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WEAK CONVERGENCE FOR APPROXIMATION OF AMERICAN OPTION PRICES

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Abstract. Based on a sequence of discretized American option price processes under the multinomial model proposed by Maller, Solomon and Szimayer [12], the sequence converges to the counterpart under the original Lévy process in distribution for almost all time. By adapting Skorokhod representation theorem, a new sequence of approximating processes with the s ame laws with the multinomial tree model defined by Maller, Solomon and Szimayer [12] is obtained. The new sequence satisfies Aldous' criterion for tightness, and the sequence of filtrations generated by the new approximation converges to the filtration generated by the representative of Lévy process weakly. By using results of Coquet and Toldo [5], we give a complete proof of the weak convergence for the approximation of American put option prices for all time. Hence the numerical approximation can be adapted in practice.

1. Introduction

We study a weak convergence for a sequence of discretized American option price processes arising from the tree-based scheme proposed by Maller, Solomon and Szimayer [12] for all time. Exponential Lévy models do not give closed form expressions for American options, and evaluation of American options by Monte-Carlo simulation is not simple to implement. Having a convergent method to approximate American option prices is indispensable. Pham [17] studies the American option with the jump-diffusion process and relates this optimal-stopping problem to a parabolic integro-differential free-boundary problem. The tree-based method (or lattice method) is more tractable to price American options in practice. Cox, Ross and Rubinstein [7] presented a binomial model to approximate the Black-Scholes model and gave the option price correspondingly. The approach by Cox, Ross and Rubinstein was extended to the finite activity case of the jump diffusion by Amin [1] and Mulinacci [14], and to the infinitely activity case by Këllezi and Webber [9]. Këllezi and Webber [9] can price the Bermudan options via a lattice method based on transition probabilities.

Ball and Torus [3] find evidence that daily stock prices are characterized by lognormally distributed jumps, and exponential Lévy process for the stock price. Applebaum [2] and Cont and Tankov [6] provide analytic examples for stock prices as exponential Lévy processes. Maller, Solomon and Szimayer [12] proposed a

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multinomial tree for a Lévy process. The approximation scheme in [12] is built by incorporating a sequence of finite time, finite state space and processes for computational convenience and practical need. But Maller, Solomon and Szimayer [12] could only show that the discrete American option price processes converge in distribution under Meyer-Zheng (MZ) topology (see Meyer and Zheng [13]), which implies the convergence only holds for t in a subset of full Lebesque measure in $[0, T]$ but not every $t \in [0, T]$. For instance, all rational times in $[0, T]$ forms a set with Lebesque measure zero and the numerical approximation may fail to converge on those times. This obstructs the implement of this approximating process for American option prices. Clearly the convergence in distribution (see Jacod and Shiryaev [8]) is stronger than Meyer-Zheng convergence. Maller, Solomon and Szimayer [12] predicted that their method does not lead convergence for *all* t *in* $[0, T]$, though "it plausibly holds" under their conditions. The main purpose of the present paper is to offer an affirmative answer to their claim. We prove the convergence for *all* t *in* $[0, T]$ *in distribution.*

More recently, Szimayer and Maller [19] proposed another path-by-path defined approximation scheme, $L_t(n)$, for a pure jump Lévy process, L_t. The sequence of discrete processes converges to the Lévy process in probability or almost surely under J_1 topology under different conditions. The proof in the last paragraph on page 1446 of Szimayer and Maller [19] makes use of Skorokhod representation theorem that requires $L_t(n)$ converge to L_t in distribution for each $t \in [0, T]$. However, the law of $X_j(n)$ in Szimayer and Maller [19] is not given explicitly and the law of $X_j(n)$ must be consistent with (A.2)–(A.5) of Maller, Solomon and Szimayer [12] in order to achieve the necessary and sufficient conditions for $L_t(n) \to L_t$ in distribution. Under the multinomial tree scheme in Maller, Solomon and Szimayer [12], we prove that the discretized American put option prices converge to the continuous time counterpart for *all* t *in* $[0, T]$ *in distribution.* We make use of the Skorokhod representation theorem, some results in Maller, Solomon and Szimayer [12] and the results of Coquet and Toldo [5].

The plan of the paper is as follows. In section 2, the Skorokhod representation theorem is used to obtain representatives of the original scheme and the pure jump Lévy process. The Snell envelopes of the discounted payoff processs under the representatives for the original approximation scheme converge to that under the representative for the original Lévy process by using a result of Coquet and Toldo [5]. Since the original processes and their representatives are equal in distribution from the Skorokhod representation theorem, we get the convergence result for the Snell envelopes of the discounted payoff processes under the approximation scheme defined by Maller, Solomon and Szimayer [12]. In section 3, we prove that the discretized American option price processes, $\pi_t(n)$, converge to the continuous time American option price process, π_t, at every time $t \in [0, T]$. This main result is proved by verifying conditions of Corollary 6 in Coquet and Toldo [5].

2. An Approximation Scheme for the Lévy Process and Their Representatives

Let $L = (L_t,\ t \geq 0)$ be a Lévy process with càdlàg paths defined on a completed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\mathbb{F}^L = (\mathcal{F}^L_t)_{t\geq 0}$ be the right continuous filtration generated by $(L_t,\ t \geq 0)$. Suppose that $\mathcal{F}_0^L$ contains all $\mathbb{P}-$null sets and that $\mathcal{F}_\infty^L = \mathcal{F}$. We assume that the Lévy triplet of $(L_t,\ t \geq 0)$ is $(\gamma,\ 0,\ \Pi)$, where $\gamma \in \mathbb{R}$ and $\Pi(\cdot)$ is a Lévy measure. We also assume that $\mathbb{E}|L_1| < \infty$.

Assume that the approximation of the Lévy process is only on the finite time interval $[0, T]$. In this present paper, the tree-based approximation scheme, $L(n) = (L_t(n),\ 0 \leq t \leq T),\ n \in \mathbb{N}$, is exactly the one proposed by Maller, Solomon and Szimayer [12]. The scheme is set up so similar as the binomial tree for the Black-Sholes model that the corresponding option price could be computed straightforward by the backward induction technique as in J. Neveu [16].

Let us recall the construction of $L(n)$ in Maller, Solomon and Szimayer [12]. The number of time steps per unit time is denoted by $N(n)$, and each time period is $\Delta t(n) = 1/N(n)$ for $n \in \mathbb{N}$. The increments of $L_t(n)$ take values of integer multiples of $\Delta(n)$. The range of the increments is determined by the number of possible steps up: $m_+(n)$, and down: $m_-(n)$.

Let us choose sequences $\{\Delta(n)\} \downarrow 0$ and $\{N(n)\} \uparrow \infty$, as $n \to \infty$, satisfying

$$\liminf_{n\to\infty} \sqrt{N(n)}\Delta(n) > 0. \tag{2.1}$$

Suppose that the sequences $m_\pm(n),\ n = 1, 2, \cdots$, satisfy

$$\lim_{n\to\infty} \Delta(n)m_\pm(n) = \infty.$$

Denote, for any $n \in \mathbb{N}$,

$$\mathcal{M}(n) = \{-m_-(n),\ \cdots,\ -1,\ 1,\ \cdots,\ m_+(n)\},$$

$$I_k(n) = ((k - \frac{1}{2})\Delta(n), (k + \frac{1}{2})\Delta(n)],\ \forall k \in \mathcal{M}(n).$$

Note that there is no 0 in $\mathcal{M}(n)$, and the union of nonoverlapping intervals $I_k(n)$ is

$$\mathcal{I}(n) = (-m_-(n) - 1/2)\Delta(n), (m_+(n) + 1/2)\Delta(n)] \setminus (-\Delta(n)/2, \Delta(n)/2].$$

Definition 2.1. For each $n \in \mathbb{N}$, let $X(n)$ be a random variable taking values in $\{k\Delta(n),\ k \in \mathcal{M}(n) \cup 0\}$. The law of $X(n)$ is given by

$$\mathbb{P}(X(n) = k\Delta(n)) = \frac{1}{N(n)}\Pi(I_k(n)),\ k \in \mathcal{M}(n),$$

and

$$\mathbb{P}(X(n) = 0) = 1 - \sum_{k\in\mathcal{M}(n)} \mathbb{P}(X(n) = k\Delta(n)).$$

Let $X_j(n),\ 1 \leq j \leq \lfloor N(n)T \rfloor$, be independent and identically distributed (i.i.d.) copies of $X(n)$. Define

$$L_t(n) = \sum_{j=1}^{\lfloor N(n)t \rfloor} (X_j(n) - a(n)) \tag{1}$$

with $a(n) = -\frac{\gamma}{N(n)} + \mathbb{E}(X(n)1_{\{|X(n)|\leq 1\}}) + b(n)$, where $b(n)$ is any non-stochastic sequence which is $o(1/N(n))$ as $n \in \mathbb{N}$.

By the claim in Maller, Solomon and Szimayer [12] and the condition (2.1), the above random variable $X(n)$ is well-defined when $n \geq n_0$ for some $n_0 \in \mathbb{N}$.

Notation. The space of càdlàg functions from $[0,T]$ to $\mathbb{R}$ is denoted by $\mathbb{D}[0,T]$. For any two càdlàg functions $X(t),\ Y(t) \in \mathbb{D}[0,T]$, the Skorokhod distance between them is defined as

$$\rho(X,Y) \triangleq \inf_{\lambda\in\Lambda}\{\sup_{0\leq t\leq T}|X(t)-Y(\lambda(t))| + \sup_{0\leq t\leq T}|t-\lambda(t)|\},$$

where Λ is the set of strictly increasing continuous functions, λ, defined on $[0,T]$ such that $\lambda(0)=0$, $\lambda(T)=T$. The topology generated by the Skorokhod distance, ρ, is called J_1-topology. A càdlàg process $Z=(Z_t,\ 0\leq t\leq T)$ can be seen as a $\mathbb{D}[0,T]$-valued random variable. If two $\mathbb{D}[0,T]$-valued random variables X and Y are equal in distribution under the Skorokhod J_1-topology, we denote it by $X \stackrel{\mathfrak{L}}{=} Y$. If a sequence of $\mathbb{D}[0,T]$-valued random variables X_n converges to X in distribution under the Skorokhod J_1-topology, we write $X_n \stackrel{\mathfrak{L}}{\to} X$. See Jacod and Shiryaev [8] for more on the J_1-topology and the equivalence.

Notation. Let X and Y be two random variables taking values in $\mathbb{R}$. If they are equal in distribution, we denote it by $X \stackrel{D}{=} Y$. If a sequence of $\mathbb{R}-$valued random variables X_n, converges in distribution to X, we write $X_n \stackrel{D}{\to} X$.

Proposition 2.2. *For the processes $L(n)=(L_t(n),\ 0\leq t\leq T),\ n\in\mathbb{N}$ and $L=(L_t,\ 0\leq t\leq T)$ defined above, there exist $\mathbb{D}[0,T]-$valued random variables $\widehat{L}(n),\ n\in\mathbb{N}$ and $\widehat{L}$ defined on a common (complete) probability space $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$ such that*

$$\widehat{L} \stackrel{\mathfrak{L}}{=} L,\ \widehat{L}(n) \stackrel{\mathfrak{L}}{=} L(n),\ n\in\mathbb{N},$$

$$\widehat{L}(n)(\omega)\to\widehat{L}(\omega) \text{ under the } J_1-\text{topology, for every } \omega\in\widehat{\Omega}.$$

Remark 2.3. Both $\widehat{L}(n),\ n\in\mathbb{N}$ and $\widehat{L}$ are called representatives of $L(n),\ n\in\mathbb{N}$ and L respectively. Theorem 3.1 of Maller, Solomon and Szimayer [12] shows that $L(n) \stackrel{\mathfrak{L}}{\to} L$ in $\mathbb{D}[0,T]$. Thus, the Proposition 2.2 is a direct conclusion of the Skorokhod representation theorem.

By Definition 2 of Coquet, Mémin and Słominski [4], a sequence of filtrations $\overline{\mathbb{F}}^n=(\overline{\mathcal{F}}^n_t)_{t\in[0,T]}$ converges weakly to a filtration $\overline{\mathbb{F}}=(\overline{\mathcal{F}}_t)_{t\in[0,T]}$, denoted by $\overline{\mathbb{F}}^n \stackrel{w}{\to} \overline{\mathbb{F}}$, if and only if, for all $B\in\overline{\mathcal{F}}_T$, the sequence of càdlàg martingales $(\mathbb{E}[1_B|\overline{\mathcal{F}}^n_\cdot])_{n\in\mathbb{N}}$ converges in probability under the Skorokhod J_1-topology in $\mathbb{D}[0,T]$ to the martingale $(\mathbb{E}[1_B|\overline{\mathcal{F}}_\cdot])$.

Lemma 2.4. *There exist random variables, $Y_j(n)$, $j=1,2,\cdots,\lfloor N(n)T\rfloor$, defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{P}})$ such that, for any $t\in[0,T]$,*

$$\widehat{L}_t(n)=\sum_{j=1}^{\lfloor N(n)t\rfloor}(Y_j(n)-a(n)), \tag{2.2}$$

where $a(n)$ is given in Definition 2.1.

Proof. Let $M(n) = (m_+(n) + m_-(n) + 1) \times \lfloor N(n)T \rfloor$ for each $n \in \mathbb{N}$. By the definition of $L_t(n)$, $L_\cdot(n)$ has $M(n)$ step function style paths, denoted by $f_1(t), f_2(t), \cdots, f_{M(n)}(t)$, $t \in [0,T]$. Then, $\sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) = 1$. By Proposition 2.2 with $\widehat{L}(n) \overset{\mathfrak{L}}{=} L(n)$ and $\widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) = \mathbb{P}(L_\cdot(n) = f_l)$ for any $l = 1, 2, \cdots, M(n)$, we have

$$\sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) = \sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) = 1.$$

That is, $\widehat{\mathbb{P}}(\widehat{L}_\cdot(n) \in \{f_1, f_2, \cdots, f_{M(n)}\}) = 1$. Hence, the paths of $\widehat{L}(n)$ are of step function style with jumps occurring only at the grid points $j\Delta t(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ with probability 1. Therefore we have

$$\widehat{L}_t(n) = \sum_{j=1}^{\lfloor N(n)t \rfloor} Z_j(n),$$

where $Z_j(n)$ are random variables defined on $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$ representing the jumps of $\widehat{L}_\cdot(n)$ occurring at the grid point $j\Delta t(n)$, $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Let $Y_j(n) = Z_j(n) + a(n)$. Hence, the required identity is obtained. □

Lemma 2.5. *For each $n \in \mathbb{N}$ and $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$,*

$$X_j(n) \overset{D}{=} Y_j(n). \tag{2.3}$$

Proof. Let $\Delta f_l(j\Delta t(n))$ be the jump of function f_l occurring at $j\Delta t(n)$, for any $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ and $l = 1, 2, \cdots, M(n)$. By the definitions of $X_j(n)$ and $Y_j(n)$ and arguments in Lemma 2.4, we get

$$\begin{aligned} \mathbb{P}(X_j(n) = k\Delta(n)) &= \sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) 1_{\{\Delta f_l(j\Delta t(n)) = k\Delta(n) - a(n)\}} \\ &= \sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) 1_{\{\Delta f_l(j\Delta t(n)) = k\Delta(n) - a(n)\}} \\ &= \widehat{\mathbb{P}}(Y_j(n) = k\Delta(n)), \end{aligned}$$

for any $k \in \mathcal{M}(n) \cup \{0\}$. Thus the result follows. □

Proposition 2.6. *Let $\mathbb{F}^{\widehat{L}(n)}$, $n \in \mathbb{N}$ be the filtrations generated by $\widehat{L}(n)$, $n \in \mathbb{N}$ and $\mathbb{F}^{\widehat{L}}$ be the right continuous filtration generated by $\widehat{L}$. Then*

$$\mathbb{F}^{\widehat{L}(n)} \overset{w}{\to} \mathbb{F}^{\widehat{L}} \text{ as } n \to \infty.$$

Proof. Proposition 2 of Coquet, Mémin and Słominski [4] states that if the sequence of càdlàg processes, $(\widehat{L}(n), n \in \mathbb{N})$, converges to the càdlàg process, $\widehat{L}$, in probability under the J_1-topology and $\widehat{L}(n)$ has independent increments for each $n \in \mathbb{N}$, then $\mathbb{F}^{\widehat{L}(n)} \overset{w}{\to} \mathbb{F}^{\widehat{L}}$. By Proposition 2.2, $\widehat{L}(n)$, for all $n \in \mathbb{N}$, and $\widehat{L}$ are all càdlàg processes and $\widehat{L}(n)(\omega) \to \widehat{L}(\omega)$ under the $J_1 -$ topology, for each $\omega \in \widehat{\Omega}$.

Therefore, in order to prove $\mathbb{F}^{\widehat{L}(n)} \xrightarrow{w} \mathbb{F}^{\widehat{L}}$, we only need to show that $\widehat{L}(n)$ has independent increments for each $n \in \mathbb{N}$.

By Lemma 2.5, $Y_j(n) \stackrel{D}{=} X_j(n)$ for all $j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Hence we have that $(Y_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ are identically distributed and

$$\begin{aligned}
&\widehat{\mathbb{P}}(Y_i(n) = k_1\Delta(n), Y_j(n) = k_2\Delta(n)) \\
&= \sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}.(n) = f_l) 1_{\{\Delta f_l(i\Delta t(n)) = k_1\Delta(n) - a(n)\}} 1_{\{\Delta f_l(j\Delta t(n)) = k_2\Delta(n) - a(n)\}} \\
&= \sum_{l=1}^{M(n)} \mathbb{P}(L.(n) = f_l) 1_{\{\Delta f_l(i\Delta t(n)) = k_1\Delta(n) - a(n)\}} 1_{\{\Delta f_l(j\Delta t(n)) = k_2\Delta(n) - a(n)\}} \\
&= \mathbb{P}(X_i(n) = k_1\Delta(n), X_j(n) = k_2\Delta(n)) \\
&= \mathbb{P}(X_i(n) = k_1\Delta(n))\mathbb{P}(X_j(n) = k_2\Delta(n)) \\
&= \widehat{\mathbb{P}}(Y_i(n) = k_1\Delta(n))\widehat{\mathbb{P}}(Y_j(n) = k_2\Delta(n)),
\end{aligned}$$

for any $i \neq j,\ 1 \leq i, j \leq \lfloor N(n)T \rfloor$ and $k_1,\ k_2 \in \mathcal{M}(n) \cup \{0\}$, where the first and the third equalities follow from the definitions, the second from Proposition 2.2, the fourth from the i.i.d. property of $(X_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ and the last from Lemma 2.5. Hence, $Y_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$ are mutually independent. Therefore $(Y_j(n))_{j=1,2,\cdots,\lfloor N(n)T \rfloor}$ are i.i.d..

Note that $\widehat{L}_t(n) - \widehat{L}_s(n) = \sum_{j=\lfloor N(n)s \rfloor + 1}^{\lfloor N(n)t \rfloor} Y_j(n)$ and $\widehat{L}_s(n) = \sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n))$.

$$\begin{aligned}
&\widehat{\mathbb{P}}\left((\widehat{L}_t(n) - \widehat{L}_s(n)) \cdot \widehat{L}_s(n)\right) \\
&= \widehat{\mathbb{P}}\left((\sum_{j=\lfloor N(n)s \rfloor + 1}^{\lfloor N(n)t \rfloor} Y_j(n)) \cdot \sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n)) \right) \\
&= \widehat{\mathbb{P}}\left(\sum_{j=\lfloor N(n)s \rfloor + 1}^{\lfloor N(n)t \rfloor} Y_j(n) \right) \cdot \widehat{\mathbb{P}}\left(\sum_{j=1}^{\lfloor N(n)s \rfloor} (Y_j(n) - a(n)) \right) \\
&= \widehat{\mathbb{P}}(\widehat{L}_t(n) - \widehat{L}_s(n)) \cdot \widehat{\mathbb{P}}(\widehat{L}_s(n)),
\end{aligned}$$

where the second equality is from the mutually independence of $Y_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$. Hence $(\widehat{L}_t(n),\ t \in [0, T])$ has independent increments for all $n \in \mathbb{N}$. Therefore our result, $\mathbb{F}^{\widehat{L}(n)} \xrightarrow{w} \mathbb{F}^{\widehat{L}}$, follows from Proposition 2 of Coquet, Mémin and Słominski [4]. □

A sequence of processes $(X_t(n),\ 0 \leq t \leq T)_{n \in \mathbb{N}}$, satisfies the Aldous' criterion for tightness if for any $\varepsilon > 0$,

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \sup_{\tau, \sigma \in \mathcal{S}_{0,T}^{X(n)}, \sigma \leq \tau \leq \sigma + \delta} \mathbb{P}(|X_\tau(n) - X_\sigma(n)| \geq \epsilon) = 0.$$

Here and later, for any process Y, $\mathcal{S}_{0,T}^Y$ denotes the set of $\mathbb{F}^Y-$stopping times taking values in $[0, T]$, where $\mathbb{F}^Y$ is the right continuous filtration generated by

Y. Next, we want to show that the sequence of processes $(\widehat{L}_t(n),\ 0 \le t \le T)_{n\in\mathbb{N}}$ satisfies the Aldous' criterion for tightness. To show that, we need to prove the following five lemmas, i.e., Lemma 2.7-2.11.

Lemma 2.7. *Let $\delta > 0$ and $\sigma,\ \tau \in \mathcal{S}_{0,T}^{\widehat{L}(n)}$ satisfying $\sigma \le \tau \le \sigma + \delta$. Then we have*
(1) $\lfloor N(n)\tau \rfloor - \lfloor N(n)\sigma \rfloor \le N(n)\delta + 1$;
(2) $II \le C_0\delta + \frac{C_0}{N(n)}$ for some constant $C_0 > 0$.

Proof. (1) Let $0 \le \varepsilon_1, \varepsilon_2 < 1$ be the numbers such that $\lfloor N(n)\tau \rfloor = N(n)\tau - \varepsilon_1$ and $\lfloor N(n)\sigma \rfloor = N(n)\sigma - \varepsilon_2$. Then we have $0 \le |\varepsilon_1 - \varepsilon_2| < 1$ and

$$\begin{aligned}\lfloor N(n)\tau \rfloor - \lfloor N(n)\sigma \rfloor =&(N(n)\tau - \varepsilon_1) - (N(n)\sigma - \varepsilon_2)\\ =&N(n)(\tau - \sigma) - (\varepsilon_1 - \varepsilon_2)\\ \le&N(n)(\tau - \sigma) + 1\\ \le&N(n)\delta + 1.\end{aligned}$$

Because $b(n),\ n \in \mathbb{N}$ is a non-stochastic sequence and independent of j,

$$\begin{aligned}(2)\ II = \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma \rfloor+1}^{\lfloor N(n)\tau \rfloor}[\frac{\gamma}{N(n)} - b(n)]\right| \le&(\frac{|\gamma|}{N(n)} + |b(n)|)\mathbb{E}(\lfloor N(n)\tau \rfloor - \lfloor N(n)\sigma \rfloor)\\ \le&(\frac{|\gamma|}{N(n)} + |b(n)|)(N(n)\delta + 1)\\ =&\delta|\gamma| + \frac{|\gamma|}{N(n)} + |b(n)|N(n)\delta + |b(n)|\\ =&\delta|\gamma| + \frac{|\gamma|}{N(n)} + o(\delta) + o\left(\frac{1}{N(n)}\right)\\ \le&\delta C_0 + \frac{C_0}{N(n)},\end{aligned}$$

for some constant $C_0 > |\gamma| > 0$, where the second inequality follows from part (1) and the second equality from $b(n) = o\left(\frac{1}{N(n)}\right)$. □

Lemma 2.8. *(Triangular Inequality) $I \le I_1 + I_2$, where*

$$\begin{aligned}I_1 &= \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma \rfloor+1}^{\lfloor N(n)\tau \rfloor}[Y_j(n)1_{\{|Y_j(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]\right|,\\ I_2 &= \mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma \rfloor+1}^{\lfloor N(n)\tau \rfloor} Y_j(n)1_{\{|Y_j(n)|>1\}}\right|.\end{aligned}$$

Lemma 2.9. *We have the following estimates.*

(1) $\mathbb{E}\left|Y_1(n)1_{\{|Y_1(n)|>1\}}\right| \le \frac{3}{N(n)}\{\int_{|x|>1}|x|\Pi(dx) + \Pi(1 - \frac{\Delta(n)}{2}, 1]\}$;
(2) $\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})^2 \le \frac{9}{N(n)}[\int_{|x|\le 1}x^2\Pi(dx) + \int_{1<|x|\le 1+\frac{\Delta(n)}{2}}x^2\Pi(dx)]$;

(3) *for all* $s \in \mathbb{Z}^+$ *satisfying* $\lfloor N(n)\sigma \rfloor + s \le \lfloor N(n)T \rfloor$,

$$\mathbb{E}\left|Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|>1\}}\right| = \mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}|;$$

(4) *for all* $s \in \mathbb{Z}^+$ *satisfying* $\lfloor N(n)\sigma \rfloor + s \le \lfloor N(n)T \rfloor$,

$$\mathbb{E}\left[Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|\le1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le1\}})\right]^2$$
$$= \mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\le1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le1\}})]^2.$$

Proof. (1) By the proof of Lemma B.1 in Appendix B of Maller, Solomon and Szimayer [12], for any $x \in I_k(n)$, $k \in \mathcal{M}(n)$, we have that $|k\Delta(n) - x| \le \Delta(n)$ and $|x| \ge \frac{\Delta(n)}{2}$. By the triangular inequality, $|k\Delta(n)| - |x| \le |k\Delta(n) - x|$. Thus, $|k\Delta(n)| - |x| \le \Delta(n)$ and so $|k\Delta(n)| \le |x| + \Delta(n) \le |x| + 2|x| = 3|x|$. If $x \in I_k(n)$, then $k\Delta(n) - \frac{\Delta(n)}{2} < x \le k\Delta(n) + \frac{\Delta(n)}{2}$. If $|k\Delta(n)| > 1$ i.e., $k\Delta(n) > 1$ or $k\Delta(n) < -1$, then $x > k\Delta(n) - \frac{\Delta(n)}{2} > 1 - \frac{\Delta(n)}{2}$ or $x \le k\Delta(n) + \frac{\Delta(n)}{2} < -(1 - \frac{\Delta(n)}{2})$ correspondingly. Hence $|x| > 1 - \frac{\Delta(n)}{2}$. Therefore,

$$\begin{aligned}
\mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}| &= \sum_{k\in\mathcal{M}(n),\ |k\Delta(n)|>1} |k\Delta(n)| \frac{1}{N(n)} \Pi(I_k(n)) \\
&\le \frac{3}{N(n)} \sum_{k\in\mathcal{M}(n),\ |k\Delta(n)|>1} \int_{I_k(n)} |x|\Pi(dx) \\
&\le \frac{3}{N(n)} \int_{|x|>1-\frac{\Delta(n)}{2}} |x|\Pi(dx) \\
&= \frac{3}{N(n)} \left[\int_{|x|>1} |x|\Pi(dx) + \int_{1-\frac{\Delta(n)}{2}<|x|\le1} |x|\Pi(dx)\right] \\
&\le \frac{3}{N(n)} \left[\int_{|x|>1} |x|\Pi(dx) + \Pi(1 - \frac{\Delta(n)}{2}, 1]\right].
\end{aligned}$$

(2) This follows from (A.6) of Maller, Solomon and Szimayer [12] and the equality after (A.6).

(3) For all $s \in \mathbb{Z}^+$ with $\lfloor N(n)\sigma \rfloor + s \le \lfloor N(n)T \rfloor$, we have

$$\begin{aligned}
&\mathbb{E}|Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|>1\}}| \\
&= \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{E}(|Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}}| | \lfloor N(n)\sigma \rfloor = j)\mathbb{P}(\lfloor N(n)\sigma \rfloor = j) \\
&= \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{E}(|Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}}|)\mathbb{P}(\lfloor N(n)\sigma \rfloor = j) \\
&=\mathbb{E}(|Y_1(n)1_{\{|Y_1(n)|>1\}}|) \sum_{j=0}^{\lfloor N(n)T\rfloor-s} \mathbb{P}(\lfloor N(n)\sigma \rfloor = j) \\
&=\mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}|,
\end{aligned}$$

where the second equality follows from the independence of $\lfloor N(n)\sigma \rfloor = j$ and $Y_{j+s}(n)1_{\{|Y_{j+s}(n)|>1\}}$.

(4) For all $s \in \mathbb{Z}^+$ with $\lfloor N(n)\sigma \rfloor + s \le \lfloor N(n)T \rfloor$, we have

$$\begin{aligned}
&\mathbb{E}\left[Y_{\lfloor N(n)\sigma\rfloor+s}(n)1_{\{|Y_{\lfloor N(n)\sigma\rfloor+s}(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})\right]^2\\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \mathbb{E}\left([Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]^2 | \lfloor N(n)\sigma\rfloor = j\right)\mathbb{P}_j\\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \mathbb{E}([Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]^2)\mathbb{P}_j\\
=&\sum_{j=0}^{\lfloor N(n)T\rfloor - s} \{\mathbb{E}[Y_{j+s}(n)1_{\{|Y_{j+s}(n)|\le 1\}}]^2 - \mathbb{E}^2(Y_1(n)1_{\{|Y_1(n)|\le 1\}})\}\mathbb{P}_j\\
=&\{\mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\le 1\}}]^2 - \mathbb{E}^2(Y_1(n)1_{\{|Y_1(n)|\le 1\}})\}\sum_{j=0}^{\lfloor N(n)T\rfloor - s}\mathbb{P}_j\\
=&\mathbb{E}([Y_1(n)1_{\{|Y_1(n)|\le 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\le 1\}})]^2),
\end{aligned}$$

where $\mathbb{P}_j = \mathbb{P}(\lfloor N(n)\sigma\rfloor = j)$. □

Lemma 2.10. *There exist $\overline{n}_1 \in \mathbb{N}$ and a positive constant C_1 such that for $n > \overline{n}_1$,*

$$I_2 \le 3C_1\delta + \frac{3C_1}{N(n)}.$$

Proof. It is pointed out by Szimayer and Maller [19] that $\mathbb{E}|L_1| < \infty$ is equivalent to $\int_{|x|>1} |x|\Pi(dx) < \infty$ by Theorem 25.3 of Sato [18]. And, $\Pi(1 - \frac{\Delta(n)}{2}, 1] \to 0$ as $n \to \infty$ implies that there exists $\overline{n}_1 \in \mathbb{N}$ such that $\Pi(1 - \frac{\Delta(n)}{2}, 1]$ is bounded for $n > \overline{n}_1$. Let C_1 be an upper bound of $\int_{|x|>1} |x|\Pi(dx) + \Pi(1 - \frac{\Delta(n)}{2}, 1] \le C_1$ for $n > \overline{n}_1$. For $n > \overline{n}_1$,

$$\begin{aligned}
I_2 =&\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)1_{\{|Y_j(n)|>1\}}]\right|\\
\le&\mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}\left|Y_j(n)1_{\{|Y_j(n)|>1\}}\right|\right)\\
\le&\mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{(\lfloor N(n)\sigma\rfloor+N(n)\delta+1)\wedge\lfloor N(n)T\rfloor}|Y_j(n)1_{\{|Y_j(n)|>1\}}|\right)\\
=&(N(n)\delta+1)\cdot\mathbb{E}|Y_1(n)1_{\{|Y_1(n)|>1\}}|\\
\le&(N(n)\delta+1)\frac{3}{N(n)}\left\{\int_{|x|>1}|x|\Pi(dx) + \Pi(1-\frac{\Delta(n)}{2}, 1]\right\}
\end{aligned}$$

$$\leq 3C_1\delta + \frac{3C_1}{N(n)},$$

where the second inequality follows from triangle inequality, the third from Lemma 2.7 (1), the fourth identity from Lemma 2.9 (3), and the fifth from Lemma 2.9 (1). Thus the result follows. □

Lemma 2.11. *There exists a positive constant C_2 such that*

$$I_1 \leq \{18C_2(\delta + \frac{1}{N(n)})\}^{\frac{1}{2}}.$$

Proof. Note that $\int\limits_{|x|\leq 1} x^2\Pi(dx) + \int\limits_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx)$ is bounded because of the definition of Lévy measure. Let constant C_2 be an upper bound of $\int\limits_{|x|\leq 1} x^2\Pi(dx) + \int\limits_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx)$. We have the following estimates.

$$\begin{aligned}
I_1^2 =& \left(\mathbb{E}|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)1_{\{|Y_j(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]|\right)^2 \\
\leq& \mathbb{E}\left(\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)1_{\{|Y_j(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right)^2 \\
=&(N(n)\delta+1)\mathbb{E}[Y_1(n)1_{\{|Y_1(n)|\leq 1\}} - \mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]^2 \\
=&(N(n)\delta+1)\{\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 - (\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}}))^2\} \\
\leq&(N(n)\delta+1)\{\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 + (\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}}))^2\} \\
\leq& 2(N(n)\delta+1)\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})^2 \\
\leq& 2(N(n)\delta+1)\cdot\frac{9}{N(n)}\left[\int_{|x|\leq 1} x^2\Pi(dx) + \int_{1<|x|\leq 1+\frac{\Delta(n)}{2}} x^2\Pi(dx)\right] \\
\leq& 18C_2(\delta + \frac{1}{N(n)}),
\end{aligned}$$

where the second inequality follows from Jensen's inequality, the third identity from Lemma 2.9(4) and the seventh from Lemma 2.9(2). □

Proposition 2.12. *The sequence of processes $(\widehat{L}_t(n),\ 0\leq t\leq T)_{n\in\mathbb{N}}$ satisfies the Aldous' criterion for tightness.*

Proof. Let $\delta > 0$ and $\sigma,\ \tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)}$ satisfying $\sigma\leq\tau\leq\sigma+\delta$. By the construction of $\widehat{L}_t(n)$ and a similar argument as in (A.7) of Maller, Solomon and Szimayer [12],

$$\begin{aligned}
&\mathbb{E}|\widehat{L}_\tau(n) - \widehat{L}_\sigma(n)| \\
=&\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}(Y_j(n) - a(n))\right|
\end{aligned}$$

$$=\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)-\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})+\frac{\gamma}{N(n)}-b(n)]\right|$$
$$\leq\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)-\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right|+\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[\frac{\gamma}{N(n)}-b(n)]\right|.$$

Define $I=\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[Y_j(n)-\mathbb{E}(Y_1(n)1_{\{|Y_1(n)|\leq 1\}})]\right|$ and

$$II=\mathbb{E}\left|\sum_{j=\lfloor N(n)\sigma\rfloor+1}^{\lfloor N(n)\tau\rfloor}[\frac{\gamma}{N(n)}-b(n)]\right|.$$

Then we have $\mathbb{E}|\widehat{L}_\tau(n)-\widehat{L}_\sigma(n)|\leq I+II$.

By Lemma 2.8 and the definition of I and II, we have

$$\mathbb{E}|\widehat{L}_\tau(n)-\widehat{L}_\sigma(n)|\leq I+II\leq I_1+I_2+II.$$

By Lemma 2.7 (2), Lemma 2.10 and Lemma 2.11, we have

$$\mathbb{E}|\widehat{L}_\tau(n)-\widehat{L}_\sigma(n)|\leq(C_0\delta+\frac{C_0}{N(n)})+(3C_1\delta+\frac{3C_1}{N(n)})+\{18C_2(\delta+\frac{1}{N(n)})\}^{\frac{1}{2}}.$$

By taking limit for $n\to\infty$, and $\delta\to 0^+$, therefore we obtain

$$\lim_{\delta\downarrow 0}\limsup_{n\to\infty}\sup_{\sigma,\tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)},\sigma\leq\tau\leq\sigma+\delta}\mathbb{E}|\widehat{L}_\tau(n)-\widehat{L}_\sigma(n)|=0.$$

□

Theorem 2.13. *Assume that $(\gamma^n(s,x),n\in\mathbb{N})$ is a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to the continuous bounded function $\gamma(s,x)$ on $[0,T]\times\mathbb{R}$. Then*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)}}\mathbb{E}(\gamma^n(\tau,\widehat{L}_\tau(n)))\to\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}}}\mathbb{E}(\gamma(\tau,\widehat{L}_\tau))\ \textit{as}\ n\to\infty.$$

Proof. It is easy to see that Proposition 2.2, Proposition 2.6 and Proposition 2.12 give the three required conditions of Corollary 6 in Coquet and Toldo [5] for $\widehat{L}(n)$ and $\widehat{L}$. Hence, we obtain that

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}(n)}}\mathbb{E}(\gamma^n(\tau,\widehat{L}_\tau(n)))\to\operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T}^{\widehat{L}}}\mathbb{E}(\gamma(\tau,\widehat{L}_\tau))\ \text{as}\ n\to\infty$$

when $(\gamma^n)_{n\in\mathbb{N}}$ is a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to a continuous bounded function, γ, on $[0,T]\times\mathbb{R}$. □

3. Convergence of American (Put) Option Prices

Having obtained the representative $\widehat{L}_t(n)$ of the approximation $L_t(n)$, we show that the snell envelope of the discounted payoff process achieves the same value. For this goal, we use results in Jacod and Shiryaev [8] and some technique lemmas of Lamberton and Pagès [11]. For an American option with discounted payoff function γ, our main purpose is to prove the weak convergence of the American option prices under the approximation $L_t(n)$ to their continuous time counterpart.

Lemma 3.1. *Let $(X_t,\ t \in [0,T])$, $(Y_t,\ t \in [0,T])$ be two cádlág processes defined on probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{P}})$, respectively. Assume that X is a process satisfying $\Delta X_t = 0$ almost surely for any $t \in [0,T]$ and that $X \stackrel{\mathfrak{L}}{=} Y$ in $\mathbb{D}[0,T]$. Then*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)),$$

where $\gamma(s,x)$ is a continuous bounded function on $[0,T] \times \mathbb{R}$.

Proof. Since γ is continuous, we have $\mathbb{F}^X = \mathbb{F}^{\gamma(\cdot, X.)}$ and $\mathbb{F}^Y = \mathbb{F}^{\gamma(\cdot,Y.)}$. Correspondingly, $\mathcal{S}^X_{0,T} = \mathcal{S}^{\gamma(\cdot,X.)}_{0,T}$ and $\mathcal{S}^Y_{0,T} = \mathcal{S}^{\gamma(\cdot,Y.)}_{0,T}$. Since γ is bounded,

$$\sup_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) < \infty; \ \sup_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)) < \infty.$$

Now we have that $(\gamma(\cdot, X.),\ \mathbb{F}^{\gamma(\cdot,X.)})$ and $(\gamma(\cdot,Y.),\ \mathbb{F}^{\gamma(\cdot,Y.)})$ are both of class D. See term (7) of Lamberton and Pagès [11] for the definition of class D. Let $C = (C_t,\ 0 \le t \le T)$ be the canonical process on $\mathbb{D}[0,T]$ and $\mathcal{T}$ be the set of the $\mathbb{F}^C$-stopping times. Let $Z^n = \gamma(\cdot,\ Y.)$, $n \in \mathbb{N}$, $Z = \gamma(\cdot,\ X.)$. Thus, for any $\tau \in \mathcal{T}$, $\{Z^n_{\tau \circ Z^n},\ n \in \mathbb{N}\}$ is uniformly integrable by the boundedness of γ (see Section 3.1 of Lamberton and Pagès [11] for $\tau \circ Z^n$ in detail). Since $Y \stackrel{\mathfrak{L}}{=} X$ and $\Delta X_t = 0$ almost surely for any $t \in [0,T]$, which implies $J(X) = \emptyset$, $Y \stackrel{\mathcal{L}([0,T])}{=} X$ by 6.3.14 of Jacod and Shiryaev [8]. Since γ is continuous, we have

$$\gamma(\cdot, Y.) \stackrel{\mathcal{L}([0,T])}{=} \gamma(\cdot, X.), \text{ i.e., } Z^n \stackrel{\mathcal{L}([0,T])}{\longrightarrow} Z. \tag{3.1}$$

By Theorem 3.2 of Lamberton and Pagès [11], we obtain that

$$\sup_{\tau \in \mathcal{S}^Z_{0,T}} \mathbb{E}(Z_\tau) \le \sup_{\tau \in \mathcal{S}^{Z^n}_{0,T}} \mathbb{E}(Z^n_\tau).$$

That is,

$$\sup_{\tau \in \mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) \le \sup_{\tau \in \mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)).$$

Note that $(Z^n,\ n \in \mathbb{N})$ is in fact a sequence of constant processes. Hence (3.1) can be written as

$$\gamma(\cdot,\ Y.) \stackrel{\mathcal{L}([0,T])}{=} \gamma(\cdot,\ X.). \tag{3.2}$$

By switching X and Y, let $Z^n = \gamma(\cdot,\ X.)$, $n \in \mathbb{N}$, $Z = \gamma(\cdot,\ Y.)$. Thus, we have $Z^n \stackrel{\mathcal{L}([0,T])}{\longrightarrow} Z$ since they are equal in finite dimensional distribution, see (3.2). By

Theorem 3.2 of Lamberton and Pagès [11] again, we have

$$\sup_{\tau\in\mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau)) \leq \sup_{\tau\in\mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)).$$

Therefore $\sup_{\tau\in\mathcal{S}^X_{0,T}} \mathbb{E}(\gamma(\tau, X_\tau)) = \sup_{\tau\in\mathcal{S}^Y_{0,T}} \mathbb{E}(\gamma(\tau, Y_\tau))$. □

Theorem 3.2. *Let $(\gamma^n(s,x), n\in\mathbb{N})$ be a sequence of continuous bounded functions on $[0,T]\times\mathbb{R}$ which uniformly converges to the continuous bounded function $\gamma(s,x)$ defined on $[0,T]\times\mathbb{R}$. Then*

$$\operatorname*{ess\,sup}_{\tau\in\mathcal{S}^{L(n)}_{0,T}} \mathbb{E}(\gamma^n(\tau, L_\tau(n))) \to \operatorname*{ess\,sup}_{\tau\in\mathcal{S}^{L}_{0,T}} \mathbb{E}(\gamma(\tau, L_\tau)) \text{ as } n\to\infty.$$

Proof. Since L is a Lévy process, $J(L)=\emptyset$ by Lemma 2.3.2 of Applebaum [2]. Since $L \overset{\mathfrak{L}}{=} \widehat{L}$ and both of $\widehat{L}$ and L are cádlág processes, by Lemma 3.1 we obtain

$$\sup_{\tau\in\mathcal{S}^L_{0,T}} \mathbb{E}(\gamma(\tau, L_\tau)) = \sup_{\tau\in\mathcal{S}^{\widehat{L}}_{0,T}} \mathbb{E}(\gamma(\tau, \widehat{L}_\tau)).$$

By the arguments in the proof of Lemma 2.4, both $L_t(n)$ and $\widehat{L}_t(n)$, $t\in[0,T]$, take only finitely many values,

$$k_1\Delta(n) - \lfloor N(n)t\rfloor a(n),\ k_2\Delta(n) - \lfloor N(n)t\rfloor a(n), \cdots, k_{m_n(t)}\Delta(n) - \lfloor N(n)t\rfloor a(n),$$

where $m_n(t) = (m_+(n)+m_-(n))\lfloor N(n)t\rfloor + 1$. We know that $L(n) \overset{\mathfrak{L}}{=} \widehat{L}(n)$. Hence, for each $i=1,2,\cdots,m_n(t)$, we obtain that

$$\begin{aligned}
\mathbb{P}(L_t(n) =& k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n))\\
=& \sum_{l=1}^{M(n)} \mathbb{P}(L_\cdot(n) = f_l) 1_{\{f_l(t) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)\}}\\
=& \sum_{l=1}^{M(n)} \widehat{\mathbb{P}}(\widehat{L}_\cdot(n) = f_l) 1_{\{f_l(t) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)\}}\\
=& \widehat{\mathbb{P}}(\widehat{L}_t(n) = k_i(n)\Delta(n) - \lfloor N(n)t\rfloor a(n)),
\end{aligned}$$

where $M(n)$ is as in Lemma 2.4. Thus $L_t(n) \overset{D}{=} \widehat{L}_t(n)$ and $\gamma(t, L_t(n)) \overset{D}{=} \gamma(t, \widehat{L}_t(n))$ for every $t\in[0,T]$ by the definition of convergence in distribution. Similarly, by taking $t = t_1, t_2, \cdots, t_m$, $m\in\mathbb{N}$, we obtain that the term $\mathbb{P}(L_{t_1}(n) = k_{i_1}(n)\Delta(n) - \lfloor N(n)t_1\rfloor a(n),\ \cdots,\ L_{t_m}(n) = k_{i_m}(n)\Delta(n) - \lfloor N(n)t_m\rfloor a(n))$ equals to $\widehat{\mathbb{P}}(\widehat{L}_{t_1}(n) = k_{i_1}(n)\Delta(n) - \lfloor N(n)t_1\rfloor a(n),\ \cdots,\ \widehat{L}_{t_m}(n) = k_{i_m}(n)\Delta(n) - \lfloor N(n)t_m\rfloor a(n))$, where $k_{i_l}(n)\Delta(n) - \lfloor N(n)t_1\rfloor a(n)$ is an possible value of $L_{t_l}(n)$. Hence $\gamma(\cdot, L(n)) \overset{\mathcal{L}([0,T])}{=} \gamma(\cdot, \widehat{L}(n))$. By Theorem 3.2 of Lamberton and Pagès [11] and the same arguments with Lemma 3.1,

$$\sup_{\tau\in\mathcal{S}^{L(n)}_{0,T}} \mathbb{E}(\gamma(\tau, L_\tau(n))) = \sup_{\tau\in\mathcal{S}^{\widehat{L}(n)}_{0,T}} \mathbb{E}(\gamma(\tau, \widehat{L}_\tau(n))).$$

Therefore our result follows by Theorem 2.13. □

From the proofs of Proposition 2.2, 2.6, 2.12 and Theorem 2.13, Lemma 3.1 and Theorem 3.2, we can see the conditions we need therein are as follows:

(1) L_t and $L_t(n)$, $n \in \mathbb{N}$ are all càdlàg processes;

(2) $L_t(n)$ has only finitely many step function style paths;

(3) $L(n) \xrightarrow{\mathfrak{L}} L$;

(4) $\Delta L_t(n) = 0$ almost surely for each $t \in [0,T]$;

(5) Jumps of $L_t(n)$, $X_j(n),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor$, are i.i.d. with law $\mathbb{P}(X_j(n) = k\Delta(n)) = \frac{1}{N(n)}\Pi(I_k(n)),\ j = 1, 2, \cdots, \lfloor N(n)T \rfloor,\ k \in \mathcal{M}(n)$ and $\mathbb{P}(X_j(n) = 0) = 1 - \sum_{k \in \mathcal{M}(n)} \mathbb{P}(X_j(n) = k\Delta(n))$.

Let process, $R = (R_t,\ 0 \le t \le T)$, where $R_t = \ln S_0 + L_t$ for each $t \in [0,T]$. Let $R(n) = (R_t(n),\ 0 \le t \le T)$ where $R_t(n) = \ln S_0(n) + L_t(n)$ for each $t \in [0,T]$, $n \in \mathbb{N}$. By these definitions, the difference of L and R is that the initial value is changed from 0 to $\ln S_0$ and that of $L(n)$ and $R(n)$ is that the initial value is changed from 0 to $\ln S_0(n)$. Hence, R and $R(n)$ satisfy the above conditions (1), (4) and (5). We know that $L(n) \xrightarrow{\mathfrak{L}} L$, $S_0(n) \xrightarrow{D} S_0$ and S_0 is independent of $L_t,\ t \in [0,T]$, $S_0(n)$ is independent of $L_t(n),\ t \in [0,T]$ for any $n \in \mathbb{N}$. Hence, $R(n) \xrightarrow{\mathfrak{L}} R$ in $\mathbb{D}[0,T]$ as $n \to \infty$. Both S_0 and $S_0(n)$, $n \in \mathbb{N}$, take only finitely many values. Thus, conditions (2) and (3) still hold for R and $R(n)$. Therefore, Proposition 2.2, 2.6, 2.12 and Theorem 2.13, Lemma 3.1 and Theorem 3.2 are true for both R and $R(n)$. Let us restate Theorem 3.2 here for R and $R(n)$. Notice that $\mathcal{S}_{0,T}^{R(n)} = \mathcal{S}_{0,T}(n)$ and that $\mathcal{S}_{0,T}^{R} = \mathcal{S}_{0,T}$.

Theorem 3.3. *Let $(\gamma^n(s,x), n \in \mathbb{N})$ be a sequence of continuous bounded functions on $[0,T] \times \mathbb{R}$ which uniformly converges to the continuous bounded function $\gamma(s,x)$ defined on $[0,T] \times \mathbb{R}$. Then*

$$\operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}(n)} \mathbb{E}(\gamma^n(\tau, R_\tau(n))) \to \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{0,T}} \mathbb{E}(\gamma(\tau, R_\tau)) \ \textit{as } n \to \infty.$$

Assume the stock price process is given by

$$S_t = S_0 e^{L_t},\ 0 \le t \le T, \tag{3.1}$$

where L_t is the Lévy process defined in §2 and $S_0 \in \mathbb{R}^+$ is an initial stock price, which is a random variable independent of $(L_t, 0 \le t \le T)$. Assume that $\mathbb{E}(S_0) < \infty$, $\mathbb{E}(e^{L_t}) < \infty$ and that a discount bond with maturity $T > 0$ and unit face value is traded. Assume the instantaneous interest rate $r > 0$ is constant for all maturities. Let $g(x)$ be the payoff function. Suppose that the option is not exercised before time t. Let $\mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]}$ be the right continuous filtration generated by $(S_t,\ t \in [0,T])$. Let $\mathcal{S}_{s_1,s_2}$ be the set of $\mathbb{F}-$stopping times taking values in $[s_1, s_2]$. The American option price can be given as the solution to the optimal stopping problem (see Myneni[15]): For $0 \le t \le T$,

$$\pi_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}(e^{-r(\tau - t)} g(S_\tau) | \mathcal{F}_t).$$

Using the discretization $L(n)$ illustrated in §2, a discrete approximation of the American option price process could be achieved. Similar to (3.1), let

$$S_t(n) = S_0(n) e^{L_t(n)},\ \text{for } 0 \le t \le T,$$

where $S_0(n) > 0$ is the starting value of the discrete stock price process independent of $(L_t(n))_{0\le t\le T}$, for each $n \in \mathbb{N}$. Assume that $S_0(n) \xrightarrow{D} S_0$, as $n \to \infty$. For computational convenience, we assume that $S_0(n)$ takes only finitely many values for each $n \in \mathbb{N}$. One example is that $S_0(n) = \{m(n) \wedge \lfloor \frac{S_0}{\Delta(n)} \rfloor\}\Delta(n)$, where $\{m(n)\Delta(n)\} \uparrow \infty$. In fact, as mentioned in the Remark 4.5 of Maller, Solomon and Szimayer [12], $S_0(n) = S_0$, a constant, is often taken in most cases. See VG and NIG examples in Maller, Solomon and Szimayer [12] and the setup of Szimayer and Maller [19].

Let $\mathbb{F}^n = (\mathcal{F}^n_t)_{t\in[0,T]}$ be the filtration generated by $(S_t(n),\ 0 \le t \le T)$ and $\mathcal{S}_{s_1,s_2}(n)$ be the set of $\mathbb{F}^n-$stopping times taking values in $[s_1, s_2]$. The discounted price process of the not-exercised option under the approximation, $L(n)$, is given by the Snell envelop

$$\pi'_t(n) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,T}(n)} \mathbb{E}(e^{-r(\tau-t)} g(S_\tau(n))|\mathcal{F}^n_t).$$

Here, $\pi'_t(n)$ is exactly the same as $\pi_t(n)$ defined in (4.4) of Maller, Solomon and Szimayer [12].

We define another discrete price process, $\pi_t(n)$, which equals $\pi'_t(n)$ eventually. Let the discrete price process $\pi_t(n)$ be defined as the following:

$$\begin{cases} \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{t,T}(n)} \mathbb{E}(e^{-r(\tau-t)} g(S_\tau(n))|\mathcal{F}^n_t),\ t = j\Delta t(n),\ j = 0, 1, \cdots, \lfloor N(n)T\rfloor \\ \pi_{j\Delta t(n)}(n),\ j\Delta t(n) \le t < (j+1)\Delta t(n) \wedge T,\ j = 0, 1, \cdots, \lfloor N(n)T\rfloor. \end{cases}$$

The term $\pi_t(n)$ is an interim value between $\pi'_t(n)$ and π_t. It is for the convenience of our later proof.

As in Lamberton [10] and Szimayer and Maller [19], the option prices can be expressed by their value functions.

Definition 3.4. For any $(t, x) \in [0, T] \times \mathbb{R}^+$, the *value function* of π_t is defined by

$$v(t, x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}} \mathbb{E}(e^{-r\tau} g(xS_0e^{L_\tau})),$$

and the value function of $\pi_t(n)$ is defined by

$$v_n(t, x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau} g(xS_0(n)e^{L_\tau(n)})),$$

for $t = j\Delta t(n)$, $j = 0, 1, 2, \cdots, \lfloor N(n)T\rfloor$ and

$$v_n(t, x) = v_n(j\Delta t(n), x),\ \text{for } j\Delta t(n) \le t < (j+1)\Delta t(n) \wedge T.$$

Remark 3.5. Notice that $\pi_t(n) = v_n(t, e^{L_t(n)})$ and $\pi_t = v(t, e^{L_t})$. By Remark 5 of Szimayer and Maller [19], for any $t = j\Delta t(n)$, $j = 0, 1, 2, \cdots, \lfloor N(n)T\rfloor$, it is easy to see that the stopping time in $\mathcal{S}_{t,T}(n)$ that maximize $v_n(t, x)$ must take values on the discrete grid $[t, T] \cap \{j\Delta t(n) : j = 0, 1, \cdots, \lfloor N(n)T\rfloor\}$.

Remark 3.6. We can use a similar idea as that used in the proof of [19, Theorem 5.1] to show that $\lim_{n\to\infty} v_n(t, x_n) = v(t, x)$ for any $(t, x) \in [0, T] \times \mathbb{R}^+$. First of all, we define a sequence of functions, $\widetilde{v}_n(t,x)$ on $[0,T]\times\mathbb{R}^+$ to be the following

$$\widetilde{v}_n(t,x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau}g(xe^{R_\tau(n)})).$$

Hence we have that

$$\widetilde{v}_n(t,x_n) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(e^{-r\tau}g(x_ne^{R_\tau(n)})) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}(n)} \mathbb{E}(\gamma^n(\tau,R_\tau(n))),$$

where $\gamma^n(\tau,y) = e^{-r\tau}g(x_ne^y)$ is continuous and bounded for $(\tau,y)\in[0,T]\times\mathbb{R}$ since g is continuous and bounded. Let $v(t,x) = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t}}\mathbb{E}(\gamma(\tau,R_\tau))$, where $\gamma(\tau,y) = e^{-r\tau}g(xe^y)$. So γ is also continuous and bounded for $(\tau,y)\in[0,T]\times\mathbb{R}$.

Lemma 3.7. *The sequence of continuous bounded functions $\gamma^n(\tau,y)$ converges to $\gamma(\tau,y)$ uniformly on $(\tau,y)\in[0,T]\times\mathbb{R}$.*

Proof. We give a proof for the sake of completeness. Let K be a fixed positive number. If $xe^y < K$, then there exists $\delta > 0$ such that $xe^y \le K-\delta$. Since $\lim\limits_{n\to\infty} x_n = x$, a positive number, there exists $n_1\in\mathbb{N}$ such that $|x_n - x| < \frac{\delta x}{K}$ for $n\ge n_1$. Then, $|x_ne^y - xe^y| = e^y|x_n - x| < \delta$ and so $x_ne^y < K$ for $n \ge n_1$.

Similarly, if $xe^y > K$, there exists $n_2\in\mathbb{N}$ such that $x_ne^y > K$ for $n\ge n_2$.

If $xe^y = K$, $e^y = \frac{K}{x}$. Hence for $n\ge\max\{n_1,\ n_2\}$, we have that

$$|(K-x_ne^y)^+ - (K-xe^y)^+| = \begin{cases} |x-x_n|e^y, & xe^y < K\\ 0, & xe^y > K\\ \frac{K}{x}(x-x_n)^+, & xe^y = K.\end{cases}$$

Since $\lim\limits_{n\to\infty} x_n = x$, for any $\varepsilon > 0$, there exists $n_3\in\mathbb{N}$ such that $|x-x_n| < \frac{x\varepsilon}{K}$, i.e., $|\frac{x_n}{x} - 1| < \frac{\varepsilon}{K}$ for $n\ge n_3$. For $n\ge\max\{n_1,n_2,n_3\}$, $|(K-x_ne^y)^+ - (K-xe^y)^+| \le \varepsilon$ uniformly for $y\in\mathbb{R}$. Therefore, $\gamma^n(s,y)\to\gamma(s,y)$ uniformly for $(s,y)\in[0,T]\times\mathbb{R}$ as $n\to\infty$. □

Proposition 3.8. *For any $\varepsilon > 0$, there exists $N\in\mathbb{N}$ such that for $n > N$, $\widetilde{v}_n(t,x_n) \le v_n(t,x_n) \le \widetilde{v}_n(t,x_n) + \varepsilon$.*

Proof. For any $(t,x)\in[0,T]\times\mathbb{R}^+$, by the definition of $v_n(t,x)$, we could write

$$v_n(t,x) = \operatorname*{ess\,sup}_{\tau\in\mathcal{S}_{0,T-t+\rho_n(t)}(n)} \mathbb{E}(e^{-r\tau}g(xe^{R_\tau(n)})),$$

where $\rho_n(t) = t - \lfloor N(n)t\rfloor\Delta t(n)$, for any $n\in\mathbb{N}$. Clearly, $0\le\rho_n(t) < \Delta t(n)$.

Let $\tau_0\in\mathcal{S}_{0,T-t+\rho_n(t)}(n)$ be the optimal stopping time of $v_n(t,x_n)$. By Remark 3.4,

$$\tau_0\in[0,T-t]\cap\{j\Delta t(n),\ j=0,1,\cdots,\lfloor N(n)(T-t)\rfloor\}$$

and

$$v_n(t,x_n) = \mathbb{E}(e^{-r\tau_0}g(x_ne^{R_{\tau_0}(n)})).$$

Taking $\tau_1 = \tau_0 \wedge (T-t)$, then $\tau_1 \in \mathcal{S}_{0,T-t}(n)$ and $0 \leq \tau_0 - \tau_1 \leq \rho_n(t) < \Delta t(n)$. Consider the following estimate.

$$\begin{aligned}
&|\mathbb{E}(e^{-r\tau_0}g(x_n e^{R_{\tau_0}(n)})) - \mathbb{E}(e^{-r\tau_1}g(x_n e^{R_{\tau_1}(n)}))| \\
\leq&|\mathbb{E}(e^{-r\tau_0}g(x_n e^{R_{\tau_0}(n)})) - \mathbb{E}(e^{-r\tau_0}g(x_n e^{R_{\tau_1}(n)}))| \\
&+|\mathbb{E}(e^{-r\tau_0}g(x_n e^{R_{\tau_1}(n)})) - \mathbb{E}(e^{-r\tau_1}g(x_n e^{R_{\tau_1}(n)}))| \\
\leq&\mathbb{E}|e^{-r\tau_0}g(x_n e^{R_{\tau_0}(n)}) - e^{-r\tau_0}g(x_n e^{R_{\tau_1}(n)})| \\
&+\mathbb{E}|e^{-r\tau_0}g(x_n e^{R_{\tau_1}(n)}) - e^{-r\tau_1}g(x_n e^{R_{\tau_1}(n)})| \\
\leq&\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| + \mathbb{E}|[e^{-r\tau_0} - e^{-r\tau_1}]g(x_n e^{R_{\tau_1}(n)})|.
\end{aligned}$$

By (2.1), there exists $C_{inf} > 0$ and $\widehat{n}_0 \in \mathbb{N}$ such that $0 < C_{inf} \leq \Delta(n)\sqrt{N(n)}$ for $n > \widehat{n}_0$. Therefore, $N(n) \geq \frac{C_{inf}^2}{\Delta^2(n)}$ for $n > \widehat{n}_0$. By $g(x) = (K-x)^+$, $0 \leq g(\cdot) \leq K$ and the definition of $L_t(n)$, τ_0 and τ_1, we have for $n > \widehat{n}_0$,

$$\begin{aligned}
\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| \leq&K\mathbb{P}(L_{\tau_0}(n) \neq L_{\tau_1}(n)) \\
\leq&K\mathbb{P}(X_{\lfloor N(n)(T-t)\rfloor}(n)1_{\{\tau_0\neq\tau_1\}} \neq 0) \\
\leq&K\mathbb{P}(X_{\lfloor N(n)(T-t)\rfloor}(n) \neq 0) \\
=&K\sum_{k\in\mathcal{M}(n)}\frac{1}{N(n)}\Pi(I_k(n)) \\
\leq&\frac{K}{C_{inf}^2}\sum_{k\in\mathcal{M}(n)}(\Delta(n))^2\Pi(I_k(n)) \\
\leq&\frac{K}{C_{inf}^2}(\Delta(n))^2\overline{\Pi}(\frac{\Delta(n)}{2}),
\end{aligned}$$

where the second inequality follows from $L_{\tau_0}(n) \neq L_{\tau_1}(n)$, which implies $\tau_0 \neq \tau_1$ and so $\tau_0 > T-t$. Hence, $\tau_0 = \lfloor N(n)(T-t)\rfloor\Delta t(n) > T-t$ and so $L_{\tau_0}(n) - L_{\tau_1}(n) = X_{\lfloor N(n)(T-t)\rfloor}(n)1_{\{\tau_0\neq\tau_1\}}$. Since $\Pi(\cdot)$ is a Lévy measure, $(\Delta(n))^2\overline{\Pi}(\frac{\Delta(n)}{2}) \to 0$ as $n \to \infty$. For any $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $(\Delta(n))^2\overline{\Pi}(\frac{\Delta(n)}{2}) < \frac{\varepsilon C_{inf}^2}{2K}$ for $n \geq n_0$. Hence,

$$\mathbb{E}|g(x_n e^{R_{\tau_0}(n)}) - g(x_n e^{R_{\tau_1}(n)})| \leq \frac{\varepsilon}{2}$$

for $n \geq \max\{n_0, \widehat{n}_0\}$. Note that

$$\begin{aligned}
\mathbb{E}|[e^{-r\tau_0} - e^{-r\tau_1}]g(x_n e^{R_{\tau_1}(n)})| \leq&K\mathbb{E}|e^{-r\tau_0} - e^{-r\tau_1}| \\
=&K\mathbb{E}|e^{-r\tau_0}(1 - e^{r(\tau_0-\tau_1)})| \\
=&K\mathbb{E}(e^{-r\tau_0}(e^{r(\tau_0-\tau_1)} - 1)) \\
\leq&K\mathbb{E}(e^{r\rho_n(t)} - 1) \\
\to&0 \text{ as } n \to \infty.
\end{aligned}$$

Hence, there exists $n_0' \in \mathbb{N}$ such that $K\mathbb{E}(e^{r\rho_n(t)} - 1) < \frac{\varepsilon}{2}$ for $n \geq n_0'$, and so

$$\mathbb{E}|(e^{-r\tau_0} - e^{-r\tau_1})g(x_n e^{R_{\tau_1}(n)})| \leq \frac{\varepsilon}{2}$$

for $n \geq n_0'$. Therefore, for $n \geq \max\{n_0, n_0', \widehat{n}_0\}$, we get

$$v_n(t, x_n) \leq \mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)})) + \varepsilon.$$

Since $\tau_1 \in \mathcal{S}_{0,T-t}(n)$, $\mathbb{E}(e^{-r\tau_1} g(x_n e^{R_{\tau_1}(n)})) \leq \widetilde{v}_n(t, x_n)$ and so,

$$v_n(t, x_n) \leq \widetilde{v}_n(t, x_n) + \varepsilon.$$

On the other hand, by the construction of $\widetilde{v}_n(t, x_n)$ and $v_n(t, x_n)$, it is easy to see that, for any $n \in \mathbb{N}$,

$$\widetilde{v}_n(t, x_n) \leq v_n(t, x_n).$$

Taking $N = \max\{n_0, n_0', \widehat{n}_0\}$, the result follows. □

Proposition 3.9. *For each $t \in [0, T]$, $\lim_{n\to\infty} \pi_t(n) = \lim_{n\to\infty} \pi_t'(n))$.*

Proof. Fix $t \in [0, T]$. For each $n \in \mathbb{N}$, let $t \in [j\Delta t(n), (j+1)\Delta t(n))$. Consider that

$$\pi_t(n) = \pi_{j\Delta t(n)}(n) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{j\Delta t(n),T}(n)} \mathbb{E}(e^{-r(\tau - j\Delta t(n))} g(S_\tau(n)) | \mathcal{F}^n_{j\Delta t(n)}).$$

Since $\mathcal{F}^n_{j\Delta t(n)} = \mathcal{F}^n_t$,

$$\pi_t'(n) e^{-r(\Delta t(n))} \leq \pi_t'(n) e^{-r(t - j\Delta t(n))} \leq \pi_t(n).$$

Let $\pi_t(n) = \mathbb{E}(e^{-r(\sigma_0 - j\Delta t(n))} g(S_{\sigma_0}(n)) | \mathcal{F}^n_t)$, where $\sigma_0 \in \mathcal{S}_{j\Delta t(n),T}(n)$ is the optimal stopping time of $\pi_t(n)$. By Remark 3.4, σ_0 only takes values in $\{k\Delta t(n) | k = j, j+1, \cdots, \lfloor N(n)T \rfloor\}$. By taking $\sigma_1 = \sigma_0 \vee t$, we obtain that $\sigma_1 \in \mathcal{S}_{t,T}(n)$, $\sigma_0 \leq \sigma_1 < \sigma_0 + \Delta t(n)$ and that $S_{\sigma_1}(n) = S_{\sigma_0}(n)$. Hence,

$$\begin{aligned}
\pi_t(n) e^{-r(\Delta t(n))} =& \mathbb{E}(e^{-r(\sigma_0 - j\Delta t(n) + \Delta t(n))} g(S_{\sigma_0}(n)) | \mathcal{F}^n_t) \\
\leq& \mathbb{E}(e^{-r(\sigma_1 - j\Delta t(n))} g(S_{\sigma_1}(n)) | \mathcal{F}^n_t) \\
\leq& \mathbb{E}(e^{-r(\sigma_1 - t)} g(S_{\sigma_1}(n)) | \mathcal{F}^n_t) \\
\leq& \pi_t'(n).
\end{aligned}$$

By the Squeeze law, the result follows. □

Theorem 3.10. *Suppose that the option is an American put option, i.e., the payoff function $g(x) = (K - x)^+$, where K is the strike price and x is the stock price when the option is exercised. Then, whenever $\{x_n\} \to x$ as $n \to \infty$, we have*

$$\lim_{n\to\infty} v_n(t, x_n) = v(t, x), \text{ for any } (t, x) \in [0, T] \times \mathbb{R}^+, \tag{3.2}$$

$$\pi_t(n) \xrightarrow{D} \pi_t, \text{ as } n \to \infty, \text{ for each } t \in [0, T], \tag{3.3}$$

$$\pi_t'(n) \xrightarrow{D} \pi_t, \text{ as } n \to \infty, \text{ for each } t \in [0, T]. \tag{3.4}$$

Proof. By Lemma 3.7 and Theorem 3.3,

$$\widetilde{v}_n(t, x_n) \to v(t, x) \text{ whenever } x_n \to x \text{ as } n \to \infty. \tag{3.5}$$

By Proposition 3.8, $|v_n(t, x_n) - \widetilde{v}_n(t, x_n)| \leq \varepsilon$ when $n > N$. Hence, $v_n(t, x_n) \to v(t, x)$ whenever $x_n \to x$ as $n \to \infty$. Thus (3.2) is proved.

Since $L(n) \xrightarrow{\mathcal{L}} L$ in $\mathbb{D}[0, T]$, $e^{L(n)} \xrightarrow{\mathcal{L}} e^L$ in $\mathbb{D}[0, T]$ as $n \to \infty$ by 6.3.8 of Jacod and Shiryaev [8]. Since L_t is almost surely continuous for any $t \in [0, T]$, $e^{L_t(n)} \xrightarrow{D} e^{L_t}$ as $n \to \infty$, for any given $t \in [0, T]$, by 6.3.14 of Jacod and Shiryaev [8]. From the Skorokhod representation theorem, it follows that there exist random variables $Z_t(n),\ n \in \mathbb{N}$ and Z_t defined on a common probability space $(\Omega^{Z_t}, \mathcal{F}^{Z_t}, \mathbb{P}^{Z_t})$, such that $Z_t(n) \overset{D}{=} e^{L_t(n)}$, $Z_t \overset{D}{=} e^{L_t}$ and $Z_t(n) \to Z_t$ for every $\omega \in \Omega^{Z_t}$, as $n \to \infty$. By (3.2), we get that

$$v_n(t, Z_t(n)) \to v(t, Z_t) \text{ for every } \omega \in \Omega^{Z_t},\ \text{ as } n \to \infty. \tag{3.6}$$

Consider that, for any fixed $t \in [0,T]$ and fixed $n \in \mathbb{N}$, $Z_t(n) \overset{D}{=} e^{L_t(n)}$ take only finitely many values. So, $v_n(t, Z_t(n)) \overset{D}{=} \pi_t(n)$. On the other hand, if $L(n) = L$ for each $n \in \mathbb{N}$, then $\widetilde{v}_n(t, x_n) = v(t, x_n)$. (3.5) implies that, for any $(t,x) \in [0,T] \times \mathbb{R}^+$, $v(t, x_n) \to v(t,x)$, whenever $\{x_n\} \to x$. So, the value function $v(t,x)$ is bounded and continuous with respect to x. Hence, $v(t, e^{L_t}) \overset{D}{=} v(t, Z_t)$, i.e., $v(t, Z_t) \overset{D}{=} \pi_t$, by the definition of convergence in distribution. Therefore, $\pi_t(n) \overset{D}{\to} \pi_t$, as $n \to \infty$, for any $t \in [0,T]$ by (3.6).

By Proposition 3.8, $\lim\limits_{n\to\infty} \mathbb{E}(f(\pi'_t(n))) = \lim\limits_{n\to\infty} \mathbb{E}(f(\pi_t(n))) = \lim\limits_{n\to\infty} \mathbb{E}(f(\pi_t))$ for any continuous bounded function $f : \mathbb{R} \to \mathbb{R}$. Therefore, $\pi'_t(n) \overset{D}{\to} \pi_t$, as $n \to \infty$, for any $t \in [0,T]$. □

Remark 3.11. In the proof of Theorem 3.10, the continuity and boundedness of the payoff function are required. Although the payoff function of a call option is not bounded, we can modify it to be a bounded one. Let the payoff function of a modified call option is of the form

$$g(x) = (x - K)^+ \wedge M,$$

where M is a (sufficiently) large positive number. Then g is continuous bounded. By a similar proof as that of Theorem 3.10, we could get the same convergence results as those of Theorem 3.10 for the modified American call option.

Corollary 3.12. *Suppose that the option is an modified American call option with the payoff function $g(x) = (x-K)^+ \wedge M$ for a (sufficiently) large positive number M. Then we have,*

$$\lim_{n\to\infty} v_n(t, x_n) = v(t,x),\ \textit{for any } (t,x) \in [0,T] \times \mathbb{R}^+,$$

$$\pi_t(n) \overset{D}{\to} \pi_t,\ \textit{as } n \to \infty,\ \textit{for each } t \in [0,T],$$

$$\pi'_t(n) \overset{D}{\to} \pi_t,\ \textit{as } n \to \infty,\ \textit{for each } t \in [0,T],$$

provided $\{x_n\} \to x$ as $n \to \infty$.

Proof. From the proofs of Lemma 3.7, Proposition 3.8, 3.9 and Theorem 3.10, we only need to show $\gamma^n(s,y) \to \gamma(s,y)$ uniformly on $(s,y) \in [0,T] \times \mathbb{R}$.

By a very similar argument as that in Lemma 3.7, we have that there exists $\widehat{n} \in \mathbb{N}$ such that, if $n \geq \widehat{n}$,

$$|(x_n e^y - K)^+ \wedge M - (xe^y - K)^+ \wedge M|$$

$$= \begin{cases} 0, & xe^y < K \\ \frac{K}{x}(x_n - x)^+, & xe^y = K \\ |x_n - x|\frac{M+K}{x}, & M+K > xe^y > K \\ |(x_n e^y - K)^+ \wedge M - M|, & xe^y = M+K \\ 0, & xe^y > M+K. \end{cases}$$

Therefore, $\gamma^n(s,y) \to \gamma(s,y)$ uniformly for $(s,y) \in [0,T] \times \mathbb{R}$ as $n \to \infty$. □

4. Conclusion

The approximation scheme proposed by Maller, Solomon and Szimayer [12] can be seen as a generalization of the binomial tree for the Black-Sholes model. The tree-based scheme makes it easier to compute American option prices in practice. Just as in Maller, Solomon and Szimayer [12], the essential advantage of the tree-based scheme is that the model and the valuation principles are easily implemented and understood without deep knowledge of the underlying financial, mathematical and probabilistic fundamentals. They proved that $\pi_t(n)$ converge to π_t for each t in a full measure set of $[0,T]$ but not every time $t \in [0,T]$. This convergence result can not satisfy practical need because we need to have a scheme to price an American option at any time.

The approximation scheme proposed by Szimayer and Maller [19] is defined path-by-path. The idea to achieve the convergence of the sequence of Snell envelopes under the approximation scheme in Szimayer and Maller [19] is to apply Theorem 5 and Corollary 6 of Coquet and Toldo [5] by verifying the conditions therein.

In this paper, we have adapted the same principle with Szimayer and Maller [19] to the approximation scheme given in the multinomial tree of Maller, Solomon and Szimayer [12]. But the directly checking the conditions of Theorem 5 and Corollary 6 of Coquet and Toldo [5] fails. We have to construct another discrete approximation model which is equal in distribution from the Skorokhod representation theorem. This relies on a basic result proved in Maller, Solomon and Szimayer [12]. The main result of this paper is that the sequence of American (put) option price processes under the multinomial tree scheme proposed by Maller, Solomon and Szimayer [12] converges to the continuous time counterpart in distribution *for all* $t \in [0,T]$. Therefore we have overcome the main difficulty in the weak convergence issue in Maller, Solomon and Szimayer [12], and *our result is strong enough to fulfill the practical need.* Our proof is not only applicable for American put options but also applicable for any option whose payoff function is continuous bounded and satisfies the statement in Lemma 3.7. For call option cases, we only discuss modified call options in Remark 3.11 and Corollary 3.12.

Research in convergence and convergence rates for the multinomial scheme is quite challenge. There are substantially technical problems to overcome in establishing convergence rates for the methods we present in this paper theoretically, so we will leave it in a future study. For pure jump Lévy processes approximated on an equally spaced time grid, Szimayer and Maller [19] established the convergence rates for different approximation schemes. It would be interesting to know if the method used in Szimayer and Maller [19] can be applied in the discrete approximation we studied in this paper and Maller, Solomon and Szimayer [12], with weak convergence results.

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