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DENSITY DEPENDENT UTILITIES WITH TRANSACTION COSTS

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Abstract. We investigate the combined singular and impulse control problem in the context of jump diffusions. Problems of this nature often arise when both fixed and proportional transaction costs are considered, for instance, in finance. We formulate and prove a verification theorem for the generalized combined singular and impulse control. This theorem establishes sufficient conditions for the existence of both the value function and optimal combined controls. An illustrative example of this result is presented.

1. Introduction

Several researchers have considered the problem of portfolio optimization in the presence of transaction costs, for example [1, 6, 7]. The inclusion of both fixed and proportional transaction costs gives rise to problems which exhibit both singular and impulse control features. Previous authors focused their work on such problems without considering the situation with jump diffusions. We seek to close this gap. In particular, using some of the earlier arguments, we develop the theory of combined singular and impulse control for Lévy processes.

This paper is further distinguished by the fact that it illustrates the application of combined singular impulse control to the problem of optimal harvesting with density dependent prices, in a framework of jump diffusions and in the presence of transaction costs, with an example motivated by Example 3.1 in [2]. In [2] the problem of optimal stochastic harvesting with density dependent prices for diffusions is discussed under the *no transaction costs* assumption. For an extensive coverage of the theory and application of singular control and impulse control as separate stochastic control techniques for Lévy processes, we refer the reader to [12] and the references provided therein.

The paper is organized as follows: we forumlate the general combined singular and impulse control problem in Section 2, followed by a discussion of the verification theorem in Section 3. The final section includes an example on the application of the theory of combined singular and impulse control for jump diffusions. This example takes both proportional and transaction costs into account. The present article is a substantial revision of [5].

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2. Background and Problem Formulation

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a filtered complete probability space satisfying the usual conditions. Assume that in the absence of interventions, the state $Y(t) \in \mathbb{R}^k$, of a given system evolves according to the following equations

$$dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}^k} \gamma(Y(t^-), z)\tilde{N}(dt, dz), \tag{2.1}$$

$$Y(0^-) = y \in \mathbb{R}^k, \tag{2.2}$$

where $b : \mathbb{R}^k \to \mathbb{R}^k$, $\sigma : \mathbb{R}^k \to \mathbb{R}^{k\times m}$ and $\gamma : \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}^{k\times d}$ are functions satisfying the conditions for the existence and uniqueness of a strong solution $Y(t)$. Here, $B(t)$ is m-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}$ and $\tilde{N}_r(.,.)$ is a compensated Poisson random measure given by $\tilde{N}_r(dt, dz) = N_r(dt, dz) - dt\nu_r(dz);\ r = 1, 2, \ldots, d$ where $\nu_r(.)$ is a Lévy measure associated with the Poisson random measure $N_r(.,.)$. A more extensive treatment of random measures and stochastic differential equations with a jump component may be found in [4, 8, 9, 11, 12, 13].

The generator $\mathcal{L}$ of $Y(t)$ is given by

$$\begin{aligned}\mathcal{L}\phi(y) = \sum_{i=1}^{k} b_i\Big(y, u(y)\Big)\frac{\partial \phi}{\partial y_i} + \frac{1}{2}\sum_{i,s=1}^{k}(\sigma\sigma^T)_{is}\Big(y, u(y)\Big)\frac{\partial^2\phi}{\partial y_i \partial y_s} \\ + \int_{\mathbb{R}^k}\sum_{r=1}^{d}\Big\{\phi(y) + \gamma^{(r)}(y, u(y), z) - \phi(y) \\ -\nabla\phi(y)^T\gamma^{(r)}\,(y, u(y), z)\Big\}\,\nu_r(dz_r),\end{aligned}$$

where $\phi \in C^2(\mathbb{R}^k)$.

Suppose that at any given point τ_j the decision maker is free to give the system an *impulse*, $\xi_j \in \mathcal{Z} \subset \mathbb{R}^p$, where $\mathcal{Z}$ is the set of all admissible impulses. The notion of an *impulse control* is defined below.

Definition 2.1. An *impulse control*, for the system described by (2.1)-(2.2), is a double sequence

$$v = (\tau_1, \tau_2, \ldots, \tau_j \ldots, \xi_1, \xi_2, \ldots, \xi_j, \ldots)_{j\leq M}, \quad M \leq \infty,$$

where $0 \leq \tau_1 \leq \tau_2 \leq \ldots$ is an increasing sequence of $\mathcal{F}_t$-stopping times and $\xi_1, \xi_2, \ldots$ are the corresponding $\mathcal{F}_{\tau_j}$-adapted impulses at these stopping times.

Let $\mathcal{S} \subset \mathbb{R}^k$ be a fixed Borel set in which we seek solutions to the problem with $\mathcal{S} \subset \bar{\mathcal{S}}^0$. Here $\mathcal{S}^0$ denotes the interior of $\mathcal{S}$ and $\bar{\mathcal{S}}^0$ is the closure of $\mathcal{S}^0$.

Given continuous functions $f : \mathcal{S} \to \mathbb{R},\ g : \mathbb{R}^k \to \mathbb{R}, \kappa = [\kappa_{ie}] \in \mathbb{R}^{k\times p}$ and $\theta = [\theta_i]$, let the profit of making an intervention with impulse $\xi \in \mathcal{Z}$ when the state is y be $K(y, \xi)$, where $K : \mathcal{S} \times \mathcal{Z} \to \mathbb{R}$.

Definition 2.2. Let $\mathcal{H}$ be the space of all measurable functions $h : \mathcal{S} \to \mathbb{R}$. The *intervention operator* $\mathcal{M} : \mathcal{H} \to \mathcal{H}$ is defined by

$$\mathcal{M}h(y) = \sup\{h(\Gamma(y, \xi)) + K(y, \xi);\ \xi \in \mathcal{Z}\}.$$

Suppose that at times $t_n \in [\tau_j, \tau_{j+1}]$ one is allowed to apply the singular control ψ for $n = 1, 2, \ldots, q$, where $\psi \in \mathbb{R}^p$ is an adapted *cádlág* process with non-negative, increasing components such that $\psi(0^-) = 0$. Let the jumps caused by the singular control ψ be denoted by

$$\triangle_\psi Y(t) = \kappa(Y(t^-))\triangle_\psi$$

and consider

$$\triangle_\psi \phi(Y(t_n)) = \phi(Y(t_n)) - \phi(Y(t_n^-))$$

to be the change in ϕ resulting from the jump $\triangle\psi(t) = \psi(t) - \psi(t^-)$ at $t = t_n$. Denote by $\mathcal{W}$ the set of all admissible combined controls $w = (v, \psi)$. Suppose that the controlled process $Y^{(w)}$ satisfies (2.4)-(2.5) given as follows

$$Y^{(w)}(0^-) = y \text{ and } Y^{(w)}(t) = Y(t), \quad 0 < t < \tau_1, \tag{2.3}$$

$$Y^{(w)}(\tau_j) = \Gamma(\check{Y}^{(w)}(\tau_j^-), \xi_j), \quad j = 1, 2, \ldots \tag{2.4}$$

$$\begin{aligned} dY^{(w)}(t) &= b(Y^{(w)}(t))dt + \sigma(Y^{(w)}(t))dB(t) + \kappa(Y^w(t))d\psi \\ &\quad + \int_{\mathbb{R}^l} \gamma(Y^{(w)}(t^-), z)\tilde{N}(dt, dz), \quad \tau_j < t < \tau_{j+1} < \tau^*, \end{aligned} \tag{2.5}$$

where

$$\tau^* = \tau^*(\omega) = \lim_{R\to\infty}(\inf\{t > 0; \mid Y^{(v)}(t) \mid \geq R\}) \leq \infty.$$

In equations (2.4)-(2.5) above, $\check{Y}^{(w)}(\tau_j^-) = Y^{(w)}(\tau_j^-) + \triangle_N Y(\tau_j)$ defines the jump in $Y^{(w)}(\tau_j)$ which stems from $N(.,.)$ and $\Gamma : \mathbb{R}^k \times \mathcal{Z} \to \mathbb{R}^k$ is a given function. Let $\tau_{\mathcal{S}} := \inf\{t \geq 0;\ Y^{(w)} \notin \mathcal{S}\}$ and consider

$$\mathcal{T} := \{\tau;\ \tau \text{ stopping times},\ 0 \leq \tau \leq \tau_{\mathcal{S}}\}.$$

Define a performance functional $J^{(w)}$, for the controlled process $Y^{(w)}$, by

$$\begin{aligned} J^{(w)}(y) \quad &= \quad E^y\Big[\int_0^{\tau_{\mathcal{S}}} f(Y^{(w)}(t))dt + g(Y^{(w)}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}} \\ &\qquad + \int_0^{\tau_{\mathcal{S}}} \theta^T(Y(t))d\psi(t) + \sum_{\tau_j \leq \tau_{\mathcal{S}}} K(Y^{(w)}(\tau_j^-), \xi_j)\Big]. \end{aligned}$$

The *combined singular and impulse control problem* for jump diffusions is to find $\Phi(y)$ and $w^* \in \mathcal{W}$ such that

$$\Phi(y) = \sup\{J^{(w)}(y); w \in \mathcal{W}\} = J^{(w^*)}(y).$$

In the next section we state and prove a verification theorem for the combined singular and impulse control problem for jump diffusions.

3. Main Results

The following two theorems constitute the main results of our paper.

Theorem 3.1. *Suppose there exists a function* $\phi : \bar{\mathcal{S}} \to \mathbb{R}$ *such that*

(i) $\phi \in C^1(\mathcal{S}^o) \cap C(\bar{\mathcal{S}})$,
(ii) $\phi \geq \mathcal{M}\phi$ *on* $\mathcal{S}^0$,
(iii) $\sum_{i=1}^k \kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y) \leq 0$ *for all* $y \in \mathcal{S}$,

(iv) $E^y\Big[\int_0^{\tau_{\mathcal{S}}} \chi_{\partial D}(Y^{(w)}(t))dt\Big] = 0$ *for all* $y \in \mathcal{S},\ w \in \mathcal{W}$,

(v) $Y^{(w)}(\tau_{\mathcal{S}}) \in \partial S$ *a.s. on* $\{\tau_{\mathcal{S}} < \infty\}$ *and* $\phi(Y^{(w)}(t)) \to g(Y^{(w)}(\tau_{\mathcal{S}})).\chi_{\{\tau_{\mathcal{S}}<\infty\}}$ *as* $t \to \tau_{\mathcal{S}}^-$ *a.s. for all* $y \in \mathcal{S},\ w \in \mathcal{W}$,

(vi) $\{\phi^-(Y^{(w)}(\tau));\ \tau \in \mathcal{T}\}$ *is uniformly integrable for all* $y \in \mathcal{S}$ *and* $w \in \mathcal{W}$,

(vii) *for all* $y \in \mathcal{S}$ *and* $w \in \mathcal{W}$.

$$E^y\Big[\int_0^{\tau_{\mathcal{S}}}\Big\{\big|\sigma^T(Y(t),u(t))\nabla\phi(Y(t))\big|^2 + \sum_{m=1}^{d}\int_{\mathbb{R}^k}\Big|\phi(Y(t)+\gamma^{(m)}) - \phi(Y(t))\Big|^2\nu_m(dz)\Big\}dt\Big] < \infty.$$

Then we have $\phi(y) \geq \Phi(y)$ *for all* $y \in \mathcal{S}$.

Proof. On the basis of an approximation argument (see for example Theorem 10.4.1 in [12]) and by applying $(iv)-(vi)$ of the above lemma, we can assume that $\phi \in C^2(\mathcal{S}) \cap C(\bar{\mathcal{S}})$.

Consider an arbitrarily chosen impulse control

$$v = (\tau_1, \tau_2, ..., \tau_j, ...; \xi_1, \xi_2, ..., \xi_j,\ .\ .\ .) \in \mathcal{V}$$

and let $\tau_0 = 0$. Applying Itô's generalized formula for semimartingales, (see for example [13], page 74, Theorem 33), between the stopping times τ_j and τ_{j+1} with $y \in \mathcal{S}$, we obtain

$$\begin{aligned}
&\phi(\check{Y}^{(w)}(\tau_{j+1}^-)) - \phi(Y^{(w)}(\tau_j)) \\
= \ & \int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y^{(w)}(t))dt \\
&+ \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial\phi}{\partial y_i}(Y^{(w)}(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y^{(w)}(t^-))d\psi_e^c(t) \\
&+ \sum_{\tau_j<t_n<\tau_{j+1}} \triangle_\psi \phi(Y^{(w)}(t_n)),
\end{aligned} \tag{3.1}$$

where $\check{Y}^{(w)}(\tau_{j+1}^-) = Y^{(w)}(\tau_{j+1}^-) + \triangle_N Y(w)(\tau_{j+1})$ and $\psi_e^c(t)$ denotes the continuous part of $\psi_e(t)$.

Taking expectations in (3.1) we get

$$\begin{aligned}
&E^y\Big[\phi(\check{Y}^{(w)}(\tau_{j+1}^-))\Big] - E^y\Big[\phi(Y^{(w)}(\tau_j))\Big] \\
= \ & E^y\Big[\int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y^{(w)}(t)))dt \\
&+ \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial\phi}{\partial y_i}(Y^{(w)}(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y^{(w)}(t^-))d\psi_e^c(t) \\
&+ \sum_{\tau_j<t_n<\tau_{j+1}} \triangle_\psi \phi(Y^{(w)}(t_n))\Big].
\end{aligned}$$

This last equation is equivalent to

$$
\begin{aligned}
&E^y\Big[\phi(Y^{(w)}(\tau_j))\Big] - E^y\Big[\phi(\check{Y}^{(w)}(\tau_{j+1}^-))\Big] \\
&= -E^y\Big[\int_{\tau_j}^{\tau_{j+1}} \mathcal{L}\phi(Y^{(w)}(t)))dt \\
&\quad + \int_{\tau_j}^{\tau_{j+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y^{(w)}(t^-))d\psi_e^c(t) \\
&\quad + \sum_{\tau_j<t_n<\tau_{j+1}} \triangle_\psi \phi(Y^{(w)}(t_n))\Big].
\end{aligned}
$$

Summing up from $j = 0$ to $j = m$ yields

$$
\begin{aligned}
\phi(y) \quad + \quad & \sum_{j=1}^{m} E^y\Big[\phi(Y^{(w)}(\tau_j)) - \phi(\check{Y}^{(w)}(\tau_j^-))\Big] - E^y\Big[\phi(Y^{(w)}(\tau_{m+1}^-))\Big] \\
= \quad & -E^y\Big[\int_0^{\tau_{m+1}} \mathcal{L}\phi(Y^{(w)}(t)))dt \\
& + \int_0^{\tau_{m+1}} \sum_{i=1}^{k} \frac{\partial \phi}{\partial y_i}(Y^{(w)}(t^-)) \sum_{e=1}^{p} \kappa_{ie}(Y^{(w)}(t^-))d\psi_e^c(t) \\
& + \sum_{0<t_n<\tau_{j+1}} \triangle_\psi \phi(Y^{(w)}(t_n))\Big].
\end{aligned}
$$

It is easy to note that

$$
\phi(Y^{(w)}(\tau_j)) \le \phi(\Gamma(Y^{(w)}(\tau_j^-), \xi_j)) + K(Y^{(w)}(\tau_j^-), \xi_j).
$$

Applying the definition of the intervention operator $\mathcal{M}$, we obtain

$$
\begin{aligned}
\phi(Y^{(w)}(\tau_j)) \quad &= \quad \phi(\Gamma(Y^{(w)}(\tau_j^-), \xi_j)) + K(Y^{(w)}(\tau_j^-), \xi_j) \\
&\le \quad \mathcal{M}\phi(Y^{(w)}(\tau_j^-))
\end{aligned}
$$

if $\tau_j < \tau_{\mathcal{S}}$ and

$$
\phi(Y^{(w)}(\tau_j)) = \phi(Y^{(w)}(\tau_{\mathcal{S}})
$$

if $\tau_j = \tau_{\mathcal{S}}$. Thus

$$
\begin{aligned}
\phi(Y^{(w)}(\tau_j)) \quad &\le \quad \phi(\Gamma(Y^{(w)}(\tau_j^-), \xi_j)) \\
&\le \quad \mathcal{M}\phi(Y^{(w)}(\tau_j^-)) - K(Y^{(w)}(\tau_j^-), \xi_j)
\end{aligned}
\tag{3.2}
$$

if $\tau_j < \tau_{\mathcal{S}}$ and

$$
\phi(Y^{(w)}(\tau_j)) = \phi(Y^{(w)}(\tau_{\mathcal{S}})
$$

if $\tau_j = \tau_{\mathcal{S}}$. From (3.2) we get

$$
\mathcal{M}\phi(Y(\tau_j^-)) - \phi(Y(\tau_j^-)) \ge \phi(Y(\tau_j)) - \phi(Y(\tau_j^-)) + K(Y(\tau_j^-), \xi_j). \tag{3.3}
$$

Applying the mean value theorem we obtain

$$\begin{aligned}\triangle_{\psi}\phi(Y^{(w)}(t_n)) &= \nabla\phi(Y^{(w)}(t_n))^T\triangle_{\psi}(Y^{(w)}(t_n)) \\ &= \sum_{i=1}^{k}\sum_{l=1}^{p}\frac{\partial\phi}{\partial y_i}Y^{(w)}(t_n^-)\kappa_{il}(Y^{(w)}(t_n^-))(\triangle_{\xi_l}t_n). \qquad (3.4)\end{aligned}$$

Now, combining (3.3) and (3.4) results in

$$\begin{aligned}\phi(y) &+ \sum_{j=1}^{m}E^y\Big[\{\mathcal{M}\phi(Y^{(w)}(\tau_j^-)) - \phi(Y^{(w)}(\tau_j^-))\}\chi_{\{\tau_j<\tau_{\mathcal{S}}\}}\Big] \\ &\geq E^y\Big[\phi(Y^{(w)}(\tau_{m+1}^-)) - \int_0^{\tau_{m+1}}\mathcal{L}\phi(Y^{(w)}(t)))dt \\ &\quad - \int_0^{\tau_{m+1}}\sum_{i=1}^{k}\frac{\partial\phi}{\partial y_i}(Y^{(w)}(t^-)\sum_{e=1}^{p}\kappa_{ie}(Y^{(w)}(t^-))d\psi_e^{(c)}(t) \\ &\quad - \sum_{0<t_n<\tau_{j+1}}\triangle_{\psi}\phi(Y^{(w)}(t_n)) + \sum_{i=1}^{k}K(Y^{(w)}(\tau_j^-),\xi_j)\Big] \\ &\geq E^y\Big[\int_0^{\tau_{m+1}}f(Y^{(w)}(t),u(t)))dt + \phi(Y^{(w)}(\tau_{m+1}^-)) \\ &\quad + \sum_{e=1}^{p}\int_0^{\tau_{m+1}}\theta_e(Y^{(w)}(t))d\psi_e(t) + \sum_{i=1}^{k}K(Y^{(w)}(\tau_j^-),\xi_j)\Big]. \qquad (3.5)\end{aligned}$$

Letting $m \to M$, we have

$$\begin{aligned}\phi(y) &\geq E^y\Big[\int_0^{\tau_{\mathcal{S}}}f(Y^w(t),u(t)))dt + g(Y^w(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}} \\ &\quad + \int_0^{\tau_{\mathcal{S}}}\theta(Y^w(t))d\psi_l(t) + \sum_{i=1}^{k}K(Y^{(w)}(\tau_j^-),\xi_j)\Big] \\ &= J^w(y) \qquad (3.6)\end{aligned}$$

for all $y \in \mathcal{S}$. □

Theorem 3.2. *Let*

$$D = \{y \in \mathcal{S};\ \max_e\{\mathcal{M}\phi(y) - \phi(y), \sum_{i=1}^{k}\kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y)\} \leq 0\}.$$

Suppose that, in addition to conditions $(i)-(vii)$ in Theorem 3.1,

(i) *there exists a function $\hat{w} = (\hat{v}, \hat{u}, \hat{\psi}) \in \mathcal{W}$ such that*

$$\mathcal{L}^{\hat{w}}\phi(y) + f(y, \hat{w}(y)) = 0,\ \forall\, y \in D$$

(ii) $Y^{\hat{w}}(t) \in \bar{D}$

(iii) $\sum_{e=1}^{p}\Big\{\sum_{i=1}^{k}\kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(Y(t^-)) + \theta_e\Big\}d\hat{\psi}_e^{(c)} = 0$ *for all* $1 \leq p$, *where* $\psi_e^{(c)}(t)$ *is the continuous part of* $\psi_e^{(c)}$.

(iv) $\lim_{R\to\infty} E^y\Big[\phi(Y^{\hat{w}}(T_R))\Big] = E^y\Big[g(Y^{\hat{w}}(T)).\chi_{\{T<\infty\}}\Big]$ *with* T_R *given by*

$$T_R = \min(\tau_{\mathcal{S}}, R),\ R < \infty$$

(vi) $\hat{\xi}(y) \in Argmax\{\phi(\Gamma(y,.)) + K(y,.)\} \in \mathcal{Z}$ *exists for all* $y \in \mathcal{S}$ *and* $\hat{\xi}(.)$ *is a Borel measurable selection.*

Then $\phi(y) = \Phi(y)$ *for all* $y \in \mathcal{S}$ *and* $\hat{w} \in \mathcal{W}$ *is an optimal combined singular impulse control.*

Proof. Assuming conditions $(i)-(vi)$ hold, we apply the reasoning to $\hat{w} = (\hat{v}, \hat{\xi})$. From (3.5) and (3.6) respectively, we get the following equalities:

$$\begin{aligned}
&\phi(y) + \sum_{j=1}^{m} E^y\Big[\{\mathcal{M}\phi(Y^{(w)}(\tau_j^-)) - \phi(Y^{(w)}(\tau_j^-))\}\chi_{\{\tau_j<\tau_{\mathcal{S}}\}}\Big]\\
&= E^y\Big[\phi(Y(\tau_{m+1}^-)) - \int_0^{\tau_{m+1}} \mathcal{L}\phi(Y^{(w)}(t)))dt\\
&\quad - \int_0^{\tau_{m+1}} \sum_{i=1}^{k} \frac{\partial\phi}{\partial y_i}(Y(t^-) \sum_{l=1}^{p} \kappa_{ie}(Y(t^-))d\hat{\psi}_e^{(c)}(t)\\
&\quad - \sum_{0<t_n<\tau_{j+1}} \triangle_\psi \phi(Y(t_n) + \sum_{i=1}^{k} K(Y^{(w)}(\tau_j^-), \xi_j)\Big]\\
&= E^y\Big[\int_0^{\tau_{m+1}} f(Y^{(w)}(t), \hat{u}(t)))dt + \phi(Y^{(w)}(\tau_{m+1}^-))\\
&\quad + \sum_{e=1}^{p} \int_0^{\tau_{m+1}} \theta_e(Y^{(w)}(t))d\psi_e(t) + \sum_{i=1}^{k} K(Y^{(w)}(\tau_j^-), \xi_j)\Big]
\end{aligned}$$

and

$$\begin{aligned}
\phi(y) &= E^y\Big[\int_0^{\tau_{\mathcal{S}}} f(Y^{\hat{w}}(t), \hat{u}(t)))dt + g(Y^{\hat{w}}(\tau_{\mathcal{S}}))\chi_{\{\tau_{\mathcal{S}}<\infty\}}\\
&\quad + \int_0^{\tau_{\mathcal{S}}} \theta(Y^{\hat{w}}(t))d\hat{\psi}_e(t) + \sum_{i=1}^{k} K(Y^{\hat{w}}(\tau_j^-), \hat{\xi}_j)\Big]\\
&= J^{\hat{w}}(y)
\end{aligned}$$

for all $y \in \mathcal{S}$. It follows that $\phi(y) = \Phi(y) = \sup\{J^{(w)}(y); w \in \mathcal{W}\} = J^{\hat{w}}(y)$. □

Remark 3.3. The above theorems and their proofs make up the verification theorem.

4. Application: Optimal Harvesting with Transaction Costs

Suppose that if there are no interventions the stochastic process $X(t)$, which might represent the remaining resources, for example some mineral resource or wildlife population at time t (with $\mu, \sigma, \beta > 0$ constants), evolves according to

$$dX(t) = \mu dt + \sigma dB(t) + \beta \int_{\mathbb{R}} z\tilde{N}(dt, dz); \quad X(0) = x > 0 \tag{4.1}$$

where $B(t)$ is 1-dimensional standard Brownian motion, $\tilde{N}(.,.)$ is a compensated Poisson random measure and $\beta z \leq 0$ for $a.a\ z(\nu)$.

Now, assume that at any time τ_j, where $j = 1, 2, ---$, the investor is free to take out an amount, ξ_j, from $X(t)$ and such a transaction incurs a cost denoted by $m(\xi_j)$, and given by

$$m(\xi_j) = \delta\xi_j + c$$

where $c \geq 0$ and $\delta \in (0, 1)$ are constants. In particular, c is a fixed transaction cost.

Suppose that the decision maker applies a combined control $w = (v, \psi)$ where $v := (\tau_1, \tau_2, ..., \tau_j ..., \xi_1, \xi_2, ..., \xi_j ...)$ is an impulse control and $\psi(t)$ is an increasing, adapted *cádlág* process representing the total amount taken out from $X(t)$ up to time t .

Let $\mathcal{W}$ be the set of all combined singular and impulse controls $w = (v, \psi)$ such that $X^{(w)}(t) \geq 0$. We call $\mathcal{W}$ the set of admissible combined singular and impulse controls.

We now assume that as a result of applying the combined singular and impulse control w the evolution of the controlled process $X(t) = X^{(w)}(t)$ is described by (4.2)-(4.4) given below

$$X^{(w)}(t) = X(t) \qquad \text{if} \quad 0 \leq t < \tau_1; \tag{4.2}$$

$$dX^{(w)}(t) = \mu dt + \sigma dB(t) + \beta\int_{\mathbb{R}} z\tilde{N}(dt, dz) \quad - \quad (1 + \delta)d\psi(t)$$
$$\text{if} \quad \tau_j \leq t < \tau_{j+1}; \tag{4.3}$$

$$X^{(w)}(\tau_j) = \check{X}^{(w)}(\tau_j^-) - (1 + \delta)\xi_j - c. \tag{4.4}$$

Define the performance criterion, $J^{(w)}(s, x)$, by

$$J^{(w)}(s, x) := E^{s,x}\Big[\int_0^{\tau} e^{-\rho(s+t)}X^{\alpha}(t)d\psi(t)\Big]$$

where $\tau = \inf\{t : X(t) \leq 0\}$ (time to exhaustion of resources), $\rho > 0$ is a discount factor, $0 <\mid \alpha \mid\leq 1$, $E(.)$ denotes expectation with respect to probability law P and $w = (v, \psi)$ represents an admissible combined singular and impulse control.

The problem is to find $\Phi(s, x)$ and $w^* = (v^*, \psi^*)$ such that

$$\Phi(s, x) = \sup_{(v,\psi)=w\in\mathcal{W}} J^{(w)}(s, x) = J^{(w^*)}(s, x)$$

4.1. Solution. In this case the singular control is ψ and the impulse control is v.

We apply Theorems 3.1 and 3.2 to solve the problem. Here we consider that $-1 \leq \alpha < 0$. Without loss of generality, we take $\alpha = -\frac{1}{2}$. In this case the performance functional, $J^w(s, x)$, is given by

$$J^{(w)}(s, x) := E\Big[\int_s^{\tau} e^{-\rho(s+t)}(X(t))^{-\frac{1}{2}}d\psi(t)\Big].$$

We observe that $K = u = g = f = 0$, $\theta = e^{-\rho s}x^{-\frac{1}{2}}$, $\kappa(s, x) = -(1+\delta)$, $\Gamma(s, x, \xi_j) = x - (1 + \delta)\xi_j - c$ and $\mathcal{S} = \{(s, x);\ x > 0\}$. It is worth noting that $\theta : \mathbb{R} \to \mathbb{R}$

is a non-increasing function (density dependent prices) and for that reason our analysis follows closely arguments presented in [2] with the necessary extensions to the jump diffusion case. Moreover, the discussion herebelow takes transaction costs into account. We examine the problem as a combined singular and impulse control whereas in [2] it is handled from the singular control angle only.

If we apply the *"take the money and run"*-strategy, $\dot{w}$, then *all* the resources are taken out immediately. Such a strategy is described by

$$\dot{w}(s) = \dot{\psi}(s) = (1-\delta)x - c.$$

The value function obtained from this strategy is

$$\Phi(s,x) = e^{-\rho s} x^{-\frac{1}{2}}[(1-\delta)x - c] = e^{-\rho s}[(1-\delta)\sqrt{x} - cx^{-\frac{1}{2}}]; \quad x > 0.$$

Apparently, this strategy is not optimal simply because it does not take into account the impact of transaction costs on total discounted gains, neither does it cater for the price increases as the resources diminish. Consequently, we seek a kind of "chattering strategy", denoted by $\tilde{w}^{(m,\eta)} = \tilde{\psi}^{(m,\eta)}$ where m is a fixed positive integer and $\eta > 0$.

At times τ_j given by

$$\tau_j = \left(s + \frac{j}{m}\eta\right) \wedge \tau : \quad j = 1, 2, ..., m$$

an amount of resources $\triangle\tilde{\psi}(\tau_j)$ given by

$$\triangle\tilde{\psi}(\tau_j) := \tilde{\xi}_j = \frac{1}{m}x$$

is taken out. This gives the expected total value of harvested resources

$$J^{(\tilde{w})}(s,x) = E^{s,x}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(X^{(\tilde{w})}(\tau_j^-))^+]^{-\frac{1}{2}}\right]\tilde{\xi}_j,$$

where $x^+ = \max\{x, 0\}$ for $x \in \mathbb{R}$.

We may present this as

$$J^{(\tilde{w})}(s,x) = E^{s,x}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(x - (1+\delta)\xi_j - c)^+]^{-\frac{1}{2}}\right]\tilde{\xi}_j.$$

Letting $\eta \to 0$ we realize that $\tau_j \to s$ for $j = 1, 2, ..., m$ and we get

$$\begin{aligned} J^{(\tilde{w}(m,0))}(s,x) : &= \lim_{\eta\to 0} J^{(\tilde{w}(m,\eta))}(s,x) \\ &= \lim_{\eta\to 0} E^{(s,x)}\left[\sum_{j=1}^{m} e^{-\rho\tau_j}[(x - \frac{j}{m}(1+\delta)x - c)^+]^{-\frac{1}{2}}\right]\frac{x}{m} \\ &= e^{-\rho s}\sum_{j=1}^{m} h(x_j)\triangle x_j. \end{aligned}$$

where $h(y) = [(x - (1+\delta)y - c)^+]^{-\frac{1}{2}}$, $x_j = \frac{jx}{m}$ and $\triangle x_j = x_{j+1} - x_j = \frac{x}{m}$.

Given $\epsilon > 0$ there exists a positive integer m such that

$$e^{-\rho s} \mid \int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}} dy - \sum_{j=1}^{m} h(x_j)\triangle x_j \mid < \epsilon.$$

By making an appropriate choice of m and η we obtain the following

$$\mid J^{(\tilde{w}(m,\eta))}(s,x) - e^{-\rho s} \int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}} dy \mid < \epsilon.$$

We conclude that

$$\lim_{\substack{m\to\infty \\ \eta\longrightarrow 0}} J^{\tilde{w}}(s,x) = e^{-\rho s} \int_0^x [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}} dy = \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}\,.$$

We call this "chattering policy" of applying $\tilde{w}^{(m,\eta)}$ in the limit as $\eta \to 0$ and $m \to \infty$ the *policy of immediate chattering down to 0.*

Let us now investigate whether the function

$$\phi(s,x) := \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}$$

satisfies the conditions of Theorems 3.1 and 3.2.

For Theorem 3.1, condition (i) holds since the function $\phi(s,x)$ is differentiable on S and continuous on the closure of S whenever $x-c>0$. Moreover,

$$\begin{aligned} \mathcal{M}\phi &= \sup_{\xi}\Big\{\phi(\Gamma(s,x,\xi)) : \quad 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\ &= \frac{2e^{-\rho s}}{1+\delta}\sup_{\xi}\Big\{\sqrt{x-(1+\delta)\xi - c} : \quad 0 \le \xi \le \frac{x-c}{1+\delta}\Big\} \\ &\le \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c} = \phi(s,x,). \end{aligned}$$

Hence, $\phi(s,x)$ satisfies condition (ii). We also have

$$\begin{aligned} \sum_{i=1}^{k}\kappa_{ie}(y)\frac{\partial\phi}{\partial y_i}(y) + \theta_e(y) &= -(1+\delta).\frac{2e^{-\rho s}}{1+\delta}.\frac{d}{dx}\Big[(x-c)^{\frac{1}{2}}\Big] + e^{-\rho s}x^{-\frac{1}{2}} \\ &= -e^{-\rho s}\Big[-(x-c)^{-\frac{1}{2}} + x^{-\frac{1}{2}}\Big] \\ &\le e^{-\rho s}\Big[-\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{x}}\Big] \\ &= 0. \end{aligned}$$

This proves that $\phi(s,x)$ satisfies condition (iii).

Using the second-order integro-partial-differential operator

$$\begin{aligned} \mathcal{L}\phi(s,x) &= \frac{\partial\phi}{\partial s} + \mu\frac{\partial\phi}{\partial x} + \frac{1}{2}\sigma^2\frac{\partial^2\phi}{\partial x^2} \\ &\quad + \int_{\mathbb{R}}\Big\{\phi(s,x+\beta z) - \phi(s,x) - \beta z\frac{\partial\phi}{\partial x}\Big\}\nu(dz), \end{aligned}$$

we obtain

$$\begin{aligned}\mathcal{L}\phi(s,x) &= \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}}+\mu(x-c)^{-\frac{1}{2}}-\frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\ &\quad +\int_{\mathbb{R}}\{2\sqrt{x+\beta z-c}-2(x-c)^{\frac{1}{2}}-\beta z(x-c)^{-\frac{1}{2}}\}\nu(dz)\Big] \\ &\le \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}}+\mu(x-c)^{-\frac{1}{2}}-\frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\ &\quad +\int_{\mathbb{R}}\{2\sqrt{(x-c)}-2(x-c)^{\frac{1}{2}}-\beta z(x-c)^{-\frac{1}{2}}\}\nu(dz)\Big] \\ &= \frac{e^{-\rho s}}{1+\delta}\Big[-2\rho(x-c)^{\frac{1}{2}}+\mu(x-c)^{-\frac{1}{2}}-\frac{1}{4}\sigma^2(x-c)^{-\frac{3}{2}} \\ &\qquad -\int_{\mathbb{R}}\beta z(x-c)^{-\frac{1}{2}}\nu(dz)\Big].\end{aligned}$$

We have applied the fact that $\beta z \le 0$. Thus

$$\begin{aligned}&\mathcal{L}\phi(s,x) \\ &\le \frac{-2\rho e^{-\rho s}}{1+\delta}(x-c)^{-\frac{3}{2}}[(x-c)^2-\frac{\mu}{2\rho}(x-c)+\frac{\sigma^2}{8\rho}+(x-c)\int_{\mathbb{R}}\beta z\nu(dz)] \\ &= \frac{-2\rho e^{-\rho s}}{1+\delta}(x-c)^{-\frac{3}{2}}\Big[(x-c)^2+\Big(\int_{\mathbb{R}}\beta z\nu(dz)-\frac{\mu}{2\rho}\Big)(x-c)+\frac{\sigma^2}{8\rho}\Big].\end{aligned}$$

So, condition (vii) holds if $x \ge c$ and

$$\Big(\int_{\mathbb{R}}\beta z\nu(dz)-\frac{\mu}{2\rho}\Big)^2 \le \frac{\sigma^2}{2\rho}.$$

The preceding results can now be summarized in the following theorem:

Theorem 4.1. *Let $X^{(w)}(t)$ be given by (4.2) – (4.4).*

(1) *Assume that $x \ge c$ and*

$$\left(\int_{\mathbb{R}}\beta z\nu(dz)-\frac{\mu}{2\rho}\right)^2 \le \frac{\sigma^2}{2\rho}. \tag{4.5}$$

Then the value function is given by

$$\Phi(s,x)=\frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}, \tag{4.6}$$

where σ and ρ are defined as before. This function is achieved in the limit if we apply the strategy $\tilde{w}^{(m,\eta)}$ described above with $\eta \to 0$ and $m \to \infty$, that is, by applying the policy of immediate chattering to 0.

(2) *Assume that $x^* \ge c$ and*

$$\left(\int_{\mathbb{R}}\beta z\nu(dz)-\frac{\mu}{2\rho}\right)^2 > \frac{\sigma^2}{2\rho}. \tag{4.7}$$

Then the value function is

$$\Phi(s,x)=\begin{cases} e^{-\rho s}A(e^{r_1x}-e^{r_2x}), & \textit{for } 0\le x<x^* \\ e^{-\rho s}(\frac{2}{1+\delta}\sqrt{x-c}-\frac{2}{1+\delta}\sqrt{x^*-c}+B), & \textit{for } x^*\le x \end{cases} \tag{4.8}$$

for some constants $A>0$, $B>0$ and $x^*>0$, where r_1 and r_2 are the solutions of the equation

$$-\rho+\mu r+\frac{1}{2}\sigma^2r^2+\int_{\mathbb{R}}\{e^{r\beta z}-1-r\beta z\}\nu(dz)=0, \tag{4.9}$$

with $r_2<0<r_1$ and $|r_2|>r_1$.

Here, the optimal policy is as follows:

(i) *If $x>x^*$, it is optimal to apply immediate chattering from x down to x^*.*

(ii) *If $0<x<x^*$, it is optimal to apply the harvesting equal to the local time of the downward reflected process $\bar{X}(t)$ at x^*.*

Proof. We need to show that the proposed value function satisfies all the conditions of Theorem 3.1. Let us first examine the case

$$\left(\int_{\mathbb{R}}\beta z\nu(dz)-\frac{\mu}{2\rho}\right)^2\le\frac{\sigma^2}{2\rho}.$$

In this case we have

$$\phi(s,x)=\frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}.$$

From the construction of $\phi(s,x)$ we can see that conditions $(i)-(iii)$ and (vii) are satisfied.

Since $X(t)$ spends no time on ∂D, then $\chi_{\partial D}X(t)=0$ $a.e$ and this leads to

$$E^y\Big[\int_0^\tau\chi_{\partial D}(Y^{(v)}(t))dt\Big]=0 \quad \text{for all } y\in\mathcal{S},\ v\in\mathcal{V}.$$

So, condition (iv) is satisfied.

In this example the boundary ∂D, of the non-intervention region D, is given by

$$\partial D=\partial D_1\cup\partial D_2$$

where $\partial D_1=\{0\}$ and $\partial D_2=\{x^*\}$. However, ∂D_1 and ∂D_2 are both Lipschitz surfaces since each of them is a singleton consisting of a constant. Hence, ∂D is also a Lipschitz surface. Thus $\phi(s,x)$ satisfies condition (v) .

For $x>c$ it can easily be verified that the function

$$\phi(s,x)=\frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}$$

is twice continuously differentiable on $\mathcal{S}\setminus\partial D$ and none of its derivatives explodes near ∂D. This establishes part (vi) of Theorem 3.1.

Recalling that $g\Big(Y^w(\tau)\Big)=0$ we note that

$$\lim_{t\to\tau}\phi(t,x)=\lim_{t\to\tau}\frac{2e^{-\rho t}}{1+\delta}\sqrt{x-c}=0=g\Big(Y^w(\tau)\Big)\quad as\quad \tau\to\infty.$$

This proves that (vii) is satisfied.

Up to this point we have proved that $\phi(s,x) \geq \Phi(s,x)$. Now, by construction of $\phi(s,x)$ we observe that

$$\mathcal{L}^{\hat{\psi}(y)}\phi + f(y, \hat{\psi}(y)) = 0 \quad for \;\; all \;\; y \in D.$$

Next, for Theorem 3.2, the preceding equation shows that condition (i) is satisfied. Conditions $(ii) - (vi)$ can be verified using similar arguments as in Example 2.14 of [8]. Thus

$$\phi(s,x) = \frac{2e^{-\rho s}}{1+\delta}\sqrt{x-c}$$

satisfies all the requirements of Theorem 3.2. Hence it is a value function for the given problem. This completes the proof of part 1.

We now turn to the proof of part 2, and suppose that

$$\left(\int_{\mathbb{R}} \beta z \nu(dz) - \frac{\mu}{2\rho}\right)^2 > \frac{\sigma^2}{2\rho}.$$

We need to show that the function $\phi(s,x)$ given by

$$\phi(s,x) = \begin{cases} e^{-\rho s} A(e^{r_1 x} - e^{r_2 x}), & \text{for } \; 0 \leq x < x^* \\ e^{-\rho s}\left(\frac{2}{1+\delta}\sqrt{x-c} - \frac{2}{1+\delta}\sqrt{x^*-c} + B\right), & \text{for } \; x^* \leq x \end{cases}$$

also satisfies the conditions of Theorems 3.1 and 3.2 where $A,\ B,\ x^*,\ r_1,\ r_2$ are as specified in Theorem 4.1.

To this end, we follow closely arguments used to prove part (b) of Theorem 3.2 in [2], with the necessary extension to cater for the jump component as well as transaction costs. First, we observe that if we apply the policy of immediate chattering from x to x^* where $0 < x^* < x$, then the value of the dividends paid out is given by

$$\begin{aligned} & e^{-\rho s}\int_0^{x-x^*} [(x-(1+\delta)y-c)^+]^{-\frac{1}{2}} dy \\ & = \frac{2e^{-\rho s}}{1+\delta}\left[\sqrt{x-c} - \sqrt{((1+\delta)x^* - \delta x - c)^+}\right]. \end{aligned}$$

This follows by the argument presented above in (4.5)-(4.9) and $\phi(s,x)$ given above. the given . To verify the conclusions of part 2 of Theorem 4.1 we observe that r_1 and r_2 are the roots of the equation

$$-\rho + \mu r + \frac{1}{2}\sigma^2 r^2 + \int_{\mathbb{R}} \{e^{r\beta z} - 1 - r\beta z\}\nu(dz) = 0.$$

Hence, by defining $\phi(s,x)$ as in (4.21) it is relatively easy to show that for $x < x^*$

$$\mathcal{L}\phi(s,x) = 0$$

and

$$\phi(s,0) = 0.$$

Combining the smooth contact principle and the requirement that $\phi(s,x)$ be C^2 at $x = x^*$, we obtain the following three equations

$$A(e^{r_1 x^*} - e^{r_2 x^*}) = B \tag{4.10}$$

$$A(r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}) = (x^*)^{-\frac{1}{2}} \tag{4.11}$$

$$A(r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}) = -\frac{1}{2}(x^*)^{-\frac{3}{2}} \tag{4.12}$$

Dividing (4.12) by (4.12) we obtain the equation

$$\frac{r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}}{r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}} = -2x^*. \tag{4.13}$$

Now, observing that

$$\lim_{x^* \to 0} \frac{r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}}{r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}} = \frac{1}{r_1 + r_2} < 0$$

and

$$\lim_{x^* \to \infty} \frac{r_1 e^{r_1 x^*} - r_2 e^{r_2 x^*}}{r_1^2 e^{r_1 x^*} - r_2^2 e^{r_2 x^*}} = \frac{1}{r_1} > 0,$$

the intermediate value theorem guarantees the existence of x^* satisfying equation (4.13). With this value of x^* we define A by (4.12) and B by (4.10). This proves the existence of a solution of the system (4.10)-(4.12) where $A > 0,\ B > 0,\ x^* > 0$. With this choice of $A > 0,\ B > 0,\ x^* > 0$ the function $\phi(s,x)$ is C^2 and we can easily verify that ϕ satisfies all the conditions in Theorems 3.1 and 3.2. Hence,

$$\phi(s,x) \geq \Phi(s,x) \qquad \text{for all } s, x. \tag{4.14}$$

Moreover, the non-intervention region D is identified to be

$$D = \{(s,x) :\ 0 < x < x^*\}.$$

In particular, since

$$\phi(s,x) = \begin{cases} e^{-\rho s} A(e^{r_1 x} - e^{r_2 x}), & \text{for } 0 \leq x < x^* \\ e^{-\rho s}\Big(\frac{2}{1+\delta}\sqrt{x-c} - \frac{2}{1+\delta}\sqrt{x^*-c} + B\Big), & \text{for } x^* \leq x, \end{cases}$$

we know that condition (i) of Theorem 3.1 holds.

Additionally, it is an established fact that the local time $\hat{\psi}$ of the downward reflected process $\bar{X}(t)$ at x^* satisfies conditions $(ii) - (vii)$ ([2, 12] and references therein).

On the basis of Theorem 3.2, we conclude that if $x \leq x^*$ then $\psi^* := \hat{\psi}$ is optimal and $\phi(s,x) = \Phi(s,x)$. Finally, if $x > x^*$ then it follows by the above choice of $\phi(s,x)$ that immediate chattering from x to x^* gives the value

$$\Phi(s,x) \geq e^{-\rho s}[\sqrt{x}(1-\delta) - cx^{-\frac{1}{2}}] + \Phi(s,x^*) \qquad \text{for all } x > x^*$$

Combining with equation (4.14), this proves that

$$\phi(s,x) = \Phi(s,x) \qquad \text{for all } (s,x)$$

and the proof of part 2 of Theorem 4.1 is complete. □

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