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APPROXIMATIONS OF FRACTIONAL STOCHASTIC DIFFERENTIAL EQUATIONS BY MEANS OF TRANSPORT PROCESSES*

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Abstract. We present strong approximations with rate of convergence for the solution of a stochastic differential equation of the form

$$dX_t = b(X_t)dt + \sigma(X_t)dB_t^H,$$

where $b \in C_b^1$, $\sigma \in C_b^2$, B^H is fractional Brownian motion with Hurst index H, and we assume existence of a unique solution with Doss-Sussmann representation. The results are based on a strong approximation of B^H by means of transport processes of Garzón et al (2009 [11]). If σ is bounded away from 0, an approximation is obtained by a general Lipschitz dependence result of Römisch and Wakolbinger (1985 [25]). Without that assumption on σ, that method does not work, and we proceed by means of Euler schemes on the Doss-Sussmann representation to obtain another approximation, whose proof is the bulk of the paper.

1. Introduction

We consider one dimensional fractional stochastic differential equations of the form

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dB_t^H, \quad t \in (0,T], \\ X_0 &= x_0, \end{aligned} \tag{1.1}$$

where B^H is fractional Brownian motion with Hurst index H, and b and σ are continuous functions. Equations of this type appear in several areas of application due to the properties of B^H (see e.g. [18] and references therein). $B^H = (B_t^H)_{t\geq 0}$ is defined for any $H \in (0,1)$ as a centered Gaussian process with covariance

$$\mathrm{Cov}(B_s^H, B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

We exclude the case $H = 1/2$, which corresponds to Brownian motion. The main properties of B^H (for $H \neq 1/2$) are self-similarity, stationarity of increments, long range dependence, k-Hölder continuity of trajectories for $k < H$, and it is neither a Markov process nor a semimartingale, hence the classical Itô calculus cannot be used for this process. See [23, 27] for background. There is now an

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extensive literature on fractional stochastic differential equations and applications. We restrict the references to a minimum for reasons of space.

In order to give a precise meaning to the stochastic differential equation (1.1) it is necessary to define stochastic integral with respect to B^H and to define solution, and then existence and uniqueness of solution can be proved under suitable conditions on H, b and σ, in particular for the solution to have a Doss-Sussmann representation (given by equation (2.9)). For example, this can be done for $H > 1/3$, b Lipschitz, and $\sigma \in C_b^2$. These questions have been treated in [21] (see also [4] and references therein). The paths of B^H are increasingly irregular as H decreases, and $H = 1/4$ is a critical value for some problems (see e.g. [5, 10, 14, 15, 22]). Hence the assumption $H > 1/4$ is often made. In this case, under some conditions on b and σ, a unique solution with Doss-Sussmann representation has been established in [1] for $1/4 < H < 1/2$ with Stratonovich integral, and in [20] for $H > 1/2$ with forward integral. In [2] it is remarked that under some assumptions on the integrand, the Stratonovich integral coincides with the forward integral (see [26] for those integrals). In this paper we take $b \in C_b^1$ and $\sigma \in C_b^2$ (C_b^i is the space of bounded functions with continuous bounded derivatives of order $\leq i$), and we assume existence of a unique solution of (1.1) with Doss-Sussmann representation; this holds at least for $H > 1/4$. The restriction $H > 1/4$ is not used in our proofs, and it seems to be unnecessary if (1.1) is treated with the theory of rough paths; moreover, in that way it may also be possible to study strong transport approximations in the multidimensional case (see [3, 17, 24, 28] and references therein concerning the theory of rough paths and related fractional stochastic diferential equations).

In [11] we obtained a strong approximation of B^H with a rate of convergence by means of the Mandelbrot-van Ness representation of B^H as a stochastic integral with respect to Brownian motion [16], and a strong approximation of Brownian motion by transport processes [13]. The result is that for each $q > 0$ and each β such that $|H - 1/2| < \beta < 1/2$, if B^n is the n-th approximation of B^H (defined in (2.6)-(2.7)), there is a positive constant C such that

$$P\left(\sup_{0\leq t\leq T} |B_t^H - B_t^n| > Cn^{-1/2+\beta}(\log n)^{5/2}\right) = o(n^{-q}) \quad \text{as} \quad n \to \infty. \tag{1.2}$$

Thus the approximation becomes better when H approaches $1/2$.

The aim of this paper is to show that under the above assumptions on b and σ, approximate solutions X^n of equation (1.1) can be constructed by means of B^n, which strongly approximate the solution X similarly as (1.2), with rate

$$n^{-1/2+\beta+\delta}(\log n)^{5/2},$$

where $\delta > 0$ is arbitrarily small, i.e., almost the same rate as (1.2). If σ is bounded away from 0, this can be done directly from (1.2) using a general Lipschitz dependence result obtained in [25] (see also [8]), without recourse to specifics of B^n. This is Theorem 3.1. Without that assumption on σ the result of [25] cannot be used (see Remark 4.1), and we proceed by applying Euler schemes to the Doss-Sussmann representation of the solution. In this case, properties of B^n are involved, and the approximation is somewhat different in the probability sense. This is Theorem 3.3, whose proof is the bulk of the paper.

We stress that the approximations of the solution of (1.1) by means of transport processes are of theoretical interest, but may not be useful for generating approximate solutions with computational efficiency. To that end there are ad hoc methods (e.g. [19] and references therein).

Section 2 contains background, Section 3 results, and Section 4 proofs.

2. Background

2.1. Approximation of fractional Brownian motion by transport processes. For each $n = 1, 2, \ldots$, a (uniform) transport process $Z^n = (Z^n(t))_{t\geq 0}$ represents the position on the real line at each time t of a particle that starts from 0 with velocity $+n$ or $-n$, with probability 1/2 each, continues with that velocity during an exponentially distributed time with parameter n^2, at the end of which it changes velocity from $\pm n$ to $\mp n$, and so on the same way, changing sign at consecutive independent exponentially holding times. Such a process can be constructed from a given Brownian motion B on a probability space using the Skorohod embedding, and it was shown in [13] that Z^n converges strongly to B uniformly on a given bounded time interval with rate $Cn^{-1/2}(\log n)^{5/2}$, as $n \to \infty$, where C is a positive constant.

The Mandelbrot-van Ness representation of B^H is given by

$$B_t^H = C_H\left(\int_{-\infty}^0 f_t(s)dB(s) + \int_0^t g_t(s)dB(s)\right), \quad 0 \leq t \leq T, \tag{2.1}$$

where

$$f_t(s) = (t-s)^{H-1/2} - (-s)^{H-1/2} \quad \text{for} \quad s < 0 \leq t, \tag{2.2}$$

$$g_t(s) = (t-s)^{H-1/2} \quad \text{for} \quad s < t, \tag{2.3}$$

B is a Brownian motion on the whole real line, and C_H is a positive constant [16]. Fix $a < 0$. After an integration by parts and a change of variable, B^H can be written as

$$\begin{aligned} B_t^H = C_H\Bigg(&\int_0^t g_t(s)dB_1(s) + \int_a^0 f_t(s)dB_2(s) + f_t(a)B_2(a) \\ &- \int_{1/a}^0 \partial_s f_t\left(\frac{1}{v}\right)\frac{1}{v^3}B_3(v)dv\Bigg), \end{aligned} \tag{2.4}$$

where B_1, B_2 and B_3 are Brownian motions given, respectively, by the restriction of B to $[0,T]$, the restriction of B to $[a,0]$, and

$$B_3(s) = \begin{cases} sB(\frac{1}{s}) & \text{if} \quad s \in [1/a, 0), \\ 0 & \text{if} \quad s = 0. \end{cases}$$

To define the approximation $B^n = (B_t^n)_{t\geq 0}$ that appears in (1.2), the idea is to approximate B_1, B_2 and B_3 by corresponding transport processes Z_1^n, Z_2^n and Z_3^n.

For $n = 1, 2, \ldots$ and $0 < \beta < 1/2$, let

$$\varepsilon_n = -n^{-\beta/|H-1/2|}. \tag{2.5}$$

For $H > 1/2$, define

$$B_t^n = C_H \left(\int_0^t g_t(s) dZ_1^n(s) + \int_a^0 f_t(s) dZ_2^n(s) + f_t(a) Z_2^n(a) + \int_{1/a}^0 \left(- \int_{1/a}^{s \wedge \varepsilon_n} \partial_s f_t\left(\frac{1}{v}\right) \frac{1}{v^3} dv \right) dZ_3^n(s) \right), \tag{2.6}$$

and for $H < 1/2$, define

$$B_t^n = C_H \left(\int_0^{(t+\varepsilon_n)\vee 0} g_t(s) dZ_1^n(s) + \int_{(t+\varepsilon_n)\vee 0}^t g_t(\varepsilon_n + s) dZ_1^n(s) + \int_a^{\varepsilon_n} f_t(s) dZ_2^n(s) + f_t(a) Z_2^n(a) + \int_{1/a}^0 \left(- \int_{1/a}^{s} \partial_s f_t\left(\frac{1}{v}\right) \frac{1}{v^3} dv \right) dZ_3^n(s) \right). \tag{2.7}$$

Note that B^n depends on β through (2.5).

The following consequence of (1.2) is obvious since q is arbitrary, hence large enough.

$$P\left(\limsup_{n\to\infty} \left\{ \sup_{0\le t\le T} |B_t^H - B_t^n| > C n^{-1/2+\beta} (\log n)^{5/2} \right\} \right) = 0, \tag{2.8}$$

where lim sup is understood in the sense of sets. This will be used for the proof of Theorem 3.3.

2.2. Doss-Sussmann representation. Under suitable assumptions on b, σ and H, the Doss-Sussmann representation of (1.1) is given by (see [1, 7, 20, 21])

$$X_t = h(Y_t, B_t^H), \tag{2.9}$$

where the function h and the process Y are the solutions of equations

$$\frac{\partial h}{\partial x_2}(x_1, x_2) = \sigma(h(x_1, x_2)), \quad h(x_1, 0) = x_1, \quad x_1, x_2 \in \mathbb{R}, \tag{2.10}$$

and

$$Y_t' = \exp\left(- \int_0^{B_t^H} \sigma'(h(Y_t, s)) ds \right) b(h(Y_t, B_t^H)), \quad Y_0 = x_0, \tag{2.11}$$

respectively. The function h has the property (see [7], Lemma 2)

$$\frac{\partial h}{\partial x_1}(x_1, x_2) = \exp\left(\int_0^{x_2} \sigma'(h(x_1, s)) ds \right), \tag{2.12}$$

which implies that

$$Y_t' = \left(\frac{\partial h}{\partial x_1}(Y_t, B_t^H) \right)^{-1} b(h(Y_t, B_t^H)), \quad Y_0 = x_0. \tag{2.13}$$

3. Results

Recall that $b \in C_b^1$, $\sigma \in C_b^2$, (2.9) holds, and B^n is defined by (2.6)-(2.7). The first theorem is a special case with the assumption that the function σ is bounded away from 0. The interest of this result is that a direct proof can be given using Fernique's theorem [9], and a general Lipschitz dependence result of [25], without involving anything special about B^n.

Let X^n be the solution of (1.1) with B^H replaced by B^n, and the integral is defined pathwise.

Theorem 3.1. *Assume σ is bounded away from 0. Let $|H-1/2| < \beta < 1/2$. Then for each $\delta > 0$ such that $\beta + \delta < 1/2$, there exist $q > 0$ and a positive constant C such that*

$$P\left(\sup_{0\le t\le T} |X_t - X_t^n| > Cn^{-1/2+\beta+\delta}(\log n)^{5/2}\right) = o(n^{-q}) \quad \textit{as} \quad n \to \infty. \tag{3.1}$$

Remark 3.2. (1) A transport approximation for equation (1.1) with Brownian motion ($H = 1/2$) and σ bounded away from 0 was studied in [12, 25]. The result is like (3.1) with $\beta = \delta = 0$. There was an error in the proof in [12], which was remedied with the method of [25]. The paper [6] gives a related result with a different type of formulation.

(2) The approximation (1.2) holds for arbitrary q, but in the approximation (3.1) δ is arbitrary and q is chosen appropriately small (see the proof). Hence a result like (3.2) below may not hold under the assumption of Theorem 3.1.

For the general result, let $X^n = h^n(Y^{n,n^2}, B^n)$, where $(h^n)_n$ is an Euler scheme approximation of equation (2.10), and $(Y^{n,m})_m$ is an Euler scheme approximation of Y^n, which is the (Doss-Sussmann) solution of equation (2.11) with B^H replaced by B^n (the precise definitions of h^n and $Y^{n,m}$ are given in (4.3) and (4.9)).

Theorem 3.3. *Let $|H - 1/2| < \beta < 1/2$. Then for each $\delta > 0$ such that $\delta < \beta$ and $\beta + \delta < 1/2$, there exists a positive constant C such that*

$$P\left(\limsup_{n\to\infty}\left\{\sup_{0\le t\le T} |X_t - X_t^n| > Cn^{-1/2+\beta+\delta}(\log n)^{5/2}\right\}\right) = 0. \tag{3.2}$$

Note that without the assumption on σ in Theorem 3.1 the same rate of convergence holds, but the probabilistic part of the result is slightly different.

4. Proofs

We write $||\cdot||_\infty$ for the sup norm on [0,T].

4.1. Proof of Theorem 3.1. We give an idea of the proof. By Fernique's theorem [9] there exists $\alpha > 0$ such that

$$P(||B^H||_\infty \ge s) \le Ke^{-\alpha s^2}, \quad s \ge 0,$$

where $K = E\exp(\alpha||B^H||_\infty^2) < \infty$. Then, applying Theorem 3 A) and Theorem 4 of [25], from (1.2) we obtain

$$P(||X - X^n||_\infty \ge n^\delta Cn^{-1/2+\beta}(\log n)^{5/2}) = o(n^{-q}) \quad \text{as} \quad n \to \infty,$$

with $\delta = q\gamma/\alpha^{1/2}$, where γ is the positive constant in Theorem 4 of [25]. Since q in (1.2) is arbitrary, it can be chosen so that δ is as small as desired, and (3.1) is obtained. □

Remark 4.1. It can be shown that for $\sigma(x) = \arctan x$, formula (40) of [25] cannot be proved because

$$\left(\frac{\partial h}{\partial x}(x,y)\right)^{-1} \geq \exp\left(\frac{|y|}{1+x^2}\right)$$

for $x < 0, y < 0$, so $(\frac{\partial h}{\partial x}(x,y))^{-1}$ is unbounded, and therefore the argument for the proof of Theorem 4 in [25] is not valid.

4.2. Proof of Theorem 3.3. Since the constant C in (2.8) does not play an essential role, for simplicity we put $C = 1$, and we will also prove the theorem with $C = 1$ in (3.2).

We start by describing the parts of the approximation. We first approximate the function h and the process Y given by (2.10) and (2.11) respectively, and then, based on those approximations, we formulate the approximation for the solution of (1.1).

The function $h : \mathbb{R}^2 \to \mathbb{R}$ satisfies

$$h(x,y) = x + \int_0^y \sigma(h(x,s))ds. \tag{4.1}$$

For each $n = 1, 2, \cdots$, we take the partition $\{y_i^n\}$ of the interval $[-n, n]$ given by $-n = y_{-n^2}^n < \cdots < y_{-1}^n < y_0^n = 0 < y_1^n < \cdots < y_{n^2}^n = n$, where for $r_n = 1/n$, and $i = 1, \cdots, n^2 - 1$,

$$y_{i+1}^n = y_i^n + r_n = \frac{i+1}{n}, \quad y_{-(i+1)}^n = y_{-i}^n - r_n = -\frac{i+1}{n}. \tag{4.2}$$

We define the functions $h^n : \mathbb{R}^2 \to \mathbb{R}$ by

$$h^n(x,y) = 0 \quad \text{if} \quad (x,y) \notin [-n,n] \times [-n,n],$$

for $(x,y) \in [-n,n] \times [-n,n]$ and $k = 0, 1, \cdots, n^2 - 1$,

$$\begin{aligned} h^n(x, y_0^n) &= x, \\ h^n(x, y_{k+1}^n) &= h^n(x, y_k^n) + r_n\sigma(h^n(x, y_k^n)), \\ h^n(x, y_{-(k+1)}^n) &= h^n(x, y_{-k}^n) - r_n\sigma(h^n(x, y_{-k}^n)), \end{aligned}$$

and by linear interpolation,

$$\begin{aligned} &h^n(x,y) = h^n(x, y_k^n) + (y - y_k^n)\sigma(h^n(x, y_k^n)) && \text{if} \quad y_k^n \leq y < y_{k+1}^n, \\ &h^n(x,y) = h^n(x, y_{-k}^n) - (y_{-k}^n - y)\sigma(h^n(x, y_{-k}^n)) && \text{if} \quad y_{-(k+1)}^n < y \leq y_{-k}^n. \end{aligned} \tag{4.3}$$

From (2.13) we have

$$Y_t = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s, B_s^H)\right)^{-1} b(h(Y_s, B_s^H))ds. \tag{4.4}$$

For each $n = 1, 2, \cdots$, we define the process Y^n as the solution of

$$Y_t^n = x_0 + \int_0^t \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n)\right)^{-1} b(h(Y_s^n, B_s^n))ds, \quad t \in [0,T]. \tag{4.5}$$

These processes exist and are unique because for each $n = 1, 2, \cdots$, the function

$$(x,t) \mapsto \left(\frac{\partial h}{\partial x_1}(x, B_t^n)\right)^{-1} b(h(x, B_t^n)), \quad t \in [0,T],$$

satisfies Lipschitz and linear growth conditions in x (see (2.12), (4.7) and Corollary 4.5 below). Moreover, they are defined on the same probability space as the Brownian motion B in (2.1), since the solution of (4.5) is given for each sample point, and B^n is defined on the same probability space as B.

We give next an Euler scheme for approximating the processes Y^n for each $n = 1, 2, \cdots$.

By (2.12), equation (4.5) can be written as

$$(Y_t^n)' = f(Y_t^n, B_t^n), \quad Y_0^n = x_0, \tag{4.6}$$

where $f(x,y)$ is defined by

$$f(x,y) = \exp\left(-\int_0^y \sigma'(h(x,u))du\right) b(h(x,y)). \tag{4.7}$$

For each $n = 1, 2, \cdots$, we define $f^n(x,y)$ by

$$f^n(x,y) = \exp\left(-\int_0^y \sigma'(h^n(x,u))du\right) b(h^n(x,y)), \tag{4.8}$$

with h^n given by (4.3).

The Euler scheme $(Y^{n,m})_m$ for equation (4.6) is defined as follows for each $m = 1, 2, \cdots$, the partition $0 = t_0 < \cdots < t_m = T$ of $[0,T]$ with $t_{i+1} = t_i + r_m$, and $r_m = T/m$:

$$\begin{aligned} Y_0^{n,m} &= x_0, \\ Y_{t_{k+1}}^{n,m} &= Y_{t_k}^{n,m} + r_m f^n(Y_{t_k}^{n,m}, B_{t_k}^n), \quad k = 0, \cdots, (m-1), \\ Y_t^{n,m} &= Y_{t_k}^{n,m} + (t - t_k) f^n(Y_{t_k}^{n,m}, B_{t_k}^n) \\ &= Y_{t_k}^{n,m} + \int_{t_k}^t f^n(Y_{t_k}^{n,m}, B_{t_k}^n) ds, \quad \text{if} \quad t_k \le t < t_{k+1}. \end{aligned} \tag{4.9}$$

Equation (1.1) has a unique solution X with representation (2.9), and we define similarly the approximation X^n by means of the Euler schemes for h and Y^n as

$$X_t^n = h^n(Y_t^{n,n^2}, B_t^n), \tag{4.10}$$

where h^n and Y^{n,n^2} are given by (4.3) and (4.9), respectively.

We then have

$$|X_t - X_t^n| \le H_1(t) + H_2(t) + H_3(t), \tag{4.11}$$

where

$$H_1(t) = |h(Y_t, B_t^H) - h(Y_t^n, B_t^n)|, \tag{4.12}$$

$$H_2(t) = |h(Y_t^n, B_t^n) - h(Y_t^{n,n^2}, B_t^n)|, \tag{4.13}$$

$$H_3(t) = |h(Y_t^{n,n^2}, B_t^n) - h^n(Y_t^{n,n^2}, B_t^n)|. \tag{4.14}$$

The proof consists in obtaining estimates involving H_1, H_2 and H_3.

We will need the following preliminary results. First a Lipschitz property of B^n.

Lemma 4.2. *Let B^n be defined by (2.6)-(2.7). Then for all n and for $t_1, t_2 \in [0,T]$,*

$$\left|B^n_{t_2} - B^n_{t_1}\right| \le Kn^{1+\beta} \left|t_2 - t_1\right|,$$

where K is a positive constant.

Proof. First we take $H > 1/2$. It suffices to check the property for each one of the four functions on the r.h.s. of (2.6), using (2.2), (2.3) and the transport processes. We will omit some of the calculations.

Note that $\partial_s f_t > 0$ by a straighforward calculation (see [11], Lemma 3.1). For $t_1 < t_2$, by the mean value theorem,

$$\begin{aligned}
&\left|\int_0^{t_2} g_{t_2}(s) dZ_1^n(s) - \int_0^{t_1} g_{t_1}(s) dZ_1^n(s)\right| \\
\le & n\left(\left|\int_0^{t_1}\left[(t_2-s)^{H-1/2} - (t_1-s)^{H-1/2}\right] ds\right| + \left|\int_{t_1}^{t_2} (t_2-s)^{H-1/2} ds\right|\right) \\
= & \frac{n}{H+1/2}\left(t_2^{H+1/2} - t_1^{H+1/2}\right) \le nT^{H-1/2}\left|t_2 - t_1\right|.
\end{aligned}$$

Again by the mean value theorem,

$$\begin{aligned}
&\left|\int_a^0 f_{t_2}(s) dZ_2^n(s) - \int_a^0 f_{t_1}(s) dZ_2^n(s)\right| \\
\le & n\int_a^0 |(t_2-s)^{H-1/2} - (t_1-s)^{H-1/2}| ds \\
= & \frac{n}{H+1/2}\left([(t_2-a)^{H+1/2} - (t_1-a)^{H+1/2}] - [t_2^{H+1/2} - t_1^{H+1/2}]\right)r \\
\le & \frac{n}{H+1/2}\left((t_2-a)^{H+1/2} - (t_1-a)^{H+1/2}\right) \le n(T-a)^{H-1/2}\left|t_2 - t_1\right|.
\end{aligned}$$

Next,

$$\begin{aligned}
&|f_{t_2}(a) Z_2^n(a) - f_{t_1}(a) Z_2^n(a)| \le n(-a)|(t_2-a)^{H-1/2} - (t_1-a)^{H-1/2}| \\
\le & n(-a)^{H-1/2}(H-1/2)\left|t_2 - t_1\right|.
\end{aligned}$$

Finally,

$$\begin{aligned}
&\left|\int_{1/a}^0 \left(-\int_{1/a}^{s\wedge\varepsilon_n} \partial_s f_{t_2}\left(\frac{1}{v}\right)\frac{1}{v^3} dv\right) dZ_3^n(s) - \int_{1/a}^0 \left(-\int_{1/a}^{s\wedge\varepsilon_n} \partial_s f_{t_1}\left(\frac{1}{v}\right)\frac{1}{v^3} dv\right) dZ_3^n(s)\right| \\
\le \quad & n(H-1/2)\int_{1/a}^0\int_{1/a}^{s\wedge\varepsilon_n} |-(t_2 - 1/v)^{H-3/2} + (t_1 - 1/v)^{H-3/2}|(-1/v)^3 dv ds \\
\le \quad & n(H-1/2)(3/2-H)\left|t_2 - t_1\right|\int_{1/a}^0\int_{1/a}^{s\wedge\varepsilon_n} (-v)^{-H-1/2} dv ds,
\end{aligned}$$

and by (2.5),

$$\int_{1/a}^{0}\int_{1/a}^{s\wedge\varepsilon_n}(-v)^{-H-1/2}dvds \leq \frac{1}{H-1/2}\int_{1/a}^{0}(-\varepsilon_n)^{-H+1/2}dv$$
$$= \frac{1}{H-1/2}n^{\beta}(-1/a).$$

Hence the result for $H>1/2$ follows.

We proceed similarly for $H<1/2$. Let

$$A_1 = \left|\int_0^{(t_2+\varepsilon_n)\vee 0} g_{t_2}(s)dZ_1^n(s) - \int_0^{(t_1+\varepsilon_n)\vee 0} g_{t_1}(s)dZ_1^n(s)\right|.$$

If $t_1+\varepsilon_n<0$ and $t_2+\varepsilon_n<0$, then $A_1=0$. If $t_1+\varepsilon_n<0$ and $t_2+\varepsilon_n>0$, then by the mean value theorem,

$$A_1 = \left|\int_0^{t_2+\varepsilon_n}(t_2-s)^{H-1/2}dZ_1^n(s)\right|$$
$$\leq n\int_0^{t_2+\varepsilon_n}(t_2-s)^{H-1/2}ds \leq n(-\varepsilon_n)^{H-1/2}(t_2+\varepsilon_n) \leq n^{1+\beta}\left|t_2-t_1\right|,$$

and if $t_1+\varepsilon_n>0$ and $t_2+\varepsilon_n>0$, again by the mean value theorem,

$$A_1 \leq n\left[\int_0^{t_1+\varepsilon_n}[(t_1-s)^{H-1/2}-(t_2-s)^{H-1/2}]ds + \int_{t_1+\varepsilon_n}^{t_2+\varepsilon_n}(t_2-s)^{H-1/2}ds\right]$$
$$\leq 2n(-\varepsilon_n)^{H-1/2}\left|t_2-t_1\right| = 2n^{1+\beta}\left|t_2-t_1\right|.$$

Let

$$A_2 = \left|\int_{(t_2+\varepsilon_n)\vee 0}^{t_2} g_{t_2}(\varepsilon_n+s)dZ_1^n(s) - \int_{(t_1+\varepsilon_n)\vee 0}^{t_1} g_{t_1}(\varepsilon_n+s)dZ_1^n(s)\right|$$

We show only one case. The others are similar. If $t_1+\varepsilon_n<0$, $t_2+\varepsilon_n>0$ and $t_1\geq t_2+\varepsilon_n$,

$$A_2 = \left|\int_{t_1}^{t_2}(t_2-\varepsilon_n-s)^{H-1/2}dZ_1^n(s) - \int_0^{t_2+\varepsilon_n}(t_1-\varepsilon_n-s)^{H-1/2}dZ_1^n(s)\right.$$
$$\left. + \int_{t_2+\varepsilon_n}^{t_1}[(t_2-\varepsilon_n-s)^{H-1/2}-(t_1-\varepsilon_n-s)^{H-1/2}]dZ_1^n(s)\right|$$
$$\leq \frac{2n}{H+1/2}[(t_2-\varepsilon_n-t_1)^{H+1/2}-(-\varepsilon_n)^{H+1/2}]$$
$$\leq 2n(-\varepsilon_n)^{H-1/2}\left|t_2-t_1\right| = 2n^{1+\beta}\left|t_2-t_1\right|.$$

Next,

$$\left|\int_a^{\varepsilon_n} f_{t_2}(s)dZ_2^n(s) - \int_a^{\varepsilon_n} f_{t_1}(s)dZ_2^n(s)\right|$$
$$\leq \frac{n}{H+1/2}\left[(t_2-\varepsilon_n)^{H+1/2}-(t_1-\varepsilon_n)^{H+1/2}\right]$$
$$\leq n(t_1-\varepsilon_n)^{H-1/2}\left|t_2-t_1\right| \leq n(-\varepsilon_n)^{H-1/2}\left|t_2-t_1\right| = n^{1+\beta}\left|t_2-t_1\right|,$$

$$\begin{aligned}|f_{t_2}(a)Z_2^n(a) - f_{t_1}(a)Z_2^n(a)| &\leq n(-a)(1/2-H)(t_1-a)^{H-3/2}|t_2-t_1| \\ &\leq n^{1+\beta}(-a)^{H-1/2}(1/2-H)|t_2-t_1|,\end{aligned}$$

$$\begin{aligned}&\left|\int_{1/a}^0 -\left(\int_{1/a}^s \partial_s f_{t_1}\left(\frac{1}{v}\right)\frac{1}{v^3}dv\right)dZ_3^n(s) - \int_{1/a}^0\left(-\int_{1/a}^s \partial_s f_{t_2}\left(\frac{1}{v}\right)\frac{1}{v^3}dv\right)dZ_3^n(s)\right| \\ &\leq n\int_{1/a}^0\int_{1/a}^s (1/2-H)\left[(t_1-1/v)^{H-3/2}-(t_2-1/v)^{H-3/2}\right](-1/v)^3 dvds \\ &\leq n\int_{1/a}^0\int_{1/a}^s (1/2-H)(3/2-H)(t_1-1/v)^{H-5/2}|t_2-t_1|(-1/v)^3 dvds \\ &\leq (3/2-H)(-1/a)^{3/2-H}n^{1+\beta}|t_2-t_1|.\end{aligned}$$

□

Corollary 4.3. *For each $n = 1, 2, \cdots$, $\|B^n\|_\infty < \infty$ a.s.*

Proof. Immediate from Lemma 4.2. □

By the assumptions on σ and b, we have the next bounds:

$$\begin{aligned}&|b(x)| \leq M_1,\ |\sigma'(x)| \leq M_2,\ |\sigma''(x)| \leq M_3, \\ &|b(x)-b(y)| \leq M_4|x-y|,\ |b'(x)| \leq M_4 \quad \text{and} \quad |\sigma(x)| \leq M_5,\end{aligned} \tag{4.15}$$

where $M_1, \cdots, M_5$ are constants. We put $\bar{M} = \max\{M_2, M_5\}$.

Next we present some properties of the function h.

Lemma 4.4. *Let h be defined by (2.10). Then*

(1)
$$\left|\frac{\partial h}{\partial x_1}(x,y)\right| \leq \exp(M_2|y|). \tag{4.16}$$

(2)
$$\left|\left(\frac{\partial h}{\partial x_1}(x,y)\right)^{-1}\right| = \left|\exp\left(-\int_0^y \sigma'(h(x,s))ds\right)\right| \leq \exp(M_2|y|). \tag{4.17}$$

(3)
$$\left|\frac{\partial}{\partial x_1}\left(\frac{\partial h}{\partial x_1}(x,y)\right)^{-1}\right| \leq M_3|y|\exp(2M_2|y|). \tag{4.18}$$

(4)
$$|h(x_1,y) - h(x_2,y)| \leq \exp(M_2|y|)|x_1-x_2|. \tag{4.19}$$

(5)
$$\left|\left(\frac{\partial h}{\partial x_1}(x_1,y)\right)^{-1} - \left(\frac{\partial h}{\partial x_1}(x_2,y)\right)^{-1}\right| \leq M_3|y|\exp(2M_2|y|)|x_1-x_2|. \tag{4.20}$$

(6)
$$|b(h(x_1,y)) - b(h(x_2,y))| \leq M_4\exp(M_2|y|)|x_1-x_2|. \tag{4.21}$$

(7)
$$|b(h(x,y_1)) - b(h(x,y_2))| \leq M_4M_5|y_1-y_2|. \tag{4.22}$$

(8)

$$\left| \left(\frac{\partial h}{\partial x_1}(x_1, y) \right)^{-1} b\left(h(x_1, y)\right) - \left(\frac{\partial h}{\partial x_1}(x_2, y) \right)^{-1} b\left(h(x_2, y)\right) \right| \le \exp(2M_2 \left| y \right|)[M_1 M_3 \left| y \right| + M_4] \left| x_1 - x_2 \right|. \tag{4.23}$$

Proof. (1) and (2) are immediate from (2.12) and (4.15).

(3) As σ'' is bounded by M_3, and by (4.16), (4.17), then

$$\begin{aligned} &\left| \frac{\partial}{\partial x_1} \left(\frac{\partial h}{\partial x_1}(x, y) \right)^{-1} \right| \\ &= \left| \exp\left(- \int_0^y \sigma'(h(x, u)) du \right) \right| \left| \int_0^y \sigma''(h(x, u)) \frac{\partial h}{\partial x_1}(x, u) du \right| \\ &\le M_3 |y| \exp(2M_2 |y|). \end{aligned}$$

(4) follows by the mean value theorem and (4.16).
(5) follows by the mean value theorem and (4.18).
(6) As b is Lipschitz, the result follows by the mean value theorem and (4.19).
(7) Using (2.10), b: Lipschitz, σ: bounded, and for some ξ between y_1 and y_2,

$$\begin{aligned} \left| b\left(h(x, y_1)\right) - b\left(h(x, y_2)\right) \right| \le M_4 \left| h(x, y_1) - h(x, y_2) \right| = M_4 \left| \frac{\partial h}{\partial x_2}(x, \xi) \right| \left| y_1 - y_2 \right| \\ = M_4 |\sigma(h(x, \xi))| |y_1 - y_2| \le M_4 M_5 |y_1 - y_2|. \end{aligned}$$

(8)

$$\left| \left(\frac{\partial h}{\partial x_1}(x_1, y) \right)^{-1} b\left(h(x_1, y)\right) - \left(\frac{\partial h}{\partial x_1}(x_2, y) \right)^{-1} b\left(h(x_2, y)\right) \right| \le A_1 + A_2, \tag{4.24}$$

where

$$\begin{aligned} A_1 &= \left| \left(\frac{\partial h}{\partial x_1}(x_1, y) \right)^{-1} - \left(\frac{\partial h}{\partial x_1}(x_2, y) \right)^{-1} \right| |b(h(x_1, y))|, \\ A_2 &= |b(h(x_1, y)) - b(h(x_2, y))| \left| \left(\frac{\partial h}{\partial x_1}(x_2, y) \right)^{-1} \right|. \end{aligned}$$

Using b bounded by M_1, (4.21), (4.17), (4.20) and (4.24) we obtain the result. □

Corollary 4.5. *Let f be defined by (4.7). Then for each $n = 1, 2, \cdots$ the function*

$$(x, t) \mapsto f(x, B_t^n)$$

satisfies Lipschitz and linear growth conditions in x.

Proof. By (2.12), Lemma 4.2 and parts 1 and 8 of Lemma 4.4 , for each $x_1, x_2 \in \mathbb{R}$ and $t \in [0, T]$,

$$\begin{aligned} &|f(x_1, B_t^n) - f(x_2, B_t^n)| \\ &\le \exp(2M_2 K n^{1+\beta} T)[M_1 M_3 K n^{1+\beta} T + M_4] \left| x_1 - x_2 \right|, \end{aligned}$$

and as b is bounded by M_1, we see that $|f(x_1, B_t^n)| \le M_1 \exp(M_2 K n^{1+\beta} T)$. □

Lemma 4.6. *Let f be defined by (4.7). Then for each $n = 1, 2, \cdots$,*

(1) $|f(x, B_t^n)| \leq M_1 \exp(M_2 \|B^n\|_\infty)$,
(2) $|f(x_1, B_t^n) - f(x_2, B_t^n)| \leq Z_1 |x_1 - x_2|$,
(3) $\left|f(x, B_{t_1}^n) - f(x, B_{t_2}^n)\right| \leq Z_2 \left|B_{t_1}^n - B_{t_2}^n\right|$,

with the random variables

$$Z_1 = \exp(2M_2 \|B^n\|_\infty)[M_1 M_3 \|B^n\|_\infty + M_4], \tag{4.25}$$

$$Z_2 = (M_1 M_2 + M_5 M_4) \exp(M_2 \|B^n\|_\infty). \tag{4.26}$$

Proof. (1) follows from (2.12), (4.17) and boundedness of b. (2) follows from (4.23). By (2.10), we have

$$\begin{aligned}\frac{\partial f(x, B_t^n)}{\partial x_2} =& - f(x, B_t^n)\sigma'(h(x, B_t^n)) \\ &+ \exp\left(-\int_0^{B_t^n} \sigma'(h(x, u))du\right) b'(h(x, B_t^n))\sigma(h(x, B_t^n)).\end{aligned}$$

Since σ, σ' and b' are bounded, and by part (1) and (4.17), then

$$\left|\frac{\partial f(x, B_t^n)}{\partial x_2}\right| \leq (M_1 M_2 + M_5 M_4) \exp(M_2 \|B^n\|_\infty),$$

and we have proved (3) by the mean value theorem. □

Now we do the approximation of h. For fixed n, we work in the square $[-n, n] \times [-n, n]$. Let $l = n + m$ for some $m > 0$, and consider the finer partition of $[-n, n]$ given by $-n = y_{-nl}^l < \cdots < y_0^l = 0 < \cdots < y_{nl}^l = n$, as in (4.2) with $r_l = 1/l$.

Lemma 4.7. *Let h and h^l be given by (4.1) and (4.3), respectively. Then for $(x, y) \in [-n, n] \times [-n, n]$ and $l > n$,*

$$\left|h(x, y) - h^l(x, y)\right| \leq \bar{M}^2 \frac{n}{l} \exp(\bar{M} n),$$

where $\bar{M} = \max\{M_2, M_5\}$ (see (4.15)).

Proof. Assume $y > 0$ (the case $y < 0$ is proved similarly).

If $0 = y_0^l < y \leq y_1^l$,

$$h(x, y) = x + \int_{y_0^l}^y \sigma(h(x, s))ds$$

and

$$h^l(x, y) = h^l(x, y_0^l) + (y - y_0^l)\sigma(h^l(x, y_0^l)) = x + \int_{y_0^l}^y \sigma(h^l(x, y_0^l))ds.$$

Then, using that σ is a Lipschitz function,

$$\begin{aligned}|h(x, y) - h^l(x, y)| &\leq \int_0^y |\sigma(h(x, s)) - \sigma(h^l(x, y_0^l))|ds \\ &\leq \bar{M} \int_0^y |h(x, s) - h^l(x, s)|ds + \bar{M} \int_0^y |h^l(x, s) - h^l(x, y_0^l)|ds.\end{aligned} \tag{4.27}$$

Since, for $y_0^l \leq s \leq y_1^l$,

$$\begin{aligned} |h^l(x,s) - h^l(x,y_0^l)| &\leq |h^l(x,y_0^l) + (s-y_0^l)\sigma(h^l(x,y_0^l)) - h^l(x,y_0^l)| \\ &\leq |s-y_0^l||\sigma(h^l(x,y_0^l))| \leq \bar{M}(y_1^l - y_0^l), \end{aligned}$$

then

$$\bar{M}\int_{y_0^l}^{y} |h^l(x,s) - h^l(x,y_0^l)|ds \leq \bar{M}^2(y_1^l - y_0^l)^2 = \bar{M}^2 r_l^2. \tag{4.28}$$

By (4.27), (4.28), and Gronwall's lemma, for $0 = y_0^l < y \leq y_1^l$,

$$\begin{aligned} |h(x,y) - h^l(x,y)| &\leq \bar{M}^2 r_l^2 \exp(\bar{M}(y-y_0^l)) \leq \bar{M}^2 r_l^2 \exp(\bar{M}(y_1^l - y_0^l)) \\ &\leq \bar{M}^2 r_l^2 \exp(\bar{M}n). \end{aligned} \tag{4.29}$$

We will prove by induction that for $k = 0, \cdots, nl-1$, if $y_k^l < y \leq y_{k+1}^l$, then

$$|h(x,y) - h^l(x,y)| \leq \bar{M}^2 r_l^2 \left[\exp(\bar{M}(y_{k+1}^l - y_0^l)) + \cdots + \exp(\bar{M}(y_{k+1}^l - y_k^l))\right]. \tag{4.30}$$

For $k = 0$ we have the result by (4.29). For $y_k^l < y \leq y_{k+1}^l$,

$$\begin{aligned} h(x,y) &= x + \int_{y_0^l}^{y_k^l} \sigma(h(x,s))ds + \int_{y_k^l}^{y} \sigma(h(x,s))ds \\ &= h(x,y_k^l) + \int_{y_k^l}^{y} \sigma(h(x,s))ds, \end{aligned}$$

and

$$h^l(x,y) = h^l(x,y_k^l) + (y-y_k^l)\sigma(h^l(x,y_k^l)) = h^l(x,y_k^l) + \int_{y_k^l}^{y} \sigma(h^l(x,y_k^l))ds.$$

By induction on k, (4.30), and σ being Lipschitz,

$$\begin{aligned} |h(x,y) - h^l(x,y)| \leq& |h(x,y_k^l) - h^l(x,y_k^l)| + \int_{y_k^l}^{y} \left|\sigma(h(x,s)) - \sigma(h^l(x,y_k^l))\right| ds \\ \leq& \bar{M}^2 r_l^2 \left[\exp(\bar{M}(y_k^l - y_0^l)) + \cdots + \exp(\bar{M}(y_k^l - y_{k-1}^l))\right] \\ &+ \bar{M}\left(\int_{y_k^l}^{y} \left|h(x,s) - h^l(x,s)\right| ds + \int_{y_k^l}^{y} \left|h^l(x,s) - h^l(x,y_k^l)\right| ds\right). \end{aligned} \tag{4.31}$$

Since, for $y_k^l \leq s \leq y_{k+1}^l$,

$$\begin{aligned} \left|h^l(x,s) - h^l(x,y_k^l)\right| &= |h^l(x,y_k^l) + (s-y_k^l)\sigma(h^l(x,y_k^l)) - h^l(x,y_k^l)| \\ &\leq |s-y_k^l||\sigma(h^l(x,y_k^l))| \leq \bar{M}(y_{k+1}^l - y_k^l), \end{aligned}$$

then

$$\int_{y_k^l}^{y} \left|h^l(x,s) - h^l(x,y_k^l)\right| ds \leq \bar{M}(y_{k+1}^l - y_k^l)^2 = \bar{M} r_l^2. \tag{4.32}$$

By (4.31), (4.32), and Gronwall's lemma,

$$\begin{aligned} &|h(x,y) - h^l(x,y)| \\ \leq& \bar{M}^2 r_l^2 \left[\exp(\bar{M}(y_k^l - y_0^l)) + \cdots + \exp(\bar{M}(y_k^l - y_{k-1}^l)) + 1\right] \exp\left(\bar{M}(y_{k+1}^l - y_k^l)\right) \\ =& \bar{M}^2 r_l^2 \left[\exp(\bar{M}(y_{k+1}^l - y_0^l)) + \cdots + \exp(\bar{M}(y_{k+1}^l - y_k^l))\right]. \end{aligned}$$

Then for all $(x,y)\in[-n,n]\times[-n,n]$ and $l>n$, there is some $k\in\{0,\cdots,nl-1\}$ such that $y_k^l<y\le y_{k+1}^l$, and by (4.30),

$$\begin{aligned}|h(x,y)-h^l(x,y)| &\le \bar{M}^2 r_l^2\left[\exp(\bar{M}(y_{k+1}^l-y_0^l))+\cdots+\exp(\bar{M}(y_{k+1}^l-y_k^l))\right]\\ &\le \bar{M}^2 r_l^2(k+1)\exp(\bar{M}n)\le \bar{M}^2 r_l^2 nl\exp(\bar{M}n)\\ &= \bar{M}^2\frac{n}{l}\exp(\bar{M}n).\end{aligned}$$

Hence the lemma is proved. □

Next we do the approximation of Y. We denote

$$\alpha_n = n^{-1/2+\beta+\delta}(\log n)^{5/2}, \tag{4.33}$$

with β,δ such that $|H-1/2|<\beta<1/2$, $0<\delta<\beta$ and $\beta+\delta<1/2$.

Proposition 4.8. *Let Y and Y^n be the processes given by (4.4) and (4.5), respectively. Then*

$$P\left(\limsup_{n\to\infty}\{\|Y-Y^n\|_\infty>\alpha_n\}\right)=0,$$

where α_n is defined by (4.33).

Proof.

$$\begin{aligned}|Y_t-Y_t^n| &= \left|\int_0^t\left(\frac{\partial h}{\partial x_1}(Y_s,B_s^H)\right)^{-1}b(h(Y_s,B_s^H))ds\right.\\ &\qquad\left.-\int_0^t\left(\frac{\partial h}{\partial x_1}(Y_s^n,B_s^n)\right)^{-1}b(h(Y_s^n,B_s^n))ds\right|\\ &\le \int_0^t I_1(s)ds+\int_0^t I_2(s)ds,\end{aligned}\tag{4.34}$$

where

$$I_1(s)=\left|\left(\frac{\partial h}{\partial x_1}(Y_s,B_s^H)\right)^{-1}-\left(\frac{\partial h}{\partial x_1}(Y_s^n,B_s^n)\right)^{-1}\right|\left|b(h(Y_s^n,B_s^n))\right|, \tag{4.35}$$

and

$$I_2(s)=\left|\left(\frac{\partial h}{\partial x_1}(Y_s,B_s^H)\right)^{-1}\right|\left|b(h(Y_s,B_s^H))-b(h(Y_s^n,B_s^n))\right|. \tag{4.36}$$

As b is bounded by M_1, then

$$\int_0^t I_1(s)ds\le M_1\int_0^t\left|\left(\frac{\partial h}{\partial x_1}(Y_s,B_s^H)\right)^{-1}-\left(\frac{\partial h}{\partial x_1}(Y_s^n,B_s^n)\right)^{-1}\right|ds. \tag{4.37}$$

Denoting

$$F(x_1,x_2)=\left(\frac{\partial h}{\partial x_1}(x_1,x_2)\right)^{-1}=\exp\left(-\int_0^{x_2}\sigma'(h(x_1,u))du\right),$$

(see (2.12)), we have

$$\left|\left(\frac{\partial h}{\partial x_1}(Y_s, B_s^H)\right)^{-1} - \left(\frac{\partial h}{\partial x_1}(Y_s^n, B_s^n)\right)^{-1}\right| \le I_3(s) + I_4(s), \tag{4.38}$$

where

$$I_3(s) = \left|F(Y_s, B_s^H) - F(Y_s^n, B_s^H)\right|,$$
$$I_4(s) = \left|F(Y_s^n, B_s^H) - F(Y_s^n, B_s^n)\right|.$$

By (4.20),

$$I_3(s) \le M_3||B^H||_\infty \exp(2M_2||B^H||_\infty)|Y_s - Y_s^n|. \tag{4.39}$$

Since σ' is bounded by M_2, and by (4.17),

$$\left|\frac{\partial F}{\partial x_2}(x_1, x_2)\right| = \left|\exp\left(-\int_0^{x_2} \sigma'(h(x_1, u))du\right)\right| |\sigma'(h(x_1, x_2))| \le M_2 \exp(M_2|x_2|).$$

By (2.8) there is measurable subset A of the underlying sample space with $P(A) = 1$, and for each $\omega \in A$ there is a positive integer $\hat{N} = \hat{N}(\omega)$ such that

$$||B^H(\omega) - B^n(\omega)||_\infty < 1 \quad \text{for all} \quad n > \hat{N}. \tag{4.40}$$

For each $\omega \in A$ fixed and $n > \hat{N}(\omega)$, by the mean value theorem, for some $\bar{r}(\omega)$ between $B_s^H(\omega)$ and $B_s^n(\omega)$, $|\bar{r}(\omega)| \le 1 + ||B^H(\omega)||_\infty$, thus (omitting ω)

$$\begin{aligned} I_4(s) &= \left|\frac{\partial F}{\partial x_2}(Y_s^n, \bar{r})\right| |B_s^H - B_s^n| \le M_2 \exp(M_2|\bar{r}|)|B_s^H - B_s^n| \\ &\le M_2 \exp(M_2(1 + ||B^H||_\infty))||B^H - B^n||_\infty \quad \text{for} \quad n > \hat{N}. \end{aligned} \tag{4.41}$$

From (4.37), (4.38), (4.39) and (4.41), for $n > \hat{N}$,

$$\begin{aligned} \int_0^t I_1(s)ds \le M_1 \int_0^t &[M_3||B^H||_\infty \exp(2M_2||B^H||_\infty)|Y_s - Y_s^n| \\ &+ M_2 \exp(M_2(1 + ||B^H||_\infty))|B_s^H - B_s^n|]ds. \end{aligned} \tag{4.42}$$

Now, by (4.17) and (4.36),

$$I_2(s) \le \exp(M_2||B^H||_\infty)(I_5(s) + I_6(s)), \tag{4.43}$$

where

$$I_5(s) = \left|b(h(Y_s, B_s^H)) - b(h(Y_s^n, B_s^H))\right|,$$
$$I_6(s) = \left|b(h(Y_s^n, B_s^H)) - b(h(Y_s^n, B_s^n))\right|.$$

From (4.21) and (4.22),

$$I_5(s) \le M_4 \exp(M_2||B^H||_\infty)|Y_s - Y_s^n|, \tag{4.44}$$

$$I_6(s) \le M_4 M_5|B_s^H - B_s^n|. \tag{4.45}$$

By (4.43), (4.44) and (4.45),

$$\begin{aligned} \int_0^t I_2(s)ds \le \int_0^t &\exp(M_2||B^H||_\infty)\left[M_4 \exp(M_2||B^H||_\infty)|Y_s - Y_s^n|\right. \\ &\left.+ M_4 M_5|B_s^H - B_s^n|\right] ds. \end{aligned} \tag{4.46}$$

Therefore, for $\omega \in A$ there is $\hat{N} = \hat{N}(\omega)$, as above, such that for $n > \hat{N}$, by (4.34), (4.42) and (4.46),

$$|Y_t - Y_t^n| \leq \alpha ||B^H - B^n||_\infty + \kappa \int_0^t |Y_s - Y_s^n| ds,$$

where α and κ are the random variables

$$\alpha = (TM_1 M_2 \exp(M_2) + TM_4 M_5) \exp(M_2 ||B^H||_\infty),$$

$$\kappa = (M_1 M_3 ||B^H||_\infty + M_4) \exp(2M_2 ||B^H||_\infty).$$

By Gronwall's lemma

$$|Y_t - Y_t^n| \leq Z ||B^H - B^n||_\infty,$$

where Z is the random variable

$$Z = \alpha \exp(T\kappa).$$

Then, for $n > \hat{N}$,

$$\|Y - Y^n\|_\infty \leq Z ||B^H - B^n||_\infty,$$

and it follows that on A,

$$\limsup_{n\to\infty} \{\|Y - Y^n\|_\infty > \alpha_n\} \subseteq \limsup_{n\to\infty} \left\{Z ||B^H - B^n||_\infty > \alpha_n\right\}.$$

Hence

$$P\left(\limsup_{n\to\infty} \{\|Y - Y^n\|_\infty > \alpha_n\}\right) \leq P\left(\limsup_{n\to\infty} \left\{Z ||B^H - B^n||_\infty > \alpha_n\right\}\right). \tag{4.47}$$

Next we show that

$$\limsup_{n\to\infty} \left\{Z ||B^H - B^n||_\infty > \alpha_n\right\} \subseteq \limsup_{n\to\infty} \left\{||B^H - B^n||_\infty > n^{-1/2+\beta} (\log n)^{5/2}\right\}.$$

Since

$$\begin{aligned} &\limsup_{n\to\infty} \left\{Z ||B^H - B^n||_\infty > n^{-1/2+\beta+\delta} (\log n)^{5/2}\right\} \\ &= \limsup_{n\to\infty} \left\{(Zn^{-\delta}) ||B^H - B^n||_\infty > n^{-1/2+\beta} (\log n)^{5/2}\right\}, \end{aligned}$$

then, by (2.8) we have

$$\begin{aligned} &P\left(\limsup_{n\to\infty} \left\{Z ||B^H - B^n||_\infty > \alpha_n\right\}\right) \\ &\leq P\left(\limsup_{n\to\infty} \left\{||B^H - B^n||_\infty > n^{-1/2+\beta} (\log n)^{5/2}\right\}\right) = 0. \end{aligned} \tag{4.48}$$

The proof is finished by (4.47) and (4.48). □

Now we do the approximation of Y^n.

Lemma 4.9. *For every n, m and $Y^{n,m}$ defined by (4.9),*

$$|Y_t^{n,m}| \leq |x_0| + TM_1 \exp(M_2 \|B^n\|_\infty), \quad t \in [0, T]. \tag{4.49}$$

Proof. As b and σ are bounded, and by (4.17), then, for f^n defined by (4.8),

$$|f^n(Y_t^{n,m}, B_t^n)| \le M_1 \exp(M_2 \|B^n\|_\infty). \tag{4.50}$$

We will first prove that for all $k = 0, \cdots, m-1$,

$$|Y_{t_k}^{n,m}| \le |x_0| + k r_m M_1 \exp(M_2 \|B^n\|_\infty). \tag{4.51}$$

For $k = 0$ it is obvious. If we assume that

$$|Y_{t_{k-1}}^{n,m}| \le |x_0| + (k-1) r_m M_1 \exp(M_2 \|B^n\|_\infty),$$

then from (4.50) and (4.9),

$$\begin{aligned} |Y_{t_k}^{n,m}| &= |Y_{t_{k-1}}^{n,m} + r_m f^n(Y_{t_{k-1}}^{n,m}, B_{t_{k-1}}^n)| \\ &\le |x_0| + (k-1) r_m M_1 \exp(M_2 \|B^n\|_\infty) + r_m M_1 \exp(M_2 \|B^n\|_\infty) \\ &= |x_0| + k r_m M_1 \exp(M_2 \|B^n\|_\infty). \end{aligned}$$

Now, if $t_k \le t < t_{k+1}$, by (4.50) and (4.51),

$$\begin{aligned} |Y_t^{n,m}| &= |Y_{t_k}^{n,m} + (t - t_k) f^n(Y_{t_k}^{n,m}, B_{t_k}^n)| \\ &\le |x_0| + k r_m M_1 \exp(M_2 \|B^n\|_\infty) + r_m M_1 \exp(M_2 \|B^n\|_\infty) \\ &= |x_0| + (k+1) r_m M_1 \exp(M_2 \|B^n\|_\infty) \\ &\le |x_0| + m r_m M_1 \exp(M_2 \|B^n\|_\infty), \end{aligned}$$

and (4.49) is obtained. □

Lemma 4.10. *There exists $N = N(\omega) > 0$ such that for every n, m and $t \in [0, T]$,*

$$(Y_t^{n,m}, B_t^n) \in [-N, N] \times [-N, N] \quad a.s. \tag{4.52}$$

Proof. Let $\omega \in A$ and $\hat{N} = \hat{N}(\omega)$ (see (4.40)), then for all $n > \hat{N}$,

$$\|B^n\|_\infty \le 1 + \|B^H\|_\infty \tag{4.53}$$

and by the Lemma 4.2, for $n \le \hat{N}$,

$$\|B^n\|_\infty \le K \hat{N}^{1+\beta} T. \tag{4.54}$$

Hence, by (4.49),

$$|Y_t^{n,m}| \le \begin{cases} |x_0| + T M_1 \exp(M_2(1 + ||B^H||_\infty)) & \text{if } \ n > \hat{N}, \\ |x_0| + T M_1 \exp(M_2 K \hat{N}^{1+\beta} T) & \text{if } \ n \le \hat{N}. \end{cases} \tag{4.55}$$

Taking

$$N = N(\omega) = \max \left\{ \begin{matrix} 1 + ||B^H||_\infty, K\hat{N}^{1+\beta}T, |x_0| + TM_1 \exp(M_2(1 + ||B^H||_\infty)), \\ |x_0| + TM_1 \exp(M_2 K \hat{N}^{1+\beta} T) \end{matrix} \right\}, \tag{4.56}$$

we then have the result by (4.53), (4.54), (4.55) and (4.56). □

Corollary 4.11. *Let N be given by Lemma 4.10. Then for all $n > N$ and $t \in [0, T]$,*

$$|h(Y_t^{n,m}, B_t^n) - h^n(Y_t^{n,m}, B_t^n)| \le \bar{M}^2 \frac{N}{n} \exp(\bar{M} N),$$

where $\bar{M} = \max\{M_2, M_5\}$.

Proof. The assertion follows from Lemma 4.7 and (4.52). □

Lemma 4.12. *For every $n > N$, m and $t \in [0, T]$,*

$$|Y_t^n - Y_t^{n,m}| \leq J_{n,m} T \left(\exp\left(Z_1 T\right)\right) \quad a.s., \tag{4.57}$$

where $J_{n,m}$ is the random variable

$$\begin{aligned} J_{n,m} =& Z_1 M_1 \exp(M_2 \left\|B^n\right\|_\infty) r_m + Z_2 K n^{1+\beta} r_m \\ &+ \bar{M}^2 \exp(M_2 \|B^n\|_\infty) \exp\left(\bar{M} N\right)(M_1 M_3 \|B^n\|_\infty + M_4) \frac{N}{n}, \end{aligned} \tag{4.58}$$

and Z_1, Z_2 and N are the random variables defined by (4.25), (4.26) and (4.56) respectively, and $r_m = T/m$.

Proof. First, if $0 = t_0 < t \leq t_1$, by (4.5), (4.7) and (4.9),

$$|Y_t^n - Y_t^{n,m}| \leq \int_{t_0}^t |f(Y_s^n, B_s^n) - f^n(Y_{t_0}^{n,m}, B_{t_0}^n)| ds, \tag{4.59}$$

and

$$\begin{aligned} &|f(Y_s^n, B_s^n) - f^n(Y_{t_0}^{n,m}, B_{t_0}^n)| \\ &\leq |f(Y_s^n, B_s^n) - f(Y_s^{n,m}, B_s^n)| + |f(Y_s^{n,m}, B_s^n) - f(Y_{t_0}^{n,m}, B_s^n)| \\ &\quad + |f(Y_{t_0}^{n,m}, B_s^n) - f(Y_{t_0}^{n,m}, B_{t_0}^n)| + |f(Y_{t_0}^{n,m}, B_{t_0}^n) - f^n(Y_{t_0}^{n,m}, B_{t_0}^n)|. \end{aligned} \tag{4.60}$$

By part (2) of Lemma 4.6,

$$|f(Y_s^n, B_s^n) - f(Y_s^{n,m}, B_s^n)| \leq Z_1 |Y_s^n - Y_s^{n,m}|. \tag{4.61}$$

Analogously, and by (4.9) and (4.50), for $s \leq t_1$,

$$\begin{aligned} |f(Y_s^{n,m}, B_s^n) - f(Y_{t_0}^{n,m}, B_s^n)| &\leq Z_1 |Y_s^{n,m} - Y_{t_0}^{n,m}| \\ &\leq Z_1 (s - t_0) |f^n(Y_{t_0}^{n,m}, B_{t_0}^n)| \leq Z_1 M_1 \exp(M_2 \left\|B^n\right\|_\infty) r_m. \end{aligned} \tag{4.62}$$

By Lemmas 4.2 and 4.6, for $t_0 < s < t_1$,

$$\begin{aligned} |f(Y_{t_0}^{n,m}, B_s^n) - f(Y_{t_0}^{n,m}, B_{t_0}^n)| &\leq Z_2 |B_s^n - B_{t_0}^n| \leq Z_2 K n^{1+\beta} (s - t_0) \\ &\leq Z_2 K n^{1+\beta} r_m. \end{aligned} \tag{4.63}$$

Moreover,

$$|f(Y_{t_0}^{n,m}, B_{t_0}^n) - f^n(Y_{t_0}^{n,m}, B_{t_0}^n)| \leq I_7 + I_8, \tag{4.64}$$

where

$$\begin{aligned} I_7 &= |b(h(Y_{t_0}^{n,m}, B_{t_0}^n))| \left| \exp\left(- \int_0^{B_{t_0}^n} \sigma'(h(Y_{t_0}^{n,m}, u)) du \right) \right. \\ &\qquad \left. - \exp\left(- \int_0^{B_{t_0}^n} \sigma'(h^n(Y_{t_0}^{n,m}, u)) du \right) \right|, \end{aligned}$$

$$I_8 = \left| \exp\left(- \int_0^{B_{t_0}^n} \sigma'(h^n(Y_{t_0}^{n,m}, u)) du \right) \right| |b(h(Y_{t_0}^{n,m}, B_{t_0}^n)) - b(h^n(Y_{t_0}^{n,m}, B_{t_0}^n))|.$$

Applying the mean value theorem for the exponential function and σ', by Lemmas 4.7 and 4.10, (4.17), and boundedness of b, σ', then, for $n > N$,

$$\begin{aligned} I_7 &\leq M_1 M_3 \exp(M_2 \|B^n\|_\infty) \int_0^{|B^n_{t_0}|} \left| h(Y^{n,m}_{t_0}, u) - h^n(Y^{n,m}_{t_0}, u) \right| du \\ &\leq M_1 M_3 \exp(M_2 \|B^n\|_\infty) \int_0^{|B^n_{t_0}|} \bar{M}^2 \frac{N}{n} \exp(\bar{M}N) du \\ &\leq M_1 M_3 \bar{M}^2 \exp(M_2 \|B^n\|_\infty) \exp(\bar{M}N) \|B^n\|_\infty \frac{N}{n}. \end{aligned} \tag{4.65}$$

By Corollary 4.11, (4.17) and b being Lipschitz, for $n > N$,

$$\begin{aligned} I_8 &\leq M_4 \exp(M_2 \|B^n\|_\infty) \left| h(Y^{n,m}_{t_0}, B^n_{t_0}) - h^n(Y^{n,m}_{t_0}, B^n_{t_0}) \right| \\ &\leq M_4 \bar{M}^2 \exp(M_2 \|B^n\|_\infty) \exp(\bar{M}N) \frac{N}{n}. \end{aligned} \tag{4.66}$$

From (4.64), (4.65) and (4.66),

$$\begin{aligned} &|f(Y^{n,m}_{t_0}, B^n_{t_0}) - f^n(Y^{n,m}_{t_0}, B^n_{t_0})| \\ &\leq \bar{M}^2 \exp(M_2 \|B^n\|_\infty) \exp(\bar{M}N)(M_1 M_3 \|B^n\|_\infty + M_4) \frac{N}{n}. \end{aligned} \tag{4.67}$$

By (4.60), (4.61), (4.62), (4.63), (4.67) and (4.58), for $s \in [0, T]$,

$$|f(Y^n_s, B^n_s) - f^n(Y^{n,m}_{t_0}, B^n_{t_0})| \leq Z_1 |Y^n_s - Y^{n,m}_s| + J_{n,m}. \tag{4.68}$$

From (4.59), (4.68), for $0 < t \leq t_1$,

$$|Y^n_t - Y^{n,m}_t| \leq \int_{t_0}^t Z_1 |Y^n_s - Y^{n,m}_s| ds + \int_{t_0}^t J_{n,m} ds,$$

and by Gronwall's lemma, for $t_0 < t \leq t_1$ and $r_m = T/m$,

$$|Y^n_t - Y^{n,m}_t| \leq J_{n,m} r_m \exp(Z_1(t_1 - t_0)). \tag{4.69}$$

If $t_1 < t \leq t_2$, then

$$\begin{aligned} |Y^n_t - Y^{n,m}_t| &= \left| Y^n_{t_1} + \int_{t_1}^t f(Y^n_s, B^n_s) ds - Y^{n,m}_{t_1} - \int_{t_1}^t f^n(Y^{n,m}_{t_1}, B^n_{t_1}) ds \right| \\ &\leq |Y^n_{t_1} - Y^{n,m}_{t_1}| + \int_{t_1}^t |f(Y^n_s, B^n_s) - f^n(Y^{n,m}_{t_1}, B^n_{t_1})| ds. \end{aligned} \tag{4.70}$$

Proceeding similarly, taking t_1 instead of t_0, from (4.68),

$$|f(Y^n_s, B^n_s) - f^n(Y^{n,m}_{t_1}, B^n_{t_1})| \leq Z_1 |Y^n_s - Y^{n,m}_s| + J_{n,m}. \tag{4.71}$$

From (4.69), (4.70) and (4.71), for $t_1 \leq t \leq t_2$,

$$\begin{aligned} |Y^n_t - Y^{n,m}_t| &\leq J_{n,m} r_m \exp(Z_1(t_1 - t_0)) + \int_{t_1}^t Z_1 |Y^n_s - Y^{n,m}_s| ds + \int_{t_1}^t J_{n,m} ds \\ &\leq J_{n,m} r_m (\exp(Z_1(t_1 - t_0)) + 1) + \int_{t_1}^t Z_1 |Y^n_s - Y^{n,m}_s| ds, \end{aligned}$$

and by Gronwall's lemma,

$$\begin{aligned} |Y_t^n - Y_t^{n,m}| &\leq J_{n,m} r_m \left(\exp\left(Z_1(t_1 - t_0)\right) + 1\right) \exp\left(Z_1(t_2 - t_1)\right) \\ &\leq J_{n,m} r_m \left(\exp\left(Z_1(t_2 - t_0)\right) + \exp\left(Z_1(t_2 - t_1)\right)\right) \\ &\leq J_{n,m} r_m 2 \left(\exp\left(Z_1 T\right)\right). \end{aligned}$$

Analogously, for $t_k < t \leq t_{k+1}$, $k = 0, \cdots, m-1$,

$$\begin{aligned} &|Y_t^n - Y_t^{n,m}| \\ &\leq J_{n,m} r_m [\exp\left(Z_1(t_{k+1} - t_0)\right) + \exp\left(Z_1(t_{k+1} - t_1)\right) + \cdots + \exp\left(Z_1(t_{k+1} - t_k)\right)] \\ &\leq J_{n,m} r_m (k+1) \left(\exp\left(Z_1 T\right)\right) \leq J_{n,m} r_m m \left(\exp\left(Z_1 T\right)\right) \\ &= J_{n,m} T \left(\exp\left(Z_1 T\right)\right), \end{aligned}$$

which finishes the proof. □

Proposition 4.13. *Let Y^n and $Y^{n,m}$ be given by (4.5) and (4.9), respectively. Then*

$$P\Big(\limsup_{n\to\infty}\Big\{\|Y^n - Y^{n,n^2}\|_\infty > \alpha_n\Big\}\Big) = 0,$$

where α_n is given by (4.33).

Proof. By Lemma 4.10, there is $N = N(\omega)$ given by (4.56) such as for n, m,

$$(Y_t^{n,m}, B_t^n) \in [-N, N] \times [-N, N] \quad a.s.$$

The random variables Z_1 and Z_2 given by (4.25) and (4.26) are bounded uniformly in n, and defining $M = \max\{M_1, M_2, M_3, M_4, M_5\}$,

$$Z_1 \leq \exp(2MN)[M^2 N + M], \quad Z_2 \leq 2M^2 \exp(MN),$$

and by (4.58),

$$\begin{aligned} J_{n,m} \leq &\exp(3MN)(M^3 N + M^2) r_m + 2M^2 K \exp(MN) n^{1+\beta} r_m \\ &+ M^2 \exp(2MN)(M^2 N + M)\frac{N}{n}. \end{aligned} \tag{4.72}$$

From (4.57) and (4.72), for $n > N$ and taking $m = n^2$, then $r_m = T/n^2$ and

$$\begin{aligned} |Y_t^n - Y_t^{n,n^2}| &\leq J_{n,m} T \left(\exp\left(Z_1 T\right)\right) \\ &\leq \Big\{\exp(3MN)(M^3 N + M^2) r_m + 2M^2 K \exp(MN) n^{1+\beta} r_m \\ &\quad + M^2 \exp(2MN)(M^2 N + M)\frac{N}{n}\Big\} T \exp\big(T \exp(2MN)[M^2 N + M]\big) \\ &\leq Z_3 \frac{1}{n^{1-\beta}}, \end{aligned}$$

where the random variable Z_3 is defined by

$$\begin{aligned} Z_3 \quad = \quad &\big\{\exp(3MN)(M^3 N + M^2) T + 2M^2 K \exp(MN) T \\ &+ M^2 N \exp(2MN)(M^2 N + M)\big\} T \exp\big(T \exp(2MN)[M^2 N + M]\big). \end{aligned}$$

Therefore, we obtain

$$P\Big(\limsup_{n\to\infty}\{\|Y^n - Y^{n,n^2}\|_\infty > \alpha_n\}\Big)$$
$$\le P\Big(\limsup_{n\to\infty}\{Z_3 > n^{1-\beta}\alpha_n\}\Big) = P\Big(\limsup_{n\to\infty}\{Z_3 > n^{1/2+\delta}(\log n)^{5/2}\}\Big) = 0.$$

□

Corollary 4.14. *Let Y and $Y^{n,m}$ be given by (2.11) and (4.9), respectively. Then*

$$P\Big(\limsup_{n\to\infty}\{\|Y - Y^{n,n^2}\|_\infty > \alpha_n\}\Big) = 0,$$

where α_n is given by (4.33).

Proof. By the proofs of Propositions 4.8 and 4.13, and replacing Z and Z_3 by $2Z$ and $2Z_3$, respectively,

$$P\Big(\limsup_{n\to\infty}\{\|Y - Y^{n,n^2}\|_\infty > \alpha_n\}\Big)$$
$$\le P\Big(\limsup_{n\to\infty}\{\|Y - Y^{n}\|_\infty > \alpha_n/2\}\Big) + P\Big(\limsup_{n\to\infty}\{\|Y^n - Y^{n,n^2}\|_\infty > \alpha_n/2\}\Big)$$
$$= 0,$$

which completes the proof. □

We define for each $n = 1, 2, \ldots,$

$$\tilde{X}^n_t = h(Y^n_t, B^n_t), \tag{4.73}$$

where B^n and Y^n are given by (2.6)-(2.7) and (4.5), respectively.

Proposition 4.15. *For any $\tilde{C} > 0$,*

$$P\Big(\limsup_{n\to\infty}\{\|X - \tilde{X}^n\|_\infty > \tilde{C}\alpha_n\}\Big) = 0,$$

where X is given by (2.9) and α_n is given by (4.33).

Proof. For convenience of notation we put $\tilde{C} = 1$. Then

$$|X_t - \tilde{X}^n_t| = |h(Y_t, B^H_t) - h(Y^n_t, B^n_t)| \le I_9(t) + I_{10}(t), \tag{4.74}$$

where

$$I_9(t) = |h(Y_t, B^H_t) - h(Y^n_t, B^H_t)|$$
$$I_{10}(t) = |h(Y^n_t, B^H_t) - h(Y^n_t, B^n_t)|.$$

By (4.19),

$$I_9(t) \le Z_4\|Y - Y^n\|_\infty, \tag{4.75}$$

where $Z_4 = \exp(M_2 \left\|B^H\right\|_\infty)$,

Proceeding similary as in the proof of (4.48), then from Proposition 4.8,

$$P\Big(\limsup_{n\to\infty}\{Z_4 \|Y - Y^n\|_\infty > \alpha_n/2\}\Big)$$
$$\le \quad P\Big(\limsup_{n\to\infty}\{\|Y - Y^n\|_\infty > n^{-1/2+\beta+\delta/2}(\log n)^{5/2}\}\Big) \;=\; 0.$$

Thus by (4.75),

$$P\Big(\limsup_{n\to\infty}\{\|I_9\|_\infty > \alpha_n/2\}\Big) = 0. \tag{4.76}$$

From (2.10), boundedness of σ, and the mean value theorem,

$$I_{10}(t) \le M_5\|B^H - B^n\|_\infty,$$

and by (2.8),

$$P\Big(\limsup_{n\to\infty}\{\|I_{10}\|_\infty > \alpha_n/2\}\Big) = 0. \tag{4.77}$$

The result follows from (4.74), (4.76) and (4.77). □

For the final step of the proof of the theorem we go to (4.11). By (2.9), (4.73), (4.12) and Proposition 4.15,

$$P\Big(\limsup_{n\to\infty}\{\|H_1\|_\infty > \alpha_n/4\}\Big) = 0. \tag{4.78}$$

By (4.52) there is $N = N(\omega)$ given by (4.56) such that for each n, m,

$$(Y_t^{n,m}, B_t^n) \in [-N, N] \times [-N, N] \quad \text{a.s.} \tag{4.79}$$

From (4.13) and (4.19),

$$H_2(t) \le \exp(M_2\|B^n\|_\infty)|Y_t^n - Y_t^{n,n^2}| \le Z_5|Y_t^n - Y_t^{n,n^2}|, \tag{4.80}$$

where $Z_5 = \exp(M_2N)$. By Proposition 4.13 and (4.80),

$$\begin{aligned} P\Big(\limsup_{n\to\infty}\{\|H_2\|_\infty > \alpha_n/4\}\Big) &\le P\Big(\limsup_{n\to\infty}\{4Z_5\|Y^n - Y^{n,n^2}\|_\infty > \alpha_n\}\Big) \\ \le \quad & P\Big(\limsup_{n\to\infty}\{\|Y^n - Y^{n,n^2}\|_\infty > n^{-1/2+\beta+\delta/2}(\log n)^{5/2}\}\Big) = 0. \end{aligned} \tag{4.81}$$

On account of (4.79) and (4.14), for all $n > N$, applying Lemma 4.7,

$$H_3(t) \le Z_6\frac{1}{n},$$

where $Z_6 = \bar{M}^2N\exp(\bar{M}N)$. Then,

$$\begin{aligned} P\Big(\limsup_{n\to\infty}\{\|H_3\|_\infty > \alpha_n\}\Big) &\le P\Big(\limsup_{n\to\infty}\Big\{Z_6 > n\alpha_n\Big\}\Big) \\ &\le P\Big(\limsup_{n\to\infty}\{Z_6 > n^{1/2+\beta+\delta}(\log n)^{5/2}\}\Big) = 0. \end{aligned} \tag{4.82}$$

The final result follows from (4.11), (4.12), (4.13), (4.14), (4.78), (4.81) and (4.82).

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