

STATIONARY DISTRIBUTIONS OF THE BERNOULLI TYPE GALTON-WATSON BRANCHING PROCESS WITH IMMIGRATION

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ABSTRACT. In this paper, we describe a decomposition of the transition probability matrix of Bernoulli type Galton-Watson branching processes with immigration. Moreover we give the limiting distribution of this process by using this decomposition.

1. Introduction

Branching processes are mathematical models which are applied to the physical and biological sciences. The most famous branching process is a Galton-Watson branching process. In this paper we consider a discrete time Galton-Watson branching process with immigration. It is known that this process is a Markov chain whose state space is a countably infinite set. Discrete time Galton-Watson branching processes with immigration have been described in, for instance, [1, 5, 6, 7].

In the preceding studies, the concrete structure of the limiting distribution and the stationary distribution of the general Galton-Watson branching process with immigration has not been fully investigated. The goal of this study is to find these distributions. In this paper, we find these distributions for the Bernoulli type Galton-Watson branching process with immigration. This is the simplest case. However, it seems that even in this simplest case, to find these distributions is complicated because the transition probability matrix of this process is an $\infty \times \infty$ matrix. Therefore, we introduce a reflecting barrier. Then the transition probability matrix is reduced to a finite matrix. For this matrix, we construct an algorithm which simplifies the eigenpolynomial (Figure 2). These distributions are derived by a matrix decomposition based on this algorithm. If $\boldsymbol{\pi} \equiv (\pi_j)_{j \in \mathbb{N}_0}$ and $\hat{\boldsymbol{\pi}} \equiv (\hat{\pi}_j)_{j \in \mathbb{N}_0}$ represent the stationary distribution and the limiting distribution, respectively, then they can be expressed as:

$$\pi_j = \hat{\pi}_j = \sum_{i=j}^{\infty} (-1)^{i+j} \lambda_{0i}^{(\infty)} \binom{i}{j}, \quad (1.1)$$

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where $\binom{i}{j}$ is a binomial coefficient and

$$\lambda_{0j}^{(\infty)} = \begin{cases} 1, & \text{if } j = 0, \\ \prod_{k=1}^j \frac{p_1^{k-1} q_1}{1-p_1^k}, & \text{if } j \neq 0. \end{cases} \quad (1.2)$$

The paper is organized as follows. In section 2, we introduce the definition of the Bernoulli type Galton-Watson branching process with immigration. In section 3, we construct the reflecting barrier to reduce the countably infinite state space of this process to the finite set. In section 4, we illustrate the algorithm which simplifies the eigenpolynomial of the transition probability matrix of the Bernoulli type Galton-Watson branching process. In addition, the reflecting characteristic which decides the eigenvalues of that matrix is obtained. In section 5, we show the matrix decomposition based on the proposed algorithm illustrated in section 4. Moreover, the limiting distribution and the stationary distribution of the Bernoulli type Galton-Watson branching process with immigration are shown in this section.

2. The Galton-Watson Branching Process

In this section, we introduce some definitions related to the Galton-Watson branching process.

Definition 2.1. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\{X_{m,t}; m \in \mathbb{N}, t \in \mathbb{N}_0\}$ be \mathbb{N}_0 -valued independent and identically distributed (i.i.d.) random variables. For $t \in \mathbb{N}_0$, \hat{Y}_t is defined by

$$\hat{Y}_{t+1} = \sum_{m=1}^{\hat{Y}_t} X_{m,t}, \quad \hat{Y}_0 = x, \quad (2.1)$$

where x is an \mathbb{N}_0 -valued random variable which is independent of $\{X_{m,t}\}$. The sequence $\{\hat{Y}_t\}_{t \in \mathbb{N}_0}$ is called the *Galton-Watson branching process*.

Definition 2.2. Let $\{I_t; t \in \mathbb{N}_0\}$ be \mathbb{N}_0 -valued i.i.d. random variables. For $t \in \mathbb{N}_0$, Y_t is defined as follows:

$$Y_{t+1} = \sum_{m=1}^{Y_t} X_{m,t} + I_t, \quad Y_0 = x. \quad (2.2)$$

The sequence $\{Y_t\}_{t \in \mathbb{N}_0}$ is called the *Galton-Watson branching process with immigration*.

Note that the Galton-Watson branching process with immigration such that $P(I_t = 0) = 1$ for each t is the same as the simple Galton-Watson branching process. It is known that the process is a Markov chain. From the above definitions, we see that its state space is a countable infinite set. Moreover, the Galton-Watson branching process has an absorbing state. That is the state 0 because clearly $P(Y_{t+1} = 0 | Y_t = 0) = 1$. On the other hand, the general Galton-Watson branching process with immigration does not have any absorbing state. In this case, the state 0 is not the absorbing state since $P(I_t = 0) < 1$. We can deduce $P(Y_{t+1} = 0 | Y_t = 0) < 1$ from the condition $P(I_t = 0) < 1$.

Definition 2.3. The *Bernoulli type Galton-Watson branching process with immigration* is defined by $\{Y_t\}_{t \in \mathbb{N}_0}$ such that $X_{m,t}$ and I_t have Binomial distributions $B(1, p_1)$ and $B(1, q_1)$, respectively, i.e., for all $m \in \mathbb{N}$ and $t \in \mathbb{N}_0$,

$$P(X_{m,t}=k) = \begin{cases} p_0, & \text{if } k=0, \\ p_1, & \text{if } k=1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad P(I_t=k) = \begin{cases} q_0, & \text{if } k=0, \\ q_1, & \text{if } k=1, \\ 0, & \text{otherwise,} \end{cases}$$

where $p_0 + p_1 = 1$ and $q_0 + q_1 = 1$.

The transition probability from i to j is denoted by P_{ij} , i.e.,

$$P_{ij} \equiv P(Y_{t+1} = j | Y_t = i).$$

For all t , $P(Y_{t+1} = j | Y_t = i) = P(Y_t = j | Y_{t-1} = i)$ is deduced from Definition 2.3. Hence, this process is time-homogeneous. Moreover, from this definition, the transition probability P_{ij} can be expressed as follows:

$$P_{ij} = \begin{cases} \binom{i}{j-1} p_0^{i-j+1} p_1^{j-1} q_1 + \binom{i}{j} p_0^{i-j} p_1^j q_0, & \text{if } 0 \leq i \text{ and } 0 \leq j \leq i + 1, \\ 0, & \text{otherwise,} \end{cases} \quad (2.3)$$

where $\binom{a}{b}$ is a binomial coefficient which is defined by

$$\binom{a}{b} \equiv \begin{cases} \frac{a!}{b!(a-b)!}, & \text{if } 0 < b < a, \\ 1, & \text{if } 0 \leq a = b \text{ or } 0 < a \text{ and } b = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we obtain the following transition probability matrix:

$$\mathbf{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & \cdots & n+1 & \cdots \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ \vdots \\ n \\ \vdots \end{matrix} & \begin{pmatrix} P_{00} & P_{01} & 0 & \cdots & 0 & \cdots \\ P_{10} & P_{11} & P_{12} & \ddots & \vdots & \\ \vdots & \vdots & \vdots & \ddots & 0 & \cdots \\ P_{n0} & P_{n1} & P_{n2} & \cdots & P_{nn+1} & \ddots \\ \vdots & \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \end{matrix}. \quad (2.4)$$

Rows and columns of this matrix are numbered from 0, and correspond to states.

The purpose of this paper is to find a probability distribution $\boldsymbol{\pi} \equiv (\pi_j)_{j \in \mathbb{N}_0}$ such that

$$\boldsymbol{\pi} \mathbf{P} = \boldsymbol{\pi}.$$

A probability distribution satisfying the above equation is called a stationary distribution. Moreover, if we represent $\hat{\pi}_i^{(t)} \equiv P(Y_t = i)$ and $\hat{\boldsymbol{\pi}}^{(t)} \equiv (\hat{\pi}_j^{(t)})_{j \in \mathbb{N}_0}$, then $\hat{\boldsymbol{\pi}}^{(\infty)} \equiv \lim_{t \rightarrow \infty} \hat{\boldsymbol{\pi}}^{(t)}$ if it exists is called a limiting distribution. In the case of the time-homogeneous Markov chain, for $t > 0$, $\hat{\boldsymbol{\pi}}^{(t)}$ can be expressed as:

$$\hat{\boldsymbol{\pi}}^{(t)} = \hat{\boldsymbol{\pi}}^{(t-1)} \mathbf{P} = \hat{\boldsymbol{\pi}}^{(t-2)} \mathbf{P}^2 = \dots = \hat{\boldsymbol{\pi}}^{(0)} \mathbf{P}^t.$$

Therefore, we see that if $\lim_{t \rightarrow \infty} \mathbf{P}^t$ exists, then so does $\hat{\boldsymbol{\pi}}^{(\infty)}$.

3. The Reflecting Barrier

The Galton-Watson branching process with immigration has the state space $\mathcal{S} = \{0, 1, 2, \dots\}$. Thus, \mathbf{P} is an $\infty \times \infty$ transition probability matrix. In this case, to find the stationary distribution the following linear simultaneous equations with infinitely many unknowns must be solved:

$$\begin{aligned}
 P_{00}\pi_0 + P_{10}\pi_1 + P_{20}\pi_2 + \dots + P_{n-10}\pi_{n-1} + \dots &= \pi_0 \\
 P_{01}\pi_0 + P_{11}\pi_1 + P_{21}\pi_2 + \dots + P_{n-11}\pi_{n-1} + \dots &= \pi_1 \\
 P_{12}\pi_1 + P_{22}\pi_2 + \dots + P_{n-12}\pi_{n-1} + \dots &= \pi_2 \\
 P_{23}\pi_2 + \dots + P_{n-13}\pi_{n-1} + \dots &= \pi_3 \\
 &\dots\dots\dots \\
 P_{n-1n}\pi_{n-1} + \dots &= \pi_n \\
 &\dots\dots\dots
 \end{aligned}$$

However, this is difficult. Therefore, we apply a reflecting barrier to the Galton-Watson branching process with immigration. By the reflecting barrier, the state space is reduced to a finite set.

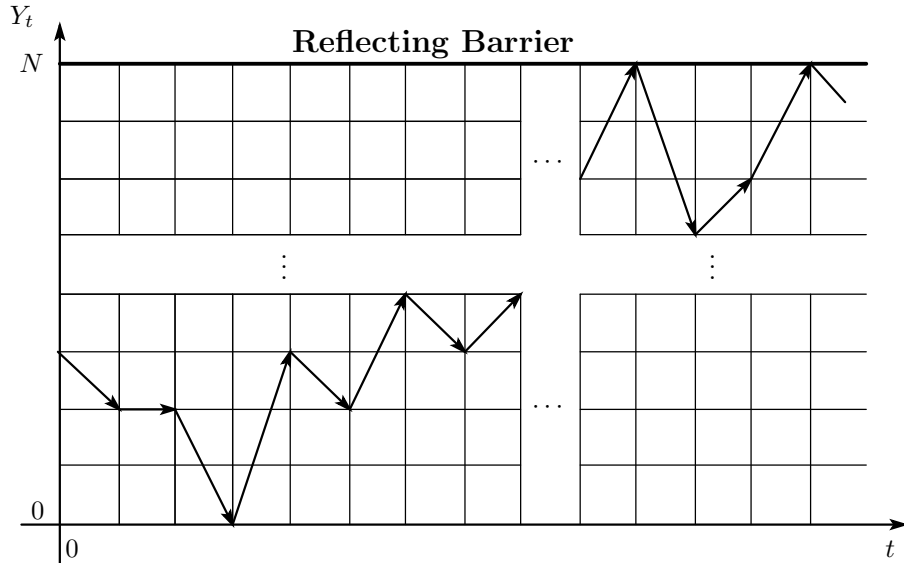


FIGURE 1. A Galton-Watson branching process with immigration which has a reflecting barrier. Arrows and lattice points represent transitions and states, respectively. In this case, there does not exist transitions from i to j , for all $i \in \{0, 1, \dots, N\}$ and $j > N$.

Example 3.1. For example, the reflecting barrier can be set by the following conditions:

- (1) $X_{m,t} \sim B(1, p_1)$.

- (2) (a) If $Y_t < N$, then $I_t \sim B(1, q_1)$.
- (b) If $Y_t = N$, then $I_t = 0$.

Then we have

$$Y_{t+1} = \begin{cases} \sum_{m=1}^{Y_t} X_{m,t} + I_t, & \text{if } Y_t < N, \\ \sum_{m=1}^{Y_t} X_{m,t}, & \text{if } Y_t = N. \end{cases}$$

Thus, if $Y_t < N$, then $Y_{t+1} \leq Y_t + 1$. On the other hand, if $Y_t = N$, then $Y_{t+1} \leq Y_t$. Moreover, the (i, j) entry of the transition probability matrix can be expressed as:

$$\hat{P}_{ij} = \begin{cases} P_{ij}, & \text{if } 0 \leq i \leq N - 1 \text{ and } 0 \leq j \leq i + 1, \\ \binom{N}{j} p_0^{N-j} p_1^j, & \text{if } i = N \text{ and } 0 \leq j \leq N. \end{cases}$$

Therefore, this Markov chain has the finite state space $\mathcal{S} = \{0, 1, \dots, N\}$.

From Example 3.1, we find that the state space can be reduced to a finite set by setting transition probabilities from N suitably. We call the state N and the set of transition probabilities from this state (which is $\{\hat{P}_{Nj}; j = 0, 1, \dots, N\}$) the *reflecting barrier* and a *reflecting characteristic*, respectively. In this case, the transition probability matrix is denoted by $\mathbf{P}(N)$. Thus:

$$\mathbf{P}(N) = \begin{pmatrix} P_{00} & P_{01} & 0 & \cdots & 0 & 0 \\ P_{10} & P_{11} & P_{12} & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ P_{N-10} & P_{N-11} & P_{N-12} & \cdots & P_{N-1N-1} & P_{N-1N} \\ \hat{P}_{N0} & \hat{P}_{N1} & \hat{P}_{N2} & \cdots & \hat{P}_{NN-1} & \hat{P}_{NN} \end{pmatrix}.$$

The reflecting barrier whose entries of reflecting characteristic are

$$\hat{P}_{Nj} = \delta_{Nj} \equiv \begin{cases} 0, & \text{if } j < N, \\ 1, & \text{if } j = N, \end{cases}$$

is an *absorbing barrier*. Then N is clearly an absorbing state.

4. Eigenvalues of the Transition Probability Matrix

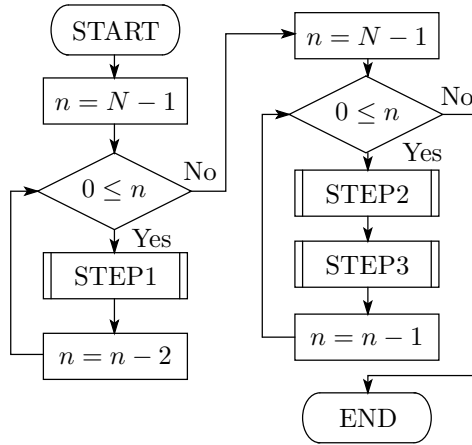
4.1. An Algorithm for Simplifying the Eigenpolynomial . In this section, we propose an algorithm for simplifying the eigenpolynomial $|\lambda \mathbf{I} - \mathbf{P}(N)|$. Flowcharts of the proposed method is shown in Figure 2. STEP1, STEP2 and STEP3 in this figure are illustrated as follows:

STEP1: Let $\mathbf{p}_i^{\text{row}}$ denote the i -th row of $\lambda \mathbf{I} - \mathbf{P}(N)$. For $i = N$ to $N - n$, $p_0 \mathbf{p}_{i-1}^{\text{row}}$ is subtracted from $\mathbf{p}_i^{\text{row}}$.

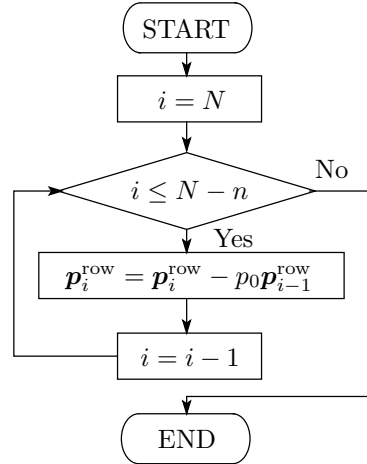
If we define

$$\mathbf{C}(N-1) \equiv \begin{matrix} & & 0 & & 1 & & \cdots & & N-1 \\ 0 & \left(\begin{array}{cccc} \binom{0}{0} p_0^0 p_1^0 & 0 & \cdots & 0 \\ \binom{1}{0} p_0^1 p_1^0 & \binom{1}{1} p_0^0 p_1^1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \binom{N-1}{0} p_0^{N-1} p_1^0 & \binom{N-1}{1} p_0^{N-2} p_1^1 & \cdots & \binom{N-1}{N-1} p_0^0 p_1^{N-1} \end{array} \right), & & & & \end{matrix}$$

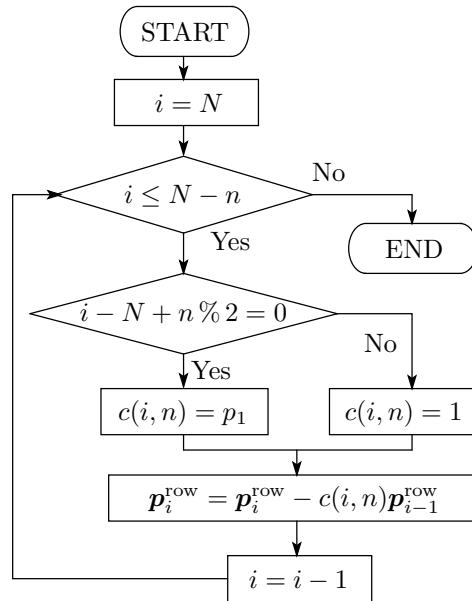
The Main Routine



STEP1



STEP2



STEP3

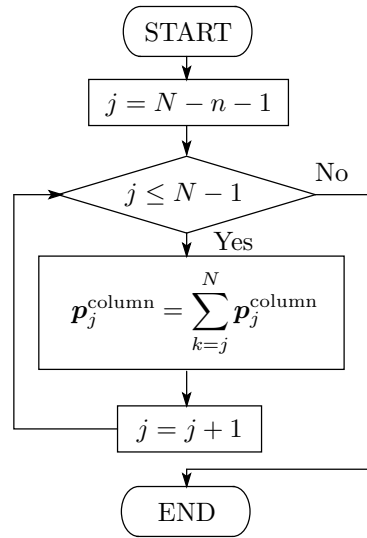


FIGURE 2. Flowcharts of the proposed method.

then the matrix $P(N)$ is expressed as:

$$P(N) = q_0 C^{\text{left}}(N) + q_1 C^{\text{right}}(N) + R, \tag{4.1}$$

where

$$C^{\text{left}}(N) \equiv \left(\begin{array}{c|c} C(N-1) & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \dots & 0 & 0 \end{array} \right), \quad C^{\text{right}}(N) \equiv \left(\begin{array}{c|c} 0 & \\ \vdots & C(N-1) \\ \hline 0 & 0 & \dots & 0 \end{array} \right)$$

and

$$R \equiv \left(\begin{array}{ccc} 0 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 0 \\ \hline \tilde{P}_{N0} & \dots & \tilde{P}_{NN} \end{array} \right).$$

Therefore, we apply STEP1 to each matrices in the equation (4.1) and λI .

Let $C^{\text{STEP1}}(N)$ be the matrix that $C^{\text{left}}(N)$ is processed by the left side of the main routine. Then the (i, j) entry of $C^{\text{STEP1}}(N)$ can be expressed as

$$C_{ij}^{\text{STEP1}}(N) = \begin{cases} \left(\binom{i}{j} - \sum_{k=1}^{\lfloor \frac{i+1}{2} \rfloor} \binom{i-k}{j-k+1} \right) p_0^{i-j} p_1^j, & \text{if } i < N \text{ and } 0 \leq j \leq i, \\ - \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N-1-k}{j-k} p_0^{N-j} p_1^j, & \text{if } i = N \text{ and } 0 \leq j < N, \\ 0, & \text{otherwise,} \end{cases}$$

where $\lfloor \cdot \rfloor$ is the floor function which is defined by $\lfloor x \rfloor \equiv \max\{n \in \mathbb{Z}; n \leq x\}$. In the above equation, if $i < N$ and $0 \leq j \leq i$, then

$$\begin{aligned} C_{ij}^{\text{STEP1}}(N) &= \left(\binom{i}{j} - \sum_{k=1}^{\lfloor \frac{i+1}{2} \rfloor} \binom{i-k}{j-k+1} \right) p_0^{i-j} p_1^j \\ &= \left(\binom{i-1}{j-1} - \sum_{k=2}^{\lfloor \frac{i+1}{2} \rfloor} \binom{i-k}{j-k+1} \right) p_0^{i-j} p_1^j \\ &= \left(\binom{i-2}{j-2} - \sum_{k=3}^{\lfloor \frac{i+1}{2} \rfloor} \binom{i-k}{j-k+1} \right) p_0^{i-j} p_1^j \\ &= \dots \\ &= \left(\binom{i - (\lfloor \frac{i+1}{2} \rfloor - 1)}{j - (\lfloor \frac{i+1}{2} \rfloor - 1)} - \binom{i - \lfloor \frac{i+1}{2} \rfloor}{j - \lfloor \frac{i+1}{2} \rfloor + 1} \right) p_0^{i-j} p_1^j \\ &= \binom{i - \lfloor \frac{i+1}{2} \rfloor}{j - \lfloor \frac{i+1}{2} \rfloor} p_0^{i-j} p_1^j. \end{aligned}$$

Moreover, for any j , the (N, j) entry of $C^{\text{STEP1}}(N)$ can be expressed as:

$$\begin{aligned} C_{Nj}^{\text{STEP1}}(N) &= - \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N-1-k}{j-k} p_0^{N-j} p_1^j \\ &= - \left(\sum_{k_1=0}^j \binom{N-1-k_1}{j-k_1} - \sum_{k_2=\lfloor \frac{N-1}{2} \rfloor+1}^j \binom{N-1-k_2}{j-k_2} \right) p_0^{N-j} p_1^j \\ &= - \left(\sum_{\hat{k}_1=0}^j \binom{N-1-j+\hat{k}_1}{\hat{k}_1} - \sum_{\hat{k}_2=0}^{j-\lfloor \frac{N-1}{2} \rfloor-1} \binom{N-1-j+\hat{k}_2}{\hat{k}_2} \right) p_0^{N-j} p_1^j. \end{aligned}$$

Noting that

$$\begin{aligned} \sum_{k=0}^m \binom{N+k}{k} &= \binom{N}{0} + \binom{N+1}{1} + \sum_{k=2}^m \binom{N+k}{k} \\ &= \binom{N+1}{0} + \binom{N+1}{1} + \sum_{k=2}^m \binom{N+k}{k} \\ &= \binom{N+2}{1} + \sum_{k=2}^m \binom{N+k}{k} \\ &= \binom{N+3}{2} + \sum_{k=3}^m \binom{N+k}{k} \\ &= \dots \\ &= \binom{N+m+1}{m}, \end{aligned}$$

we see that

$$- \sum_{k=0}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N-1-k}{j-k} p_0^{N-j} p_1^j = - \left(\binom{N}{j} - \binom{N-\lfloor \frac{N-1}{2} \rfloor-1}{j-\lfloor \frac{N-1}{2} \rfloor-1} \right) p_0^{N-j} p_1^j.$$

Therefore, we obtain the following:

$$C_{ij}^{\text{STEP1}} = \begin{cases} \binom{i-\lfloor \frac{i+1}{2} \rfloor}{j-\lfloor \frac{i+1}{2} \rfloor} p_0^{i-j} p_1^j, & \text{if } i < N \text{ and } \lfloor \frac{i+1}{2} \rfloor \leq j \leq i, \\ \left(\binom{N-\lfloor \frac{N-1}{2} \rfloor-1}{j-\lfloor \frac{N-1}{2} \rfloor-1} - \binom{N}{j} \right) p_0^{N-j} p_1^j, & \text{if } i = N \text{ and } 0 \leq j < N, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, if i is an odd number, then $C_{ij}^{\text{STEP1}} = p_1 C_{i-1, j-1}^{\text{STEP1}}$. For example, in the case of $N = 5$ we have

$$\begin{aligned} &C^{\text{STEP1}}(5) \\ &= \begin{pmatrix} \binom{0}{0} p_0^0 p_1^0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \binom{0}{0} p_0^0 p_1^1 & 0 & 0 & 0 & 0 \\ 0 & \binom{1}{0} p_0^1 p_1^1 & \binom{1}{1} p_0^0 p_1^2 & 0 & 0 & 0 \\ 0 & 0 & \binom{1}{0} p_0^1 p_1^2 & \binom{1}{1} p_0^0 p_1^3 & 0 & 0 \\ 0 & 0 & \binom{2}{0} p_0^2 p_1^2 & \binom{2}{1} p_0^1 p_1^3 & \binom{2}{2} p_0^0 p_1^4 & 0 \\ -\binom{5}{0} p_0^5 p_1^0 & -\binom{5}{1} p_0^4 p_1^1 & -\binom{5}{2} p_0^3 p_1^2 & \left(\binom{2}{0} - \binom{5}{3} \right) p_0^1 p_1^3 & \left(\binom{2}{1} - \binom{5}{4} \right) p_0^1 p_1^4 & 0 \end{pmatrix}. \end{aligned}$$

STEP2: For $i = N$ to $N - n$, $c(n, i) \mathbf{p}_{i-1}^{\text{row}}$ is subtracted from $\mathbf{p}_i^{\text{row}}$, where $c(n, i)$ is defined by

$$c(n, i) \equiv \begin{cases} p_1, & \text{if } i - N + n \text{ is the even number,} \\ 1, & \text{otherwise.} \end{cases}$$

In the flowchart of STEP2, $a \% b$ means the remainder of a/b .

STEP3: Let $\mathbf{p}_j^{\text{column}}$ denote the j -th column of $\lambda \mathbf{I} - \mathbf{P}(N)$. For $j = N - n - 1$ to $N - 1$, $\mathbf{p}_j^{\text{column}}$ is replaced by $\sum_{k=j}^N \mathbf{p}_k^{\text{column}}$.

For given n from the main routine, let $\mathbf{C}^{\text{STEP2}}(n)$ and $\mathbf{C}^{\text{STEP3}}(n)$ represent matrices which are calculated by STEP2 and STEP3, respectively. By setting $\mathbf{C}^{\text{STEP3}}(N) \equiv \mathbf{C}^{\text{STEP1}}(N)$, the (i, j) entry of $\mathbf{C}^{\text{STEP2}}(n)$ can be expressed as follows:

$$\begin{aligned} C_{ij}^{\text{STEP2}}(n) &= \begin{cases} p_1^i, & \text{if } i < N - n \text{ and } i = j; \\ p_1^{m(n)} (C_{i-m(n), j-m(n)}^{\text{STEP1}} - c(n, i) C_{i-m(n)-1, j-m(n)}^{\text{STEP1}}), & \text{if } N - n \leq i < N; \\ C_{Nj}^{\text{STEP3}}(n+1) - \binom{\lfloor \frac{n}{2} \rfloor}{N - \lfloor \frac{n}{2} \rfloor} p_0^{N-j} p_1^j, & \text{if } i = N \text{ and } 0 \leq j < N; \\ 0, & \text{otherwise;} \end{cases} \\ &= \begin{cases} p_1^i, & \text{if } i < N - n \text{ and } i = j, \\ \binom{i-\hat{m}(n)}{j-\hat{m}(n)} p_0^{i-j} p_1^j - \binom{i-\hat{m}(n)}{j-\hat{m}(n)+1} p_0^{i-j-1} p_1^{j+1}, & \text{if } N - n \leq i < N \text{ and } c(n, i) = p_1, \\ \binom{i-\hat{m}(n)}{j-\hat{m}(n)} p_0^{i-j} p_1^j - \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} p_0^{i-j-1} p_1^j, & \text{if } N - n \leq i < N \text{ and } c(n, i) = 1, \\ C_{Nj}^{\text{STEP3}}(n+1) - \binom{\lfloor \frac{n}{2} \rfloor}{N - \lfloor \frac{n}{2} \rfloor} p_0^{N-j} p_1^j, & \text{if } i = N \text{ and } 0 \leq j < N, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

where $\hat{m}(n) \equiv \lfloor \frac{i+1}{2} \rfloor + m(n)$ and $m(n) \equiv N - n - 1$. Moreover, from the flowchart of STEP3, we have

$$C_{ij}^{\text{STEP3}}(n) = \begin{cases} p_1^i, & \text{if } i < m(n) \text{ and } i = j, \\ \sum_{k=j}^{N-1} C_{ik}^{\text{STEP2}}(n), & \text{if } m(n) \leq i < N \text{ and } m(n) \leq j < N, \\ \hat{C}_j(n), & \text{if } i = N \text{ and } 0 \leq j < N, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\hat{C}_j(n) = \begin{cases} -\binom{N}{j} p_1^N - \sum_{k=j}^{N-1} \binom{k}{j} \binom{N}{k} p_0^{N-k} p_1^k, & \text{if } j \leq m(n), \\ -\binom{N-j+m(n)}{j} p_1^N - \sum_{k=j}^{N-1} \binom{k-j+m(n)}{m(n)} \binom{N}{k} p_0^{N-k} p_1^k, & \text{if } m(n) < j < N - \lfloor \frac{n}{2} \rfloor, \\ \binom{\lfloor \frac{n}{2} \rfloor}{j - N + \lfloor \frac{n}{2} \rfloor} p_0^{N-j} p_1^j - \binom{N-j+m(n)}{j} p_1^N - \sum_{k=j}^{N-1} \binom{k-j+m(n)}{m(n)} \binom{N}{k} p_0^{N-k} p_1^k, & \text{if } N - \lfloor \frac{n}{2} \rfloor \leq j < N. \end{cases}$$

Since

$$\begin{aligned} & \binom{i-\hat{m}(n)}{j-\hat{m}(n)} p_0^{i-j} p_1^j - \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} p_0^{i-j-1} p_1^j \\ &= \left(\binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} + \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)-1} \right) p_0^{i-j} p_1^j - \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} p_0^{i-j-1} p_1^j, \end{aligned}$$

if $N - n \leq i < N$ and $c(n, i) = 1$, then $C_{ij}^{\text{STEP3}}(n) = \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} p_0^{i-j} p_1^j$. Thus $C_{ij}^{\text{STEP3}}(n)$ is expressed as follows:

$$\begin{aligned}
 & C_{ij}^{\text{STEP3}}(n) \\
 &= \begin{cases} p_1^i, & \text{if } i \leq m(n) \text{ and } i = j; \\ \binom{i-\hat{m}(n)}{j-\hat{m}(n)} p_0^{i-j} p_1^j, & \text{if } m(n) < i < N \text{ and } c(n, i) = p_1; \\ \binom{i-\hat{m}(n)-1}{j-\hat{m}(n)} p_0^{i-j} p_1^j, & \text{if } m(n) < i < N \text{ and } c(n, i) = 1; \\ \hat{C}_j(n), & \text{if } i = N \text{ and } 0 \leq j < N; \\ 0, & \text{otherwise;} \end{cases} \\
 &= \begin{cases} p_1^i, & \text{if } i \leq m(n) \text{ and } i = j, \\ p_1^{m(n)+1} C_{i-(m(n)+1)j-(m(n)+1)}^{\text{STEP1}}(n), & \text{if } m(n) < i \leq N \text{ and } m(n) < j \leq N, \\ \hat{C}_j(n), & \text{if } i = N \text{ and } 0 \leq j < N, \\ 0, & \text{otherwise.} \end{cases}
 \end{aligned}$$

Therefore, we obtain following matrix:

$$\begin{aligned}
 & C^{\text{STEP3}}(n) \\
 &= \left(\begin{array}{cccc|cccc} p_1^0 & 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ 0 & p_1^1 & \ddots & \vdots & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & p_1^{m(n)} & 0 & \cdots & 0 & 0 \\ \hline & \mathbf{O} & & & p_1^{m(n)+1} C^{\text{STEP1}}(n) & & & \\ \hline \hat{C}_0(n) & \hat{C}_1(n) & \cdots & \hat{C}_{m(n)}(n) & \hat{C}_{m(n)+1}(n) & \cdots & \hat{C}_{N-1}(n) & 0 \end{array} \right),
 \end{aligned}$$

where \mathbf{O} is the zero matrix. As a result of the right side of the main routine, we can obtain

$$C^{\text{STEP3}} \equiv C^{\text{STEP3}}(0) = \begin{pmatrix} p_1^0 & 0 & \cdots & 0 & 0 \\ 0 & p_1^1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & p_1^{N-1} & 0 \\ \hat{C}_0 & \hat{C}_1 & \cdots & \hat{C}_{N-1} & 0 \end{pmatrix},$$

where for $j \in \{0, 1, \dots, N - 1\}$,

$$\hat{C}_j \equiv \hat{C}_j(0) = -\binom{N}{j} p_1^N - \sum_{k=j}^{N-1} \binom{k}{j} \binom{N}{k} p_0^{N-k} p_1^k.$$

Similarly, if we apply the proposed algorithm to $C^{\text{right}}(N)$ and \mathbf{R} , then their (i, j) entries become, respectively,

$$\begin{cases} p_1^i, & \text{if } i < m(n) \text{ and } j = i, i + 1, \\ \hat{C}_0, & \text{if } i = N \text{ and } j = 0, \\ \hat{C}_{j-1} + \hat{C}_j, & \text{if } i = N \text{ and } 1 \leq j < N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \begin{cases} \sum_{k=j}^N \binom{k}{j} \hat{P}_{Nk}, & \text{if } i = N, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, if this method is applied to identity matrix, it is restored. Therefore, we have the following:

Theorem 4.1. *The eigenpolynomial of $\mathbf{P}(N)$ can be expressed as*

$$|\lambda \mathbf{I} - \mathbf{P}(N)| = \begin{vmatrix} \lambda - p_1^0 & -p_1^0 q_1 & 0 & \cdots & 0 \\ 0 & \lambda - p_1^1 & -p_1^1 q_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda - p_1^{N-1} & -p_1^{N-1} q_1 \\ \tilde{P}_{N0} & \tilde{P}_{N1} & \cdots & \tilde{P}_{N,N-1} & \lambda + \tilde{P}_{NN} \end{vmatrix}, \quad (4.2)$$

where

$$\begin{aligned} \tilde{P}_{Nj} &= -q_0 \hat{C}_j - q_1 (\hat{C}_{j-1} + \hat{C}_j) - \sum_{k=j}^N \binom{k}{j} \hat{P}_{Nk} \\ &= -\hat{C}_j - q_1 \hat{C}_{j-1} - \sum_{k=j}^N \binom{k}{j} \hat{P}_{Nk} \end{aligned}$$

and

$$\hat{C}_j = \begin{cases} -\binom{N}{j} p_1^N - \sum_{k=j}^{N-1} \binom{k}{j} \binom{N}{k} p_0^{N-k} p_1^k, & \text{if } j < N, \\ 0, & \text{if } j = N. \end{cases}$$

4.2. Decision of Eigenvalues . We construct the reflecting barrier for which $\tilde{P}_{N0} = \tilde{P}_{N1} = \cdots = \tilde{P}_{N,N-1} = 0$ and $\tilde{P}_{NN} = -\hat{\lambda}$. Then the eigenvalues of $\mathbf{P}(N)$ become $\lambda_0 = p_1^0, \lambda_1 = p_1^1, \dots, \lambda_{N-1} = p_1^{N-1}$ and $\lambda_N = \hat{\lambda}$. The reflecting characteristic $\{\hat{P}_{Nj}; j = 0, 1, \dots, N\}$ satisfying this condition can be deduced from these $N + 1$ equations:

$$\begin{aligned} \hat{C}_0 + \binom{0}{0} \hat{P}_{N0} + \binom{1}{0} \hat{P}_{N1} + \binom{2}{0} \hat{P}_{N2} + \cdots + \binom{N-1}{0} \hat{P}_{N,N-1} + \binom{N}{0} \hat{P}_{NN} &= 0 \\ q_1 \hat{C}_0 + \hat{C}_1 + \binom{1}{1} \hat{P}_{N1} + \binom{2}{1} \hat{P}_{N2} + \cdots + \binom{N-1}{1} \hat{P}_{N,N-1} + \binom{N}{1} \hat{P}_{NN} &= 0 \\ q_1 \hat{C}_1 + \hat{C}_2 + \binom{2}{2} \hat{P}_{N2} + \cdots + \binom{N-1}{2} \hat{P}_{N,N-1} + \binom{N}{2} \hat{P}_{NN} &= 0 \\ \dots\dots\dots & \\ q_1 \hat{C}_{N-2} + \hat{C}_{N-1} + \binom{N-1}{N-1} \hat{P}_{N,N-1} + \binom{N}{N-1} \hat{P}_{NN} &= 0 \\ q_1 \hat{C}_{N-1} + \binom{N}{N} \hat{P}_{NN} &= \hat{\lambda}. \end{aligned} \quad (4.3)$$

This set of equations can be expressed as

$$\begin{pmatrix} \hat{C}_0 \\ \hat{C}_1 \\ \hat{C}_2 \\ \vdots \\ \hat{C}_{N-1} \\ 0 \end{pmatrix} + q_1 \begin{pmatrix} 0 \\ \hat{C}_0 \\ \hat{C}_1 \\ \vdots \\ \hat{C}_{N-2} \\ \hat{C}_{N-1} \end{pmatrix} + \begin{pmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{N-1}{0} & \binom{N}{0} \\ 0 & \binom{1}{1} & \binom{2}{1} & \cdots & \binom{N-1}{1} & \binom{N}{1} \\ 0 & 0 & \binom{2}{2} & \cdots & \binom{N-1}{2} & \binom{N}{2} \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \binom{N-1}{N-1} & \binom{N}{N-1} \\ 0 & 0 & \cdots & 0 & 0 & \binom{N}{N} \end{pmatrix} \begin{pmatrix} \hat{P}_{N0} \\ \hat{P}_{N1} \\ \hat{P}_{N2} \\ \vdots \\ \hat{P}_{N,N-1} \\ \hat{P}_{NN} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ \hat{\lambda} \end{pmatrix}.$$

In Figure 3, we show flowcharts of the algorithm to find solutions of Eq.(4.3).

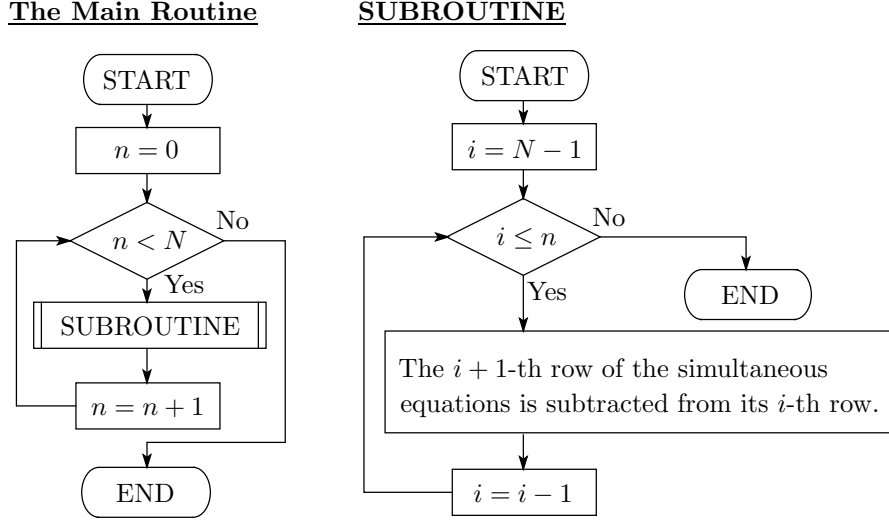


FIGURE 3. Flowcharts of the algorithm to find the solutions of Eq.(4.3).

Using this algorithm, Eq.(4.3) becomes

$$\begin{pmatrix} \hat{C}_0^{(0)} \\ \hat{C}_1^{(1)} \\ \hat{C}_2^{(2)} \\ \vdots \\ \hat{C}_{N-1}^{(N-1)} \\ 0 \end{pmatrix} + q_1 \begin{pmatrix} -\hat{C}_0^{(0)} \\ \hat{C}_0^{(0)} - \hat{C}_1^{(1)} \\ \hat{C}_1^{(1)} - \hat{C}_2^{(2)} \\ \vdots \\ \hat{C}_{N-2}^{(N-2)} - \hat{C}_{N-1}^{(N-1)} \\ \hat{C}_{N-1}^{(N-1)} \end{pmatrix} + \begin{pmatrix} \hat{P}_{N0} \\ \hat{P}_{N1} \\ \hat{P}_{N2} \\ \vdots \\ \hat{P}_{NN-1} \\ \hat{P}_{NN} \end{pmatrix} = \begin{pmatrix} (-1)^N \binom{N}{0} \hat{\lambda} \\ (-1)^{N-1} \binom{N}{1} \hat{\lambda} \\ (-1)^{N-2} \binom{N}{2} \hat{\lambda} \\ \vdots \\ (-1)^1 \binom{N}{N-1} \hat{\lambda} \\ \hat{\lambda} \end{pmatrix},$$

where $\hat{C}_j^{(n)}$ is deduced from the following recurrence relations:

$$\hat{C}_j^{(n)} = \begin{cases} \hat{C}_j - \hat{C}_{j+1}^{(0)}, & \text{if } n = 0 \text{ and } j < N, \\ \hat{C}_j^{(n-1)} - \hat{C}_{j+1}^{(n)}, & \text{if } 0 < n \leq j \text{ and } j < N, \\ 0, & \text{if } j = N. \end{cases}$$

In particular,

$$\hat{C}_j^{(j)} = \begin{cases} -\binom{N}{j} p_0^{N-j} p_1^j - \sum_{k=j}^{N-1} (-1)^{k-j} \binom{k}{j} \binom{N}{k} p_1^N, & \text{if } j < N - 1, \\ \hat{C}_{N-1}, & \text{if } j = N - 1. \end{cases}$$

Therefore, we obtain solutions of Eq.(4.3):

$$\begin{pmatrix} \hat{P}_{N0} \\ \hat{P}_{N1} \\ \hat{P}_{N2} \\ \vdots \\ \hat{P}_{NN-1} \\ \hat{P}_{NN} \end{pmatrix} = \begin{pmatrix} (-1)^N \binom{N}{0} \hat{\lambda} - q_0 \hat{C}_0^{(0)} \\ (-1)^{N-1} \binom{N}{1} \hat{\lambda} - q_0 \hat{C}_1^{(1)} - q_1 \hat{C}_0^{(0)} \\ (-1)^{N-2} \binom{N}{2} \hat{\lambda} - q_0 \hat{C}_2^{(2)} - q_1 \hat{C}_1^{(1)} \\ \vdots \\ (-1)^1 \binom{N}{N-1} \hat{\lambda} - q_0 \hat{C}_{N-1}^{(N-1)} - q_1 \hat{C}_{N-2}^{(N-2)} \\ \hat{\lambda} - q_1 \hat{C}_{N-1}^{(N-1)} \end{pmatrix}.$$

It means that the following theorem holds.

Theorem 4.2. *Suppose the matrix $\mathbf{P}(N)$ has the reflecting barrier whose reflecting characteristic is*

$$\hat{P}_{Nj} = \begin{cases} (-1)^N \binom{N}{0} \hat{\lambda} - q_0 \hat{C}_0^{(0)}, & \text{if } j = 0, \\ (-1)^{N-j} \binom{N}{j} \hat{\lambda} - q_0 \hat{C}_j^{(j)} - q_1 \hat{C}_{j-1}^{(j-1)}, & \text{if } 0 < j < N, \\ \hat{\lambda} - q_1 \hat{C}_{N-1}^{(N-1)}, & \text{if } j = N, \end{cases} \quad (4.4)$$

where

$$\hat{C}_j^{(j)} = \begin{cases} -\binom{N}{j} p_0^{N-j} p_1^j - \sum_{k=j}^{N-1} (-1)^{k-j} \binom{k}{j} \binom{N}{k} p_1^N, & \text{if } j < N-1, \\ \hat{C}_{N-1} = -\binom{N}{N-1} p_1^{N-1}, & \text{if } j = N-1. \end{cases} \quad (4.5)$$

Then $\mathbf{P}(N)$ has eigenvalues $p_1^0, p_1^1, \dots, p_1^{N-1}$ and $\hat{\lambda}$.

Note that the matrix $\mathbf{P}(N)$ with the reflecting characteristic derived by this method does not necessarily satisfy conditions of a probability matrix. In fact, in the case of $p_0 = q_0 = p$, $p_1 = q_1 = q$, $p < q$ and $\hat{\lambda} = q^N$, $\mathbf{P}(N)$ does not become a probability matrix because

$$\hat{P}_{Nj} = \binom{N+1}{j} q^j (p^{N+1-j} + (-1)^{N-j} q^{N+1-j}).$$

Now, we define

$$\mathbf{Q}(N) \equiv \begin{pmatrix} P_{00} & P_{01} & 0 & \cdots & 0 \\ P_{10} & P_{11} & P_{12} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ P_{N-10} & P_{N-11} & P_{N-12} & \cdots & P_{N-1N} \\ Q_{N0} & Q_{N1} & Q_{N2} & \cdots & Q_{NN} \end{pmatrix} \quad (4.6)$$

where $\{Q_{Nj}; j = 0, 1, \dots, N\}$ is the reflecting characteristic of Theorem 4.2. Moreover, the matrix $\bar{\mathbf{Q}}(N)$ is defined by

$$\bar{\mathbf{Q}}(N) \equiv \left(\begin{array}{cccc|cc} P_{00} & P_{01} & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \\ P_{N-10} & P_{N-11} & \cdots & P_{N-1N} & 0 & \cdots \\ Q_{N0} & Q_{N1} & \cdots & Q_{NN} & 0 & \cdots \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \cdots & \vdots & \vdots & \ddots \end{array} \right). \quad (4.7)$$

Then we have

$$\mathbf{P} - \bar{\mathbf{Q}}(N) = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \ddots & \vdots & \vdots & \\ 0 & \cdots & 0 & 0 & \cdots \\ P_{N0} - Q_{N0} & \cdots & P_{NN} - Q_{NN} & 0 & \cdots \\ P_{N+10} & \cdots & P_{N+1N} & P_{N+1N+1} & \ddots \\ \vdots & \cdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Since

$$\lim_{N \rightarrow \infty} (P_{Nj} - Q_{Nj}) = 0 \quad \text{and} \quad \lim_{N \rightarrow \infty} P_{N+1k} = 0,$$

we have

$$\lim_{N \rightarrow \infty} \|\mathbf{P} - \bar{\mathbf{Q}}(N)\|_{\infty} = 0, \quad (4.8)$$

where the matrix norm $\|\cdot\|_{\infty}$ is defined by

$$\|\mathbf{X}\|_{\infty} \equiv \max_{\|\mathbf{a}\|_{\infty}=1} \|\mathbf{X}\mathbf{a}\|_{\infty} \quad \text{and} \quad \|\mathbf{a}\|_{\infty} \equiv \max_k |a_k|, \quad \mathbf{a} \in \mathbb{R}^N.$$

Thus the matrix $\mathbf{Q}(N)$ is a good approximation for \mathbf{P} .

5. The Stationary Distribution

5.1. The Matrix Decomposition Based on the Proposed Algorithm. We consider a matrix representation of the algorithm described in Section 4.1. It can be inferred from Theorem 4.1 that if this representation is applied to the matrix $\mathbf{Q}(N)$ in Section 4.2, then we obtain the following matrix:

$$\mathbf{\Lambda}(N) \equiv \begin{pmatrix} p_1^0 & p_1^0 q_1 & & \mathbf{O} \\ & p_1^1 & \ddots & \\ & & \ddots & p_1^{N-1} q_1 \\ \mathbf{O} & & & p_1^N \end{pmatrix}. \quad (5.1)$$

We first construct the matrix corresponding to the process of STEP1. For n given from the main routine, this process is represented by left-multiplying by

$$\mathbf{S}^{\text{STEP1}}(N, n) \equiv \mathbf{I} - p_0 \mathbf{U} \mathbf{U}^{m(n)} \mathbf{t} \mathbf{U}^{m(n)},$$

where \mathbf{I} is the identity matrix and the matrix \mathbf{U} is defined by

$$\mathbf{U} \equiv \begin{pmatrix} 0 & 1 & \cdots & N \\ 0 & \begin{pmatrix} 0 & & & \mathbf{O} \\ 1 & 0 & & \\ \vdots & \ddots & \ddots & \\ N & \mathbf{O} & 1 & 0 \end{pmatrix} \end{pmatrix}.$$

Therefore, the (i, j) entry of the upper matrix is expressed as

$$S_{ij}^{\text{STEP1}}(N, n) = \begin{cases} 1, & \text{if } i = j, \\ -p_0, & \text{if } m(n) < i \text{ and } i - 1 = j, \\ 0, & \text{otherwise.} \end{cases} \quad (5.2)$$

For any $\mathbf{X} \equiv (X_{ij})_{0 \leq i, j \leq N}$, its rows or columns are shifted by using \mathbf{U} . In fact,

$$\begin{aligned} \mathbf{UX} &= \begin{pmatrix} 0 & & \mathbf{O} \\ 1 & 0 & \\ & \ddots & \ddots \\ \mathbf{O} & & 1 & 0 \end{pmatrix} \begin{pmatrix} X_{00} \cdots X_{0N} \\ X_{10} \cdots X_{1N} \\ \vdots \quad \ddots \quad \vdots \\ X_{N0} \cdots X_{NN} \end{pmatrix} = \begin{pmatrix} 0 & \cdots & 0 \\ X_{00} & \cdots & X_{0N} \\ \vdots & \ddots & \vdots \\ X_{N-10} & \cdots & X_{N-1N} \end{pmatrix}, \\ {}^t\mathbf{UX} &= \begin{pmatrix} 0 & 1 & & \mathbf{O} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \mathbf{O} & & & 0 \end{pmatrix} \begin{pmatrix} X_{00} \cdots X_{0N} \\ X_{10} \cdots X_{1N} \\ \vdots \quad \ddots \quad \vdots \\ X_{N0} \cdots X_{NN} \end{pmatrix} = \begin{pmatrix} X_{10} \cdots X_{1N} \\ \vdots \quad \ddots \quad \vdots \\ X_{N0} \cdots X_{NN} \\ 0 \cdots 0 \end{pmatrix}, \\ \mathbf{XU} &= \begin{pmatrix} X_{00} \cdots X_{0N} \\ X_{10} \cdots X_{1N} \\ \vdots \quad \ddots \quad \vdots \\ X_{N0} \cdots X_{NN} \end{pmatrix} \begin{pmatrix} 0 & & \mathbf{O} \\ 1 & 0 & \\ & \ddots & \ddots \\ \mathbf{O} & & 1 & 0 \end{pmatrix} = \begin{pmatrix} X_{01} \cdots X_{0N} & 0 \\ X_{11} \cdots X_{1N} & 0 \\ \vdots \quad \ddots \quad \vdots & \vdots \\ X_{N1} \cdots X_{NN} & 0 \end{pmatrix} \end{aligned}$$

and

$$\mathbf{X}^t\mathbf{U} = \begin{pmatrix} X_{00} \cdots X_{0N} \\ X_{10} \cdots X_{1N} \\ \vdots \quad \ddots \quad \vdots \\ X_{N0} \cdots X_{NN} \end{pmatrix} \begin{pmatrix} 0 & 1 & & \mathbf{O} \\ & 0 & \ddots & \\ & & \ddots & 1 \\ \mathbf{O} & & & 0 \end{pmatrix} = \begin{pmatrix} 0 & X_{00} \cdots X_{0N-1} \\ 0 & X_{10} \cdots X_{1N-1} \\ \vdots & \vdots \quad \ddots \quad \vdots \\ 0 & X_{N0} \cdots X_{NN-1} \end{pmatrix}.$$

The left-multiplying $\mathbf{Q}(N)$ by $\mathbf{S}^{\text{STEP1}}(N, n)$ gives

$$\begin{aligned} &\mathbf{S}^{\text{STEP1}}(N, n)\mathbf{Q}(N) \\ &= (\mathbf{I} - p_0\mathbf{UU}^{m(n)}{}^t\mathbf{U}^{m(n)})\mathbf{Q}(N) = \mathbf{Q}(N) - p_0\mathbf{UU}^{m(n)}{}^t\mathbf{U}^{m(n)}\mathbf{Q}(N) \\ &= \begin{matrix} 0 \\ \vdots \\ m(n) \\ m(n)+1 \\ \vdots \\ N-1 \\ N \end{matrix} \begin{pmatrix} \mathbf{p}_0^{\text{row}} \\ \vdots \\ \mathbf{p}_{m(n)}^{\text{row}} \\ \mathbf{p}_{m(n)+1}^{\text{row}} - p_0\mathbf{p}_{m(n)}^{\text{row}} \\ \vdots \\ \mathbf{p}_{N-1}^{\text{row}} - p_0\mathbf{p}_{N-2}^{\text{row}} \\ \mathbf{q}_N^{\text{row}} - p_0\mathbf{p}_{N-1}^{\text{row}} \end{pmatrix} \text{ where } \mathbf{q}_N^{\text{row}} = (Q_{N0} \quad Q_{N1} \quad \cdots \quad Q_{NN}). \end{aligned}$$

This is equal to the result of STEP1 for given n . Therefore, the matrix representation corresponding to the left side of the main routine is given by

$$\begin{aligned} & \mathbf{S}^{\text{STEP1}}(N) \\ & \equiv \prod_{\substack{k=0 \\ \text{left}}}^{\lfloor \frac{N-1}{2} \rfloor} \mathbf{S}^{\text{STEP1}}(N, N-1-2k) \\ & \equiv \mathbf{S}^{\text{STEP1}}(N, N-1-2\lfloor \frac{N-1}{2} \rfloor) \cdots \mathbf{S}^{\text{STEP1}}(N, N-3) \mathbf{S}^{\text{STEP1}}(N, N-1). \end{aligned}$$

The (i, j) entry of $\mathbf{S}^{\text{STEP1}}(N)$ can be expressed as

$$S_{ij}^{\text{STEP1}}(N) = \begin{cases} (-1)^{i+j} \binom{\lfloor \frac{i+1}{2} \rfloor}{\lfloor \frac{i+1}{2} \rfloor - i + j} p_0^{i-j}, & \text{if } i - \lfloor \frac{i+1}{2} \rfloor \leq j \leq i, \\ 0, & \text{otherwise.} \end{cases} \quad (5.3)$$

Next, we consider the matrix corresponding to the process of STEP2. For given n , this process is represented by the left-multiplying by the following matrix:

$$\mathbf{S}^{\text{STEP2}}(N, n) \equiv \mathbf{I} - \mathbf{U} \mathbf{U}^{m(n)} \mathbf{R}(N, p_1) \mathbf{U}^{m(n)}$$

where the matrix $\mathbf{R}(N, p_1)$ is defined by

$$\mathbf{R}(N, p_1) \equiv (R_{ij})_{0 \leq i, j \leq N}, \quad R_{ij} \equiv \begin{cases} p_1, & \text{if } i = j \text{ and } i \text{ is the even number,} \\ 1, & \text{if } i = j \text{ and } i \text{ is the odd number,} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, the (i, j) entry of the matrix $\mathbf{S}^{\text{STEP2}}(N, n)$ is expressed as:

$$S_{ij}^{\text{STEP2}}(N, n) = \begin{cases} 1, & \text{if } i = j, \\ -p_1, & \text{if } m(n) < i \text{ and } i - m(n) \text{ is the odd number,} \\ -1, & \text{if } m(n) < i \text{ and } i - m(n) \text{ is the even number,} \\ 0, & \text{otherwise.} \end{cases} \quad (5.4)$$

Finally, we construct the matrix corresponding to the process of STEP3. For given n , it is represented by the right-multiplying by

$$\mathbf{S}^{\text{STEP3}}(N, n) \equiv \mathbf{U}^{(N-m(n)+1)} \mathbf{U}^{(N-m(n)+1)} + \mathbf{U}^{m(n)} \mathbf{T} \mathbf{U}^{m(n)},$$

where the matrix \mathbf{T} is defined by

$$\mathbf{T} \equiv \begin{pmatrix} 1 & & & \mathbf{O} \\ \vdots & \ddots & & \\ 1 & \cdots & 1 & \end{pmatrix}.$$

The right-multiplying by \mathbf{T} means the summation from right to left, i.e., for any $\mathbf{X} \equiv (X_{ij})_{0 \leq i, j \leq N}$,

$$\mathbf{X} \mathbf{T} = \begin{pmatrix} \sum_{j=0}^N X_{0j} & \sum_{j=1}^N X_{0j} & \cdots & \sum_{j=N-1}^N X_{0j} & X_{0N} \\ \sum_{j=0}^N X_{1j} & \sum_{j=1}^N X_{1j} & \cdots & \sum_{j=N-1}^N X_{1j} & X_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sum_{j=0}^N X_{Nj} & \sum_{j=1}^N X_{Nj} & \cdots & \sum_{j=N-1}^N X_{Nj} & X_{NN} \end{pmatrix}.$$

Then we have

$$\begin{aligned} &\mathbf{S}^{\text{STEP3}}(N, n) \\ &= \begin{matrix} 0 \\ \vdots \\ m(n) - 1 \end{matrix} \begin{pmatrix} 0 & \cdots & m(n) - 1 & & & & & \\ 1 & & & & \mathbf{O} & & & \\ & \ddots & & & & & & \\ & & 1 & & & & & \\ \mathbf{O} & & & & \mathbf{O} & & & \end{pmatrix} + \begin{matrix} m(n) \\ \vdots \\ N \end{matrix} \begin{pmatrix} \mathbf{O} & & & & & & m(n) & \cdots & N \\ & & & & & 1 & & & \\ & & & & & \vdots & & \ddots & \\ \mathbf{O} & & & & & 1 & & \cdots & 1 \end{pmatrix} \\ &= \begin{matrix} 0 \\ \vdots \\ m(n) - 1 \\ m(n) \\ \vdots \\ N \end{matrix} \begin{pmatrix} 0 & \cdots & m(n) - 1 & m(n) & \cdots & N \\ 1 & & & & & \mathbf{O} \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & \vdots & \ddots & \\ \mathbf{O} & & & 1 & \cdots & 1 \end{pmatrix}. \end{aligned}$$

Since STEP2 and STEP3 are represented by left- and right-multiply matrices, respectively, we define

$$\begin{aligned} \mathbf{S}^{\text{STEP2}}(N) &\equiv \prod_{\substack{k=0 \\ \text{left}}}^{N-1} \mathbf{S}^{\text{STEP2}}(N, N - 1 - k) = \prod_{\ell=0}^{N-1} \mathbf{S}^{\text{STEP2}}(N, m(\ell)) \\ &= \mathbf{S}^{\text{STEP2}}(N, 0) \cdots \mathbf{S}^{\text{STEP2}}(N, N - 2) \mathbf{S}^{\text{STEP2}}(N, N - 1) \end{aligned}$$

and

$$\begin{aligned} \mathbf{S}^{\text{STEP3}}(N) &\equiv \prod_{k=0}^{N-1} \mathbf{S}^{\text{STEP3}}(N, N - 1 - k) = \prod_{\substack{k=0 \\ \text{left}}}^{N-1} \mathbf{S}^{\text{STEP3}}(N, k) \\ &= \mathbf{S}^{\text{STEP3}}(N, N - 1) \cdots \mathbf{S}^{\text{STEP3}}(N, 1) \mathbf{S}^{\text{STEP3}}(N, 0) \\ &= \begin{pmatrix} \binom{0}{0} & 0 & \cdots & 0 \\ \binom{1}{0} & \binom{1}{1} & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ \binom{N}{0} & \binom{N}{1} & \cdots & \binom{N}{N} \end{pmatrix}. \end{aligned} \tag{5.5}$$

Then the following holds:

$$\mathbf{S}^{\text{STEP2}}(N) \mathbf{S}^{\text{STEP1}}(N) \mathbf{Q}(N) \mathbf{S}^{\text{STEP3}}(N) = \Lambda(N). \tag{5.6}$$

In the equation,

$$\begin{aligned}
 & \mathbf{S}^{\text{STEP2}}(N)\mathbf{S}^{\text{STEP1}}(N) \\
 &= \left(\prod_{k=0}^{N-1} \mathbf{S}^{\text{STEP2}}(N, m(k)) \right) \begin{pmatrix} \binom{0}{0} & 0 & \cdots & 0 \\ * & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & 1 \end{pmatrix} \\
 &= \left(\prod_{k=0}^{N-2} \mathbf{S}^{\text{STEP2}}(N, m(k)) \right) \begin{pmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ (-1)^1 \binom{1}{0} & (-1)^2 \binom{1}{1} & 0 & \cdots & 0 \\ * & * & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ * & * & \cdots & * & 1 \end{pmatrix} \\
 &\dots\dots\dots \\
 &= \left(\prod_{k=0}^{N-(\ell+1)} \mathbf{S}^{\text{STEP2}}(N, m(k)) \right) \begin{pmatrix} \binom{0}{0} & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots & \ddots & \vdots \\ (-1)^\ell \binom{\ell}{0} & \cdots & (-1)^{2\ell} \binom{\ell}{\ell} & 0 & \cdots & 0 \\ * & \cdots & * & 1 & \ddots & \vdots \\ \vdots & \ddots & \vdots & \ddots & \ddots & 0 \\ * & \cdots & * & \cdots & * & 1 \end{pmatrix} \\
 &\dots\dots\dots \\
 &= \begin{pmatrix} \binom{0}{0} & 0 & 0 & \cdots & 0 \\ (-1)^1 \binom{1}{0} & (-1)^2 \binom{1}{1} & 0 & \cdots & 0 \\ (-1)^2 \binom{2}{0} & (-1)^3 \binom{2}{1} & (-1)^4 \binom{2}{2} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ (-1)^N \binom{N}{0} & (-1)^{N+1} \binom{N}{1} & (-1)^{N+2} \binom{N}{1} & \cdots & (-1)^{2N} \binom{N}{N} \end{pmatrix}.
 \end{aligned}$$

Thus, we see that

$$\begin{aligned}
 (\mathbf{S}^{\text{STEP2}}(N)\mathbf{S}^{\text{STEP1}}(N)) \mathbf{S}^{\text{STEP3}}(N) &= \mathbf{I}, \\
 \mathbf{S}^{\text{STEP3}}(N) (\mathbf{S}^{\text{STEP2}}(N)\mathbf{S}^{\text{STEP1}}(N)) &= \mathbf{I},
 \end{aligned}$$

namely, $\mathbf{S}^{\text{STEP2}}(N)\mathbf{S}^{\text{STEP1}}(N)$ is the inverse matrix of $\mathbf{S}^{\text{STEP3}}(N)$.

Theorem 5.1. *The matrix $\mathbf{Q}(N)$ in (4.6) can be decomposed as*

$$\mathbf{Q}(N) = \mathbf{S}(N)\mathbf{\Lambda}(N)\mathbf{S}^{-1}(N) \tag{5.7}$$

where

$$S_{ij}(N) = \begin{cases} \binom{i}{j}, & \text{if } j \leq i \leq N, \\ 0, & \text{otherwise,} \end{cases} \quad S_{ij}^{-1}(N) = \begin{cases} (-1)^{i+j} \binom{i}{j}, & \text{if } j \leq i \leq N, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\Lambda_{ij}(N) = \begin{cases} p_1^i, & \text{if } i = j \leq N, \\ p_1^i q_1, & \text{if } i + 1 = j \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Moreover, we also have the following:

Theorem 5.2. *The $\infty \times \infty$ matrix \mathbf{P} in Eq.(2.4) can be decomposed as*

$$\mathbf{P} = \mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1} \tag{5.8}$$

where \mathbf{S} , $\mathbf{\Lambda}$, and \mathbf{S}^{-1} are given by

$$S_{ij} = \begin{cases} \binom{i}{j}, & \text{if } j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

$$S_{ij}^{-1} = \begin{cases} (-1)^{i+j} \binom{i}{j}, & \text{if } j \leq i, \\ 0, & \text{otherwise,} \end{cases}$$

$$\Lambda_{ij} = \begin{cases} p_1^i, & \text{if } i = j, \\ p_1^i q_1, & \text{if } i + 1 = j, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. We prove the entry for which $j \leq i + 1$. In other cases, it is clear that $P_{ij} = 0$. The right-hand side of Eq.(5.8) is expressed as

$$\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1} = \mathbf{\Phi}^{\text{left}} + q_1 \mathbf{\Phi}^{\text{right}}$$

where $\mathbf{\Phi}^{\text{left}}$ and $\mathbf{\Phi}^{\text{right}}$ are given by

$$\mathbf{\Phi}^{\text{left}} \equiv \begin{pmatrix} \binom{0}{0} p_1^0 & & & \mathbf{O} \\ \vdots & \ddots & & \\ \binom{n}{0} p_1^0 & \cdots & \binom{n}{n} p_1^n & \\ \vdots & & \vdots & \ddots \end{pmatrix} \mathbf{S}^{-1},$$

$$\mathbf{\Phi}^{\text{right}} \equiv \begin{pmatrix} 0 & \binom{0}{0} p_1^0 & & \mathbf{O} \\ \vdots & \vdots & \ddots & \\ 0 & \binom{n}{1} p_1^1 & \cdots & \binom{n}{n} p_1^n \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \mathbf{S}^{-1}$$

because

$$\begin{aligned}
& \mathbf{S} \mathbf{\Lambda} \mathbf{S}^{-1} \\
&= \begin{pmatrix} \binom{0}{0} p_1^0 & & & & \mathbf{O} \\ \binom{1}{0} p_1^0 & \binom{1}{1} p_1^0 q_1 & & & \\ \vdots & \vdots & \ddots & & \\ \binom{n}{0} p_1^0 & \binom{n}{1} p_1^0 q_1 & \cdots & \binom{n}{n} p_1^n & \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \begin{pmatrix} p_1^0 & p_1^0 q_1 & & & \mathbf{O} \\ & p_1^1 & \ddots & & \\ & & \ddots & p_1^{n-1} q_1 & \\ & & & p_1^n & \ddots \\ \mathbf{O} & & & & \ddots \end{pmatrix} \mathbf{S}^{-1} \\
&= \begin{pmatrix} \binom{0}{0} p_1^0 & \binom{0}{0} p_1^0 q_1 & & & \mathbf{O} \\ \binom{1}{0} p_1^0 & \binom{1}{0} p_1^0 q_1 + \binom{1}{1} p_1^1 & \ddots & & \\ \vdots & \vdots & \ddots & \binom{n-1}{n-1} p_1^{n-1} q_1 & \\ \binom{n}{0} p_1^0 & \binom{n}{0} p_1^0 q_1 + \binom{n}{1} p_1^1 & \cdots & \binom{n}{n-1} p_1^{n-1} q_1 + \binom{n}{n} p_1^n & \binom{n}{n} p_1^n q_1 \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \mathbf{S}^{-1} \\
&= \left\{ \begin{pmatrix} \binom{0}{0} p_1^0 & & & & \mathbf{O} \\ \binom{1}{0} p_1^0 & \binom{1}{1} p_1^1 & & & \\ \vdots & \vdots & \ddots & & \\ \binom{n}{0} p_1^0 & \binom{n}{1} p_1^1 & \cdots & \binom{n}{n} p_1^n & \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} + q_1 \begin{pmatrix} \binom{0}{0} p_1^0 & & & & \mathbf{O} \\ \binom{1}{0} p_1^0 & \binom{1}{1} p_1^1 & & & \\ \vdots & \vdots & \ddots & & \\ \binom{n}{0} p_1^0 & \binom{n}{1} p_1^1 & \cdots & \binom{n}{n} p_1^n & \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \right\} \mathbf{S}^{-1}.
\end{aligned}$$

Noting that

$$\begin{aligned}
\frac{\binom{i}{j+k}}{\binom{i}{j}} \binom{j+k}{j} &= \frac{i!}{(j+k)!(i-j-k)!} \frac{j!(i-j)!}{i!} \frac{(j+k)!}{j!(j+k-j)!} \\
&= \frac{(i-j)!}{(i-j-k)!k!} = \binom{i-j}{k},
\end{aligned}$$

we see that

$$\begin{aligned}
\Phi_{ij}^{\text{left}} &= \sum_{k=j}^i (-1)^{j+k} \binom{i}{k} \binom{k}{j} p_1^k = \sum_{\ell=0}^{i-j} (-1)^{2j+\ell} \binom{i}{j+\ell} \binom{j+\ell}{j} p_1^{j+\ell} \\
&= \binom{i}{j} p_1^j \sum_{\ell=0}^{i-j} (-1)^{2j+\ell} \frac{\binom{i}{j+\ell}}{\binom{i}{j}} \binom{j+\ell}{j} p_1^\ell = \binom{i}{j} p_1^j \sum_{\ell=0}^{i-j} (-1)^{2j+\ell} \binom{i-j}{\ell} p_1^\ell \\
&= \binom{i}{j} p_1^j (1-p_1)^{i-j} = \binom{i}{j} p_1^j p_0^{i-j}.
\end{aligned}$$

Moreover, we obtain

$$\begin{aligned}
 \Phi_{ij}^{\text{right}} &= \sum_{k=j-1}^i (-1)^{j+k} \binom{i}{k} \binom{k+1}{j} p_1^k \\
 &= \sum_{\ell=-1}^{i-j} (-1)^{2j+\ell+1} \binom{i}{\ell+j} \binom{j+\ell+1}{j} p_1^{j+\ell} \\
 &= \sum_{\ell=-1}^{i-j} (-1)^{\ell+1} \binom{i}{\ell+j} \left(\binom{j+\ell}{j-1} + \binom{j+\ell}{j} \right) p_1^{j+\ell} \\
 &= \sum_{\ell_1=-1}^{i-j} (-1)^{\ell_1+1} \binom{i}{\ell_1+j} \binom{j+\ell_1}{j-1} p_1^{j+\ell_1} + \sum_{\ell_2=-1}^{i-j} (-1)^{\ell_2+1} \binom{i}{\ell_2+j} \binom{j+\ell_2}{j} p_1^{j+\ell_2} \\
 &= \binom{i}{j-1} p_1^{j-1} \sum_{\ell_1=-1}^{i-j} (-p_1)^{\ell_1+1} \binom{i-j+1}{\ell_1+1} + \binom{i}{j-1} \binom{j-1}{j} p_1^{j-1} - \binom{i}{j} p_1^j p_0^{i-j} \\
 &= \binom{i}{j-1} p_1^{j-1} p_0^{i-j+1} - \binom{i}{j} p_1^j p_0^{i-j}.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \Phi_{ij}^{\text{left}} + q_0 \Phi_{ij}^{\text{right}} &= \sum_{k=j}^i (-1)^{j+k} \binom{i}{k} \binom{k}{j} p_1^k + q_1 \sum_{k=j-1}^i (-1)^{j+k} \binom{i}{k} \binom{k+1}{j} p_1^k \\
 &= \binom{i}{j} p_1^j p_0^{i-j} + q_1 \left(\binom{i}{j-1} p_1^{j-1} p_0^{i-j+1} - \binom{i}{j} p_1^j p_0^{i-j} \right) \\
 &= q_0 \binom{i}{j} p_1^j p_0^{i-j} + q_1 \binom{i}{j-1} p_1^{j-1} p_0^{i-j+1} = P_{ij}.
 \end{aligned}$$

Thus, the proof is completed. \square

5.2. The Stationary Distribution of the Transition Probability Matrix P .

We start with the following.

Definition 5.3. The generating function of the sequence $\{x_t\}_{t \in \mathbb{N}_0}$ is defined by

$$X(s) \equiv \mathcal{G}[x_t; s] \equiv \sum_{t=0}^{\infty} x_t s^t. \quad (5.9)$$

In this definition, we suppose that s is included in the region of convergence, thus, $|X(s)| < \infty$. It is known that the generating function has following basic properties.

Theorem 5.4. Consider a real sequence $\{x_t\}_{t \in \mathbb{Z}}$ with $x_t = 0$ for $t < 0$. Then we have following properties:

- (1) *Left Shift:* $\mathcal{G}[x_{t+k}] = s^{-k} \left\{ X(s) - \sum_{\ell=0}^{k-1} x_\ell s^\ell \right\}$.
- (2) *Right Shift:* $\mathcal{G}[x_{t-k}] = s^k X(s)$.
- (3) *Final Value Theorem:* if both $\lim_{s \rightarrow 1} X(s)$ and $\lim_{t \rightarrow \infty} x_t$ exist, then

$$\lim_{s \rightarrow 1} \frac{1-s}{s} X(s) = \lim_{t \rightarrow \infty} x_t.$$

It is known that the generating function whose s to replace z^{-1} is also called the Z-transform. For this theorem, we refer to [2]. Let $\lambda_{ij}^{(k)}$ represent the (i, j) entry of Λ^k . Then we obtain

$$\lambda_{ij}^{(k+1)} = \begin{cases} p_1^j \lambda_{ij}^{(k)}, & \text{if } i = j, \\ p_1^{j-1} (q_1 \lambda_{ij-1}^{(k)} + p_1 \lambda_{ij}^{(k)}), & \text{if } i < j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \lambda_{ij}^{(0)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise,} \end{cases}$$

because for any $k \geq 0$,

$$\begin{aligned} \Lambda^{k+1} &= \Lambda^k \Lambda \\ &= \begin{pmatrix} \lambda_{00}^{(k)} & \lambda_{01}^{(k)} & \cdots & \lambda_{0i}^{(k)} & \cdots \\ & \lambda_{11}^{(k)} & \cdots & \lambda_{1i}^{(k)} & \cdots \\ & & \ddots & \vdots & \\ & & & \lambda_{ii}^{(k)} & \cdots \\ \mathbf{O} & & & & \ddots \end{pmatrix} \begin{pmatrix} p_1^0 & p_1^0 q_1 & & & \mathbf{O} \\ & p_1^1 & \ddots & & \\ & & \ddots & p_1^{i-1} q_1 & \\ & & & p_1^i & \ddots \\ \mathbf{O} & & & & \ddots \end{pmatrix} \\ &= \begin{pmatrix} p_1^0 \lambda_{00}^{(k)} & p_1^0 (q_1 \lambda_{00}^{(k)} + p_1 \lambda_{01}^{(k)}) & \cdots & p_1^i (q_1 \lambda_{0i-1}^{(k)} + p_1 \lambda_{0i}^{(k)}) & \cdots \\ & p_1^1 \lambda_{11}^{(k)} & \cdots & p_1^i (q_1 \lambda_{1i-1}^{(k)} + p_1 \lambda_{1i}^{(k)}) & \cdots \\ & & \ddots & \vdots & \\ & & & p_1^i \lambda_{ii}^{(k)} & \cdots \\ \mathbf{O} & & & & \ddots \end{pmatrix}. \end{aligned}$$

Theorem 5.5. Let $\Lambda^\infty \equiv \lim_{k \rightarrow \infty} \Lambda^k$ and let $\lambda_{ij}^{(\infty)}$ denote the (i, j) entry of Λ^∞ . Then we have

$$\lambda_{ij}^{(\infty)} = \begin{cases} 1, & \text{if } i = j = 0, \\ \prod_{k=1}^j \frac{p_1^{k-1} q_1}{1 - p_1^k}, & \text{if } i = 0 \text{ and } 0 < j, \\ 0, & \text{otherwise.} \end{cases} \quad (5.10)$$

Proof. We first prove the case of $i = j$. By Theorem 5.4, for all $i \geq 0$, the generating function of $\lambda_{ii}^{(k)}$ with respect to k can be derived as follows:

$$\begin{aligned} \mathcal{G}[\lambda_{ii}^{(k+1)}; s] &= \mathcal{G}[p_1^i \lambda_{ii}^{(k)}; s]; \\ s^{-1} (\Lambda_{ii}(s) - \lambda_{ii}^{(0)}) &= p_1^i \Lambda_{ii}(s); \\ \Lambda_{ii}(s) &= \frac{1}{1 - p_1^i s}. \end{aligned}$$

Moreover, by the final value theorem in Theorem 5.4, we can obtain

$$\lim_{t \rightarrow \infty} \lambda_{ii}^{(t)} = \lim_{s \rightarrow 1} \frac{1-s}{s} \Lambda_{ii}(s) = \lim_{s \rightarrow 1} \frac{1-s}{1-p_1^i s} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Next, we consider the case of $i < j$. As in the case of the main diagonal components, for $i < j$, we can obtain

$$\begin{aligned} \mathcal{G}[\lambda_{ij}^{(k+1)}; s] &= \mathcal{G}[p_1^{j-1}(q_1\lambda_{ij-1}^{(k)} + p_1\lambda_{ij}^{(k)}); s]; \\ s^{-1}(\Lambda_{ij}(s) - \lambda_{ij}^{(0)}) &= p_1^{j-1}q_1\Lambda_{ij-1}(s) + p_1^j\Lambda_{ij}(s); \\ \Lambda_{ij}(s) &= \frac{p_1^{j-1}q_1s}{1 - p_1^j s}\Lambda_{ij-1}(s). \end{aligned}$$

Moreover, $\Lambda_{ij}(s)$ can be expressed as follows:

$$\Lambda_{ij}(s) = \frac{1}{1 - p_1^i s} \prod_{k=i+1}^j \frac{p_1^{k-1}q_1s}{1 - p_1^k s}.$$

Therefore, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \lambda_{ij}^{(t)} &= \lim_{s \rightarrow 1} \frac{1-s}{s} \Lambda_{ij}(s) = \lim_{s \rightarrow 1} \frac{1-s}{1 - p_1^i s} \prod_{k=i+1}^j \frac{p_1^{k-1}q_1s}{1 - p_1^k s} \\ &= \begin{cases} \prod_{k=1}^j \frac{p_1^{k-1}q_1}{1 - p_1^k}, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Finally, it is obvious that $\lambda_{ij}^{(\infty)} = 0$ in the case of $j < i$. Therefore, Theorem 5.5 is satisfied. \square

From Theorem 5.5, we see that

$$\mathbf{P}^\infty = \mathbf{S}\mathbf{\Lambda}^\infty\mathbf{S}^{-1} = \mathbf{S} \begin{pmatrix} \lambda_{00}^{(\infty)} & \lambda_{01}^{(\infty)} & \cdots & \lambda_{0j}^{(\infty)} & \cdots \\ 0 & 0 & \cdots & 0 & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \mathbf{S}^{-1}.$$

Therefore, the k -th entry of the limiting distribution $\hat{\boldsymbol{\pi}} = (\hat{\pi}_j)_{j \in \mathbb{N}_0}$ is expressed as follows:

$$\hat{\pi}_k = \sum_{i=k}^{\infty} \lambda_{0i}^{(\infty)} (-1)^{i+k} \binom{i}{k}.$$

Moreover, from

$$\mathbf{P}^\infty \mathbf{P} = (\mathbf{S}\mathbf{\Lambda}^\infty\mathbf{S}^{-1})(\mathbf{S}\mathbf{\Lambda}\mathbf{S}^{-1}) = \mathbf{S}\mathbf{\Lambda}^\infty\mathbf{S}^{-1},$$

we see that $\hat{\boldsymbol{\pi}}$ is also the stationary distribution. Therefore, we obtain the following theorem.

Theorem 5.6. *Assume that $0 < p_0 \leq 1$ and $0 \leq q_0 \leq 1$. Then the transition probability matrix \mathbf{P} given by Equation (2.4) has a unique stationary distribution $\boldsymbol{\pi} = (\pi_j)_{j \in \mathbb{N}_0}$ and, for each j , π_j is expressed as*

$$\pi_j = \sum_{i=j}^{\infty} (-1)^{i+j} \lambda_{0i}^{(\infty)} \binom{i}{j}, \tag{5.11}$$

where

$$\lambda_{0j}^{(\infty)} = \begin{cases} 1, & \text{if } j = 0, \\ \prod_{k=1}^j \frac{p_1^{k-1} q_1}{1-p_1^k}, & \text{if } j \neq 0. \end{cases} \quad (5.12)$$

Moreover, π is also the limiting distribution.

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