

STOCHASTIC JACOBIANS IN AFFINE TERM-STRUCTURE MODELS: A LOCAL PROPERTY

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ABSTRACT. Affine diffusions are popular models for risk-factors in mathematical finance and, in particular, form the basis of several term-structure models. We consider the stochastic Jacobian of the stochastic flow associated with an affine diffusion and prove that the conditional expectation under the forward measure of the stochastic Jacobian is deterministic in a neighborhood of the maturity time of a zero-coupon bond. This local result corrects a lemma of Elliott and van der Hoek [7] to the extent possible using the method of proof originally proposed. The local result corresponds to the fact that Riccati equations, which must normally be solved numerically to implement financial models based on affine processes, have solutions in a neighborhood of the boundary condition but not necessarily over a fixed interval.

1. Introduction

Affine diffusions constitute a popular class of multi-factor models in mathematical finance which include Gaussian processes and several popular interest rate and commodity price models as special cases. If the risk-free interest rate is an affine function of such factors it has been shown that the zero-coupon bond price is an exponential affine function of the affine diffusion with coefficients solving Riccati-type ordinary differential equations as in [6]. Similar results hold for the forward price and futures price when the risky asset price is an exponential affine function of the factors as in [1]. In the Gaussian case it is possible to derive the bond price, futures price, and forward price formulae by considering the stochastic flow associated with the affine diffusion and using the fact that the derivative of the flow with respect to the initial condition, the Jacobian of the stochastic flow, is deterministic as in [7, 10].

In the case of a general (non-Gaussian) affine diffusion the Jacobian of the stochastic flow is not deterministic. As such the techniques applied to the Gaussian case to characterize the bond price, futures price, and forward price cannot be applied without modification. By considering the dynamics of the stochastic Jacobian under the forward measure Elliott and van der Hoek [7] showed that the

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bond price satisfies an equation resembling a system of linear ordinary differential equations which, provided the conditional expectation of the stochastic Jacobian is deterministic, can be solved to derive the bond price. An approximation lemma was presented by Elliott and van der Hoek [7] to demonstrate that the conditional expectation under the forward measure of the stochastic Jacobian is deterministic and solves a nonlinear integral equation with two parameters. Unfortunately, there is a mistake in the proof of this lemma presented in [7] for the general multi-dimensional case. In the one dimensional case, or in the multi-dimensional case with decoupled components, corresponding to the Cox, Ingersoll, and Ross [2] interest rate model the approximation lemma is not required as the flow (semi-group) property can be employed to prove that the conditional expectation of the Jacobian of the stochastic flow is deterministic.

In order to address the general multi-dimensional case a method, based on forward-backward stochastic differential equations, was introduced in [9, 11] that shows as a corollary to the main results that the conditional expectation of the Jacobian of the stochastic flow is deterministic. However, a sufficient condition for the main result of [11] is that the Riccati-type differential equations have unique solution over a fixed time interval corresponding to the time to maturity of the bond. This sufficient condition weakens the results of [7] which claimed that Riccati equations were not needed as a consequence of their methods.

Riccati equations, as nonlinear ordinary differential equations, are generally not *a priori* solvable over a fixed interval but have a unique solution in a neighborhood of the boundary (initial or terminal) condition. As such, combined with our previous results in [11], it is not surprising that the properties of the conditional expectation of the stochastic Jacobian are related closely to the solvability of the Riccati differential equations.

In this paper we prove a local version of the approximation lemma of Elliott and van der Hoek [7]. We discuss the affine model in relation to interest rate models in Section 2. Stochastic flows, Jacobians, and their relationship to deriving a formula for zero-coupon bond prices are discussed in Section 3. The main result and proof is presented in Section 4. Section 5 concludes.

2. Affine Factor Models

As is done in much of the mathematical finance literature, a notable exception being [4], we shall begin our analysis on the risk neutral probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, Q)$ for $0 \leq t \leq T^*$ where T^* is the investment horizon, \mathcal{F}_t is a right-continuous and complete filtration, and Q is the risk-neutral (martingale) measure. A continuous stochastic process X_t taking values in R^n models risk factors in the economy. In this paper we consider an affine diffusion

$$dX_t = (AX_t + \tilde{B})dt + S \text{diag}\left(\sqrt{\alpha_i + \beta_i X_t}\right)dB_t, \quad (2.1)$$

where B is an n -dimensional (\mathcal{F}_t, Q) -Brownian motion, A is an $(n \times n)$ -matrix of scalars, \tilde{B} is an $(n \times 1)$ -vector of scalars, for each $i \in \{1, \dots, n\}$ the α_i are scalars, for each $i \in \{1, \dots, n\}$ the $\beta_i = (\beta_{i1}, \dots, \beta_{in})$ are $(1 \times n)$ -vectors taking values in R^n , and S is a non-singular $(n \times n)$ -matrix. Duffie and Kan [6] consider solutions

to equation (2.1) taking values in the open set $D \subset R^n$ defined by

$$D := \{x \in R^n : \alpha_i + \beta_i x > 0, \quad i \in \{1, \dots, n\}\}.$$

Further, Duffie and Kan [6] show that if, for all i , the conditions:

(A-I) for all x such that $\alpha_i + \beta_i x = 0$, $\beta_i(Ax + \tilde{B}) > \beta_i S S' \beta_i' / 2$;

(A-II) for all j , if $(\beta_i S)_j \neq 0$, then $\alpha_i + \beta_i x = \alpha_j + \beta_j x$

are satisfied then there exists a unique strong solution X_t to the SDE (2.1) that takes values in D . Further, for this solution, and for all i , $\alpha_i + \beta_i X_t$ is strictly positive for all t almost surely.

Assumption 2.1. Throughout we shall assume that conditions (A-I) and (A-II) hold.

Adopting the notation of [1] we may write

$$S \text{diag}(\alpha_i + \beta_i x) S' = k_0 + \sum_{i=1}^n k_i x^{(i)} \tag{2.2}$$

for symmetric $(n \times n)$ matrices k_i .

We shall follow the methodology of Duffie and Kan [6] and assume that the riskless interest rate is a function of an R^n -valued, $\{\mathcal{F}_t\}$ -adapted state process X_t . That is, $r_t = r(X_t)$, for some function $r : R^n \rightarrow R$. As remarked by Duffie and Kan [6], the set D is open and convex since it is the intersection of open half-spaces. Therefore, the separating hyperplane theorem can be applied to prove the existence of a strictly positive non-constant interest rate process $r_t = r(X_t)$ which is an affine transformation of X_t . That is, we have:

Assumption 2.2. The short rate process is given by $r_t = r(X_t)$ where, for $x \in D$,

$$r(x) = R'x + k > 0.$$

Here R is an $(n \times 1)$ -column vector and k is a scalar.

For example, as in [6], we may set $r(x) = \sum_{i=1}^n \gamma_i (\alpha_i + \beta_i x)$ for scalars $\gamma_i \geq 0$ not all equal zero.

The price of the zero-coupon bond is given by

$$P(t, T) = E_Q[\exp(-\int_t^T r(X_u) du) | \mathcal{F}_t] \tag{2.3}$$

at time t for maturity $T \leq T^*$. It was shown by Duffie and Kan [6] (see also [11, 12]) that the zero-coupon bond price is an exponential affine function of X_t :

$$P(t, T) = e^{U(t)X_t + p(t)}$$

provided $U(t) = [U_1(t), \dots, U_n(t)]$ solves the system of (coupled) Riccati-type ordinary differential equations

$$\frac{dU_j(u)}{du} + \sum_{i=1}^n U_i(u) A_{ij} + \frac{1}{2} U(u) k_j [U(u)]' - R' = 0, \quad j = 1, \dots, n, \tag{2.4}$$

for all $u \in [0, T]$ with terminal condition $U(T) = 0$ and $p(t)$ given by

$$p(t) = - \int_t^T \left(k - \frac{1}{2}U(u)k_0 [U(u)]' - U(u)\tilde{B} \right) du. \tag{2.5}$$

A motivation of the stochastic flows method of [7] appears to be avoiding the solution of the Riccati equations (2.4). We next discuss the stochastic flows method and prove our main result.

3. Stochastic Flows and Jacobians

We shall consider, as in [7], the stochastic flow associated with the factors process (2.1). However, our notation for the flow shall differ from that used by [7]. For $0 \leq t \leq s < T$ write $X_s^{t,x}$ for the flow associated with the solution of equation (2.1) such that $X_t^{t,x} = x$, so that

$$X_s^{t,x} = x + \int_t^s (AX_v^{t,x} + \tilde{B})dv + \int_t^s Sdiag(dB_v)vec(\sqrt{\alpha_i + \beta_i X_v^{t,x}}), \tag{3.1}$$

where

$$vec(\sqrt{\alpha_i + \beta_i X_v^{t,x}}) = (\sqrt{\alpha_1 + \beta_1 X_v^{t,x}}, \dots, \sqrt{\alpha_n + \beta_n X_v^{t,x}})'$$

For $x \in D$ let $\zeta(x, \omega)$ be the explosion time of the SDE (3.1) as in [13, pp. 247-248]. As pointed out by Grasselli and Tebaldi [8] the admissibility conditions of Dai and Singleton [3] imply, by [5, Theorem 2.7], the existence of a solution to the SDE (3.1) for all $s \geq t$. Further, since the coefficient of the stochastic integral in (3.1) is locally Lipschitz with respect to $X_v^{t,x}$ we have $\zeta(x, \omega) = \infty$, for all $x \in D$. We also have (see [13, Theorem V.39]) that for $x \in D$, the map $x \rightarrow X_u^{t,x}$ is almost surely differentiable and the Jacobian matrix of partial derivatives with respect to x satisfies the equation

$$\begin{aligned} (\partial_x X_u^{t,x}) &= I + \int_t^u A(\partial_x X_v^{t,x}) dv \\ &+ \frac{1}{2} \int_t^u Sdiag(dB_v)diag((\alpha_i + \beta_i X_v^{t,x})^{-1/2})C(\partial_x X_v^{t,x}), \end{aligned} \tag{3.2}$$

where C is the $(n \times n)$ -matrix whose rows are the vectors β_1, \dots, β_n .

The forward measure is often a useful tool in mathematical finance for the simplification of various pricing problems. Elliott and van der Hoek [7] use the forward measure in their characterization of the zero-coupon bond price. We recall the definition of the forward measure:

Definition 3.1. The *forward measure*, P^T , is defined on \mathcal{F}_T by

$$P^T(A) := E_Q[\Lambda_T 1_A] \tag{3.3}$$

where

$$\Lambda_T = \frac{dP^T}{dQ} \Big|_{\mathcal{F}_T} := \{P(0, T, X_0)\}^{-1} \exp\left(-\int_0^T r(X_u^{0, X_0})du\right). \tag{3.4}$$

One of the main results which allows for the characterization of bond prices as an exponential affine function of the factors process is the following:

Theorem 3.2 (Elliott and van der Hoek [7]). *If $E_T[\cdot]$ denotes expectation under P^T , then*

$$\partial_x P(t, T, X_t) = P(t, T, X_t) E_T[Y_t | \mathcal{F}_t], \tag{3.5}$$

where

$$Y_t = L(t, T, X_t) \tag{3.6}$$

and

$$L(t, T, x) := - \int_t^T (\partial_x X_u^{t,x})' R du. \tag{3.7}$$

Provided the conditional expectation in Theorem 3.2 is deterministic the linear ordinary differential equation (3.5) can be solved to obtain the bond price. Elliott and van der Hoek [7] state that, under the forward measure, the conditional expectation with respect to \mathcal{F}_t of the Jacobian of the stochastic flow evaluated at $x = X_t$:

$$\hat{D}_{ts} := E_T[(\partial_x X_s^{t,x})|_{x=X_t} | \mathcal{F}_t], \quad \text{for } 0 \leq t \leq s \leq T, \tag{3.8}$$

is deterministic. This would imply, by equations (3.7) and (3.6), that the conditional expectation in equation (3.5) is deterministic. A first step toward characterizing either expectation as deterministic is to write the dynamics of processes with respect to the forward measure. In order to do this Elliott and van der Hoek [7] employ Girsanov's Theorem to construct a Brownian motion with respect to the forward measure.

Theorem 3.3 (Elliott and van der Hoek [7]). *The process (B^T) defined by*

$$B_t^T := B_t - \int_0^t \text{diag}(\sqrt{\alpha_i + \beta_i X_u}) S' E_T[Y_u | \mathcal{F}_u] du \tag{3.9}$$

is a standard Brownian motion with respect to (P^T, \mathcal{F}_t) .

The dynamics for the i -th component, B_t^{iT} , of B^T can be written in differential form as

$$dB_t^{iT} = dB_t^i + \sqrt{\alpha_i + \beta_i X_t} R' \left(\int_t^T \hat{D}_{tv} dv \right) S e_i dt. \tag{3.10}$$

Equation (3.10) can be used to write the dynamics of the Jacobian matrix of partial derivatives of the stochastic flow, evaluated at $x = X_t$, under the forward measure. Taking the P^T conditional expectation of these dynamics with respect to \mathcal{F}_t gives that \hat{D}_{tu} satisfies

$$\begin{aligned} \hat{D}_{tu} &= I + \int_t^u A \hat{D}_{tv} dv \\ &- \frac{1}{2} \sum_{i=1}^n \int_t^u S \text{diag}(e_i) C E_T[R' \left(\int_v^T \hat{D}_{vv_1} dv_1 \right) S e_i (\partial_x X_v^{t,x})|_{x=X_t} | \mathcal{F}_t] dv \end{aligned} \tag{3.11}$$

almost surely.

4. Main Result and Proof

Based on equation (3.11) it is stated in Elliott and van der Hoek [7, Lemma 4.3] that \hat{D}_{tu} is deterministic for $0 \leq t \leq u \leq T$. The proof given in [7] proceeds by constructing a sequence of deterministic processes which are supposed to converge to \hat{D}_{tu} given by equation (3.11). However, there is a mistake in the proof provided in [7]. The difficulty with the proposed approximation comes from an upper bound which is assumed to be constant in [7] but which actually grows at a rate that makes the application of Gronwall's inequality ineffective. It is possible to prove a local version of this result by modifying the original proof.

Proposition 4.1. *There exists $\delta_1 > 0$ such that for $s \in [T - \delta_1, T]$ and $t \in [s, T]$ the process \hat{D}_{st} is deterministic.*

Proof. The first step of the proof is to identify $\delta_1 > 0$. The second step is to construct a sequence of deterministic integral equations the solutions of which converge to \hat{D}_{st} for $s \in [T - \delta_1, T]$ and $t \in [s, T]$.

For an $n \times n$ matrix $A = [a_{ij}]$ define the supremum norm

$$\|A\| = \max_{1 \leq i, j \leq n} |a_{i,j}|$$

and the following quantities

$$\begin{aligned} \Delta^1 &= e^{\|A\|(T-s)} \\ M_1 &= \frac{1}{2} \|S\|^2 \|R\| \|C\| \Delta^1 \\ D_1 &= \sup_{s \leq v_1 \leq T} E_T[\|D_{sv_1}\| | \mathcal{F}_s] < \infty \\ D_2 &= \sup_{0 \leq s \leq t \leq T} \|\hat{D}_{st}\| < \infty \\ K_1 &= \frac{1}{2} \sup_{0 \leq s \leq v_1 \leq v_2 \leq T} \|E_T[S \text{diag}(R' \hat{D}_{v_1 v_2} S) C D_{sv_1} | \mathcal{F}_s]\| < \infty \\ K_2 &= \frac{K_1}{2} \Delta^1 \\ K_3 &= \frac{1}{2} \|S\|^2 \|R\| \|C\| D_1 \\ K &= K_2 K_3 e^{(\|A\| + M_1)(T-s)}. \end{aligned}$$

Note that for any $s \in [0, T]$ the sequence $\{a_N\}$ defined, for $N \geq 2$ by

$$a_N = \frac{K^{(N-2)} (T-s)^{2(N-1)}}{3 \cdot 5 \cdots (2N-3)}$$

with $a_1 = \Delta^1$ converges to zero as N tends to infinity. Therefore, there exists some $N^* \geq 1$ such that

$$a_{N^*} = \max_{N \geq 1} a_N.$$

Define

$$\gamma = a_{N^*} + D_2. \tag{4.1}$$

Then there exists $\delta_1 > 0$ so that

$$\gamma - \Delta^1 \leq \frac{2 \left\{ \log \frac{K_1}{2} + \|A\| \delta_1 \right\}}{\|S\|^2 \|R\| \|C\| \delta_1^2}.$$

For $s \in [T - \delta_1, T]$ and $t \in [s, T]$ define

$$\Delta_{st}^1 = I + \int_s^t A \Delta_{sv_1}^1 dv_1$$

so that $\|\Delta_{st}^1\| \leq \Delta^1$. Hence

$$\begin{aligned} & \|\hat{D}_{st} - \Delta_{st}^1\| \\ &= \left\| \int_s^t A(\hat{D}_{sv_1} - \Delta_{sv_1}^1) dv_1 - \frac{1}{2} \int_s^t \int_{v_1}^T E_T[S \text{diag}(R' \hat{D}_{v_1 v_2} S) C D_{sv_1} | \mathcal{F}_s] dv_2 dv_1 \right\| \\ &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^1\| dv_1 \\ &+ \frac{1}{2} \int_s^t \int_{v_1}^T \|E_T[S \text{diag}(R' \hat{D}_{v_1 v_2} S) C D_{sv_1} | \mathcal{F}_s]\| dv_2 dv_1. \end{aligned}$$

Then

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^1\| &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^1\| dv_1 + K_1 \int_s^t \int_{v_1}^T dv_2 dv_1 \\ &= \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^1\| dv_1 + K_1 \int_s^t (T - v_1) dv_1 \\ &= \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^1\| dv_1 - \frac{1}{2} K_1 [(T - t)^2 - (T - s)^2] \\ &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^1\| dv_1 + \frac{1}{2} K_1 (T - s)^2. \end{aligned}$$

So, by Gronwall's inequality, we have

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^1\| &\leq \frac{1}{2} K_1 (T - s)^2 \exp(\|A\|(t - s)) \\ &\leq \frac{1}{2} K_1 (T - s)^2 \exp(\|A\|(T - s)) \\ &= K_2 (T - s)^2. \end{aligned}$$

Since this inequality holds for $T - \delta_1 \leq s \leq t \leq T$ we have, for $T - \delta_1 \leq s \leq v_1 \leq v_2 \leq T$,

$$\|\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^1\| \leq K_2 (T - v_1)^2. \tag{4.2}$$

Define, for $n \geq 2$ and $T - \delta_1 \leq s \leq t \leq T$,

$$\Delta_{st}^n = I + \int_s^t A \Delta_{sv_1}^n dv_1 - \frac{1}{2} \int_s^t \int_{v_1}^T S \text{diag}(R' \Delta_{v_1 v_2}^{n-1} S) C \Delta_{sv_1}^n dv_2 dv_1.$$

Then for $n = 2$

$$\begin{aligned}
\|\hat{D}_{st} - \Delta_{st}^2\| &= \left\| \int_s^t A(\hat{D}_{sv_1} - \Delta_{sv_1}^2) dv_1 \right. \\
&\quad \left. - \frac{1}{2} \int_s^t \int_{v_1}^T \left(E_T[S \text{diag}(R' \hat{D}_{v_1 v_2} S)] C D_{sv_1} | \mathcal{F}_s \right) - S \text{diag}(R' \Delta_{v_1 v_2}^1 S) C \Delta_{sv_1}^2 \right\| dv_2 dv_1 \\
&\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 + \frac{1}{2} \int_s^t \int_{v_1}^T \|E_T[S \text{diag}(R' \hat{D}_{v_1 v_2} S)] C D_{sv_1} | \mathcal{F}_s \\
&\quad - S \text{diag}(R' \Delta_{v_1 v_2}^1 S) C \Delta_{sv_1}^2 + S \text{diag}(R' \Delta_{v_1 v_2}^1 S) C \hat{D}_{sv_1} \\
&\quad - S \text{diag}(R' \Delta_{v_1 v_2}^1 S) C \hat{D}_{sv_1}\| dv_2 dv_1 = \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 \\
&\quad + \frac{1}{2} \int_s^t \int_{v_1}^T \|E_T[S \text{diag}(R' (\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^1) S)] C D_{sv_1} | \mathcal{F}_s\| \\
&\quad + S \text{diag}(R' \Delta_{v_1 v_2}^1 S) C (\hat{D}_{sv_1} - \Delta_{sv_1}^2)\| dv_2 dv_1 \leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T E_T[\|\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^1\| \|D_{sv_1}\| | \mathcal{F}_s] dv_2 dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \|\Delta_{v_1 v_2}^1\| \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_2 dv_1.
\end{aligned}$$

So, by equation (4.2),

$$\begin{aligned}
\|\hat{D}_{st} - \Delta_{st}^2\| &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T E_T[K_2(T - v_1)^2 \|D_{sv_1}\| | \mathcal{F}_s] dv_2 dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \|\Delta_{v_1 v_2}^1\| \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_2 dv_1.
\end{aligned}$$

Applying the bounds $E_T[\|D_{sv_1}\| | \mathcal{F}_s] \leq D_1$ and $\|\Delta_{v_1 v_2}^1\| \leq \Delta^1$ gives that

$$\begin{aligned}
\|\hat{D}_{st} - \Delta_{st}^2\| &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T K_2(T - v_1)^2 D_1 dv_2 dv_1 \\
&\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \Delta^1 \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_2 dv_1 \\
&\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1 + K_2 K_3 \int_s^t \int_{v_1}^T (T - v_1)^2 dv_2 dv_1 \\
&\quad + M_1 \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^2\| dv_1. \tag{4.3}
\end{aligned}$$

Note, by Fubini's theorem,

$$\int_s^t \int_{v_1}^T (T - v_1)^2 dv_2 dv_1 \leq \frac{(T - s)^4}{3} \quad (4.4)$$

so that, by equation (4.3) and Gronwall's inequality, we have

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^2\| &\leq \frac{K_2 K_3 (T - s)^4}{3} e^{(\|A\| + M_1)(t-s)} \leq \frac{K_2 K_3 (T - s)^4}{3} e^{(\|A\| + M_1)(T-s)} \\ &= \frac{K(T - s)^4}{3} \end{aligned}$$

Suppose that for $n = N - 1$

$$\|\hat{D}_{st} - \Delta_{st}^{N-1}\| \leq \frac{K^{N-2} (T - s)^{2(N-1)}}{3 \cdot 5 \cdots (2N - 3)}. \quad (4.5)$$

In particular, for $s \leq v_1 \leq v_2 \leq T$, we have

$$\|\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^{N-1}\| \leq \frac{K^{N-2} (T - v_1)^{2(N-1)}}{3 \cdot 5 \cdots (2N - 3)}. \quad (4.6)$$

By the triangle inequality, equation (4.6), and the definition of γ we have

$$\|\Delta_{v_1 v_2}^{N-1}\| \leq \|\Delta_{v_1 v_2}^{N-1} - \hat{D}_{v_1 v_2}\| + \|\hat{D}_{v_1 v_2}\| \leq \gamma. \quad (4.7)$$

Similar to the case of $n = 2$ for $n = N$ we have

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^N\| &= \left\| \int_s^t A(\hat{D}_{sv_1} - \Delta_{sv_1}^N) dv_1 \right. \\ &\quad \left. - \frac{1}{2} \int_s^t \int_{v_1}^T \left(E_T[Sdiag(R' \hat{D}_{v_1 v_2} S) C D_{sv_1} | \mathcal{F}_s] - Sdiag(R' \Delta_{v_1 v_2}^{N-1} S) C \Delta_{sv_1}^N \right) dv_2 dv_1 \right\| \\ &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 + \frac{1}{2} \int_s^t \int_{v_1}^T \|E_T[Sdiag(R' \hat{D}_{v_1 v_2} S) C D_{sv_1} | \mathcal{F}_s] \\ &\quad - Sdiag(R' \Delta_{v_1 v_2}^{N-1} S) C \Delta_{sv_1}^N + Sdiag(R' \Delta_{v_1 v_2}^{N-1} S) C \hat{D}_{sv_1} \\ &\quad - Sdiag(R' \Delta_{v_1 v_2}^{N-1} S) C \hat{D}_{sv_1}\| dv_2 dv_1 = \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 \\ &\quad + \frac{1}{2} \int_s^t \int_{v_1}^T \|E_T[Sdiag(R' (\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^{N-1}) S) C D_{sv_1} | \mathcal{F}_s] \\ &\quad + Sdiag(R' \Delta_{v_1 v_2}^{N-1} S) C (\hat{D}_{sv_1} - \Delta_{sv_1}^N)\| dv_2 dv_1 \leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T E_T[\|\hat{D}_{v_1 v_2} - \Delta_{v_1 v_2}^{N-1}\| \|D_{sv_1}\| | \mathcal{F}_s] dv_2 dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \|\Delta_{v_1 v_2}^{N-1}\| \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_2 dv_1. \end{aligned}$$

Applying the estimate (4.6) we obtain

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^N\| &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T E_T \left[\frac{K^{N-2}(T-v_1)^{2(N-1)}}{3 \cdot 5 \cdots (2N-3)} \|D_{sv_1}\| \|\mathcal{F}_s\| \right] dv_2 dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \|\Delta_{v_1 v_2}^{N-1}\| \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_2 dv_1. \end{aligned}$$

Applying the estimate (4.7) we find

$$\begin{aligned} \|\hat{D}_{st} - \Delta_{st}^N\| &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \frac{K^{N-2}(T-v_1)^{2(N-1)}}{3 \cdot 5 \cdots (2N-3)} D_1 dv_2 dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \int_s^t \int_{v_1}^T \gamma \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_2 dv_1 \\ &\leq \|A\| \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1 + \frac{K_3 K^{N-2}}{3 \cdot 5 \cdots (2N-3)} \int_s^t \int_{v_1}^T (T-v_1)^{2(N-1)} dv_2 dv_1 \\ &\quad + \frac{1}{2} \|S\|^2 \|R\| \|C\| \gamma (T-s) \int_s^t \|\hat{D}_{sv_1} - \Delta_{sv_1}^N\| dv_1. \end{aligned} \tag{4.8}$$

Note, by Fubini's theorem, for $N \geq 2$

$$\int_s^t \int_{v_1}^T (T-v_1)^{2(N-1)} dv_2 dv_1 \leq \frac{(T-s)^{2N}}{(2N-1)}$$

so that, by equation (4.8) and Gronwall's inequality, we have

$$\begin{aligned} &\|\hat{D}_{st} - \Delta_{st}^N\| \\ &\leq \frac{K_3 K^{N-2} (T-s)^{2N}}{3 \cdot 5 \cdots (2N-1)} \exp \left(\left(\|A\| + \frac{1}{2} \|S\|^2 \|R\| \|C\| \gamma (T-s) \right) (T-s) \right). \end{aligned} \tag{4.9}$$

However, as $\delta_1 > 0$ was chosen such that

$$\gamma - \Delta^1 \leq \frac{2 \left\{ \log \frac{K_1}{2} + \|A\| \delta_1 \right\}}{\|S\|^2 \|R\| \|C\| \delta_1^2}$$

and since $0 < (T-s) \leq \delta_1$ we have that

$$\gamma - \Delta^1 \leq \frac{2 \left\{ \log \frac{K_1}{2} + \|A\| (T-s) \right\}}{\|S\|^2 \|R\| \|C\| (T-s)^2}$$

Therefore,

$$\frac{1}{2} \|S\|^2 \|R\| \|C\| (\gamma - \Delta^1) (T-s)^2 \leq \left\{ \log \frac{K_1}{2} + \|A\| (T-s) \right\}$$

and taking the exponential gives

$$e^{\frac{1}{2} \|S\|^2 \|R\| \|C\| (\gamma - \Delta^1) (T-s)^2} \leq \frac{K_1}{2} e^{\|A\| (T-s)} = K_2. \tag{4.10}$$

So that finally equations (4.9) and (4.10) give that

$$\|\hat{D}_{st} - \Delta_{st}^N\| \leq \frac{K_3 K^{N-2} (T-s)^{2N}}{3 \cdot 5 \cdots (2N-1)} K_2 = \frac{K^{N-1} (T-s)^{2N}}{3 \cdot 5 \cdots (2N-1)}.$$

Since $n = N$ was arbitrary we have, by induction, that for all $n \geq 2$ and $T - \delta_1 \leq s \leq t \leq T$

$$\|\hat{D}_{st} - \Delta_{st}^n\| \leq \frac{K^{n-1} (T-s)^{2n}}{3 \cdot 5 \cdots (2n-1)}. \quad (4.11)$$

Since the right hand side of equation (4.11) approaches zero as n tends to infinity we obtain the desired result. \square

5. Conclusion

We have shown that the conditional expectation under the forward measure of the Jacobian of the stochastic flow associated with an affine diffusion is deterministic in a neighborhood of the maturity time of the zero coupon bond. This local result addresses a mistake in the proof of a global result stated by Elliott and van der Hoek [7], to the extent possible using the method of proof originally proposed. This result is not strong enough to prove that the zero-coupon bond price is exponential affine without solving Riccati-type ordinary differential equations. It was shown in [11] using different methods that the solvability of the Riccati equations over a fixed time interval $[0, T]$ is a sufficient condition for the conditional expectation of the stochastic Jacobian to be deterministic on the entire interval $[0, T]$. Nevertheless, we believe that the local result of this paper is of independent interest in showing the similarity in the solvability of Riccati-type differential equations over a fixed time interval and the properties of the conditional expectation of the stochastic Jacobian.

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