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# INTEGRATION BY PARTS FORMULA AND THE STEIN LEMMA ON ABSTRACT WIENER SPACE

HUI-HSIUNG KUO AND YUH-JIA LEE*

Abstract. We prove the integration by parts formula for abstract Wiener measures under much weaker conditions than those assumed in [11]. Thus the formula holds for a much larger class of functions. For applications, we characterize the abstract Wiener measures, extend the Stein lemma to infinite dimensions, and evaluate some functional integrals.

## 1. Introduction

The first infinite dimensional integration by parts formula was obtained by Cameron [2] in 1951 for the classical Wiener space. Later in 1964, Donsker [5] also proved this formula and applied it to study Fréchet-Volterra differential equations. In 1974 Kuo [11] generalized the formula to abstract Wiener space (Gross [8, 9], see also the book [12]) for abstract Wiener measures and applied the formula to study Fourier-Wiener transform and to evaluate some functional integrals.

The purpose of this paper is to prove the integration by parts formula in [11] under much weaker conditions. In fact, we also simplify the proof. This will be done in Section 2. As an interesting application of this formula, we will extend the Stein lemma [1, 3, 14, 15] to infinite dimensions in Section 3. Then in Section 4 we will apply this integration by parts formula to evaluate some functional integrals on an abstract Wiener space.

## 2. Integration by Parts Formula

Let $(i, H, B)$ be an abstract Wiener space, i.e., $H$ is a real Hilbert space with norm $|\cdot|$ and inner product $\langle\cdot,\cdot\rangle$, $B$ is the completion of $H$ with respect to a measurable norm $\|\cdot\|$ on $H$, and $i : H \hookrightarrow B$ is the inclusion map. The dual space $B^*$ of $B$ is embedded in $H$ by using the Riesz representation theorem to identify the dual space $H^*$ of $H$ with $H$. It follows from this identification that

$$(y, x) = \langle y, x\rangle, \quad \forall\, y \in B^*,\ x \in H,$$

where $(\cdot,\cdot)$ is the bilinear pairing of $B^*$ and $B$.

Let $p_t$ denote the Gaussian measure on $(i, H, B)$ with mean 0 and variance $t$. If $y \in B^*$, then $(y, x)$ is defined for all $x \in B$ and $(y,\cdot)$ is a normal random

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variable with mean 0 and variance $t|y|^2$ on the probability space $(B, p_t)$. Suppose $h \in H$. Take a sequence $\{y_n\}$ in $B^*$ such that $y_n \to h$ in $H$. Then $\{(y_n, \cdot)\}$ is a Cauchy sequence in $L^2(B, p_t)$. The limit of this sequence is well defined, i.e., it is independent of the choice of the sequence $\{y_n\}$. For simplicity, we will use $\langle h, \cdot \rangle$ to denote this limit. Then $\langle h, \cdot \rangle$, defined almost everywhere on $(B, p_t)$, is a normal random variable with mean 0 an variance $t|h|^2$.

Let $f$ be a function defined on an open subset $U$ of $B$ with values in a real Banach space $W$. The *H-derivative* $D_H f(x)$ of $f$ at $x \in U$ belongs to the Banach space $\mathcal{L}(H, W)$ of bounded linear operators from $H$ into $W$. A function $f$ from $U$ into $W$ is called a *$C_H^1$-function* if $D_H f$ is a continuous function from $U$ into $\mathcal{L}(H, W)$. The $k$-th $H$-derivative $D_H^k f(x)$ of $f$ at $x \in U$ is a bounded $k$-linear mapping from $H \times \cdots \times H$ ($k$ factors) into $W$. In particular, when $W = \mathbb{R}$, the second $H$-derivative $D_H^2 f(x)$ of $f$ at $x$ can be identified as a bounded linear operator on $H$. When $D_H^2 f(x)$ is a trace class operator on $H$, its trace is known as the *Gross Laplacian* of $f$:

$$\Delta_G f(x) = \operatorname{trace}\left[D_H^2 f(x)\right].$$

Now we can state the integration by parts formula from [11] on an abstract Wiener space $(i, H, B)$.

**Theorem 2.1** (Kuo [11])**.** *Let $f$ be a $C_H^1$-function from $B$ into a real Hilbert space $K$ satisfying the following conditions:*

(a) *$\|f(\cdot)\|_K \in L^2(p_t)$ with $\|\cdot\|_K$ being the norm on $K$;*

(b) *There exist constants $r > 0$ and $M > 0$ such that*

$$\int_B \|f(x+h)\|_K \, dp_t(x) < M, \quad \forall\, |h| < r;$$

(c) *The function*

$$\sup_{|h|<r} \|D_H f(x+h\|_{\mathcal{L}(H,K)}, \quad x \in B,$$

*is $p_t$-integrable.*

*Then for any $h \in H$, we have*

$$\int_B \langle h, x \rangle f(x) \, dp_t(x) = t \int_B [D_H f(x)](h) \, dp_t(x). \tag{2.1}$$

The conditions in Theorem 2.1 are too strong. We will prove in Theorem 2.6 (see below) that the integration by parts formula in Equation (2.1) holds under much weaker conditions. For the proof of Theorem 2.6, we need to prepare several lemmas. We start with a generalization of the Stein Lemma (see [1, 3, 14, 15].)

**Lemma 2.2.** *Let $f$ and $g$ be $C^1$-functions on $\mathbb{R}$ with $g(\pm\infty) = 0$. Then the following equality holds:*

$$\int_{-\infty}^{\infty} f'(x) g(x) \, dx = -\int_{-\infty}^{\infty} f(x) g'(x) \, dx, \tag{2.2}$$

*where the existence of either improper integral implies that of the other. Assume further that $f'g$ and $fg'$ are Lebesgue integrable. Then Equation* (2.2) *also holds in the sense of Lebesgue integral.*

*Proof.* We may assume that $\int_{-\infty}^{\infty} f(x)g'(x)\,dx$ exists since the proof of the other case is similar. Write the improper integral $\int_{-\infty}^{\infty} f'(x)g(x)\,dx$ as

$$\int_{-\infty}^{\infty} f'(x)g(x)\,dx = \int_{-\infty}^{0} f'(x)g(x)\,dx + \int_{0}^{\infty} f'(x)g(x)\,dx. \tag{2.3}$$

For the first improper integral in the right-hand side, we take the limit and change the order of integration of a continuous function on a finite region to get

$$\begin{aligned}
\int_{-\infty}^{0} f'(x)g(x)\,dx &= \int_{-\infty}^{0} f'(x)\Big(\int_{-\infty}^{x} g'(y)\,dy\Big)\,dx \\
&= \lim_{a\to\infty} \int_{-a}^{0} f'(x)\Big(\int_{-a}^{x} g'(y)\,dy\Big)\,dx \\
&= \lim_{a\to\infty} \int_{-a}^{0} \Big(\int_{y}^{0} f'(x)\,dx\Big)g'(y)\,dy \\
&= \lim_{a\to\infty} \int_{-a}^{0} \big(f(0)-f(y)\big)g'(y)\,dy \\
&= \lim_{a\to\infty} \Big(f(0)\big(g(0)-g(-a)\big) - \int_{-a}^{0} f(y)g'(y)\,dy\Big) \\
&= f(0)g(0) - \int_{-\infty}^{0} f(x)g'(x)\,dx, \qquad (2.4)
\end{aligned}$$

which shows that $\int_{-\infty}^{0} f'(x)g(x)\,dx$ exists. Similarly, for the second improper integral, we have

$$\int_{0}^{\infty} f'(x)g(x)\,dx = -f(0)g(0) - \int_{0}^{\infty} f(x)g'(x)\,dx, \tag{2.5}$$

which implies that $\int_{0}^{\infty} f'(x)g(x)\,dx$ exists. Hence by Equation (2.3) the improper integral $\int_{-\infty}^{\infty} f'(x)g(x)\,dx$ also exists. Then Equation (2.2) follows from Equations (2.4) and (2.5). To prove the second assertion, suppose $f'g$ and $g'f$ are Lebesgue integrable. Then by the Lebesgue dominated covergence theorem,

$$\begin{aligned}
\int_{-\infty}^{\infty} f'(x)g(x)\,dx &= \lim_{a,b\to\infty} \int_{-a}^{b} f'(x)g(x)\,dx \\
= -\lim_{a,b\to\infty} \int_{-a}^{b} f(x)g'(x)\,dx &= -\int_{-\infty}^{\infty} f(x)g'(x)\,dx,
\end{aligned}$$

which shows that Equation (2.2) holds in the sense of Lebesgue integral. □

**Lemma 2.3.** *Let $f$ and $g$ be $C^n$-functions on $\mathbb{R}$ with $g^{(j)}(\pm\infty) = 0$ for all $j = 0, 1, \ldots, n-1$. Then the following equality holds:*

$$\int_{-\infty}^{\infty} f^{(n)}(x)g(x)\,dx = (-1)^n \int_{-\infty}^{\infty} f(x)g^{(n)}(x)\,dx, \tag{2.6}$$

*where the existence of either improper integral implies that of the other. Assume further that $f^{(n)}g$ and $fg^{(n)}$ are Lebesgue integrable. Then Equation* (2.6) *also holds in the sense of Lebesgue integral.*

*Proof.* We only prove the first assertion for the improper integral case since the second one for the Lebesgue integral case can be handled by the same arguments as those used in the proof of Lemma 2.2.

We may assume that $\int_{-\infty}^{\infty} f(x)g^{(n)}(x)\,dx$ exists since the proof of the other case is similar. Apply Lemma 2.2 to the functions $f$ and $g^{(n-1)}$ to conclude that $\int_{-\infty}^{\infty} f'(x)g^{(n-1)}(x)\,dx$ exists and

$$\int_{-\infty}^{\infty} f'(x)g^{(n-1)}(x)\,dx = -\int_{-\infty}^{\infty} f(x)g^{(n)}(x)\,dx.$$

Then again apply Lemma 2.2 to the functions $f'$ and $g^{(n-2)}$ to conclude that $\int_{-\infty}^{\infty} f''(x)g^{(n-2)}(x)\,dx$ exists and

$$\int_{-\infty}^{\infty} f''(x)g^{(n-2)}(x)\,dx = -\int_{-\infty}^{\infty} f'(x)g^{(n-1)}(x)\,dx.$$

Repeat the argument until finally when we apply Lemma 2.2 to the functions $f^{(n-1)}$ and $g$ to conclude that $\int_{-\infty}^{\infty} f^{(n)}(x)g(x)\,dx$ exists and

$$\int_{-\infty}^{\infty} f^{(n)}(x)g(x)\,dx = -\int_{-\infty}^{\infty} f^{(n-1)}(x)g'(x)\,dx.$$

Putting the above equalities together, we immediately obtain Equation (2.6). □

For the next theorem, we recall the Hermite polynomial of degree $n$ defined by

$$H_n(x) = (-1)^n e^{x^2/2} D_x^n e^{-x^2/2}.$$

**Theorem 2.4.** *Let $\mu$ be the standard Gaussian measure on $\mathbb{R}$, namely, $d\mu(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\,dx$. Let $f$ be a $C^N$-function on $\mathbb{R}$ such that*

$$\int_{-\infty}^{\infty} f(x)H_n(x)\,d\mu(x) < \infty, \quad \forall\, 1 \le n \le N.$$

*Then we have*

$$\int_{-\infty}^{\infty} f^{(n)}(x)\,d\mu(x) = \int_{-\infty}^{\infty} f(x)H_n(x)\,d\mu(x), \quad \forall\, 1 \le n \le N. \tag{2.7}$$

*Proof.* For each $n = 1, 2, \ldots, N$, apply Lemma 2.3 to the functions $f(x)$ and $g(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$. Note that

$$g^{(n)} = (-1)^n H_n(x) \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

Hence the resulting equality from Equation (2.6) becomes Equation (2.7). □

Now we go back to an abstract Wiener space $(i, H, B)$. We first prove a special case of Theorem 2.6 (see below) for $K = \mathbb{R}$.

**Theorem 2.5.** *Let $p_t$ be the Gaussian measure on an abstract Wiener space $(i, H, B)$ with variance $t$. Let $f : B \to \mathbb{R}$ be a $C_H^1$-function such that $|D_H f(\cdot)| \in L^1(B, p_t)$. Then for any $h \in H$, we have*

$$\int_B \langle h, x\rangle f(x)\,dp_t(x) = t \int_B \langle D_H f(x), h\rangle\,dp_t(x). \tag{2.8}$$

*Proof.* For $h \neq 0$, put $e = h/|h|$. Let $P$ denote the projection $Px = \langle x, e\rangle e$ and let $Qx = x - Px$. Then $p_t$ can be decomposed as a product measure $p_t = \mu_t \times \mu_t'$. Here $\mu_t$ is the Gaussian measure on $\mathbb{R}$ with mean 0 and variance $t$ and $\mu_t'$ is the Gaussian measure on $QB$ with mean 0 and variance $t$. Then we have

$$\begin{aligned}\int_B \langle h, x\rangle f(x)\, dp_t(x) &= \int_{QB}\int_{-\infty}^{\infty} |h|\,\lambda\, f(\lambda e + v)\, d\mu_t(\lambda) d\mu_t'(v)\\ &= t\int_{QB}\int_{-\infty}^{\infty} |h|\, \frac{d}{d\lambda}\Big\{f(\lambda e + v)\Big\}\, d\mu_t(\lambda) d\mu_t'(v)\\ &= t\int_{QB}\int_{-\infty}^{\infty} \langle D_H f(\lambda e + v), h\rangle\, d\mu_t(\lambda) d\mu_t'(v)\\ &= t\int_B \langle D_H f(x), h\rangle\, dp_t(x),\end{aligned}$$

where the second equality follows from Lemma 2.2. □

Next we state the main theorem of this paper, which assumes much weaker conditions than those in Theorem 2.1.

**Theorem 2.6.** *Let $(i, H, B)$ be an abstract Wiener space and $f$ a $C_H^1$-function from $B$ into a real Hilbert space $K$ such that $\int_B \|D_H f(x)\|_{\mathcal{L}(H,K)}\, dp_t(x) < \infty$. Then for any $h \in H$, we have*

$$\int_B \langle h, x\rangle f(x)\, dp_t(x) = t\int_B [D_H f(x)](h)\, dp_t(x), \tag{2.9}$$

*where the integrals are $K$-valued Bochner integrals.*

*Proof.* Without loss of generality, we may assume that the space $K$ is separable by replacing $K$, if necessary, with the closure of $f(B)$. Under this assumption, the function $f$ can be taken to be either strongly measurable or weakly measurable and the integrals in Equation (2.9) exist as Bochner integrals.

For $\Phi \in K^*$, the composition $\Phi \circ f$ is a $C_H^1$-function from $B$ into $\mathbb{R}$ and

$$\langle D_H(\Phi \circ f)(x), h\rangle = \big(\Phi \circ [D_H f(x)]\big)(h), \quad h \in H. \tag{2.10}$$

Then we have

$$\begin{aligned}\int_B \big|D_H(\Phi \circ f)(x)\big|\, dp_t(x) &= \int_B \big|\Phi \circ [D_H f(x)]\big|\, dp_t(x)\\ &\le \int_B \|\Phi\|_{K^*} \|D_H f(x)\|_{\mathcal{L}(H,K)}\, dp_t(x)\\ &= \|\Phi\|_{K^*}\int_B \|D_H f(x)\|_{\mathcal{L}(H,K)}\, dp_t(x)\\ &< \infty \qquad \text{(by assumption).}\end{aligned}$$

Hence we can apply Theorem 2.5 and use Equation (2.10) to show that

$$\begin{aligned}\int_B \langle h, x\rangle \Phi(f(x))\, dp_t(x) &= t\int_B \langle D_H(\Phi \circ f)(x), h\rangle\, dp_t(x)\\ &= t\int_B \big(\Phi \circ [D_H f(x)]\big)(h)\, dp_t(x), \quad \forall\, \Phi \in K^*,\end{aligned}$$

or equivalently,

$$\Phi\Big\{\int_B \Big(\langle h,x\rangle f(x) - t\big[D_H f(x)\big](h)\Big)\, dp_t(x)\Big\} = 0, \quad \forall\, \Phi \in K^*.$$

Since $K^*$ separates points of $K$, we can conclude that

$$\int_B \Big(\langle h,x\rangle f(x) - t\big[D_H f(x)\big](h)\Big)\, dp_t(x) = 0,$$

which gives Equation (2.9). □

The next theorem follows from Theorem 2.6. We will use $\|T\|_{\mathrm{HS}}$ to denote the Hilbert–Schmidt norm of an operator $T$.

**Theorem 2.7.** *Let $(i, H, B)$ be an abstract Wiener space and $A$ a bounded operator from $B$ into a real Hilbert space $K$. Suppose $f$ is a $C_H^1$-function from $B$ into $K$ satisfying the following conditions:*

(a) *$D_H f(x)$ is a Hilbert–Schmidt operator from $H$ into $K$ and*

$$\int_B \|D_H f(x)\|_{\mathrm{HS}}\, dp_t(x) < \infty;$$

(b) $\int_B \|f(x)\|_K |Ax|_K\, dp_t(x) < \infty$.

*Then we have*

$$\int_B \langle f(x), Ax\rangle_K\, dp_t(x) = t\int_B \mathrm{trace}_K\left[D_H f(x)(A|_H)^*\right] dp_t(x),$$

*where $\langle\cdot,\cdot\rangle_K$ denotes the inner product of $K$,* $\mathrm{trace}_K$ *denotes the trace of a trace class operator of $K$, and $A|_H$ denotes the restriction of the operator $A$ to $H$.*

*Remark* 2.8. We point out that if $A$ is a bounded operator from $B$ to $K$, then its restriction $A|_H$ to $H$ is a Hilbert–Schmidt operator from $H$ into $K$. This is a fact from Corollary 4.4 (page 85 [12]), which follows from Theorem 4.6 (page 83 [12]) due to V. Goodman.

Moreover, we have the next theorem which shows that Theorem 2 in [11] is valid under a much weaker condition.

**Theorem 2.9.** *Let $u$ and $v$ be $C_H^1$-functions from $B$ into real Banach spaces $F$ and $G$, respectively. Assume that the following condition holds:*

$$\int_B \Big\{\|D_H u(x)\|_{\mathcal{L}(H,F)}\, |v(x)|_G + |u(x)|_F\, \|D_H v(x)\|_{\mathcal{L}(H,G)}\Big\}\, dp_t(x) < \infty,$$

*where $|\cdot|_G$ and $|\cdot|_F$ denote the norms on $G$ and $F$, respectively. Then for any bounded bilinear map $\Phi$ from $F\times G$ into a real Hilbert space $K$, we have*

$$\begin{aligned}&\int_B \Phi\big(u(x), [D_H v(x)](h)\big)\, dp_t(x)\\ &\quad = \int_B \Big\{\frac{1}{t}\langle h,x\rangle \Phi\big(u(x), v(x)\big) - \Phi\big([D_H u(x)](h), v(x)\big)\Big\}\, dp_t(x).\end{aligned}$$

*Proof.* Just apply Theorem 2.5 to the function $f(x) = \Phi(u(x), v(x))$. □

We can use Theorem 2.5 to generalize another formulation of integration by parts formula given in Theorem 3.7 [13].

**Theorem 2.10.** *Let $p_t$ be the Gaussian measure on an abstract Wiener space $(i, H, B)$ with mean 0 ana variance $t$. Suppose $f : B \to B^*$ is an $H$-differentiable function satisfying the following conditions:*

(a) $\|D_H f(\cdot)\|_{\mathrm{tr}} \in L^1(B, p_t)$, *($\|\cdot\|_{\mathrm{tr}}$: the trace class norm);*
(b) *There exists a constant $\alpha > 1$ such that $\int_B |f(x)|^{\alpha}_{B^*}\, dp_t(x) < \infty$, ( $|\cdot|_{B^*}$: the norm on $B^*$).*

*Then the following equality holds:*

$$\int_B \big(f(x), x\big)\, dp_t(x) = t \int_B \mathrm{trace}_{_H} \big[D_H f(x)\big]\, dp_t(x), \tag{2.11}$$

*where $(\cdot,\cdot)$ denotes the bilinear pairing of $B^*$ and $B$ and $\mathrm{trace}_{_H}$ denotes the trace of a trace class operator on $H$.*

*Proof.* We need a fact from the book [12] (page 66, Corollary 4.2). Let $(i, H, B)$ be an abstract Wiener space. Then there exist another abstract Wiener space $(i_0, H, B_0)$ and an increasing sequence $\{P_n\}_{n=1}^{\infty}$ of finite dimensional orthogonal projections converging strongly to the identity in H such that:

(1) The $B_0$-norm is stronger than the $B$-norm, (hence $B_0 \subset B$).
(2) Each $P_n$ extends by continuity to a projection $\widetilde{P}_n$ of $B_0$.
(3) $\widetilde{P}_n$ converges strongly to the identity in $B_0$ with respect to $B_0$-norm.

Moreover, let $p_t$ and $\widetilde{p}_t$ denote the Gaussian measures on $B$ and $B_0$, respectively, with mean 0 and variance $t$. Then the equality

$$\int_B g(x)\, dp_t(x) = \int_{B_0} \widetilde{g}(x)\, d\widetilde{p}_t(x) \tag{2.12}$$

holds for all nonnegative measurable or $p_t$-integrable functions $g$ on $B$. Here $\widetilde{g}$ denotes the the restriction of $g$ to $B_0$.

Now let $f$ be a function given as in the theorem. Apply Equation (2.12) to the function $g(x) = (f(x), x)$ to get

$$\int_B (f(x), x)\, dp_t(x) = \int_{B_0} (\widetilde{f}(x), x)\, d\widetilde{p}_t(x). \tag{2.13}$$

Since $\widetilde{P}_n$ is the extension of $P_n$ by continuity, we see that $\widetilde{P}_n(B_0)$ must be the closure of $P_n(H)$ with respect to the $B_0$-norm. On the other hand, note that $P_n(H)$ is finite dimensional. Hence we can conclude that $\widetilde{P}_n(B_0) = P_n(H)$.

Let $\{e_j\}_{j=1}^{\infty}$ be an orthonormal basis for $H$. Then

$$\begin{aligned}\int_{B_0} (\widetilde{f}(x), \tilde{P}_n x)\, d\tilde{p}_t(x) &= \sum_{j=1}^{\infty} \int_{B_0} \langle \widetilde{f}(x), e_j\rangle \langle \widetilde{P}_n x, e_j\rangle\, d\widetilde{p}_t(x) \\ &= \sum_{j=1}^{\infty} \int_{B_0} \langle \widetilde{f}(x), e_j\rangle \langle x, P_n e_j\rangle\, d\widetilde{p}_t(x)\end{aligned}$$

Apply Theorem 2.6 to show that

$$\begin{aligned}\int_{B_0} (\widetilde{f}(x), \tilde{P}_n x)\, d\tilde{p}_t(x) &= t\sum_{j=1}^{\infty} \int_{B_0} \langle D_H \widetilde{f}(x)(P_n e_j), e_j\rangle\, d\tilde{p}_t(x) \\ &= t\int_{B_0} \mathrm{trace}_H \big[D_H \widetilde{f}(x)\circ P_n\big]\, d\tilde{p}_t(x). \qquad (2.14)\end{aligned}$$

Note that for all $x \in B_0$,

$$\begin{aligned}\big|\mathrm{trace}_H[D_H\widetilde{f}(x)\circ P_n]\big| &\le \|D_H f(x)\|_{\mathrm{tr}}, \\ \big|(\widetilde{f}(x), \widetilde{P}_n x)\big| \le |\widetilde{f}(x)|_{B^*}\|\widetilde{P}_n x\|_0 &\le |\widetilde{f}(x)|_{B^*}\|x\|_0.\end{aligned}$$

By the Fernique theorem [7] and the condition (b), we have

$$\begin{aligned}\int_{B_0} |\widetilde{f}(x)|_{B^*}\|x\|_0\, d\,\widetilde{p}_t(x) &\le \left\{\int_{B_0} |\widetilde{f}(x)|_{B^*}^{\alpha}\, d\,\widetilde{p}_t(x)\right\}^{1/\alpha} \left\{\int_{B_0} \|x\|_0^{\alpha'}\, d\,\widetilde{p}_t(x)\right\}^{1/\alpha'} \\ &< \infty, \qquad (2.15)\end{aligned}$$

where $1/\alpha + 1/\alpha' = 1$.

With condition (a) and Equation (2.15), we can apply the Lebesgue dominated convergence theorem and the same arguments as those in [13] to let $n \to \infty$ in Equation (2.14) to obtain

$$\begin{aligned}\int_{B_0} (\widetilde{f}(x), x)\, d\tilde{p}_t(x) &= t\int_{B_0} \mathrm{trace}_H \big[D_H\widetilde{f}(x)\big]\, d\tilde{p}_t(x) \\ &= t\int_{B} \mathrm{trace}_H \big[D_H f(x)\big]\, dp_t(x). \qquad (2.16)\end{aligned}$$

Thus Equation (2.11) follows from Equations (2.13) and (2.16). □

## 3. Infinite Dimensional Stein Lemma

We can apply Theorem 2.5 to obtain a generalization of the Stein lemma [1, 3, 14, 15] to infinite dimensional spaces. Let $B$ be a real separable Banach space. A probability measure $\mu$ on $B$ is called a *Gaussian measure with mean* 0 if for each $\xi$ in the dual space $B^*$ of $B$, the random variable $(\xi, \cdot)$ on $(B, \mu)$ is Gaussian with mean 0.

**Theorem 3.1.** *Let $X$ be a random variable with values in a real separable Banach space $B$. The distribution $\mu$ of $X$ on $B$ is a Gaussian measure with mean* 0 *if and only if there exists a real separable Hilbert space $H$ such that $(i, H, B)$ is an abstract Wiener space and the equality*

$$\int_B \Big\{\langle h, x\rangle f(x) - \langle D_H f(x), h\rangle\Big\}\, d\mu(x) = 0, \quad \forall\, h \in H, \qquad (3.1)$$

*holds for all $C_H^1$-functions $f$ such that $|D_H f(\cdot)| \in L^1(B, \mu)$.*

*Proof.* We first prove the necessity part. Suppose the distribution $\mu$ of $X$ is a Gaussian measure on $B$ with mean 0. By the Kuelbs Theorem [10], there exists a real separable Hilbert space $H$ such that $(i, H, B)$ is an abstract Wiener space and $\mu = p_1$, the Gaussian measure on $B$ with mean 0 and variance 1. Then we can apply Theorem 2.5 with $t = 1$ to obtain Equation (3.1) from Equation (2.8).

Next we prove the sufficiency part. Assume that $(i, H, B)$ is an abstract Wiener space and the distribution $\mu$ of $X$ satisfies Equation (3.1). Let $\xi \in B^*$ be fixed. The characteristic function of the random variable $(\xi, X)$ is given by

$$\theta(\lambda) = Ee^{i\lambda(\xi,X)} = \int_B e^{i\lambda(\xi,x)}\, d\mu(x), \quad \lambda \in \mathbb{R}.$$

The assumption on $\mu$ allows us to differentiate $\theta$ to get

$$\theta'(\lambda) = i \int_B (\xi, x) e^{i\lambda(\xi,x)}\, d\mu(x). \tag{3.2}$$

Use Equation (3.1) for $h = \xi$ and $f(x) = \cos(\lambda(\xi, x))$ to get

$$\int_B (\xi, x) \cos(\lambda(\xi, x))\, d\mu(x) = -\int_B \lambda|\xi|^2 \sin(\lambda(\xi, x))\, d\mu(x). \tag{3.3}$$

Similarly, with $h = \xi$ and $f(x) = \sin(\lambda(\xi, x))$ in Equation (3.1), we have

$$\int_B (\xi, x) \sin(\lambda(\xi, x))\, d\mu(x) = \int_B \lambda|\xi|^2 \cos(\lambda(\xi, x))\, d\mu(x). \tag{3.4}$$

It follows from Equations (3.2), (3.3), and (3.4) that

$$\theta'(\lambda) = -\lambda|\xi|^2\theta(\lambda),$$

which together with the condition $\theta(0) = 1$ yields the equality

$$\theta(\lambda) = e^{-\frac{1}{2}\lambda^2|\xi|^2}.$$

Hence we have proved that

$$\int_B e^{i\lambda(\xi,x)}\, d\mu(x) = e^{-\frac{1}{2}\lambda^2|\xi|^2}, \quad \forall\, \lambda \in \mathbb{R},\ \xi \in B^*.$$

This equality implies that $\mu$ is a Gaussian measure on $B$ with mean 0. Thus the theorem is proved. □

More generally, we have the next theorem, which can be easily proved by using Theorem 2.7 and similar arguments as those in the proof of Theorem 3.1.

**Theorem 3.2.** *Let $X$ be a random variable taking values in a real separable Banach space $B$. Then the distribution $\mu$ of $X$ on $B$ is a Gaussian measure with mean 0 if and only if there exists a real separable Hilbert space $H$ such that $(i, H, B)$ is an abstract Wiener space and the equality*

$$\int_B \Big\{ \langle f(x), Ax\rangle_K - \mathrm{trace}_K \big[D_H f(x)(A|_H)^*\big] \Big\}\, d\mu(x) = 0$$

*holds for all bounded operators $A$ from $B$ into a real separable Hilbert space $K$ and for all $C_H^1$-functions $f$ from $B$ into $K$ satisfying the following conditions:*

(a) *$D_H f(x)$ is a Hilbert–Schmidt operator from $H$ into $K$ and*

$$\int_B \|D_H f(x)\|_{\mathrm{HS}}\, d\mu(x) < \infty;$$

(b) $\int_B \|f(x)\|_K |Ax|_K\, d\mu(x) < \infty.$

On the other hand, if we apply Theorem 2.10 instead of Theorem 2.5 or 2.7, then the infinite dimensional Stein lemma can be reformulated as given by the next theorem.

**Theorem 3.3.** *Let $X$ be a random variable with values in a real separable Banach space $B$. The distribution $\mu$ of $X$ on $B$ is a Gaussian measure with mean 0 if and only if there exists a real separable Hilbert space $H$ such that $(i, H, B)$ is an abstract Wiener space and the equality*

$$\int_B \Big\{ \big(f(x), x\big) - \mathrm{trace}_H \big[D_H f(x)\big] \Big\} \, d\mu(x) = 0$$

*holds for all $C_H^1$-functions $f : B \to B^*$ satisfying the following conditions:*

(a) $\|D_H f(\cdot)\|_{\mathrm{tr}} \in L^1(B, \mu)$*;*

(b) *There exists a constant $\alpha > 1$ such that $\int_B |f(x)|_{B^*}^{\alpha} \, d\mu(x) < \infty$.*

## 4. Evaluation of Function Integrals on Wiener Space

Let $(i, H, B)$ be an abstract Wiener space and let $\mu$ denote the Gaussian measure on $B$ with mean 0 and variance 1. The following formula was derived in [4]:

$$\begin{aligned} &\int_B \exp\Big[\frac{z}{2}(Ax, x) + (h, x)\Big] \, d\mu(x) \\ &\quad = \big\{ \det(I - zA|_H) \big\}^{-1/2} \exp\Big[\frac{1}{2}\Big\langle (I - zA\big|_H)^{-1} h, h \Big\rangle\Big], \end{aligned} \tag{4.1}$$

where $z$ is a complex number, $\Re(z) < 1$, $h \in B^*$, $A \in \mathcal{L}(B, B^*)$, and $A\big|_H$ is a self-adjoint operator of $H$. Note that $A\big|_H$ is a trace class operator of $H$ in view of a theorem (page 83 [12]) due to V. Goodman. Hence $\det(I - zA\big|_H)$ exists.

We now show that the integration by parts formula can be used to derive the formula in Equation (4.1). Define a function $U$ on $H$ by

$$U(h) = \int_B \exp\Big[\frac{z}{2}(Ax, x) + (h, x)\Big] \, d\mu(x), \quad h \in H. \tag{4.2}$$

The Fréchet derivative $U'(h)$ of $U$ at $h$ is easily checked to be given by

$$\langle U'(h), k \rangle = \int_B (k, x) \exp\Big[\frac{z}{2}(Ax, x) + (h, x)\Big] \, d\mu(x), \quad k \in H. \tag{4.3}$$

Now let $k \in B^*$. We can check that the following function

$$f(x) = k \exp\Big[\frac{z}{2}(Ax, x) + (h, x)\Big]$$

satisfies the conditions in Theorem 2.10. Hence we can apply the integration by parts formula in Equation (2.11) to the function $f$. From the resulting equality and Equation (4.3), we get

$$\langle U'(h), k \rangle = \int_B \big(f(x), x\big) \, d\mu(x) = \int_B \mathrm{trace}_H \big[D_H f(x)\big] \, d\mu(x). \tag{4.4}$$

Direct computation shows that the $H$-derivative of $f$ is given by

$$D_H f(x)(\cdot) = k\Big[z(Ax, \cdot) + \langle h, \cdot \rangle\Big] \exp\Big[\frac{z}{2}(Ax, x) + (h, x)\Big],$$

which yields the following trace:

$$\operatorname{trace}_H D_H f(x) = \Big[z(Ak, x) + \langle h, k\rangle\Big] \exp\left[\frac{z}{2}(Ax, x) + (h, x)\right]. \tag{4.5}$$

It follows from Equations (4.4) and (4.5) that

$$\begin{aligned}\langle U'(h), k\rangle &= \int_B \Big[z(Ak, x) + \langle h, k\rangle\Big] \exp\left[\frac{z}{2}(Ax, x) + (h, x)\right] d\mu(x) \\ &= z\big\langle U'(h), Ak\big\rangle + U(h)\langle h, k\rangle.\end{aligned}$$

Thus the function $U$ satisfies the differential equation

$$\big\langle U'(h), (I - zA)k\big\rangle = U(h)\langle h, k\rangle. \tag{4.6}$$

By replacing $A$ with $\widetilde{A} = [A + (A\big|_H)^*]/2$, if necessary, we may assume that the operator $A|_H$ is self-adjoint. Then we can replace $k$ with $(I - zA)^{-1}k$ in Equation (4.6) and use the assumption that $A|_H$ is self-adjoint to get

$$\big\langle U'(h), k\big\rangle = U(h)\big\langle (I - zA\big|_H)^{-1}h, k\big\rangle, \quad k \in B^*.$$

Therefore, we have the differential equation

$$U'(h) = U(h)(I - zA\big|_H)^{-1}h, \quad h \in H.$$

The solution of this differential equation is given by

$$U(h) = U(0) \exp\left[\frac{1}{2}\big\langle (I - zA\big|_H)^{-1}h, h\big\rangle\right].$$

But from Equation (4.2) we see that

$$U(0) = \int_B \exp\left[\frac{z}{2}(Ax, x)\right] d\mu(x) = \det(I - zA\big|_H)^{-1/2}.$$

Therefore,

$$U(h) = \det(I - zA\big|_H)^{-1/2} \exp\left[\frac{1}{2}\big\langle (I - zA\big|_H)^{-1}h, h\big\rangle\right]. \tag{4.7}$$

In view of Equations (4.2) and (4.7) we have derived the formula in Equation (4.1).

**Example 4.1.** Let $\mathcal{C} = C[0,1]$ be the classical Wiener space with the Wiener measure $W$. Let $z \in \mathbb{C}$ and $\Re(z) < \pi^2$. It is shown in [4] that

$$\int_{\mathcal{C}} \exp\left[\frac{z}{2}\int_0^1 \left(x(t) - \int_0^1 x(s)ds\right)^2 dt\right] dW(x) = \sqrt{\sqrt{z}\csc\sqrt{z}}. \tag{4.8}$$

We give a different method to derive this formula by using Equation (4.1). Let $\mathcal{C}'$ denote the Cameron–Matin space, namely, the space consisting of all functions $f \in \mathcal{C}$ which are absolutely continuous with $f' \in L^2[0,1]$. Then $(i, \mathcal{C}', \mathcal{C})$ is an abstract Wiener space with standard Gaussian measure $W$. It is well known that $\mathcal{C}^*$ can be identified as the space

$$\mathcal{C}^* = \Big\{h \in \mathcal{C}' : h' \text{ is right continuous and } h'(1) = 0\Big\}$$

Define an operator $A : \mathcal{C} \to \mathcal{C}^*$ by

$$(Ax)(t) = -\int_0^t \left\{ \int_0^s x(v)dv - s\int_0^1 x(v)dv \right\} ds, \quad x \in \mathcal{C}.$$

Then we have

$$\langle Ax, x\rangle = \int_0^1 \left( x(t) - \int_0^1 x(s)ds \right)^2 dt.$$

Note that $(Ax)'(\cdot)$ is a continuous function and $(Ax)'(1) = 0$. Hence $Ax \in \mathcal{C}^*$ and $A \in \mathcal{L}(\mathcal{C}, \mathcal{C}^*)$. It is easy to see that $A\big|_{\mathcal{C}'}$ is a self-adjoint operator of $\mathcal{C}'$. Moreover, we can use the same arguments as those on page 50 of the book [12] to compute the eigenvalues of $A\big|_{\mathcal{C}'}$ as given by

$$\frac{1}{n^2\pi^2}, \quad n = 1, 2, 3, \ldots$$

Thus the determinant of $I - zA\big|_{\mathcal{C}'}$ is given by

$$\det(I - zA\big|_{\mathcal{C}'}) = \prod_{n=1}^{\infty} \Big(1 - \frac{z}{n^2\pi^2}\Big) = \frac{1}{\sqrt{z}} \sin\sqrt{z}, \quad \Re(z) < \pi^2.$$

Putting this value into Equation (4.1) with $h = 0$, we immediately obtain the formula in Equation (4.8).

**Example 4.2.** For $z \in \mathbb{C}$ with $\Re(z) < \pi^2/4$ and $h \in \mathcal{C}'$, we have

$$\begin{aligned} &\int_{\mathcal{C}} \exp\left\{ i\int_0^1 \big(h(1) - h(t)\big)\, dy(t) + \frac{z}{2}\int_0^1 y(t)^2\, dt \right\} dW(y) \\ &= \sqrt{\sec\sqrt{z}} \exp\left\{ -\frac{1}{2}\int_0^1\int_0^1 K(s,t;\sqrt{z})h'(t)h'(s)\, dsdt \right\}, \end{aligned} \tag{4.9}$$

where the kernel function $K$ is defined by

$$K(s,t;z) = \frac{\sec z}{2z}\Big\{ \sin\big(z(1 - |s-t|)\big) - \sin\big(z(1 - |s+t|)\big) \Big\}. \tag{4.10}$$

To derive the formula in Equation (4.9), put

$$\widetilde{h}(t) = \int_0^t (h(1) - h(s))\, ds,$$

$$(Ay)(t) = \int_0^1 (t \wedge s) y(s)\, ds.$$

Then $\widetilde{h} \in \mathcal{C}^*$, $A \in \mathcal{L}(\mathcal{C}, \mathcal{C}^*)$, and we have

$$\begin{aligned} &\int_{\mathcal{C}} \exp\left\{ i\int_0^1 (h(1) - h(t))\, dy(t) + \frac{z}{2}\int_0^1 y(t)^2\, dt \right\} dW(y) \\ &= \int_{\mathcal{C}} \exp\Big\{ i(\widetilde{h}, y) + \frac{z}{2}(Ay, y) \Big\} dW(y). \end{aligned}$$

Then apply the formula in Equation (4.1) to get

$$\int_{\mathcal{C}} \exp\left\{ i\int_0^1 (h(1)-h(t))\,dy(t) + \frac{z}{2}\int_0^1 y(t)^2\,dt \right\} dW(y)$$
$$= \left\{ \det(I - zA\big|_{\mathcal{C}'}) \right\}^{-1/2} \exp\left[ -\frac{1}{2}\left\langle (I - z\,A\big|_{\mathcal{C}'})^{-1}\widetilde{h}, \widetilde{h} \right\rangle \right]. \tag{4.11}$$

From page 50 of the book [12], the eigenvalues of $A\big|_{\mathcal{C}'}$ are given by

$$\frac{4}{[(2n-1)\pi)]^2}, \quad n = 1, 2, 3, \ldots$$

Therefore, we have

$$\det(I - zA\big|_{\mathcal{C}'}) = \prod_{n=1}^{\infty} \Big(1 - \frac{4z}{[(2n-1)\pi)]^2}\Big) = \cos\sqrt{z}, \quad \Re(z) < \pi^2/4. \tag{4.12}$$

Next we need to evaluate $\langle (I - z\,A\big|_{\mathcal{C}'})^{-1}\widetilde{h}, \widetilde{h}\rangle$ in Equation (4.11). Put

$$u(t) = \big( (I - z\,A\big|_{\mathcal{C}'})^{-1}\widetilde{h} \big)(t).$$

Then we have $\big((I - z\,A\big|_{\mathcal{C}'})u\big)(t) = \widetilde{h}(t)$, namely,

$$u(t) - z\int_0^1 (t\wedge s)u(s)\,ds = \int_0^t (h(1) - h(s))\,ds,$$

which can be differentiated twice to produce the following differential equation with a boundary condition:

$$u''(t) + zu(t) = -h'(t), \quad u(0) = u'(1) = 0.$$

By using the variation of parameters method we can derive the solution

$$u(t) = \frac{1}{\sqrt{z}}\left\{ \sec\sqrt{z}\,\sin(\sqrt{z}\,t)\int_0^1 \cos(\sqrt{z}(1-s))h'(s)\,ds - \int_0^t \sin(\sqrt{z}(t-s))h'(s)\,ds \right\}.$$

Therefore,

$$\left\langle (I - z\,A\big|_{\mathcal{C}'})^{-1}\widetilde{h}, \widetilde{h} \right\rangle = \int_0^1 u'(t)\widetilde{h}'(t)\,dt$$
$$= \int_0^1\int_0^1 K(s,t;\sqrt{z})h'(s)h'(t)\,dsdt, \tag{4.13}$$

where the kernel function $K$ is given by Equation (4.10). Upon putting Equations (4.12) and (4.13) into the right-hand side of Equation (4.11), we obtain the formula in Equation (4.9).

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Hui-Hsiung Kuo: Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803-4918, USA

*E-mail address*: `kuo@math.lsu.edu`

Yuh-Jia Lee: Department of Applied Mathematics, National University of Kaohsiung, Kaohsiung, Taiwan 811

*E-mail address*: `yjlee@nuk.edu.tw`