

## CORNISH-FISHER EXPANSIONS FOR POISSON AND NEGATIVE BINOMIAL PROCESSES

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ABSTRACT. Negative binomial and Poisson processes  $\{X(t), t \geq 0\}$  have cumulants proportional to the rate  $\lambda$ . So, formal Cornish-Fisher expansions in powers of  $\lambda^{-1/2}$  are available for their cumulative distribution function and quantiles. If  $T$  is an independent random variable with cumulants proportional to  $\gamma$  then the cumulants of  $X(T)$  are proportional to  $\gamma$ ; so, Cornish-Fisher expansions in powers of  $\gamma^{-1/2}$  are available for its cumulative distribution function and quantiles. When  $T$  has a gamma distribution, alternative expansions in powers of  $\gamma^{-1}$  are available for the probability mass function of  $X(T)$ .

### 1. Introduction

Let  $\{X(t), t \geq 0\}$  be a point process and  $T$  be a random variable independent of it. The aim of this note is derive *exact* expressions for the cumulative distribution function, probability mass function and quantiles of  $X(T)$  for a class of point processes, a class that includes the Poisson and negative binomial processes. We believe that this is first time exact expressions have been derived for the stated functions and for the stated class of point processes.

These results are useful, for example, for modeling inventory demand when  $X(t)$  has independent increments and the distribution of  $X(t + \delta) - X(\delta)$  does not depend on  $\delta$ . In inventory modeling  $X(T)$  is the lead-time demand corresponding to a lead-time  $T$ . Its distribution is used to determine when to order stock.

The results will also be useful in other prominent application areas of Poisson processes, some of which are: electrical and electronic engineering, operations research, management science, computer science, physics, industrial engineering, economics, manufacturing engineering, neurosciences, medicine, civil engineering, environmental sciences, astronomy, mechanical engineering, geochemistry, optics, materials science, business, ecology, and educational research.

The results will also be useful for negative binomial processes which are used for modeling events for each risk type in many chronic disease processes, vehicle

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accident frequencies, simulation of storm occurrences, minimum flows in perennial streams, recurrent events under event-dependent censoring, effect of socio-economic status, arrests for drug use, frequencies of weeds in arable fields, surveillance of rare health events, and reliability.

Let  $X$  have a negative binomial distribution on  $\{0, 1, 2, \dots\}$  with parameters  $K$  and  $p$ . Then its cumulants are proportional to  $K$ . It follows that formal Cornish-Fisher expansions (Fisher and Cornish, 1960) in powers of  $K^{-1/2}$  are available for its cumulative distribution function. These are given in Section 3. A similar result holds if  $X$  is Poisson with mean  $K$ .

Now suppose that  $X(t)$  is a negative binomial or Poisson process with  $K = \lambda t$ . If  $T$  has cumulants proportional to  $\gamma$  then the cumulants of  $X(T)$  are also proportional to  $\gamma$ . So, Cornish-Fisher expansions in powers of  $\gamma^{-1/2}$  are available for the cumulative distribution function of  $X(T)$ . This is done in Section 4 for a negative binomial process and in Section 5 for a Poisson process. When  $T$  has a gamma distribution, an alternative expansion in powers of  $\gamma^{-1}$  is available for the probability mass function of  $X(T)$ . The method applies to *any* process  $X(t)$  with cumulants proportional to  $t$ .

The theory of Cornish-Fisher expansions in powers of  $n^{-1/2}$  for lattice variables  $X$  with cumulant proportional to  $n$  is given in Section 2. This theory is referred to throughout Sections 3 to 5.

For a given function, say  $\omega(\cdot)$ , we shall denote its  $k$ th derivative by  $w^{(k)}(\cdot)$  throughout.

## 2. Expansions for the Distribution of the Mean of Lattice Variables

Suppose that  $\Omega_n$  is a random variable with values in  $\mathbb{Z}$ , the set of integers, and cumulants of magnitude  $n$ :

$$\kappa_r(\Omega_n) = n\chi_r \quad (2.1)$$

for  $r \geq 1$ . Then it may be represented as

$$\Omega_n = \sum_{i=1}^n X_i,$$

where  $\{X_i\}$  are independent and identically distributed in  $\mathbb{Z}$  with cumulants  $\{\chi_r\}$ . Let  $\Phi$  and  $\phi$  be the cumulative distribution function and probability density function of a standard normal variable. For  $j \geq 1$ , let  $H_j(y)$  be the  $j$ th Hermite polynomial, listed for  $j \leq 10$  in equation (6.23) of Kendall and Stuart (1977):

$$H_j(y) = \phi(y)^{-1}(-d/dy)^j \phi(y).$$

Let  $\{B_r\}$  be the Bernoulli numbers. Define  $\{S_j\}$  as functions of period 1 with

$$S_j(x) = \sum_{r=0}^j B_r x^{j-r} / \{r!(j-r)!\}$$

for  $0 \leq x < 1$  and  $j \geq 0$ . So, for  $0 \leq x < 1$ ,  $S_0(x) = 1$ ,  $S_1(x) = x - 1/2$ ,  $S_2(x) = x^2/2 - x/2 + 1/12$ , and so on. So,  $S_1$  jumps  $-1$  at each integer while the

other  $\{S_j\}$  are continuous. For  $\chi = (\chi_3, \chi_4, \dots)$ , define

$$P_{r,i}(\chi) = (m!)^{-1} \sum \prod_{k=1}^m \chi_{j_k+2} / (j_k + 2)!$$

for  $i - r = 2m \geq 1$  even, where  $\sum$  sums over integers  $j_1 \geq 1, \dots, j_m \geq 1$  adding to  $r$ , and  $P_{r,i}(\chi) = 0$  otherwise. Set

$$l_r = \chi_2^{-r/2} \chi_r, \mathbf{l} = (l_3, l_4, \dots), P_{r,i} = P_{r,i}(\mathbf{l}) = \chi_2^{-r/2} P_{r,i}(\chi).$$

Set  $\Lambda_n(y) = P(n^{-1/2}(\Omega_n - n\chi_1) \leq \chi_2^{1/2}y)$ .

**Theorem 2.1.** *Suppose that (2.1) holds with  $\chi_2 > 0$  and  $\chi_r$  finite for  $1 \leq r \leq R+2$  with  $R \geq 0$ . Suppose that  $\mathbb{Z}$  is the minimal lattice for  $\Omega_n$ . For  $r \geq 1$  and  $0 \leq j \leq r$ , set*

$$p_{r,j}(y) = \sum_{i=r+2}^{3r} P_{r,i} H_{j+i-1}(y),$$

$$h_r(y) = \sum_{j=0}^r S_j \left( n\chi_1 + n^{1/2} \chi_2^{1/2} y \right) \chi_2^{-j/2} p_{r,j}(y).$$

Then

$$\sup_y \left| \Lambda_n(y) - \Phi(y) + \phi(y) \sum_{r=1}^R n^{-r/2} h_r(y) \right| = o\left(n^{-R/2}\right) \tag{2.2}$$

as  $n \rightarrow \infty$ .

*Proof.* Represent  $\Omega_n$  as  $\sum_{i=1}^n X_i$  with  $\{X_i\}$  independent and identically distributed in  $\mathbb{Z}$  with cumulants  $\{\chi_r\}$ . Now apply Theorem 23.1 of Bhattacharya and Ghosh (1976). This gives (2.2) with

$$\phi(y) h_r(y) = \sum_{\alpha=0}^r (-1)^\alpha S_\alpha \left( n\chi_1 + n^{1/2} x \right) \beta_{r-\alpha}^{(\alpha)}(x),$$

where  $x = \chi_2^{1/2}y$ ,  $\beta_i(x)$  is  $P_i(-\Phi_{0,V} : \chi)(x)$  of equation (7.17) with  $V = \chi_2$ , so  $\beta_i(x) = \tilde{P}_r(-d/dx : \chi)\Phi(y)$  for  $\tilde{P}_r(\xi : \chi)$  of equation (7.3). Also  $\tilde{P}_r(\xi : \chi) = \sum_i P_{r,i}(\chi)\xi^i$ .  $\square$

Note that  $h_r(y)$  depends on  $n$ . Writing (2.2) as  $\sup_y |\Lambda_n(y) - \Lambda_{n,R}(y)| = o(n^{-R/2})$ , we see that for  $n\chi_1 + (n\chi_2)^{1/2}y$  an integer,  $h_r(y)$  and  $\Lambda_{n,R}(y)$  jump by

$$\Delta h_r(y) = -\chi_2^{-1/2} p_{r,1}(y) = -\chi_2^{-1/2} \sum_{i=r+2}^{3r} P_{r,i} H_i(y)$$

and

$$\Delta \Lambda_{n,R}(y) = -\phi(y) \sum_{r=1}^R n^{-r/2} \Delta h_r(y).$$

From Corollary 22.3 of Bhattacharya and Ghosh (1976), we obtain

**Theorem 2.2.** *Suppose that the conditions of Theorem 2.1 hold. Set  $y_{i,n} = n^{1/2}(i - n\chi_1)$  and*

$$q_{n,R}(x) = \phi\left(\chi_2^{-1/2}x\right) (n\chi_2)^{-1/2} \sum_{r=0}^R n^{-r/2} p_{r,1}\left(\chi_2^{-1/2}x\right).$$

Then

$$\sup_i \left(1 + |y_{i,n}|^{R+2}\right) |P(\Omega_n = i) - q_{n,R}(y_{i,n})| = o\left(n^{-(1+R)/2}\right)$$

and

$$\sum_{i=-\infty}^{\infty} |P(\Omega_n = i) - q_{n,R}(y_{i,n})| = o\left(n^{-R/2}\right).$$

Since  $\Lambda_n(y)$  is discontinuous,  $\Lambda_n^{-1}(y)$  may take any value between its right - and left - continuous inverse. A formal expansion for  $\Lambda_n^{-1}(y)$  is given by the following Cornish-Fisher type theorem with  $\epsilon = n^{-1/2}$ .

**Theorem 2.3.** *Suppose that*

$$P(y) \approx \Phi(y) - \phi(y) \sum_{r=1}^{\infty} \epsilon^r h_r(y). \quad (2.3)$$

Then

$$\Phi^{-1}(P(y)) \approx y - \sum_{r=1}^{\infty} \epsilon^r f_r(y) \quad (2.4)$$

and

$$P^{-1}(\Phi(y)) \approx y + \sum_{r=1}^{\infty} \epsilon^r g_r(y) \quad (2.5)$$

for some functions  $f_r(\cdot)$  and  $g_r(\cdot)$ .

*Proof.* Use Taylor's expansion. □

For the case of *continuous* variables with limit  $\Phi(y)$  non-normal, equations (46) and (47) of Hill and Davis (1968) give analogs of (2.4), (2.5). From their equation (12) we now obtain an explicit formula for  $f_r$ . This uses the partial ordinary Bell polynomials defined by

$$\left(\sum_{i=1}^{\infty} \epsilon^i h_i\right)^k = \sum_{r=k}^{\infty} \epsilon^r \widehat{B}_{r,k}(\mathbf{h}) \quad (2.6)$$

for  $\mathbf{h} = (h_1, h_2, \dots)$ . They are tabled on page 309 of Comtet (1974) for  $r \leq 10$ .

**Theorem 2.4.** *Suppose that (2.3) holds, where the cumulative distribution function  $\Phi$  is arbitrary but infinitely differentiable with probability density function  $\phi$ . Then (2.4) holds with*

$$f_r(y) = \sum_{k=1}^r (-1)^{k-1} c_k(y) \widehat{B}_{r,k}(\mathbf{h}(y)) / k!,$$

where  $c_1(y) = 1$ ,  $c_{r+1}(y) = (r\psi(y) + D)c_r(y)$  for  $r \geq 1$ ,  $\psi(y) = -D \ln \phi(y)$  and  $D = d/dy$ . Furthermore, (2.5) holds with

$$g_r(y) = \sum_{k=1}^r (-1)^{k-1} D_{(k)} \widehat{B}_{r,k}(\mathbf{h}(y)) / k!,$$

where

$$D_{(k)} = \prod_{i=1}^{k-1} [i\psi(y) - D], \quad \prod_{i=1}^{k-1} a_i = a_1 \cdots a_{k-1}.$$

*Proof.* Apply (2.6) to equations (12) and (23) of Hill and Davis (1968). □

### 3. Expansions for a Negative Binomial

Consider the negative binomial variable:

$$P(X = i) = p^K (-q)^i \binom{-K}{i} = p^K q^i \binom{K + i - 1}{K - 1} \tag{3.1}$$

for  $i = 0, 1, \dots$ . This is the probability of needing  $K + i$  Binomial ( $p$ ) trials in order to get  $K$  successes. Here,  $q = 1 - p$ , the probability of a failure. Provided  $0 \leq q < 1$ , (3.1) remains a probability mass function for any value of  $K$ . This distribution is sometimes denoted as  $NBD(P, K)$ , where  $P = q/p$ , that is  $p = 1/(1 + P)$ .

By equation (5.37) of Kendall and Stuart (1977, page 138) its cumulants generating function is

$$\ln \{ \mathbb{E} [\exp(sX)] \} = Kf(s),$$

where

$$f(s) = \ln p - \ln \{ 1 - q \exp(s) \} = \sum_{r=1}^{\infty} \chi_r s^r / r!. \tag{3.2}$$

By Faa di Bruno's rule its  $r$ th cumulant is

$$\kappa_r(X) = K\chi_r,$$

where

$$\chi_r = \sum_{k=1}^r (k-1)! P^k C_{r,k}, \quad C_{r,k} = \sum_{\mathbf{n}}^{r,k} m(\mathbf{n}),$$

where

$$m(\mathbf{n}) = r! / \prod_{i=1}^r (i!^{n_i} n_i!),$$

the partition function, and  $\sum_{\mathbf{n}}^{r,k}$  sums over  $\{n_i \geq 0, \sum_{i=1}^r i n_i = r, \sum_{i=1}^r n_i = k\}$ . This may easily be calculated using  $C_{r,k} = B_{r,k}(\mathbf{1})$ , where  $B_{r,k}(\mathbf{x})$  is the partial

exponential Bell polynomial tabled for  $r \leq 12$  on page 307 of Comtet (1974). One obtains

$$\begin{aligned}\chi_1 &= P, \quad \chi_2 = P(1 + P), \quad \chi_3 = P(1 + P)(1 + 2P), \\ \chi_4 &= P(1 + P)(1 + 6P + 6P^2), \\ \chi_5 &= P(P + 1)(2P + 1)(12P^2 + 12P + 1), \\ \chi_6 &= P(P + 1)(120P^4 + 240P^3 + 150P^2 + 30P + 1),\end{aligned}$$

and so on. These can be put in the form given by equation (5.38) of Kendall and Stuart (1977) for  $r \leq 4$ :  $\chi_1 = q/p$ ,  $\chi_2 = q/p^2$ ,  $\chi_3 = q(1 + p)/p^3$  and  $\chi_4 = q(1 + 4q + q^2)/p^4$ . Expansions in powers of  $K^{-1/2}$  for the cumulative distribution function, probability mass function and quantiles of  $X$  are given in Theorem 2.1, Theorem 2.2 and Theorems 2.3-2.4, respectively, with  $(n, \Omega_n) = (K, X)$ .

#### 4. Expansions for a Negative Binomial Process

Suppose that  $X = X(t)$  satisfies (3.1) with  $p = p(t)$ ,  $K = K(t)$ . Then we have a negative binomial process. For three examples with  $K$  not depending on  $t$ , see the birth processes in Bailey (1964). Here, we assume  $p$  (or  $P$ ) does not depend on  $t$  and  $K(t) = \lambda t$ , where the rate  $\lambda$  is constant. The marginals of the process are defined by requiring that  $X(t)$  has independent increments. For example, for  $s < t$ ,  $X(t) - X(s)$  is negative binomial with parameters  $K = \lambda(t - s)$  and  $p$  or  $P$ .

Suppose that  $T$  is a random variable independent of  $X(\cdot)$  with cumulants

$$\kappa_r(T) = \gamma a_r \tag{4.1}$$

for  $r \geq 1$ , where  $a_r$  does not depend on  $\gamma$ . Then  $X(T)$  has cumulant generating function

$$\ln \{\mathbb{E}[\exp(sX(T))]\} = \ln \{\mathbb{E}[\exp(T\lambda f(s))]\} = \sum_{r=1}^{\infty} \kappa_r(T) (\lambda f(s))^r / r!$$

for  $f(s)$  of (3.2).

Expressions for the cumulative distribution function of  $X(T)$  may be obtained by numerically inverting its Laplace transform ( $s$  real) or Fourier transform ( $s$  complex). Its  $r$ th cumulant is

$$\kappa_r(X(T)) = \sum_{k=1}^r \lambda^k \kappa_k(T) B_{r,k}(\boldsymbol{\chi}),$$

where  $B_{r,k}(\cdot)$  was defined in Section 3. Now suppose (4.1). Then

$$\kappa_r(X(T)) = \gamma A_r, \tag{4.2}$$

where

$$A_r = \sum_{k=1}^r \lambda^k a_k B_{r,k}(\boldsymbol{\chi}). \tag{4.3}$$

Expansions in powers of  $\gamma^{-1/2}$  for the cumulative distribution function, probability mass function and quantiles of  $X(T)$  are given in Theorem 2.1, Theorem 2.2 and Theorems 2.3-2.4, respectively, with  $(n, \Omega_n, \boldsymbol{\chi}) = (\gamma, X(T), \mathbf{A})$ .

**Example 4.1.** Let  $T$  have probability density function  $c^\gamma t^{\gamma-1} \exp(-ct)/\Gamma(\gamma)$  for  $0 < t < \infty$ . Then

$$\mathbb{E}[\exp(sX(T))] = (1 - \nu f(s))^{-\gamma}$$

for  $\nu = \lambda/c$  and  $f(s)$  of (3.2). Since (4.1) holds with  $a_r = c^{-r}(r-1)!$ , we obtain (4.3) with

$$\begin{aligned} A_1 &= \nu P, \\ A_2 &= \nu P(1 + P) + \nu^2 P^2, \\ A_3 &= \nu P(1 + P)(1 + 2P) + 3\nu^2 P^2(1 + P) + 2\nu^3 P^3, \\ A_4 &= \nu \chi_4 + \nu^2 P^2(1 + P)(7 + 11P) + 12\nu^3 P^3(1 + P) + 6\nu^4 P^4, \\ A_5 &= \nu \chi_5 + 5\nu^2 P^2(1 + P)(3 + 12P + 10P^2) + 10\nu^3 P^3(1 + P)(5 + 7P) \\ &\quad + 60\nu^4 P^4(1 + P) + 24\nu^5 P^5, \\ A_6 &= \nu \chi_6 + 3\nu^2 P^2(7 + 70P + 145P^2 + 110P^3 + 32P^4) \\ &\quad + 30\nu^3 P^3(1 + P)(6 + 20P + 15P^2) + 30\nu^4 P^4(1 + P)(13 + 17P) \\ &\quad + 360\nu^5 P^5(1 + P) + 120\nu^6 P^6. \end{aligned}$$

An alternative expansion for the probability density function is sometimes available using the following lemma.

**Lemma 4.2.** By <http://functions.wolfram.com/06.11.06.0015.01>,

$$\begin{aligned} \ln \left\{ \frac{\Gamma(\xi + c)}{\Gamma(\xi)} \right\} &= \ln \Gamma(\xi + c) - \ln \Gamma(\xi) \\ &= \left( \xi + c - \frac{1}{2} \right) \ln(\xi + c) - \left( \xi - \frac{1}{2} \right) \ln \xi - c \\ &\quad + \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)\xi^{2k-1}} \\ &\quad - \sum_{k=1}^{\infty} \frac{B_{2k}}{2k(2k-1)(\xi + c)^{2k-1}}, \end{aligned}$$

where  $B_k$  denotes the Bernoulli numbers. By <http://functions.wolfram.com/06.05.06.0016.02>,

$$\frac{\Gamma(\xi + c)}{\Gamma(\xi)} = \xi^c \sum_{k=0}^{\infty} \frac{(-1)^k (-c)_k}{k!} B_k^{(c+1)}(c) \xi^{-k} = \xi^c \sum_{k=0}^{\infty} \sigma_k \xi^{-k},$$

where  $(c)_k = c(c+1) \cdots (c+k-1)$  and  $B_k(\cdot)$  denotes the  $k$ th order Bernoulli polynomial.

It follows from this lemma that (3.1) can be written

$$\begin{aligned} P(X = i) &= (q^i/i!) p^K \Gamma(K+i)/\Gamma(K) \\ &= (q^i/i!) p^K \{ \sigma_0 K^i + \sigma_1 K^{i-1} + \sigma_2 K^{i-2} + \cdots \} \end{aligned}$$

and that

$$P(X(T) = i) = (q^i/i!) \{ \sigma_0 a_i + \sigma_1 a_{i-1} + \sigma_2 a_{i-2} + \cdots \}, \tag{4.4}$$

where  $a_i = \mathbb{E}[p^{\lambda T}(\lambda T)^i]$ .

**Example 4.3.** (Continuation of Example 4.1). Note that

$$a_i = (c\lambda^{-1} - \ln p)^{-i} (1 - \lambda c^{-1} \ln p)^{-\gamma} \Gamma(i + \gamma) / \Gamma(\gamma)$$

since  $c\lambda^{-1} - \ln p > 0$ . The ratio of successive terms is  $a_{i-1}/a_i \sim \epsilon = (c\lambda^{-1} - \ln p)\gamma^{-1}$ . So, (4.4) may be viewed as a power series in  $\epsilon$ . Provided  $\epsilon$  is not too large, the partial sum for (4.4) should give a reasonable approximation for fixed  $i$ .

## 5. Expansions for Poisson Processes

Suppose that  $X$  is Poisson with mean  $K$ . Then its cumulant generating function is  $Kf(s)$  with  $f(s)$  given by

$$f(s) = \exp(s) - 1 = \sum_{r=1}^{\infty} \chi_r s^r / r!$$

for  $\chi_r = 1$ .

Expansions in powers of  $K^{-1/2}$  for the cumulative distribution function and quantiles of  $X$  are given in Theorem 2.1 and Theorems 2.3-2.4, respectively, with  $(n, \Omega_n, \chi) = (K, X, \mathbf{1})$ .

Now suppose that  $X = X(t)$ ,  $K = \lambda t$  and that  $T$  is an independent random variable with cumulants as in (4.1). Then  $X(T)$  has cumulants given by (4.2) with  $\chi = \mathbf{1}$ . So, expansions in powers of  $\gamma^{-1/2}$  for its cumulative distribution function and quantiles are given in Theorem 2.1 and Theorems 2.3-2.4, respectively, with  $(n, \Omega_n, \chi) = (\gamma, X(T), \mathbf{A})$ , where  $A_r$  is given by (4.3) with  $\chi = \mathbf{1}$ .

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