

A STOCHASTIC LAGRANGIAN PARTICLE MODEL AND NONLINEAR FILTERING FOR THREE DIMENSIONAL EULER FLOW WITH JUMPS

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ABSTRACT. In this paper we will introduce a stochastic Lagrangian particle model with jumps for the three dimensional Euler flow and study the associated nonlinear filtering problem. We apply results from backward integro-differential equation problem to prove uniqueness of solution to the Zakai equations.

1. Introduction

Nonlinear estimation of turbulence and vortical structures has many applications in engineering and in geophysical sciences. This paper is devoted to the mathematical study of associated stochastic differential equations, in particular the unique solvability theorem of Fujisaki-Kallianpur-Kunita(FKK) and Zakai equation. We will formulate a stochastic vortex model with jumps as the signal process and study the associated nonlinear filtering problem.

Popa and Sritharan [14] studied the nonlinear filtering problem for a general finite dimensional jump-diffusion processes. The uniqueness of solution to Zakai equation is also obtained with the help of a-priori moment estimate of the signal. Amirdjanova and Kallianpur [1] considered a signed measure-valued stochastic partial differential equation for the vorticity process based on a Itô-Lévy evolution of a system of N randomly moving point vortices. The corresponding Fujisaki-Kallianpur-Kunita(FKK) equation of the optimal filter is also derived in [1]. In Sritharan and Xu [20], the uniqueness to the solution of Zakai equation is proved for general diffusions with unbounded observations. The goal of this paper is to use a unique solvability result from the theory of parabolic integro-differential equation to prove the uniqueness of the solution to the Zakai equation. The result is an extension of the result obtained by Rozovskii [16] to signal process of Itô-Lévy type with small and large jump noises.

In this paper, we consider the particle trajectory model in 3D Euler flow using the Lagrangian representation. This is different from the problem studied in [20], where a regularized system of point-vortices in \mathbb{R}^2 is considered with Gaussian

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noise. Our model simulates the random behavior of three dimensional Euler flow by studying the evolution of flow particles with initial position in \mathbb{R}^3 . In section 2 we study the solvability of the stochastic Lagrangian particle model. A moment estimate for the solution is given in section 3. Section 4 is devoted to an introduction to the associated nonlinear filtering problem. Solvability of a parabolic integro-differential problem is considered in section 5. This result is crucial in proving the uniqueness of solution to measured valued Zakai equations given in section 6.

2. Stochastic vortex model and its well-posedness

Let us suppose that a complete probability space (Ω, \mathcal{F}, P) is equipped with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. We introduce a stochastic Lagrangian representation for moving particles in three dimensional Euler flow as follows. Denote $X(t) = (X^i(t))$ $i = 1, \dots, N$ as the location for N number of particles in \mathbb{R}^3 with initial data $X^i(0)$ and $u(x, t)$ as the velocity field. We consider:

$$\begin{aligned} dX(t) = & u(X(t), t)dt + \sigma(X(t))dW(t) + \int_{|z| < 1} F(X(t-), z)\tilde{\mathcal{N}}(dt, dz) \\ & + \int_{|z| \geq 1} G(X(t-), z)\mathcal{N}(dt, dz), \end{aligned} \quad (2.1)$$

with

$$u(x, t) = K * \omega_t(x) = \int_{\mathbb{R}^3} K(x - y)\omega(y, t)dy. \quad (2.2)$$

Here K is given by

$$K(x) = \frac{1}{4\pi|x|^3} \begin{pmatrix} 0 & x_3 & -x_2 \\ -x_3 & 0 & x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$$

and ω is the vorticity field.

For each $x, y \in \mathbb{R}^{3N}$ we introduce the $3N \times 3N$ matrix

$$a(x, y) = \sigma(x)\sigma(y)^T, \quad (2.3)$$

and define the matrix seminorm as

$$\|a\| = \sum_{i=1}^{3N} |a_i^i|. \quad (2.4)$$

Assumptions:

- (1) $W(t) = (W_1(t), \dots, W_{3N}(t)), t \geq 0$, is a $3N$ -dimensional \mathcal{F}_t -adapted standard Brownian motion.
- (2)

$$\begin{aligned} \tilde{\mathcal{N}}(ds, dx)^T &= (\tilde{\mathcal{N}}_1(ds, dx_1), \dots, \tilde{\mathcal{N}}_{3N}(ds, dx_{3N})) \\ &= (\mathcal{N}_1(ds, dx_1) - \nu_1(dx_1)ds, \dots, \mathcal{N}_{3N}(ds, dx_{3N}) - \nu_{3N}(dx_{3N})ds), \end{aligned}$$

where $\{\mathcal{N}_j\}$ are independent Poisson random measures with Lévy measure ν_j .

- (3) $\sigma : \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N}$ and $F, G : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \rightarrow \mathbb{R}^{3N}$ are measurable functions and there exist two functions $K_1, K_2 : \mathbb{R}^{3N} \times \mathbb{R}^{3N} \rightarrow \mathbb{R}$, such that, for all $y_1, y_2, y \in \mathbb{R}^{3N}$, $K_1(y_1, y_2), K_2(y, y) \leq C_1$, and

$$\|a(y_1, y_1) - 2a(y_1, y_2) + a(y_2, y_2)\| + \int_{|z|<1} |F(y_1, z) - F(y_2, z)|^2 \nu(dz) \leq K_1(y_1, y_2)|y_1 - y_2|, \tag{2.5}$$

$$\|a(y, y)\| + \int_{|z|<1} |F(y, z)|^2 \nu(dz) \leq K_2(y, y)(1 + |y|), \tag{2.6}$$

and

$$\int_{|z|\geq 1} |G(y, z)|^p \nu(dz) \leq K_2(y, y)(1 + |y|^p) \quad \text{for any integer } p \geq 1. \tag{2.7}$$

The mapping $y \rightarrow G(y, z)$ is continuous for all $z \geq 1$.

- (4) $\{X^i(0)\}_{i=1}^{3N}$ are square integrable \mathcal{F}_0 -adapted random variables independent of W and \mathcal{N} .

We need the following lemma to establish the unique solvability of equation (2.1). The proof of this lemma for a bounded domain in two dimensions can be found in (Lemma 1.4 of Kato [10]). For the whole space \mathbb{R}^3 , Rautmann has given a proof in [15] (Proposition 2.1).

Lemma 2.1. *The following two inequalities hold for some constant C_2 and C_3 :*

$$\|u(\cdot, t)\|_\infty = \|K(\cdot) * \omega(t)\|_\infty \leq C_2 \|\omega(\cdot, t)\|_\infty, \tag{2.8}$$

$$\int_{\mathbb{R}^3} |K(x, y) - K(x', y)| dy \leq C_3 \varphi(|x - x'|), \tag{2.9}$$

with

$$\varphi(r) = \begin{cases} r(1 - \ln r) & \text{if } r < 1, \\ 1 & \text{if } r \geq 1. \end{cases} \tag{2.10}$$

It follows from (2.8) that if

$$\omega(\cdot, t) \in L^\infty(\mathbb{R}^3),$$

then $u = K * \omega$ is a vector field uniformly bounded and continuous in x , but may not satisfy the Lipschitz condition. However, from (2.9),

$$|K * \omega(x) - K * \omega(y)| \leq C_4 \|\omega(\cdot, t)\|_\infty \varphi(|x - y|) \tag{2.11}$$

for some constant C_4 . Condition (2.11) is called a quasi-Lipschitz condition, and corrects the usual Lipschitz condition by a logarithmically diverging factor. Although weaker than the Lipschitz condition, it is sufficient to prove the existence and uniqueness for the solutions of the stochastic differential equation (2.1).

The Beale-Kato-Majda criterion [3] for three dimensional Euler flow states that a solution to the Euler equation is classical up to time T_* if

$$\int_0^T \|\omega(\cdot, t)\|_\infty dt < \infty \quad \text{for } T < T_*. \tag{2.12}$$

In the main theorem below we are going to assume that

$$\sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty < \infty,$$

which is a stronger condition than (2.12).

Theorem 2.2. *Assume (1)-(5) and $\sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty < \infty$, then there exists a unique solution $X = (X(t))$ to the SDE (2.1) and (2.2) with the initial condition satisfying $E|X(0)|^2 < \infty$. The process X is adapted and càdlàg.*

Proof. To prove this theorem we first use an iteration method for the equation driven by continuous noise with small jumps only. The Burkholder-Davis-Gundy inequality for stochastic integrals with respect to Wiener and compensated Poisson noises are also useful for the proof (the case of $p = 1$ shown by Davis [4], Theorem 1 is used here). Readers should refer to Applebaum [2] (4.16, page 257) for the quadratic variation of Lévy integrals with respect to the compensated Poisson measure.

$$E \sup_{0 \leq s \leq t} \left| \int_0^s \sigma(x(\tau))dW(\tau) \right| \leq C_3 E \left(\int_0^t \|a(x(s))\| ds \right)^{1/2}$$

and

$$\begin{aligned} E \sup_{0 \leq s \leq t} \left| \int_0^s \int_{|z| < 1} F(x(\tau-), z) \tilde{N}(d\tau, dz) \right| \\ \leq C_3 E \left(\int_0^t \int_{|z| < 1} |F(x(s-), z)|^2 \mathcal{N}(ds, dz) \right)^{1/2}. \end{aligned}$$

First define a stochastic process $(Y(t), t \geq 0)$ which satisfies the following SDE:

$$dY(t) = u(Y(t))dt + \sigma(Y(t))dW(t) + \int_{|z| < 1} F(Y(t-), z) \tilde{N}(dt, dz) \quad (2.13)$$

with initial condition $Y(0) = Y_0$ such that $E|Y_0| < \infty$ and $E|u(Y_0)| < \infty$.

Its Picard iteration can be written as

$$dY_{n+1}(t) = u(Y_n(t))dt + \sigma(Y_n(t))dW(t) + \int_{|z| < 1} F(Y_n(t-), z) \tilde{N}(dt, dz). \quad (2.14)$$

For each $1 \leq i \leq 3N$, $n \in \mathbb{N}$ and $t \geq 0$, we have

$$\begin{aligned} Y_{n+1}^i(t) - Y_n^i(t) &= \int_0^t [u^i(Y_n(s)) - u^i(Y_{n-1}(s))] ds \\ &\quad + \int_0^t [\sigma^i(Y_n(s)) - \sigma^i(Y_{n-1}(s))] dW(s) \\ &\quad + \int_0^t \int_{|z| < 1} [F^i(Y_n(s-), z) - F^i(Y_{n-1}(s-), z)] \tilde{N}(ds, dz). \end{aligned} \quad (2.15)$$

When $n = 0, t \geq 0,$

$$\begin{aligned}
 & |Y_1(t) - Y_0(t)| \\
 &= \sum_{i=1}^{3N} \left| \int_0^t u^i(Y(0)) ds + \int_0^t \sigma^i(Y(0)) dW(s) \right. \\
 &\quad \left. + \int_0^t \int_{|z|<1} F^i(Y(0), z) \tilde{\mathcal{N}}(ds, dz) \right| \\
 &\leq \sum_{i=1}^{3N} \left[\left| \int_0^t u^i(Y(0)) ds \right| + \left| \int_0^t \sigma^i(Y(0)) dW(s) \right| \right. \\
 &\quad \left. + \left| \int_0^t \int_{|z|<1} F^i(Y(0), z) \tilde{\mathcal{N}}(ds, dz) \right| \right]
 \end{aligned} \tag{2.16}$$

Take expectation and apply Burkholder-Davis-Gundy and Cauchy’s inequalities on (2.16), we obtain

$$\begin{aligned}
 E \sup_{0 \leq s \leq t} |Y_1(t) - Y_0(t)| &\leq tE|u(Y(0))| + t^{\frac{1}{2}} \left[E \|a(Y(0), Y(0))\|^2 \right]^{\frac{1}{2}} \\
 &\quad + t^{\frac{1}{2}} \left[E \int_{|z|<1} |F(Y(0), z)|^2 \nu(dz) \right]^{\frac{1}{2}}.
 \end{aligned} \tag{2.17}$$

Applying the growth condition for a and F deduces that

$$E \sup_{0 \leq s \leq t} |Y_1(s) - Y_0(s)| \leq tE|u(Y_0)| + t^{\frac{1}{2}} C_2 \left(1 + E|Y(0)| \right)^{\frac{1}{2}}. \tag{2.18}$$

We now consider the case for general $n \in \mathbb{N}.$ Arguing as above we obtain

$$\begin{aligned}
 & E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\
 &\leq \sum_{i=1}^{3N} \left[E \left(\int_0^t \sup_{0 \leq \tau \leq s} |u^i(Y_n(\tau)) - u^i(Y_{n-1}(\tau))| d\tau \right) \right. \\
 &\quad + E \sup_{0 \leq s \leq t} \left| \int_0^s [\sigma_j^i(Y_n(\tau)) - \sigma_j^i(Y_{n-1}(\tau))] dW_j(\tau) \right| \\
 &\quad \left. + E \sup_{0 \leq s \leq t} \left| \int_0^s \int_{|z|<1} [F^i(Y_n(\tau-), z) - F^i(Y_{n-1}(\tau-), z)] \tilde{\mathcal{N}}(ds, dz) \right| \right].
 \end{aligned} \tag{2.19}$$

The second term on the right hand side is estimated by Burkholder-Davis-Gundy as

$$\begin{aligned}
 & \sum_{i=1}^{3N} E \sup_{0 \leq s \leq t} \left| \int_0^s [\sigma_j^i(Y_n(\tau)) - \sigma_j^i(Y_{n-1}(\tau))] dW_j(\tau) \right| \\
 &\leq C \left[E \left(\int_0^t \|a(Y_n(s), Y_n(s)) - 2a(Y_n(s), Y_{n-1}(s)) \right. \right. \\
 &\quad \left. \left. + a(Y_{n-1}(s), Y_{n-1}(s))\|^2 ds \right) \right]^{\frac{1}{2}}.
 \end{aligned} \tag{2.20}$$

Using Burkholder-Davis-Gundy and Cauchy's inequalities again, the last term of (2.19) becomes

$$\begin{aligned}
& \sum_{i=1}^{3N} E \sup_{0 \leq s \leq t} \left| \int_0^s \int_{|z| < 1} [F^i(Y_n(\tau-), z) - F^i(Y_{n-1}(\tau-), z)] \tilde{\mathcal{N}}(d\tau, dz) \right| \\
& \leq CE \left(\int_0^t \int_{|z| < 1} |F(Y_n(s-), z) - F(Y_{n-1}(s-), z)|^2 \mathcal{N}(ds, dz) \right)^{\frac{1}{2}} \\
& \leq C \left[E \left(\int_0^t \int_{|z| < 1} |F(Y_n(s-), z) - F(Y_{n-1}(s-), z)|^2 \mathcal{N}(ds, dz) \right) \right]^{\frac{1}{2}} \\
& = C \left[E \left(\int_0^t \int_{|z| < 1} |F(Y_n(s-), z) - F(Y_{n-1}(s-), z)|^2 \nu(dz) ds \right) \right]^{\frac{1}{2}}.
\end{aligned} \tag{2.21}$$

Thus by lemma 2.1 and Lipschitz continuity of F and quasi-Lipschitz condition on u , we get

$$\begin{aligned}
& E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\
& \leq C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty \int_0^t E \sup_{0 \leq \tau \leq s} \varphi(|Y_n(\tau) - Y_{n-1}(\tau)|) ds \\
& \quad + 2C \cdot C_1^{\frac{1}{2}} \left(\int_0^t E \sup_{0 \leq \tau \leq s} |Y_n(\tau) - Y_{n-1}(\tau)| ds \right)^{\frac{1}{2}}.
\end{aligned} \tag{2.22}$$

Now using Young's inequality and

$$\varphi(r) \leq -(\ln \epsilon)r + \epsilon, \quad \text{for } 0 < \epsilon < 1, \tag{2.23}$$

(2.22) becomes

$$\begin{aligned}
& E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\
& \leq \left(C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty L_\epsilon + \frac{1}{\delta} \right) \int_0^t E \sup_{0 \leq \tau \leq s} |Y_n(\tau) - Y_{n-1}(\tau)| ds \\
& \quad + C^2 C_1 \delta + C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty T \epsilon,
\end{aligned} \tag{2.24}$$

where $L_\epsilon = -\ln \epsilon$ and $\delta > 0$.

Inequality (2.24) can be iterated to obtain

$$\begin{aligned}
& E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\
& \leq \left(C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty T \epsilon + C^2 C_1 \delta \right) \sum_{k=1}^{n-2} \frac{\left(\frac{1}{\delta} + C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty L_\epsilon \right)^k T^k}{k!} \\
& \quad + \frac{T^{n-1} \left(\frac{1}{\delta} + C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty L_\epsilon \right)^{n-1}}{(n-1)!} E \sup_{0 \leq s \leq t} |Y_1(s) - Y_0(s)|.
\end{aligned} \tag{2.25}$$

Since

$$E \sup_{0 \leq s \leq t} |Y_1(s) - Y_0(s)| \leq C_5 < \infty, \tag{2.26}$$

we have

$$\begin{aligned} & E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\ & \leq \left(C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty T\epsilon + C^2 C_1 \delta \right) \exp\left(\frac{1}{\delta} - C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty \ln \epsilon\right) \\ & \quad + C_5 \frac{T^{n-1} \left(\frac{1}{\delta} - C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty \ln \epsilon\right)^{n-1}}{(n-1)!}. \end{aligned} \tag{2.27}$$

Taking $n \rightarrow \infty$, the last term in (2.27) vanishes because $n!$ grows faster than d^n for any $d > 0$ and thus

$$\begin{aligned} & \lim_{n \rightarrow \infty} E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \\ & \leq \left(C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty T\epsilon + C^2 C_1 \delta \right) \exp\left(\frac{1}{\delta} - C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty \ln \epsilon\right). \end{aligned}$$

Now choose ϵ and δ such that

$$\begin{cases} C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty T\epsilon + C^2 C_1 \delta = \exp(-2N) \\ \frac{1}{\delta} - C_4 \sup_{0 \leq t \leq T} \|\omega(\cdot, t)\|_\infty \ln \epsilon = N, \end{cases}$$

then

$$\lim_{n \rightarrow \infty} E \sup_{0 \leq s \leq t} |Y_{n+1}(s) - Y_n(s)| \leq \exp(-N) \rightarrow 0 \quad \text{as } N \rightarrow \infty. \tag{2.28}$$

Therefore, $(Y_n(t), n \in \mathbb{N})$ is a almost surely uniformly convergent Cauchy sequence and hence is a.s. uniformly convergent to some $Y(t), t \geq 0$, which implies $Y_n(t)$ converges to $Y(t)$ in $L^2(\Omega)$. Following (2.11), $\mathbf{u}^i(Y_n(t))$ is also a.s. uniformly convergent, and thus $L^2(\Omega)$ convergent to $\mathbf{u}^i(Y(t))$. L^2 convergence of $\sigma_j^i(Y_n(t))$ and $\int_{|z|<1} F^i(Y_n(t), z) \tilde{\mathcal{N}}(t, dz)$ follows from Itô isometry and (2.5). In fact, for the Lévy integral,

$$\begin{aligned} & E \left(\left| \int_0^t \int_{|z|<1} (F^i(Y_n(s), z) - F^i(Y(s), z)) \tilde{\mathcal{N}}(ds, dz) \right|^2 \right) \\ & = E \left(\int_0^t \int_{|z|<1} |F^i(Y_n(s), z) - F^i(Y(s), z)|^2 \nu(dz) ds \right) \\ & \leq C_1 E \left(\int_0^t |Y_n(s) - Y(s)| ds \right) \rightarrow 0. \end{aligned}$$

Therefore, X satisfies (2.1) and is adapted and Cádłág. Similarly, we are able to show that there exists a unique solution to equation (2.13).

Now we will recover equation (2.1) by adding large jumps. Using interlacing techniques from Applebaum ([2] Theorem 6.2.9), we show that (2.1) has a unique solution that is cádłág. Let $\{t_n, n \in \mathbb{N}\}$ be the arrival times for the jumps of the

compound Poisson process $\{P(t), t \geq 0\}$, where each $P(t) = \int_{|z| \geq 1} z\mathcal{N}(t, dz)$. We then construct a solution to (2.1) as follows:

$$\begin{aligned} X(t) &= Y(t) && \text{for } 0 \leq t < t_1, \\ X(t_1) &= Y(t_1-) + G(Y(t_1-), \Delta P(t_1)) && \text{for } t = t_1, \\ X(t) &= X(t_1) + Y(t) - Y(t_1) && \text{for } t_1 < t < t_2, \\ X(t_2) &= X(t_2-) + G(X(t_2-), \Delta P(t_2)) && \text{for } t = t_2, \end{aligned} \tag{2.29}$$

and so on, recursively. Here Y is the unique solution to (2.13) with initial condition $Y(0) = X(t_1)$. X is clearly adapted, càdlàg and solves (2.1). Uniqueness follows from the proof above and the interlacing structure. \square

3. A moment estimate

In this section we prove an a-priori estimate for the moments of signal process $X(t)$ given by equation (2.1).

Let us first recall the multidimensional Itô’s formula (Theorem 1.16 in [13]).

Theorem 3.1. *let $X(t) \in \mathbb{R}^n$ be an Itô-Lévy process of the form*

$$\begin{aligned} dX(t) &= \alpha(t, X(t))dt + \sigma(t, X(t))dW_t + \int_{|z| < 1} \gamma(t-, z, X(t-))\tilde{\mathcal{N}}(dt, dz) \\ &+ \int_{|z| \geq 1} \xi(t-, z, X(t-))\mathcal{N}(dt, dz), \end{aligned} \tag{3.1}$$

where $\alpha : [0, T] \times \Omega \rightarrow \mathbb{R}^n, \sigma : [0, T] \times \Omega \rightarrow \mathbb{R}^{n \times m}, \gamma : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^{n \times l}$ and $\xi : [0, T] \times \mathbb{R}^n \times \Omega \rightarrow \mathbb{R}^{n \times r}$ are adapted processes such that the integrals exist. Here W_t is a m -dimensional Brownian motion and

$$\begin{aligned} \tilde{\mathcal{N}}(dt, dz)^T &= (\tilde{\mathcal{N}}_1(dt, dz_1), \dots, \tilde{\mathcal{N}}_l(dt, dz_l)) \\ &= (\mathcal{N}_1(dt, dz_1) - \nu_1(dz_1)dt, \dots, \mathcal{N}_l(dt, dz_l) - \nu_l(dz_l)dt), \end{aligned} \tag{3.2}$$

where $\{\mathcal{N}_j\}$ are independent Poisson random measures with Lévy measures $\nu_j, j = 1, \dots, l$.

When written out in detail component number $i = 1, \dots, n$ of $X(t)$ in (3.1), $X_i(t)$ gets the form

$$\begin{aligned} dX_i(t) &= \alpha_i(t, X_i(t))dt + \sum_{j=1}^m \sigma_{ij}(t, X_i(t))dW_j(t) \\ &+ \sum_{j=1}^l \int_{|z| < 1} \gamma_{ij}(t-, z_j, X_i(t-))\tilde{\mathcal{N}}_j(dt, dz_j) \\ &+ \sum_{j=1}^r \int_{|z| \geq 1} \xi_{ij}(t-, z_j, X_i(t-))\mathcal{N}_j(dt, dz_j). \end{aligned} \tag{3.3}$$

Let $f \in C^{1,2}([0, T] \times \mathbb{R}^n; \mathbb{R})$ and set $Y(t) = f(t, X(t))$. Then

$$\begin{aligned}
 dY(t) &= \frac{\partial f}{\partial t} dt + \sum_{i=1}^n \frac{\partial f}{\partial x_i} (\alpha_i dt + \sigma_i dW(t)) + \frac{1}{2} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} dt \\
 &\quad + \sum_{k=1}^l \int_{|z|<1} \{f(t, X(t-) + \gamma^{(k)}(t, z_k)) - f(t, X(t-)) \\
 &\quad - \sum_{i=1}^n \gamma_i^{(k)}(t, z_k) \frac{\partial f}{\partial x_i}(X(t-))\} \nu_k(dz_k) dt \\
 &\quad + \sum_{k=1}^l \int_{|z|<1} \{f(t, X(t-) + \gamma^{(k)}(t, z_k)) - f(t, X(t-))\} \tilde{\mathcal{N}}_k(dt, dz_k) \\
 &\quad + \sum_{k=1}^r \int_{|z|\geq 1} \{f(t, X(t-) + \xi^{(k)}(t, z_k)) - f(t, X(t-))\} \mathcal{N}_k(dt, dz_k),
 \end{aligned} \tag{3.4}$$

where $\gamma^{(k)}, \xi^{(k)} \in \mathbb{R}^n$ are column number k of the matrices $\gamma = [\gamma_{ik}]$ and $\xi = [\xi_{ik}]$, $\gamma_i^{(k)} = \gamma_{ik}$ and $\xi_i^{(k)} = \xi_{ik}$ are i -th coordinates number i of $\gamma^{(k)}$ and $\xi^{(k)}$, respectively.

Lemma 3.2. Assume $X(t)$ satisfies (2.1) with $E|X(0)|^{2m} \leq C_0$, where C_0 is a positive constant independent of m . Then for any $0 < T < \infty$

$$E|X(t)|^{2m} < \infty, \quad \text{for } m = \frac{1}{2}, 1, \frac{3}{2}, 2, \dots, \quad \text{and } t \in [0, T]. \tag{3.5}$$

Proof. We first prove the case for $m = 1$ (In this proof we use c as a general constant). Rewriting (2.1) in its integral form:

$$\begin{aligned}
 X(t) &= \int_0^t u(X(s), s) ds + \int_0^t \sigma(X(s)) dW(s) + \int_0^t \int_{|z|<1} F(X(s-), z) \tilde{\mathcal{N}}(ds, dz) \\
 &\quad + \int_0^t \int_{|z|\geq 1} G(X(s-), z) \mathcal{N}(ds, dz),
 \end{aligned} \tag{3.6}$$

we have

$$\begin{aligned}
 |X(t)|^2 &\leq \left| \int_0^t u(X(s), s) ds \right|^2 + \left| \int_0^t \sigma(X(s)) dW(s) \right|^2 \\
 &\quad + \left| \int_0^t \int_{|z|<1} F(X(s-), z) \tilde{\mathcal{N}}(ds, dz) \right|^2 + \left| \int_0^t \int_{|z|\geq 1} G(X(s-), z) \mathcal{N}(ds, dz) \right|^2.
 \end{aligned} \tag{3.7}$$

Let $\{T_N\}$ be a sequence of stopping times reducing above stochastic integrals as uniformly integrable martingales. Taking expectation of the above inequality with stopping time T_N , then (2.6)-(2.8), the Burkholder-Davis-Gundy inequality and Cauchy's inequality yield

$$E|X(T_N)|^2 \leq c + c \int_0^{t \wedge T_N} E(1 + |X(s)|^2) ds. \tag{3.8}$$

Applying the Gronwall’s inequality and we find

$$E|X(T_N)|^2 \leq c \exp(cT). \tag{3.9}$$

It is obvious from the Cauchy’s inequality that

$$E|X(T_N)| \leq \left(E|X(T_N)|^2 \right)^{\frac{1}{2}} < c. \tag{3.10}$$

Fatou’s lemma for conditional expectations deduces that, for $m = \frac{1}{2}$ and 1:

$$\begin{aligned} E|X(t)|^{2m} &= E\left[|X(t)|^{2m} | \mathcal{F}_0\right] \leq \lim_{N \rightarrow \infty} E\left[|X(T_N)|^{2m} | \mathcal{F}_0\right] \\ &= \lim_{N \rightarrow \infty} E|X(T_N)|^{2m} < \infty. \end{aligned} \tag{3.11}$$

Now from the multidimensional Itô’s formula to function $f = |x|^{2m} = (x_1^2 + \dots + x_n^2)^m$ for $m = \frac{3}{2}, 2, \dots$,

$$\frac{\partial f}{\partial x_i} = m(x_1^2 + \dots + x_n^2)^{m-1} 2x_i, \tag{3.12}$$

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = 2m(x_1^2 + \dots + x_n^2)^{m-1} \delta_{ij} + m(m-1)(x_1^2 + \dots + x_n^2)^{m-2} 2x_i 2x_j, \tag{3.13}$$

and

$$\begin{aligned} |X(t)|^{2m} &= |X(0)|^{2m} + \int_0^t 2m|X(s)|^{2m-2} \sum_{i=1}^n X_i(s) u^i(X(s)) ds \\ &\quad + \int_0^t m|X(s)|^{2m-2} \sum_{i=1}^n (\sigma \sigma^T)_{ii} ds \\ &\quad + \int_0^t 2m(m-1)|X(s)|^{2m-4} \sum_{i,j=1}^n (\sigma \sigma^T)_{ij} X_i(s) X_j(s) ds \\ &\quad + \int_0^t \sum_{k=1}^l \int_{|z|<1} \left\{ |X(s-) + F^k(s, z_k)|^{2m} - |X(s-)|^{2m} \right. \\ &\quad \left. - 2m|X(s-)|^{2m-2} \sum_{i=1}^n F_i^k(s, z_k) X_i(s-) \right\} \nu_k(dz_k) ds + M_t \\ &\quad + \int_0^t \sum_{k=1}^r \int_{|z|\geq 1} \left\{ |X(s-) + G^{(k)}(s, z_k)|^{2m} - |X(s-)|^{2m} \right\} \mathcal{N}_k(ds, dz_k), \end{aligned} \tag{3.14}$$

where

$$\begin{aligned} M_t &= \int_0^s 2m|X(s)|^{2m-2} \sum_{i=1}^n X_i(s) \sum_{j=1}^n \sigma_{ij} dW_j \\ &\quad + \int_0^t \sum_{k=1}^l \int_{|z|<1} \left\{ |X(s-) + F^k(s, z_k)|^{2m} - |X(s-)|^{2m} \right\} \tilde{\mathcal{N}}_k(ds, dz_k). \end{aligned} \tag{3.15}$$

Taking expectation of the above integration with stopping time T_N and using the growth assumptions from Section 2, we get

$$E|X(T_N)|^{2m} \leq c + c \int_0^{t \wedge T_N} E|X(s)|^{2m-1} ds + c \int_0^{t \wedge T_N} E|X(s)|^{2m} ds. \tag{3.16}$$

Keep increasing m by $\frac{1}{2}$ and using estimates (3.9) (3.10), we finally get

$$E|X(T_N)|^{2m} \leq [c + cT \exp(cT) + \dots + cT^{2m-1} \exp(cT)] \exp(cT). \tag{3.17}$$

Using (3.17) and Fatou’s lemma for conditional expectation ($m = \frac{3}{2}, 1, \dots$),

$$\begin{aligned} E|X(t)|^{2m} &= E[|X(t)|^{2m} | \mathcal{F}_0] \leq \lim_{N \rightarrow \infty} E[|X(T_N)|^{2m} | \mathcal{F}_0] \\ &= \lim_{N \rightarrow \infty} E|X(T_N)|^{2m} < \infty. \end{aligned} \tag{3.18}$$

□

4. Nonlinear filtering problem

We are interested in the estimation of the \mathbb{R}^{3N} -valued stochastic process $X(t) = (X_1(t), \dots, X_N(t))$ whose dynamics is described by equation (2.1) which can not be observed directly. The observation process $Y(t), t \geq 0$ which is related to $X(t)$ as follows:

$$Y(t) = Y(0) + B(t) + \int_0^t h(X(s)) ds. \tag{4.1}$$

Here

- The observation process $Y(t)$ is assumed to be a \mathbb{R}^m -valued stochastic process ($m < 3N$).
- $B(t)$ is a m -dimensional \mathcal{F}_t -adapted standard Brownian motion.
- $h(\cdot)$ is continuous with quadratic growth $|h(x)| \leq C(1 + |x|^2)$, for all $t \in [0, T]$.
- For each $s \in [0, T]$, the σ -fields $\sigma\{h(X(u)), B(u); 0 < u < s\}$ and $\sigma\{B(v) - B(u); s < u < v < T\}$ are independent.

The nonlinear filtering problem is to calculate the following conditional expectation

$$\pi_t(\varphi) = E[\varphi(X_t) | \mathcal{Y}_t], \tag{4.2}$$

where \mathcal{Y}_t is the σ -algebra generated by the back measurements $Y(s), 0 \leq s \leq t$. In fact one can prove that (4.2) is the least square estimate for $\varphi(X_t)$ given \mathcal{Y}_t . $\pi_t(\varphi)$ satisfies a nonlinear stochastic differential equation, called the Fujisaki-Kallianpur-Kunita(FKK) equation [6]. The idea then is to use Girsanov theorem to analyze $\rho_t(\varphi)$, the unnormalized conditional density, which is related to $\pi_t(\varphi)$ by Kallianpur-Striebel formula and satisfies a linear stochastic differential equation.

Theorem 4.1 (Girsanov). *Assume that $\psi(\cdot)$ is a \mathbb{R}^m -valued \mathcal{F}_t -predictable process such that*

$$E\left(\int_0^T |\psi(s)|^2 ds\right) < \infty, \tag{4.3}$$

and

$$E\left(\exp\left(-\int_0^T \psi(s)^T dW(s) - \frac{1}{2} \int_0^T |\psi(s)|^2 ds\right)\right) = 1. \tag{4.4}$$

Then the process

$$\tilde{W}(t) = W(t) - \int_0^t \psi(s) ds, \quad t \in [0, T] \tag{4.5}$$

is a m -dimensional Brownian motion with respect to $\{\mathcal{F}_t\}_{t \geq 0}$ on the probability space $(\Omega, \mathcal{F}, \tilde{P})$ where

$$d\tilde{P}(\omega) = \exp\left(-\int_0^T \psi(s)^T dW(s) - \frac{1}{2} \int_0^T |\psi(s)|^2 ds\right) dP(\omega). \tag{4.6}$$

Here $|\cdot|$ denotes the standard Euclidean norm.

Following Girsanov’s theorem, we define

$$\Lambda_t = \tilde{E}\left[\frac{dP}{d\tilde{P}}|\mathcal{Y}_t\right], \tag{4.7}$$

where \tilde{E} denotes expectation under the reference measure \tilde{P} .

We have the representation:

$$E[f(X(t))|\mathcal{Y}_t] = \frac{\tilde{E}[f(X(t))\Lambda_t|\mathcal{Y}_t]}{\tilde{E}[\Lambda_t|\mathcal{Y}_t]}, \tag{4.8}$$

which is the Kallianpur-Striebel’s formula.

Lemma 4.2. *With the equivalent measure $P \sim \tilde{P}$ defined in the Girsanov’s theorem, we have the following:*

- (1) Under \tilde{P} , $Y(t)$ is a standard Brownian motion.
- (2)

$$\Lambda_t = 1 + \sum_{k=1}^m \int_0^t \Lambda_{s-} h_k(X(s-)) dY_k(s). \tag{4.9}$$

- (3) Under \tilde{P} , the process $X(t)$ and $Y(t)$ are independent.
- (4) $E^0[\Lambda_t|\mathcal{F}_t] = 1$.
- (5) The restrictions of \tilde{P} and P to the \mathcal{F}_t are the same.

Proof. See Wong and Hajek [21], page 231-236. □

Definition 4.3. Let $X(t) \in \mathbb{R}^n$ be a jump diffusion. Then the generator of X is defined on the function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$Af(x) = \lim_{t \rightarrow 0+} \frac{1}{t} \left\{ E^x[f(X(t))] - f(x) \right\}, \tag{4.10}$$

where $E^x[f(X(t))] = E[f(X(t))|X(0) = x]$. The set of all functions f for which limit (4.10) exists is denoted $D(A)$.

Theorem 4.4. *Suppose $f \in C_0^2(\mathbb{R}^n)$ and $X(t)$ is given by (3.1). Then $Af(x)$ exists and is given by*

$$\begin{aligned}
 Af(x) &= \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x)u(x) + \frac{1}{2} \sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x)\sigma^i(x)\sigma^j(x) \\
 &+ \int_{\mathbb{R}^n} \sum_{i=1}^n \left\{ f(x + \xi^i(x, z)) - f(x) - \frac{\partial f}{\partial x_i} \xi^i(x, z) 1_{\{|z| < 1\}}(z) \right\} \nu_i(dz_i).
 \end{aligned}
 \tag{4.11}$$

Proof. See Skorohod [17], p.170. □

The following two propositions give the equations for which we will prove the uniqueness results in the section 6. Their proofs can be found as Theorem 4.5 and Corollary 4.7 in Popa and Sritharan [14].

Proposition 4.5. *The conditional measure P_t is defined by $P_t f = E[f(X(t))|\mathcal{Y}_t]$, which satisfies the equation*

$$P_t f = P_0 f + \int_0^t P_s A f ds + \sum_{k=1}^n \int_0^t [P_s(f h_k) - P_s(f)P_s(h_k)] d\nu_s^k \tag{4.12}$$

with $\nu_t^k = Y_k(t) - \int_0^t P_s(h_k) ds$ as the innovation process and $f \in D(A)$. Equation (4.12) is called the Fujisaki-Kallianpur-Kunita(FKK) equation.

Proposition 4.6. *The unnormalized conditional distribution is defined by $p_t(f) = \tilde{E}[f(X(t))\Lambda_t|\mathcal{Y}_t]$, which satisfies*

$$p_t(f) = p_0(f) + \int_0^t p_s(Af) ds + \sum_{k=1}^m \int_0^t p_s(f h_k) dY_k(s). \tag{4.13}$$

Equation (4.13) is called the measure-valued Zakai equation and again $f \in D(A)$.

5. Integro-differential operator and backward parabolic equation

In this section we will consider a backward parabolic Cauchy problem whose solvability is crucial in proving the uniqueness of solution to the Zakai equation in section 6.

We denote by $C_b([0, T], \mathbb{R}^{3N})$ the space of bounded continuous functions and let $W_p^{2,1}$ be the class of functions with L^p -bounded spatial derivatives up to second order and first order time derivative. H^s is a Hilbert space equipped with inner product of derivatives up to order s . Denote \mathcal{S} as the space of functions with quadratic decay:

$$\mathcal{S} = \left\{ f \in C_b([0, T]; \mathbb{R}^{3N}) \mid f(t) \cdot (1 + |x|^2) < \infty \right\}. \tag{5.1}$$

Consider

$$-\frac{\partial \varphi(x, t)}{\partial t} = A_b \varphi(x, t), \quad t < T \quad x \in \mathbb{R}^m, \tag{5.2}$$

$$\varphi(x, T) = \beta(x). \tag{5.3}$$

Let $b \in \mathcal{S}$ and be non-negative and set

$$A_b \psi(x, t) := A \psi(x, t) + b(t)h(x)\psi(x, t). \tag{5.4}$$

Here A is the infinitesimal generator of signal process X_t satisfying (2.1).

Before we study the solvability problem of (5.2), let us introduce the integro-differential operator in the whole space \mathbb{R}^{3N} as follows:

$$I\phi(x, t) = \int_F \{ \phi(x + \xi(x, t, z), t) - \phi(x, t) - \xi(x, t, z) \cdot \nabla \phi(x, t) 1_{\{|z| < 1\}} \} \nu(dz), \quad (5.5)$$

where $\nu(\cdot)$ is a σ -finite measure on the measurable space (F, \mathcal{F}) .

Definition 5.1. We say that $I\phi(\cdot, t)$ in (5.5) is *at most of order γ* , $0 < \gamma \leq 1$, if there exists a \mathcal{F} -measurable and positive function $\xi_0(x, t, z)$ and a constant C_γ such that for every x, t , and z

$$\int_F (\xi_0(x, t, z))^q \nu(dz) \leq C_\gamma, \quad \forall q \in [\gamma, 1]. \quad (5.6)$$

We assume the jump coefficients $\xi(x, t, z)$ is measurable for (x, t, z) in $\mathbb{R}^{3N} \times [0, T] \times F$ with order γ , $0 < \gamma \leq 1$. Garroni and Menaldi (Lemma 2.1.14 in [7]) proved the following property of (5.5):

Lemma 5.2. *Assume the integro-differential operator (5.5) has at most order γ , $0 < \gamma \leq 1$. Then for $0 < \gamma \leq 1$ we have*

$$\|I\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} \leq \epsilon \|\nabla \phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} + C(\epsilon) \|\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})}, \quad (5.7)$$

for every smooth function ϕ and any t in $[0, 1]$, $1 < p < \infty$. ϵ here is any positive number.

We also refer to Garroni and Menaldi (Proposition 1.1.8 in [7]) for the following interpolation inequality.

Proposition 5.3. *Let $\phi \in W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$, $1 < p < \infty$. Then the following inequality holds*

$$\|\partial^l \phi\|_{L^p([0, T] \times \mathbb{R}^{3N})} \leq \eta \|\phi\|_{W_p^{2,1}([0, T] \times \mathbb{R}^{3N})} + c\eta^{-1} \|\phi\|_{L^p([0, T] \times \mathbb{R}^{3N})}, \quad l \leq 2. \quad (5.8)$$

Here η is any positive number, $\partial^l = \partial_{t_s}^l = \partial_t^r \partial_x^s$ with $l = 2r + |s|$ and $|s| = s_1 + s_2 + \dots + s_{3N}$.

Denote by \mathcal{L} the linear parabolic differential operator

$$\mathcal{L}f = \frac{\partial f}{\partial t} + \frac{1}{2} a_{ij}(x, t) \frac{\partial^2 f}{\partial x_i \partial x_j} + v_i(x, t) \frac{\partial f}{\partial x_i} + b(t)h(x)f. \quad (5.9)$$

We assume that this operator is uniformly elliptic, namely

$$a_{ij}(x, t) \kappa_i \kappa_j \geq c|\kappa|^2, \quad \forall \kappa \in \mathbb{R}^{3N}, \quad c > 0, \quad (5.10)$$

and consider in the whole space the backward Cauchy problem:

$$\mathcal{L}\varphi(x, t) = 0 \text{ in } \mathbb{R}^{3N} \times (0, T), \quad \varphi(x, T) = \beta(x) \text{ in } \mathbb{R}^{3N}. \quad (5.11)$$

The following theorem is a special case of Theorem 5 in Solonnikov [18], it gives the existence and uniqueness results for differential version of (5.2) in the Sobolev spaces $W_p^{2m,m}([0, T] \times \mathbb{R}^{3N})$, m a positive integer.

Theorem 5.4. For $u(x, 0) \in H^s$, $s > 5/2$ and $\beta \in W_p^{2-\frac{2}{p}}(\mathbb{R}^{3N})$ with $1 < p < \infty$, $p \neq 3/2$, then problem (5.11) has a unique solution $\varphi \in W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$ and

$$\|\varphi\|_{W_p^{2,1}([0,T] \times \mathbb{R}^{3N})} \leq C\|\beta\|_{W_p^{2-\frac{2}{p}}(\mathbb{R}^{3N})}. \tag{5.12}$$

We can rewrite operator A_b in terms of \mathcal{L} and I as:

$$A_b\psi(x, t) := A\psi(x, t) + b(t)h(x)\psi(x, t) = \mathcal{L}\psi(x, t) + I\psi(x, t). \tag{5.13}$$

The existence and uniqueness to the solution of original problem (5.2) can be shown based on the theorem on problem (5.11) using fixed point argument. Special care needs to be taken for the integro-differential operator (5.5). The following result is Theorem 3.6 in Garroni and Menaldi [7].

Theorem 5.5. For $u(x, 0) \in H^s$, $s > 5/2$, let I be with order γ , $0 < \gamma \leq 1$. $\beta \in W_p^{2-\frac{2}{p}}(\mathbb{R}^{3N})$, $1 < p < \infty$, $p \neq 3/2$, then problem (5.2) has a unique solution $\varphi \in W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$ and

$$\|\varphi\|_{W_p^{2,1}([0,T] \times \mathbb{R}^{3N})} \leq C\|\beta\|_{W_p^{2-\frac{2}{p}}(\mathbb{R}^{3N})}. \tag{5.14}$$

Proof. The proof of this theorem is due to Garroni and Menaldi [7] and we give it here for completeness.

By virtue of Lemma 5.2, $I\phi \in L^p([0, T] \times \mathbb{R}^{3N})$ for $\phi \in W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$. So, for any fixed ϕ in $W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$, we can solve the following differential equation problem

$$\begin{cases} \varphi \in W_p^{2,1}([0, T] \times \mathbb{R}^{3N}), \\ \mathcal{L}\varphi = -I\phi \text{ in } [0, T] \times \mathbb{R}^{3N}, \\ \varphi(\cdot, T) = \beta \text{ in } \mathbb{R}^{3N}. \end{cases} \tag{5.15}$$

From Lemma 5.2 we deduce

$$\|I\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} \leq \epsilon\|\nabla\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} + c(\epsilon)\|\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})}. \tag{5.16}$$

Since

$$\|\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} \leq \|\phi(\cdot, 0)\|_{L^p(\mathbb{R}^{3N})} + \int_0^t \|\partial_r\phi(\cdot, \tau)\|_{L^p(\mathbb{R}^{3N})} d\tau, \tag{5.17}$$

we have

$$\|\phi\|_{L^p([0,T] \times \mathbb{R}^{3N})} \leq 2T^{\frac{1}{p}}\|\phi(\cdot, 0)\|_{L^p(\mathbb{R}^{3N})} + 2T\|\partial_t\phi\|_{L^p([0,T] \times \mathbb{R}^{3N})}, \tag{5.18}$$

and by Proposition 5.3 we deduce

$$\begin{aligned} \|\nabla\phi\|_{L^p([0,T] \times \mathbb{R}^{3N})} &\leq \eta\|\phi\|_{W_p^{2,1}([0,T] \times \mathbb{R}^{3N})} + c\eta^{-1}\left[T^{\frac{1}{p}}\|\phi(\cdot, 0)\|_{L^p(\mathbb{R}^{3N})} \right. \\ &\quad \left. + T\|\partial_t\phi\|_{L^p([0,T] \times \mathbb{R}^{3N})}\right]. \end{aligned} \tag{5.19}$$

Consider now function ϕ such that $\phi(\cdot, 0) = 0$ a.e. in \mathbb{R}^{3N} . Taking into account above inequalities we obtain

$$\|I\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} \leq [\epsilon\eta + c(\epsilon)]\|\phi\|_{W_p^{2,1}([0,T] \times \mathbb{R}^{3N})} + \epsilon c\eta^{-1}T\|\partial_t\phi\|_{L^p([0,T] \times \mathbb{R}^{3N})}. \tag{5.20}$$

We can choose ϵ, η and $T_0 \leq T$ such that

$$\|I\phi(\cdot, t)\|_{L^p(\mathbb{R}^{3N})} \leq \frac{1}{2} \|\phi\|_{W_p^{2,1}([0, T_0] \times \mathbb{R}^{3N})}. \tag{5.21}$$

Consider now two functions $\phi_i \in W_p^{2,1}([0, T_0] \times \mathbb{R}^{3N})$, $i = 1, 2$, and denote by $\Lambda\phi_i$ the corresponding solutions of problems (5.15) with the right side $I\phi_i$. From Theorem 5.4 and the above inequality we derive

$$\|\Lambda\phi_1 - \Lambda\phi_2\|_{W_p^{2,1}([0, T_0] \times \mathbb{R}^{3N})} \leq \frac{1}{2} \|\phi_1 - \phi_2\|_{W_p^{2,1}([0, T_0] \times \mathbb{R}^{3N})}. \tag{5.22}$$

Hence there exists a positive time T_0 such that Λ is a contraction in the complete metric space $\{\nu \in W_p^{2,1}([0, T_0] \times \mathbb{R}^{3N}), \nu(\cdot, 0) = \beta \in \mathbb{R}^{3N}\}$ and so admits one and only one fixed point. Thus problem (5.2)-(5.3) has one and only one solution in $W_p^{2,1}([0, T] \times \mathbb{R}^{3N})$, for any fixed T . \square

Remark 5.6. In both of the above theorems, we assume that the initial velocity $u(x, 0) \in H^s$ with $s > 5/2$. This is exactly the condition for local time existence of classical solutions to three dimensional Euler equation. Here classical solution means that $u(x, t) \in H^s$ for any $0 \leq t \leq T$.

6. Uniqueness of solution to Zakai equation

In this section we prove the uniqueness of measure-valued solutions for the FKK and Zakai equations adapting an idea due to Rozovskii [16]. This method of proving existence and uniqueness is also applied in infinite dimensional context [19], [9].

Proposition 6.1. *Both FKK and Zakai equations have measure-valued solutions P_t and p_t respectively. These solutions are connected by the relations*

$$P_t = p_t / \langle p_t, 1 \rangle, \tag{6.1}$$

and

$$p_t = P_t \exp \left\{ \int_0^t \langle P_s, h(s) \rangle dY(s) - \frac{1}{2} \int_0^t |\langle P_s, h(s) \rangle|^2 ds \right\}. \tag{6.2}$$

Equation (6.2) is the Kallianpur-Striebel formula.

Proof. See Proposition 3.1 in Rozovskii [16]. \square

The proof of uniqueness of the solution to Zakai equation below follows the idea from Theorem 3.2 in Sritharan and Xu [20]. In our case it relies on the solvability of the backward parabolic Cauchy problem presented in section 5.

Theorem 6.2. *Assume $a(x), f(x)$ and $h(x)$ are locally Lipschitz in x , h satisfies*

$$E \left(\int_0^t |h(X(s))|^2 ds \right) < \infty, \quad t \in [0, T], \tag{6.3}$$

where X is solution to the signal process (2.1), then the measure valued solution to the Zakai equation (4.13) is unique.

Proof. First let us note that the growth condition on $h(\cdot)$ and the moment estimate on $X(t)$ will imply (6.3). Now assume μ_t is a measure valued solution to the Zakai equation, choose $\eta(x)$ as the unique solution to the backward Kolmogorov equation (5.2), define $\eta_n = \alpha_n * \eta$, where $\alpha_n(t, x)$ is a mollifier in x and t such that $\eta_n \rightarrow \eta$ uniformly and $\eta_n \in C^{1,2}([0, T] \times \mathbb{R}^{3N})$, then

$$\begin{aligned} \langle \mu_t, \eta_n(t) \rangle &= \langle \mu_0, \eta_n(0) \rangle + \int_0^t \left\langle \mu_s, \frac{\partial}{\partial s} \eta_n(s) \right. \\ &\quad \left. + L\eta_n(s) \right\rangle ds + \int_0^t \langle \mu_s, h\eta_n(s) \rangle dY_s. \end{aligned} \tag{6.4}$$

Define

$$q_t := \exp \left(\int_0^t b(s) dY_s - \frac{1}{2} \int_0^t |b(s)|^2 ds \right), \tag{6.5}$$

$$p_t^{-1} := \exp \left(- \int_0^t h(X(s))^T dY_s + \frac{1}{2} \int_0^t |h(X(s))|^2 ds \right), \tag{6.6}$$

$$\gamma_t := q_t p_t^{-1} = \exp \left(\int_0^t [b(s) - h(X(s))] dY_s - \frac{1}{2} \int_0^t |h(X(s))|^2 ds \right) \tag{6.7}$$

It can be proved using truncation function technique (see Sritharan and Xu [20] for details) that q_t, p_t^{-1} and γ_t are exponential martingales[5], now apply Itô's formula to q_t, p_t^{-1} and γ_t respectively, we have

$$dq_t = q_t b(t) h(X(t)) dt + q_t b(t) dB_t, \tag{6.8}$$

$$dp_t^{-1} = -|h(X(t))|^2 p_t^{-1} dt - h(X(t)) p_t^{-1} dB_t, \tag{6.9}$$

$$d\gamma_t = -\gamma_t |h(X(t))|^2 dt + \gamma_t (b(t) - h(X(t))) dB_t. \tag{6.10}$$

Thus by Itô's product rule,

$$\begin{aligned} \langle \mu_t, \eta_n(t) \rangle \gamma_t &= \langle \mu_0, \eta_n(0) \rangle + \int_0^t \left\langle \mu_s, \frac{\partial}{\partial s} \eta_n(s) + L_b \eta_n(s) \right\rangle \gamma_s ds \\ &\quad + \int_0^t \langle \mu_s, \eta_n(s) \rangle b(s) \gamma_s dB_s. \end{aligned} \tag{6.11}$$

Now take $n \rightarrow \infty$ and the first integral on the right side vanishes because of strong convergence and solvability result in Theorem 5.5. Hence we have the following

$$\langle \mu_t, \eta(t) \rangle \gamma_t = \langle \mu_0, \eta(0) \rangle + \int_0^t \langle \mu_s, \eta(s) \rangle b(s) \gamma_s dB_s. \tag{6.12}$$

Since μ_t is the solution of the Zakai equation, by the exponential martingale property of γ_t , the last integral is a martingale. Now take expectation on both sides, we get

$$E(\langle \mu_{T_0}, \beta \rangle \gamma_{T_0}) = E\eta(0, X_0). \tag{6.13}$$

By Feynman-Kac formula,

$$E\eta(0, X) = E \left[\beta(X(T_0)) \exp \int_0^{T_0} h(X(s)) b(s) ds \right], \tag{6.14}$$

where X is the signal process from (2.1).

By Girsanov’s theorem

$$\begin{aligned} E\left[\beta(X(T_0)) \exp\left(\int_0^{T_0} h(X(s))b(s)ds\right)\right] &= E[\beta(X(T_0))q_{T_0}] \\ &= \tilde{E}[\beta(X(T_0))p_{T_0}q_{T_0}] \\ &= \tilde{E}\left[\tilde{E}[\beta(X(T_0))p_{T_0}|\mathcal{Y}_{T_0}]q_{T_0}\right]. \end{aligned} \tag{6.15}$$

The first equality holds because of the independence of X_t and B_t . In fact, by Itô’s formula,

$$\begin{aligned} E\left[\beta(X(T_0)) \exp\left(\int_0^{T_0} h(X(s))b(s)ds\right)\right] &= E\left[\beta(X(T_0)) \exp\left(\int_0^{T_0} h(X(s))b(s)ds\right)\tilde{q}_t\right] \\ &= E\left[\beta(X(T_0)) \exp\left(\int_0^{T_0} b(s)dY_s - \frac{1}{2}\int_0^{T_0} |b(s)|^2 ds\right)\right] \\ &= E[\beta(X(T_0))q_{T_0}]. \end{aligned}$$

Here

$$\tilde{q}_t = \exp\left(\int_0^t b(s)dB_s - \frac{1}{2}\int_0^t |b(s)|^2 ds\right), \tag{6.16}$$

and we used the martingale property of \tilde{q}_t .

On the other hand,

$$E(\langle \mu_{T_0}, \beta \rangle \gamma_{T_0}) = \tilde{E}(\langle \mu_T, \beta \rangle q_{T_0}), \tag{6.17}$$

hence

$$\tilde{E}\left[\tilde{E}[\beta(x(T_0))p_{T_0}|\mathcal{Y}_{T_0}]q_{T_0}\right] = \tilde{E}(\langle \mu_{T_0}, \beta \rangle q_{T_0}). \tag{6.18}$$

Note that $Y(t)$ is a Wiener martingale on $(\Omega, \mathcal{F}, \tilde{P})$. It is a fundamental theorem from the Malliavin calculus showing that $\{q_{T_0} := q_{T_0}(b), b \in C([0, T], \mathbb{R}^m)\}$ is total in $L_2(\Omega, Y_{T_0}, \tilde{P})$ (Lemma 1.1.2 in Nualart [12]), which means that if $\beta \in L_2(\Omega, Y_{T_0}, P)$ and $\tilde{E}[\beta q_{T_0}(b)] = 0$ for all $b \in C([0, T]; \mathbb{R}^m)$, then $\beta = 0$ P-a.s. Therefore

$$\langle \mu_{T_0}, \beta \rangle = \tilde{E}[\beta(x(T_0))p_{T_0}|\mathcal{Y}_{T_0}] \quad P - a.s. \tag{6.19}$$

The proof is complete. □

Remark 6.3. The unique solvability of FKK equation follows from this theorem and the Kallianpur-Striebel formula in Proposition 4.1. In Popa and Sritharan [14], a direct method of showing unique solvability of the FKK equation is applied to Itô-Lévy processes with Lipschitz coefficients. Our proof used a different approach without assuming the boundedness assumption (5.9) in [14].

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