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ON THE VALUE OF STOCHASTIC DIFFERENTIAL GAMES

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Abstract. We consider a two player, zero sum stochastic differential game based on a formulation given by Fleming and Souganidis. The saddle point property is introduced, and it is proved that the unique uniformly continuous bounded viscosity solution of the upper Isaacs PDE with boundary condition satisfies such a property. Also, it is shown that approximately optimal Markov strategies can be constructed for both players.

1. Introduction

In this paper we consider two-player, zero sum stochastic differential games on a finite time horizon. Our formulation of the stochastic differential game is the same as in Fleming-Souganidis [8]. However, we consider a different point of view, which is based on a saddle point property and approximately optimal control strategies for the maximizing and minimizing players. We do not assume that the Isaacs minimax condition holds. Hence, both upper and lower values of the stochastic differential game must be considered.

In Section 2, we recall from [8] the Elliott-Kalton definition of upper and lower game value functions $V_+(t,x)$ and $V_-(t,x)$. They are viscosity sense solutions to the upper and lower Isaacs PDE's with boundary condition at the final time T. See [8, Theorem 2.6], also [11],[12]. Throughout the paper, we will consider the upper differential game. Corresponding results for the lower game are obtained by replacing the payoff J by $-J$. At an intuitive level, the maximizing player has an "instantaneous information advantage", which is described formally at the end of Section 2 in terms of Markov control policies.

For deterministic differential games, the viscosity solution property of the upper value function V_+ can be obtained in a straightforward way from a dynamic programming principle. See [4], also [9, Sections 11.5, 11.6]. For stochastic differential games technical problems related to measurability issues are encountered with this approach. See [8, p.299] . To avoid these difficulties, another indirect argument was used in [8]. It made use of a somewhat artificial subclass of Elliott-Kalton strategies, called r-strategies. In the present paper, we will characterize the upper value function by different methods, which do not involve r-strategies or a continuous time dynamic programming principle.

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In the Elliott-Kalton definition of upper value, the minimizing player chooses a control u and the maximizing player chooses a strategy β. The asymmetric roles of the minimizing and maximizing players in this definition has sometimes been criticized within the game theory community. In Section 3 we consider a different formulation in which both players choose strategies. However, in the upper game the strategies α chosen by the minimizer must be restricted to some subclass of Elliott-Kalton strategies. We choose the subclass of strictly progressive strategies, as in Definition 3.1. We then seek pairs of admissible strategies α_ε, β_ε which are approximately optimal according to Definition 3.2 of the saddle point property. Theorem 3.4 states that α_ε, β_ε can be chosen as approximately Markov strategies, in the sense of Definition 3.3.

The definition of approximately optimal Markov strategy was suggested by a formalism for describing optimal control policies mentioned at the end of Section 2. It uses partitions π of the time interval $[t, T]$ on which the stochastic differential game is played. For the minimizer, the control u_s on each subinterval $[t_j, t_{j+1}]$ of π depends on the state at the left endpoint t_j. For the maximizer, the control z_s on $[t_j, t_{j+1}]$ depends on both the state at time t_j and on the minimizer choice of u_s. In Section 4 we review a construction from [8, Section 2] which leads to approximately Markov policies for the minimizer with the required saddle point property. A different construction is needed in Section 5 to find corresponding approximately Markov policies for the maximizer. Similar constructions were used by Świech [15, Section 2] to obtain sub and super optimality principles corresponding to smooth sub and super solutions to Isaacs PDE's. Sections 4 and 5 provide discrete time approximations $V^\pi(t,x)$ and $W^\pi(t,x)$ to the unique bounded, uniformly continuous viscosity solution $v(t,x)$ to the upper Isaacs PDE with boundary condition, as the mesh size of the partition π tends to 0. This convergence result is proved by a method due to Souganidis [13] [14]. Lemma 5.1 and Corollary 5.3 show that $W^\pi(t,x)$ satisfies a uniform Lipschitz condition in x and uniform Holder condition in t. The corresponding result for $V^\pi(t,x)$ [8, Lemma 2.4] is easily obtained by standard methods, which cannot be used for $W^\pi(t,x)$.

The results in Sections 4 and 5 lead easily in Section 6 to a proof of the main Theorem 3.4. In particular, they show that $v(t,x) = V_+(t,x)$, where V_+ is the Elliott-Kalton upper value function.

A quite different approach to stochastic differential games and viscosity solutions to Isaacs PDE's was taken in Buckdahn-Li [3]. It uses the theory of backward SDEs. An interesting connection between certain "tug-of-war" stochastic differential games and the infinity Laplace equation was considered in Atar-Budhiraja [1] and Barron-Evans-Jensen [2].

2. Stochastic Differential Games

Given $T > 0$ a finite time horizon and $t \in [0, T)$, let $(\Omega_t, \mathcal{F}, \mathbb{P}_t)$ be the canonical probability space, defined as

$$\Omega_t = \{\omega \in C([t,T]; \mathbb{R}^k) \ : \ \omega_t = 0\}, \tag{2.1}$$

the σ-algebra $\mathcal{F}$ is the family of Borel sets completed with respect to the Wiener measure $\mathbb{P}_t$ and the underlying filtration $\mathcal{F}_s$, $t \le s \le T$, is generated by the

Brownian paths; see [10]. The stochastic game will be formulated in this space. Let W_t be a Brownian motion defined as the coordinate map in this filtered space taking values in $\mathbb{R}^k$, and u_s and z_s stochastic processes taking values in some compact subsets U and Z of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively. Consider a stochastic dynamical system for which the state process evolves according to the stochastic differential equation

$$dX_s^{t,x} = f(s, X_s^{t,x}, u_s, z_s)ds + \sigma(s, X_s^{t,x}, u_s, z_s)dW_s, \quad 0 \le t \le s \le T, \tag{2.2}$$

with initial condition $X_t^{t,x} = x \in \mathbb{R}^d$.

We make the following assumptions on the coefficients. The function $f : [0,T] \times \mathbb{R}^d \times U \times Z \to \mathbb{R}^d$ and the $d \times k$ matrix $\sigma : [0,T] \times \mathbb{R}^d \times U \times Z \to \mathcal{M}^{d,k}$ are bounded, continuous, and Lipschitz continuous with respect to t, x uniformly for $(u,z) \in U \times Z$.

The payoff is defined as

$$J(t,x;u,z) = \mathbb{E}\left\{ \int_t^T L(s, X_s^{t,x}, u_s, z_s)ds + g(X_T^{t,x}) \right\}, \tag{2.3}$$

with $L : [0,T] \times \mathbb{R}^d \times U \times Z \to \mathbb{R}$ being bounded, continuous, and Lipschitz continuous with respect to t, x uniformly for $(u,z) \in U \times Z$ and $g : \mathbb{R}^d \to \mathbb{R}$ being bounded and Lipschitz continuous. The player 1 controlling u_s is trying to minimize J, while player 2 is trying to maximize J controlling z_s.

Given $0 \le t \le s \le T$ we define an admissible control process $u : [t,s] \times \Omega_t \to U$ (respectively $z : [t,s] \times \Omega_t \to Z$) for player I (resp. player II) on $[t,s]$ as an $\mathcal{F}_r$ progressively measurable process taking values in U (resp. Z), for $r \in [t,s]$. The set of admissible controls for player I (resp. II) is denoted by $\mathcal{U}(t,s)$ (resp. $\mathcal{Z}(t,s)$). For every $u \in \mathcal{U}(t,s)$ and $z \in \mathcal{Z}(t,s)$, there is a pathwise unique strong solution to the SDE (2.2) with given initial data. We identify the control processes which are equal for almost all ω and almost everywhere in the interval $[t,s]$; the elements of the same equivalent class are identified as $u \approx \tilde{u}$ and $z \approx \tilde{z}$, respectively. When $s = T$, notation is simplified writing $\mathcal{U}(t)$ and $\mathcal{Z}(t)$.

Definition 2.1. An *Elliott-Kalton strategy* for the maximizing player II is a mapping β from $\mathcal{U}(t)$ into $\mathcal{Z}(t)$ such that if $u \approx \tilde{u}$ on $[t,s]$, then $\beta(u) \approx \beta(\tilde{u})$ on $[t,s]$ for every $s \in [t,T]$. The set of these strategies is denoted as $\Delta_{EK}(t)$. The set of Elliott-Kalton strategies $\alpha : \mathcal{Z}(t) \to \mathcal{U}(t)$ for the minimizing player I can be defined in a similar way, and is denoted by $\Gamma_{EK}(t)$.

Given the initial conditions (t,x), the upper and lower value of the stochastic differential game (SDG) are defined by

$$V_+(t,x) = \sup_{\Delta_{EK}(t)} \inf_{\mathcal{U}(t)} J(t,x;u,\beta(u)) \tag{2.4}$$

and

$$V_-(t,x) = \inf_{\Gamma_{EK}(t)} \sup_{\mathcal{Z}(t)} J(t,x;\alpha(z),z). \tag{2.5}$$

Formally, each of the above values of the game has associated an Isaacs PDE. For the upper value it has the form

$$v_t + H^+(D^2v, Dv, x, t) = 0, \quad v(T,x) = g(x), \tag{2.6}$$

with $Dv,\ D^2v$ the gradient and matrix of second order partial derivatives of $v(t,\cdot)$,

$$H^+(A,p,x,t) \ := \ \min_{u\in U}\max_{z\in Z} F(A,p,x,t;u,z), \tag{2.7}$$

$$F(A,p,x,t;u,z) := \left[\mathrm{trace}(\frac{1}{2}a(t,x,u,z)A) + f(t,x,u,z)\cdot p + L(t,x,u,z)\right] \tag{2.8}$$

and $a=\sigma\cdot\sigma'$. The lower Isaacs PDE is similar, replacing the second term on the left side of (2.7) by $\max_{z\in Z}\min_{u\in U} F(A,p,x,t;u,z)$.

If for all $t\in[0,T),\ x\in\mathbb{R}^d,\ p\in\mathbb{R}^d$ and $A\in\mathcal{M}^{d,d}$ the identity

$$\min_{u\in U}\max_{z\in Z} F(A,p,x,t;u,z) = \max_{z\in Z}\min_{u\in U} F(A,p,x,t;u,z)$$

holds, it is said that the *Isaacs minimax condition* holds, and it can be verified that in that case $V_+(t,x)=V_-(t,x)$.

Throughout we shall consider the upper value of the SDG, but analogous results can be obtained for the lower value. In this case, the maximizing player has the information advantage. To illustrate this situation the class of Markov control policies for the players can be considered. A control policy for the minimizing player is a function $\underline{\mathrm{u}}:[0,T]\times\mathbb{R}^d\to U$ while a Markov policy for the maximizing player is a function $\underline{\mathrm{z}}:[0,T]\times\mathbb{R}^d\times U\to Z$. When $\underline{\mathrm{u}}$ is Lipschitz continuous there exists a strong solution to the SDE (2.2) for each control $z\in\mathcal{Z}(t)$ of the maximizing player; similarly, when $\underline{\mathrm{z}}$ is Lipschitz continuous, for each admissible control $u\in\mathcal{U}(t)$ of the minimizing player a strong solution exists for (2.2).

Formally, from the Isaacs equation (2.6) it is possible to define a control policy for each player $\underline{\mathrm{u}}^*,\underline{\mathrm{z}}^*$ such that:

$$\underline{\mathrm{u}}^*(t,x)\in\arg\min_{u\in U}\max_{z\in Z} F(D^2V_+,DV_+,x,t;u,z),$$

and

$$\underline{\mathrm{z}}^*(t,x,u)\in\arg\max_{z\in Z} F(D^2V_+,DV_+,x,t;u,z),$$

which would satisfy a saddle point property. Here D^2V_+,DV_+ are evaluated at (t,x). In general the argument above cannot be made rigorous. However, it was the main motivation to study the existence of approximately Markov control strategies which are nearly optimal in the sense to be explained in Section 3.

3. Control Strategies, Saddle Point Property

In Definition 2.1 the Elliott-Kalton strategies were introduced, as well as the definition of upper and lower value functions. According to this definition, for the lower value the minimizing controller has advantage in the information available at each time s. Now we shall reduce the class of admissible strategies for this player to the smaller class $\Gamma_S(t)$, which eliminates this advantage. The definition of $\Gamma_S(t)$ is essentially the same as the one in [9, p.392] for deterministic differential games.

Definition 3.1. A strategy $\alpha\in\Gamma_{EK}(t)$ for the minimizing player is *strictly progressively measurable* if for each strategy $\beta\in\Delta_{EK}(t)$ of the maximizing player the equation

$$u=\alpha(z)\quad z=\beta(u)$$

has a solution $\hat{u}$, $\hat{z}$. The set of strictly progressively measurable strategies is denoted by $\Gamma_S(t)$.

The same argument as in [9, p.393] easily shows that

$$V_+(t,x) \le \inf_{\alpha \in \Gamma_S(t)} \sup_{z \in \mathcal{Z}(t)} J(t,x;\alpha(z),z). \tag{3.1}$$

If the infimum on the r.h.s. is taken over $\Gamma_{EK}(t)$ instead of $\Gamma_S(t)$, then it gives the lower value $V_-(t,x)$, which can be strictly less than $V_+(t,x)$ when the Isaacs minimax condition does not hold.

Motivated by the definition of upper value in (2.4) and inequality (3.1), we introduce the

Definition 3.2. The *saddle point property* for the upper game is said to hold if there exists a real valued function $V(t,x)$ such that for each $\varepsilon > 0$ and $(t,x) \in [0,T] \times \mathbb{R}^d$ there exist $\alpha_\varepsilon \in \Gamma_S(t)$, $\beta_\varepsilon \in \Delta_{EK}(t)$ such that

(i) $V(t,x) - \varepsilon \le \inf_{u \in \mathcal{U}(t)} J(t,x;u,\beta_\varepsilon(u))$
(ii) $\sup_{z \in \mathcal{Z}(t)} J(t,x;\alpha_\varepsilon(z),z) \le V(t,x) + \varepsilon$.

Definition 3.3. (a) $\alpha \in \Gamma_{EK}(t)$ is an *approximately Markov strategy* for the minimizing player if there exists a partition $\pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ of $[0,T]$ with $t = t_i$ and Borel measurable functions $\zeta_j : \mathbb{R}^d \to U$, for $j = i, i+1, \ldots, N-1$, such that

$$\alpha(z)_s = \zeta_j(X^{t,x}_{t_j}), \;\; t_j \le s < t_{j+1}.$$

(b) $\beta \in \Delta_{EK}(t)$ is an *approximately Markov strategy* for the maximizing player if there exists such a partition π and Borel measurable functions $\eta_j : \mathbb{R}^d \times U \to Z$ such that

$$\beta(u)_s = \eta_j(X^{t,x}_{t_j}, u_s), \;\; t_j \le s < t_{j+1}.$$

(c) For such a partition π, define the class of strategies $\alpha \in \Gamma_\pi(t) \subset \Gamma_{EK}(t)$ such that $\alpha(z)_s$ is constant on each interval $[t_j, t_{j+1})$.

By solving the SDE (2.2) on successive intervals $[t_j, t_{j+1}]$, it can be seen using an induction argument that $\Gamma_\pi(t) \subset \Gamma_S(t)$. For deterministic games the complete argument is presented in [9, p.392], while for the stochastic case an analogous argument can be used. Moreover, if α is an approximately Markov strategy constructed as in part (a), then $\alpha \in \Gamma_\pi(t)$.

The main result of the paper is the following theorem, the proof of which is delayed until Section 6 after some technical preliminaries in Sections 4 and 5.

Theorem 3.4. *The saddle point property holds with $V = V_+ = v$, where $v(t,x)$ is the unique bounded, uniformly continuous viscosity solution to the upper Isaacs PDE with boundary condition (2.6). Moreover, α_ε, β_ε can be chosen as approximately Markov control strategies.*

Remark 3.5. Our methods rely on time discretizations. It would be interesting to find an approach to the saddle point property for which time discretizations are avoided and the class $\Gamma_S(t)$ is replaced by some other class $\tilde{\Gamma}(t)$ of strategies for the minimizer. For the deterministic differential games considered in [7], $\tilde{\Gamma}(t)$ is the class of Elliott-Kalton strategies α such that $\alpha(z)$ is a right continuous function on $[t,T]$ with left limits at each $s \in [t,T]$. See [7, Section 3].

4. Piecewise Constant Minimizing Controls

In this section we shall review some results from [8, Section 2] which will be needed. Let $\pi = \{0 = t_0, t_1, \ldots, t_N = T\}$ be a partition of $[0,T]$ and let $\|\pi\| = \max_j(t_{j+1} - t_j)$. For $t < \tau$ define the operator G_1 on the set $C_b^{0,1}(\mathbb{R}^d)$ of bounded, Lipschitz continuous functions on $\mathbb{R}^d$ by

$$G_1(t,\tau)\varphi(x) = \inf_{u\in U} \sup_{z\in\mathcal{Z}(t,\tau)} \mathbb{E}\left\{\varphi(X_\tau^{t,x}) + \int_t^\tau L(s, X_s^{t,x}, u, z_s)ds\right\}$$

for $0 \le t \le \tau \le T$. Here X_s corresponds to the solution of (2.2) when the first player chooses the constant control $u_s = u$ and the second player chooses control $z \in \mathcal{Z}(t,\tau)$. This operator maps the set $C_b^{0,1}(\mathbb{R}^d)$ into itself. The operators $G_1(t,\tau)$ are the same as in [8, formula (2.1)] with *infsup* replacing *supinf*.

On the other hand, define recursively backward in time

$$V^\pi(t,x) = \begin{cases} g(x),\ t = T \\ \prod_{j=i}^{N-1} G_1(t_j, t_{j+1})g(x),\ \text{if } t = t_i < T. \end{cases} \tag{4.1}$$

Then, for $j = i, \ldots, N-1$,

$$V^\pi(t_j,x) = \inf_{u\in U} \sup_{z\in\mathcal{Z}(t_j,t_{j+1})} \mathbb{E}_{t_j,x}\left\{V^\pi(t_{j+1}, X_{t_{j+1}}^{t_j,x}) + \int_{t_j}^{t_{j+1}} L(s, X_s^{t,x}, u, z_s)ds\right\}, \tag{4.2}$$

with $V_\pi(T,x) = g(x)$.

Given a time t, consider partitions π such that $t_i = t$ for some i. Let

$$\mathcal{U}_\pi(t) = \{u \in \mathcal{U}(t) \ \mid\ u_s = u_{t_j},\ \text{ for } \ s \in [t_j, t_{j+1}),\ \ j = i, \cdots, N-1\}.$$

It was proved in [8, Proposition 2.3] that

$$V^\pi(t,x) = \max_{\Delta_{EK}(t)} \min_{\mathcal{U}_\pi(t)} J(t,x;u,\beta(u)). \tag{4.3}$$

The proof of this proposition also gives $\alpha_\varepsilon \in \Gamma_\pi$ such that

$$\sup_{z\in\mathcal{Z}(t)} J(t,x;\alpha_\varepsilon(z),z) \le V^\pi(t,x) + \varepsilon/2. \tag{4.4}$$

Moreover, α_ε is of the form required for an approximately Markov strategy, namely,

$$\alpha_\varepsilon(z)(s) = \zeta_{\varepsilon j}(X_{t_j}^{t,x}),\ \ t_j \le s < t_{j+1},$$

with $\zeta_{\varepsilon j}$ Borel measurable; see formula (2.10) of [8].

From [8, Proposition 2.5], $V^\pi \to v$ uniformly on compact sets as $\|\pi\| \to 0$, where $v(t,x)$ is the unique bounded, uniformly continuous viscosity solution to the upper Isaacs PDE with boundary condition g. Since $\mathcal{U}_\pi(t) \subset \mathcal{U}(t)$, (4.3) implies that $V_+ \le V^\pi$. Hence, $V_+ \le v$; in fact, $V_+ = v$. This is a consequence of Theorem 3.4 and also of [8, Theorem 2.6].

Remark 4.1. The same proof of [8, Proposition 2.3] gives $\beta_\varepsilon \in \Delta_{EK}(t)$ such that

$$V^\pi(t,x) - \varepsilon/2 \leq \inf_{u\in\mathcal{U}_\pi(t)} J(t,x;u,\beta_\varepsilon(u)).$$

However, this is not good enough for our purposes. In Section 6, we will find a corresponding inequality in which $\inf_{u\in\mathcal{U}_\pi(t)}$ in the above display is replaced by $\inf_{u\in\mathcal{U}(t)}$, and V^π is replaced by W^π defined in Section 5.

5. Another Discrete Time Approximation

On the same partition $\pi = \{0 = t_0 < t_1 < \cdots < t_N = T\}$ define $W^\pi(t_j,x)$ by backward induction on j, as follows

$$\begin{aligned} W^\pi(T,x) &= g(x), \qquad (5.1)\\ W^\pi(t_j,x) &= \sup_{h\in H}\inf_{u\in\mathcal{U}(t_j,t_{j+1})} \mathbb{E}\Big\{ W^\pi(t_{j+1}, X^{t_j,x}_{t_{j+1}}) \\ &\quad + \int_{t_j}^{t_{j+1}} L(s, X^{t_j,x}_s, u_s, h(u_s))ds \Big\}, \end{aligned}$$

with $H := \{h : U \to Z \mid h \text{ is Borel measureable}\}$.

Lemma 5.1. *If $\|\pi\|$ is small enough, there exists M such that for all $x,\ x' \in \mathbb{R}^d$,*

$$|W^\pi(t_j,x) - W^\pi(t_j,x')| \leq M\|x-x'\|. \qquad (5.2)$$

Proof. To slightly simplify the argument, additionally to the assumptions already made on the coefficients of the SDE (2.2), assume that f and σ are also C^1 in the variables t,x. The same Lipschitz bound as in (5.2) then holds under the assumptions on f, σ in Section 2, by making smooth approximations to f, σ with uniformly bounded first order partial derivatives in t,x. Let $X^{t_j,x}_s$ and $\tilde{X}^{t_j,x'}_s$ be the solutions to (2.2) on $[t_j,t_{j+1}]$, and define $\eta_s := X^{t_j,x}_s - \tilde{X}^{t_j,x'}_s$. By using the mean value theorem,

$$d\eta_s = A_s\eta_s ds + B_s\eta_s dW_s,$$

with $A_s = \int_0^1 f_x(s,X^\lambda_s,u_s,z_s)ds$ and $B_s = \int_0^1 \sigma_x(s,X^\lambda_s,u_s,z_s)ds$, and $X^\lambda_s = (1-\lambda)X^{t_j,x}_s - \lambda\tilde{X}^{t_j,x'}_s$. Using the Ito's differential rule,

$$d\|\eta_s\|^2 = 2\eta_s\cdot d\eta_s + \|B_s\eta_s\|^2 ds,$$

and hence for some constant C, $\mathbb{E}\|\eta_s\|^2 \leq \|x-x'\|^2 + C\mathbb{E}\int_{t_j}^s \|\eta_r\|^2 dr$. Gronwall's inequality implies that

$$\begin{aligned} \mathbb{E}\|\eta_s\|^2 &\leq \|x-x'\|^2(1 + C(s-t_j)e^{C(s-t_j)}) \\ &\leq \|x-x'\|^2(1 + C_1(s-t_j)), \end{aligned}$$

for $\|\pi\|$ small enough. Since $(1+a)^{\frac{1}{2}} \leq 1 + \frac{1}{2}a$ for $a > 0$, Cauchy-Schwartz implies that

$$\mathbb{E}\|\eta_s\| = \mathbb{E}\|X^{t_j,x}_s - \tilde{X}^{t_j,x'}_s\| \leq \|x-x'\|(1 + \frac{C_1}{2}(s-t_j)), \qquad (5.3)$$

for $t_j \leq s \leq t_{j+1}$.

Now, take $u \in \mathcal{U}(t_j,t_{j+1})$ and $h \in H$, and let $z_s := h(u_s)$. Using backward induction on j, let Λ_N be the Lipschitz constant for the function g in (5.1). If Λ_{j+1}

is a Lipschitz constant for $W^\pi(t_{j+1},\cdot)$ and K_1 a Lipschitz constant for $L(s,\cdot,u,z)$, we get from (5.1) and (5.3) that

$$|W^\pi(t_j,x') - W^\pi(t_j,x)| \leq \Big[K_1(t_{j+1}-t_j)(1+\frac{C_1}{2}(t_{j+1}-t_j)) + \Lambda_{j+1}(1+\frac{C_1}{2}(t_{j+1}-t_j))\Big]\|x'-x\|.$$

For $\|\pi\|$ small enough this implies the Lipschitz property of $W^\pi(t_j,\cdot)$, with constant $\Lambda_j = \Lambda_{j+1}(1+\frac{C_1}{2}(t_{j+1}-t_j)) + 2K_1(t_{j+1}-t_j)$, which yields a uniform bound $|\Lambda_j| \leq M$. □

Lemma 5.2. *Given $t=t_i$, $x\in\mathbb{R}^d$ and $\varepsilon>0$, there exist $u_\varepsilon\in\mathcal{U}(t)$ and $\beta_\varepsilon\in\Delta_{EK}(t)$ with the following properties.*

(i) $\beta_\varepsilon(u)(s) = \eta_{\varepsilon j}(X^{t,x}_{t_j},u_s)$, $t_j\leq s<t_{j+1}$, where $\eta_{\varepsilon j}:\mathbb{R}^d\times U\to Z$ is Borel measurable. For $i\leq j\leq N$,
(ii)

$$W^\pi(t,x) - \frac{\varepsilon}{2} \leq \inf_{u\in\mathcal{U}(t,t_j)} \mathbb{E}\left\{W^\pi(t_j,X^{t,x}_{t_j}) + \int_t^{t_j} L(s,X^{t,x}_s,u_s,\beta_\varepsilon(u)_s)ds\right\} \leq W^\pi(t,x); \quad (5.4)$$

(iii) $\mathbb{E}\left\{W^\pi(t_j,X^{t,x}_{t_j}) + \int_t^{t_j} L(s,X^{t,x}_s,(u_\varepsilon)_s,\beta_\varepsilon(u_\varepsilon)_s)ds\right\} \leq W^\pi(t,x)+\varepsilon/2.$

Proof. Given $\delta>0$, to be specified later, choose a partition of $\mathbb{R}^d$ into Borel sets $\{A_1,A_2,\ldots\}$ of diameter less than δ and choose $y_k\in A_k$. Replacing x by y_k in (5.1), given $\gamma=\gamma(\varepsilon)>0$ take $h_{jk}\in H$ such that

$$W^\pi(t_j,y_k)-\gamma \leq \inf_{u\in\mathcal{U}(t_j,t_{j+1})}\mathbb{E}\Big[W^\pi(t_{j+1},X^{t_j,y_k}_{t_{j+1}}) + \int_{t_j}^{t_{j+1}} L(s,X^{t_j,y_k}_s,u_s,h_{jk}(u_s))ds\Big] \leq W^\pi(t_j,y_k) \quad (5.5)$$

Now, for $s\in[t_j,t_{j+1})$, define $\beta_\varepsilon(u)_s := \eta_{\varepsilon j}(X^{t,x}_{t_j},u_s)$, with $\eta_{\varepsilon j}(x,u)=\sum_k h_{jk}(u)\cdot I_{A_k}(x)$. Thus, β_ε is defined successively on intervals $[t_j,t_{j+1})$.

Let $K=1+\frac{C_1}{2}\|\pi\|$, with C_1 the constant on the right side of (5.3) and recall that K_1 denotes the Lipschitz constant of $L(t,\cdot,u,z)$. For $j=i$ and $x\in A_k$, using Lemma 5.1 and (5.5),

$$\begin{aligned}
&W^\pi(t,x)\leq W^\pi(t,y_k)+M\delta\\
&\leq \inf_{u\in\mathcal{U}(t,t_{i+1})}\mathbb{E}\left\{W^\pi(t_{i+1},X^{t,y_k}_{t_{i+1}}) + \int_t^{t_{i+1}} L(s,X^{t,y_k}_s,u_s,\beta_\varepsilon(u_s))ds\right\}+M\delta+\gamma\\
&\leq \inf_{u\in\mathcal{U}(t,t_{i+1})}\mathbb{E}\left\{W^\pi(t_{i+1},X^{t,x}_{t_{i+1}}) + \int_t^{t_{i+1}} L(s,X^{t,x}_s,u_s,\beta_\varepsilon(u_s))ds\right\}+M_1\delta+\gamma, \quad (5.6)
\end{aligned}$$

where $M_1 = 2M+\|\pi\|KK_1$.

If (5.6) holds for j with constant on the right side equal to $j(\delta M_1+\gamma)$, in order to verify it for $j+1$ we need to take some conditional expectations and introduce more notation. Given $j\in\{i+1,\ i+2,\dots,N-1\}$, each $\omega\in\Omega_t$ is identified with the pair $(\omega_{1j},\omega_{2j})$, with $\omega_{1j}=\omega|_{[t,t_j]}$ and $\omega_{2j}=(\omega-\omega_{1j})|_{[t_j,T]}$. Also, the Wiener space is identified as the product space $(\Omega_t,\mathbb{P}_t)=(\Omega_{t,t_j}\times\Omega_{t_j},\mathbb{P}_{1j}\times\mathbb{P}_{2j})$.

Given $u\in\mathcal{U}(t,t_{j+1})$,

$$\begin{aligned}
&\mathbb{E}\bigg\{W^\pi(t_{j+1},X^{t,x}_{t_{j+1}})+\int_t^{t_{j+1}}L(s,X^{t,x}_s,u_s,\beta_\varepsilon(u)_s)ds\bigg\}\\
&=\mathbb{E}\left\{\int_t^{t_j}L(s,X^{t,x}_s,u_s,\beta_\varepsilon(u)_s)ds+\mathbb{E}_{t_j,X^{t,x}_{t_j}}\left[W^\pi(t_{j+1},X^{t_j,X^{t,x}_{t_j}}_{t_{j+1}})\right.\right.\\
&\qquad\left.\left.+\int_{t_j}^{t_{j+1}}L(s,X^{t_j,X^{t,x}_{t_j}}_s,u_s(\omega_1,\omega_2),\beta_\varepsilon(u)_s(\omega_1,\omega_2))ds\right]\right\}
\end{aligned}\tag{5.7}$$

The term within the square brackets is not less that $W^\pi(t_j,X^{t,x}_{t_j}(\omega_1))-\gamma-\delta M_1$, by (5.6), and using the induction hypothesis, it follows that the left side of (5.7) is greater or equal $W(t,x)-(\gamma+\delta M_1)(j+1)$. Taking $\gamma=\frac{\varepsilon}{4N}$ and $\delta=\frac{\varepsilon}{4M_1N}$ the left side of part (ii) is obtained. The right side of part (ii) follows directly from (5.7) and using again an induction argument.

Given $\gamma>0$ and $s\in[t_j,t_{j+1})$, if $X_{t_j}=x\in A_k$, from part (ii) we can choose $u_{jk}\in\mathcal{U}(t_j,t_{j+1})$ such that

$$\mathbb{E}\left\{W^\pi(t_{j+1},X^{t_j,x}_{t_{j+1}})+\int_{t_j}^{t_{j+1}}L(s,X^{t_j,x}_s,u_{\varepsilon s},\beta_\varepsilon(u_\varepsilon)_s)ds\right\}\le W^\pi(t_j,x)+\gamma.$$

This defines recursively u_ε on each interval $[t_j,t_{j+1})$. Proceeding by induction in j, from (5.7) with $u=u_\varepsilon$, part (iii) is obtained taking $\gamma=\frac{\varepsilon}{2N}$. □

Corollary 5.3. *For $t=t_i,\ \tau=t_j$,*

$$|W^\pi(t,x)-W^\pi(\tau,x)|\le K(\tau-t)^{\frac12},\tag{5.8}$$

for some constant K.

Proof. Take u_ε and β_ε as in Lemma 5.2 and let $X^{t,x}_s$ the solution of (2.2), with $X_t=x$, $u=u_\varepsilon$ and $z=\beta_\varepsilon(u_\varepsilon)$. Then,

$$\begin{aligned}
|W^\pi(t,x)-W^\pi(\tau,x)| &\le \|L\|(\tau-t)+\mathbb{E}|W^\pi(\tau,X^{t,x}_\tau)-W^\pi(\tau,x)|+\varepsilon\\
&\le \|L\|(\tau-t)+M\mathbb{E}|X^{t,x}_\tau-x|+\varepsilon\\
&\le K(\tau-t)^{\frac12}+\varepsilon,
\end{aligned}$$

with $\varepsilon>0$ arbitrary. In the last inequality we have used the estimate $\mathbb{E}|X^{t,x}_\tau-x|\le K_1|\tau-t|+K_2(\tau-t)^{\frac12}$, with $K_1=\|f\|$ and $K_2=\|\sigma\|$. □

Now, for $t<\tau$, define an operator $G_2(t,\tau)$ on the set $C^{0,1}_b(\mathbb{R}^d)$ by

$$G_2(t,\tau)\varphi(x)=\sup_{h\in H}\inf_{u\in\mathcal{U}(t,\tau)}\mathbb{E}\left\{\varphi(X^{t,x}_\tau)+\int_t^\tau L(s,X^{t,x}_s,u_s,h(u_s))ds\right\}.$$

Notice that in Section 4 the operator $G_1(t,\tau)$ was defined similarly.

Lemma 5.4. *Given $\varphi \in C_b^2(\mathbb{R}^d)$ let $H^+(D^2\varphi, D\varphi, x, t; u, z)$, with H^+ as in (2.7) and $D^2\varphi$ and $D\varphi$ evaluated in x. Then,*

$$\lim_{\delta \to 0} \| \frac{1}{\delta}[G_2(t, t+\delta)\varphi(x) - \varphi(x)] - H^+(D^2\varphi, D\varphi, x, t)\| = 0,$$

where $\|\cdot\|$ is the sup norm.

The proof is similar to that for the corresponding result with G_1 instead of G_2. See [8, p.309]. It relies on the Ito differential rule and the property that

$$\min_{u \in U} \max_{z \in Z} F(A, p, x, t; u, z) = \sup_{h \in H} \inf_{u \in U} F(A, p, x, t; u, h(u)).$$

See [9, p. 377, 384].

Note that

$$W^\pi(t, x) = \begin{cases} g(x), \; t = T \\ \prod_{j=i}^{N-1} G_2(t_j, t_{j+1})g(x), \text{ if } t = t_i < T. \end{cases}$$

If G_2 is replaced by G_1, then we get the corresponding expression (4.1) for $V^\pi(t, x)$.

Theorem 5.5. $\lim_{\|\pi\| \to 0} W^\pi(t, x) = v(t, x)$, *uniformly on compact sets, where v is the unique bounded, uniformly continuous viscosity solution to the upper Isaacs PDE (2.6).*

Theorem 5.5 can be proved in the same way as for Proposition 2.5 in [8], using Lemmas 5.1, 5.4 and Corollary 5.3; see also [9, p.393]. The method is due to Souganidis [13] [14]. Observe that the strategy β_ε obtained in part (ii) of Lemma 5.2 has the approximately Markov property.

Remark 5.6. By using a method of Barles and Perthame, the uniform Lipschitz and Holder estimates for W^π in Lemma 5.1 and Corollary 5.3 could have been avoided. However, with the Barles-Perthame method discontinuous viscosity sub and super solutions must be considered. See [9, Section 9.5]. In the proof of a comparison principle for discontinuous viscosity sub and super solutions, the fact that a bounded, uniformly continuous viscosity solution $v(t, x)$ exists to the upper Isaacs PDE with boundary condition is used. See [9, Section 7.8] for fist order PDEs.

6. Proof of the Main Result

As already mentioned in Section 4, $V_+ \leq v$, where $v(t, x)$ is the unique bounded, uniformly continuous solution to the upper Issacs PDE with boundary condition g at time T. In (ii) of Lemma 5.2, let $t_i = t,\; j = N,\; t_j = T$. Then,

$$W^\pi(t, x) - \varepsilon/2 \leq \inf_{u \in \mathcal{U}(t)} J(t, x; u, \beta_\varepsilon(u)).$$

From Theorem 5.5,

$$v(t, x) - \varepsilon \leq \inf_{u \in \mathcal{U}(t)} J(t, x; u, \beta_\varepsilon(u)). \tag{6.1}$$

Since the right side is not greater that $V_+(t, x)$ and ε is arbitrary, $v(t, x) = V_+(t, x)$, and hence part (i) of Definition 3.2 is satisfied.

Part (ii) of Definition 3.2 requires strictly progressive α_ε. From (4.4) we know that there exists such a strategy with

$$\sup_{z\in\mathcal{Z}(t)} J(t,x;\alpha_\varepsilon(z),z) \le V^\pi(t,x) + \varepsilon/2,$$

and also that $V^\pi(t,x)$ tends uniformly on compact sets to $v(t,x)$ as $\|\pi\| \to 0$. Taking π small enough we get part (ii) of the saddle point property. Moreover, α_ε and β_ε are approximately Markov strategies.

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