem (or appropriate result in Folland [9], Proposition 2.27), we can move the differentiation inside the integral and inside the expectation operator on the right-hand side in (4.16). Next, differentiating, we obtain straightforwardly the results since $Y^1(t) = \partial X^x(t)/\partial x$. We follow exactly the same argument for $X_\varepsilon^x(t)$. This proves the result. \square

Finally, we state our result on robustness:

Proposition 4.6. *Let $u\hat{f}(u) \in L^1(\mathbb{R})$. For $0 \leq t \leq T$, it holds that*

$$\lim_{\varepsilon \downarrow 0} \frac{\partial}{\partial x} \mathbb{E}[f(X_\varepsilon^x(t))] = \frac{\partial}{\partial x} \mathbb{E}[f(X^x(t))].$$

Proof. Cauchy-Schwarz gives:

$$\begin{aligned} & \left| \mathbb{E}\left[Y_\varepsilon^1(t)e^{-iuX_\varepsilon^x(t)} - Y^1(t)e^{-iuX^x(t)}\right] \right| \\ & \leq \mathbb{E}\left[|Y_\varepsilon^1(t) - Y^1(t)|\right] + \mathbb{E}\left[|Y^1(t)||e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|\right] \\ & \leq \mathbb{E}\left[|Y_\varepsilon^1(t) - Y^1(t)|^2\right]^{1/2} + \mathbb{E}\left[|Y^1(t)|^2\right]^{1/2} \mathbb{E}\left[|e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|^2\right]^{1/2} \\ & \leq CG^2(\varepsilon) + \tilde{C} \mathbb{E}\left[|e^{-iuX_\varepsilon^x(t)} - e^{-iuX^x(t)}|^2\right]^{1/2} \end{aligned}$$

In the last estimation, we have used Prop. 4.4 where C, \tilde{C} are two positive constants independent of ε . Moreover, the function $\exp(-iux)$ is Lipschitz continuous, which

is seen from the polar coordinate representation, and thus the final term is also majorised by a constant times $G^2(\varepsilon)$ by Prop 3.3. Hence,

$$\lim_{\varepsilon \downarrow 0} \mathbb{E} \left[Y_\varepsilon^1(t) e^{-iuX_\varepsilon^x(t)} \right] = \mathbb{E} \left[Y^1(t) e^{-iuX^x(t)} \right].$$

By appealing to Prop. 4.4 again, we see that $\mathbb{E}[Y_\varepsilon^1(t) \exp(-iuX_\varepsilon^x(t))]$ can be bounded uniformly in ε , and hence by dominated convergence the Proposition follows. \square

Note that the above results applying the Fourier method hold also for the case $\beta = 0$. In particular, this tells that even in the singular case, i.e. when the process $X(t)$ does not have any continuous martingale part, the delta for the approximative option price based on $X_\varepsilon^x(t)$ and calculated based on Malliavin differentiation with respect to the Brownian component will converge to the true value.

We remark that there is no requirement of continuity of f in the above arguments. However, the integrability restriction excludes unbounded functions f , like for instance those coming from option pricing. However, we can easily deal with such situation by introducing a damped function f in the following manner. Define for $d > 0$, the function

$$g_d(y) = e^{-dy} f(y). \tag{4.22}$$

Assuming that $g_d \in L^1(\mathbb{R})$ and $\widehat{g}_d \in L^1(\mathbb{R})$ for some $d > 0$, we can apply the above results to g_d . To translate to f , observe that

$$f(y) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{d-iu} \widehat{g}_d(u) du$$

and

$$\widehat{g}_d(u) = \widehat{f}(u + id).$$

Hence, Prop. 4.6 holds for any f such that there exists $d > 0$ for which we have the following assumptions

$$(d - iu)\widehat{f}(u - id) \in L^1(\mathbb{R}), \quad e^{dy} P_{X_\varepsilon(t)}(dy) \in L^1(\mathbb{R}) \quad \text{and} \quad e^{dy} P_{X(t)}(dy) \in L^1(\mathbb{R}).$$

We consider two examples.

Example 4.7. Let f be the payoff from a call option written on an asset with price defined as $S(t) = S(0) \exp(X(t))$, ($S(0) > 0$). Then, with $x = \ln S(0)$, we have

$$f(y) = \max(e^y - K, 0)$$

where $K > 0$ is the strike price at expiration time T . For $d > 1$, we have that $g_d \in L^1(\mathbb{R})$. Moreover,

$$\widehat{g}_d(u) = \frac{K e^{(iu-d) \ln K}}{(iu-d)(iu-d+1)},$$

which is in $L^1(\mathbb{R})$. By a direct calculation, we find that

$$(d - iu)\widehat{f}(u + id) = \frac{K^{1+iu-d}}{1 + iu - d},$$

which belongs to $L^1(\mathbb{R})$. Hence, Prop. 4.6 ensures that the approximation $X_\varepsilon(T)$ gives a delta which converges to the delta resulting from the model with $X(T)$.

Example 4.8. We consider now a *digital option* written on an asset with price defined as $S(t) = S(0) \exp(X(t))$. Then, with $x = \ln S(0)$, we have

$$f(y) = 1_{\{e^y > B\}}, \quad B \in \mathbb{R}_+.$$

For $d > 0$, we have that $g_d \in L^1(\mathbb{R})$. Moreover,

$$\widehat{g}_d(u) = \frac{-B^{iu-d}}{iu-d},$$

which is in $L^1(\mathbb{R})$. By a direct calculation, we find that

$$(d - iu)\widehat{f}(u + id) = B^{iu-d},$$

which belongs to $L^1(\mathbb{R})$.

5. Appendix: Computation of the Delta for Diffusions Driven by Brownian Motion

In this section, we review the method of Fournié et. al [10] to derive the stochastic weights for calculating the Greeks using Monte Carlo simulations. Let $DomD^W$ be the set of Malliavin differentiable random variables for Gaussian processes and D^W the Malliavin operator. We consider the case when the underlying price process is a Markov diffusion $S(t) \in DomD^W$ of the form

$$\begin{cases} dS(t) = \mu(S(t))dt + \sigma(S(t))dW(t), \\ S(0) = x, \quad x > 0, \end{cases} \quad (5.1)$$

where $W(t)$ is a Brownian motion. Assume that μ and σ are continuously differentiable functions with bounded derivatives. We associate to the process $S(t)$, a process $V(t)$ given by:

$$\begin{cases} dV(t) = \mu'(S(t))V(t)dt + \sigma'(S(t))V(t)dW(t), \\ V(0) = 1, \end{cases} \quad (5.2)$$

This $V(t)$ is called the first variation process for $S(t)$ and we have

$$V(t) = \frac{\partial S(t)}{\partial x}.$$

Proposition 5.1. [10] *Let $S(t)$ be a process of the form (5.1). Then for all $t \geq 0$,*

$$D_s^W S(t) = V(t)V(s)^{-1}\sigma(S(s))\mathbf{1}_{\{s \leq t\}}, \quad s \geq 0.$$

Proof. We have

$$S(t) = x + \int_0^t \mu(S(u))du + \int_0^t \sigma(S(u))dW(u).$$

Thus the derivative of $S(t)$ at time s is given by

$$\begin{aligned} D_s^W S(t) &= D_s^W \left(\int_0^t \mu(S(u))du \right) + D_s^W \left(\int_0^t \sigma(S(u))dW(u) \right) \\ &= \int_s^t D_s^W \left(\mu(S(u)) \right) du + \int_s^t D_s^W \left(\sigma(S(u)) \right) dW(u) + \sigma(S(s)) \\ &= \int_s^t \mu'(S(u))D_s^W S(u)du + \int_s^t \sigma'(S(u))D_s^W S(u)dW(u) + \sigma(S(s)). \end{aligned}$$

Take $Z(t) = D_s^W S(t)$, this represents the equation of the derivative of $S(t)$ at time s fixed. For $t \geq s$,

$$\begin{cases} dZ(t) = \mu'(S(t))Z(t)dt + \sigma'(S(t))Z(t)dW(t), \\ Z(s) = \sigma(S(s)). \end{cases}$$

The processes $Z(t)$ and $V(t)$ verify the same differential equations with different initial conditions, therefore

$$Z(t) = \lambda V(t)\mathbf{1}_{\{s \leq t\}}, \quad t \geq s,$$

where $\lambda = \sigma(S(s))V(s)^{-1}$. Then

$$D_s^W S(t) = V(t)V(s)^{-1}\sigma(S(s))\mathbf{1}_{\{s \leq t\}}.$$

□

Proposition 5.2. [10] *Let $f(S(T)) \in L^2(\Omega)$ and $S(t)$ be a process of the form (5.1). Define*

$$\Gamma = \left\{ a \in L^2[0, T] \mid \int_0^T a(t)dt = 1 \right\}$$

and

$$\pi = \int_0^T a(t)V(t)\sigma^{-1}(S(t))dW(t).$$

If $a \in \Gamma$ and $(\mathbb{E}[\pi^2])^{1/2} < \infty$, then

$$\frac{\partial}{\partial x} \mathbb{E} [f(S(T))] = \mathbb{E} [f(S(T))\pi].$$

Proof. First, assume that $f \in C_K^\infty(\mathbb{R})$, the set of infinitely differentiable functions with compact support, then

$$\begin{aligned} \Delta &= \frac{\partial}{\partial x} \mathbb{E} [f(S(T))] = \mathbb{E} \left[\frac{\partial}{\partial x} f(S(T)) \right] = \mathbb{E} \left[f'(S(T)) \frac{\partial S(T)}{\partial x} \right] \\ &= \mathbb{E} [f'(S(T))V(T)], \end{aligned}$$

where $Y(t)$ is the first variation process of $S(t)$. We want to write the last expression $\mathbb{E} [f'(S(T))V(T)]$ as $\mathbb{E} [f(S(T))\delta(\eta)]$, where $\delta(\eta)$ is the Skorohod integral of a certain $\eta \in L^2(\Omega \times [0, T])$ with respect to the Brownian motion $W(t)$. By the integration by parts formula, we have

$$\begin{aligned} \mathbb{E} [f(S(T))\delta(\eta)] &= \mathbb{E} \left[\int_0^T D_s^W (f(S(T))\eta(s))ds \right] \\ &= \mathbb{E} \left[\int_0^T f'(S(T))D_s^W (S(T))\eta(s)ds \right] \\ &= \mathbb{E} \left[f'(S(T)) \int_0^T V(T)(V(s))^{-1}\sigma(S(s))\mathbf{1}_{\{s \leq t\}}\eta(s)ds \right]. \end{aligned}$$

So $\eta(s)$ should verify the following equation

$$V(T) = \int_0^T V(T)(V(s))^{-1}\sigma(S(s))\mathbf{1}_{\{s \leq t\}}\eta(s)ds. \tag{5.3}$$

Then, for some $a \in \Gamma$, we have

$$\eta(t) = a(t)V(t)\sigma(S(t))^{-1}. \quad (5.4)$$

Therefore

$$\Delta = \mathbb{E} \left[f(S(T)) \int_0^T a(t)V(t)\sigma^{-1}(S(t))dW(t) \right].$$

Now, let $f(S(T)) \in L^2(\Omega)$. Then $f(x) \in L^2(\mathbb{R}, P_{S(T)})$, where $P_{S(T)}$ is the probability density of $S(T)$. Therefore

$$\exists (f_n)_{n \in \mathbb{N}} \in C_K^\infty(\mathbb{R}) \text{ such that } \lim_{n \rightarrow \infty} f_n = f, \quad \text{the limit is in } L^2(\mathbb{R}, P_{S(T)}).$$

We denote by

$$u(x) = \mathbb{E}[f(S(T))] \quad \text{and} \quad u_n(x) = \mathbb{E}[f_n(S(T))].$$

As the convergence in L^2 implies the convergence in L^1 , $(u_n)_{n \in \mathbb{N}}$ converges point wise to u and for $x \in \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} u_n(x) = u(x).$$

As $f_n \in C_K^\infty(\mathbb{R})$, then

$$\frac{\partial}{\partial x} \mathbb{E}[f_n(S(T))] = \mathbb{E}[f_n(S(T))\pi].$$

We denote by $g(x) = \mathbb{E}[f(S(T))\pi]$. By Cauchy-Schwartz inequality, we have

$$|g(x) - \frac{\partial}{\partial x} u_n(x)| = |\mathbb{E}[(f - f_n)\pi]| \leq \left(\mathbb{E}[\pi^2] \right)^{1/2} \psi_n(x). \quad (5.5)$$

where $\psi_n(x) = \left(\mathbb{E}[(f - f_n)^2] \right)^{1/2}$. The convergence of u_n implies the convergence of ψ_n to 0 point wise when n tends to infinity. Therefore the sequence $\left(\frac{\partial}{\partial x} u_n(x) \right)_{n \in \mathbb{N}}$ converges point wise to $g(x)$. As the function $\left(\mathbb{E}[\pi^2] \right)^{1/2}$ is finite, then the equation (5.5) shows that the convergence is uniform in every compact $K \in \mathbb{R}$. Therefore the function u is differentiable and it's derivative is equal to g . Then the result holds for $f(S(T)) \in L^2(\Omega)$. \square

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