Communications on Stochastic Analysis
Vol. 5, No. 4 (2011) 633-645
Serials Publications
www.serialspublications.com

CONSISTENT PRICE SYSTEMS FOR BOUNDED PROCESSES

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ABSTRACT. In a recent paper by Guasoni, Rásonyi, and Schachermayer [9], the conditional full support (CFS) condition is introduced. It is shown that the CFS property of continuous processes in the state space $(0, \infty)$ implies the existence of consistent price systems (CPSs), a tool that plays the role of a martingale measure for markets with proportional transaction costs. In this paper we generalize this result and show that the CFS property implies CPSs for general (bounded and unbounded) continuous processes. To do this, we give an equivalent formulation of the CFS property and by using it we construct CPSs for general continuous processes. In particular, our equivalent formulation of the CFS property enables us to give simpler proofs for some of the results that are discussed in [9].

1. Introduction

Consider a financial market with one risk-free asset with price process B_t and a risky asset with price process Y_t which is assumed to be continuous. For simplicity assume $B_t \equiv 1$, which corresponds to taking the bond price as a numéraire. The price processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ that satisfies the usual conditions of right-continuity and saturatedness. Assume $\mathcal{F}_0$ is trivial and $\mathcal{F}_T = \mathcal{F}$.

In case the market $(1, Y)$ is frictionless, the existence of an equivalent martingale measure for Y solves both hedging and arbitrage problems in the market (see [6]). In contrast, if there are proportional transaction costs in the market $(1, Y)$, a much more relaxed condition on Y than requiring that Y admits a martingale measure is sufficient to handle hedging and arbitrage problems in the market $(1, Y)$, as demonstrated in the recent papers [9, 10]. More specifically, the authors showed that if Y can be approximated by martingales (under equivalent changes of measure) under the supremum norm, then the market $(1, Y)$ is arbitrage-free (see Theorem 1.11 of [10]). Also, if such approximating martingales exist, then the superreplication price of any European-style vanilla contingent claims tend to the value of the convex envelope of the contract function at the initial price as the size ϵ of the proportional transaction costs goes to zero.

These approximating martingales are called consistent price systems (CPSs). The concept of CPSs dates back to the seminal paper [11]. The name "consistent price system" appeared first in [17]. Other papers that studied CPSs include

Received 2011-4-20; Communicated by F. Viens.
2000 *Mathematics Subject Classification.* Primary 60G99; Secondary 62P05.
Key words and phrases. Consistent pricing systems, no-arbitrage, transaction costs, full support, conditional full support.

[5, 12, 13, 14, 2, 17, 4, 15, 10]. Here we recall the definition of CPSs. A strictly positive process Y_t admits an ϵ–CPS for $\epsilon > 0$ if there exists an equivalent measure $\tilde{P} \sim P$ and a $(\mathbb{F}, \tilde{P})$-martingale $\tilde{Y}_t$ such that

$$\frac{1}{1+\epsilon} \leq \sup_{t\in[0,T]} \frac{Y_t}{\tilde{Y}_t} \leq 1+\epsilon \ \text{ a.s.}$$

As stated earlier, CPSs are the main tool in solving hedging and arbitrage problems in markets with proportional transaction costs. Therefore it is crucial to study their existence.

After [9] reported the CFS condition as a sufficient condition for CPSs, a number of subsequent papers studied the CFS property for certain class of processes. For example, [3] showed the CFS property for general Gaussian moving average processes, [7] studied the CFS property of Gaussian moving average processes with stationary increments, and [16] showed the CFS property for some stochastic volatility models.

Here we recall the CFS property. To this end, let X_t be an adapted continuous process that takes values in the open interval (a, b). Let $C_x([\mu, \nu], (a, b))$ be the space of continuous functions defined on $[\mu, \nu]$ and taking values in (a, b) with $f(\mu) = x$. As usual, this space is endowed with the uniform topology. We say that the continuous process X_t has the CFS property in $C_x([0, T], (a, b))$ if

$$\operatorname{supp} \mathrm{Law}(X_\theta; t \leq \theta \leq T | \mathcal{F}_t) = C_{X_t}([t, T], (a, b)) \ \text{ a.s.},$$

where "supp" denotes the support (the smallest closed set of probability one). It is clear that the CFS property is invariant under composition with homeomorphisms. Namely, if $F : (a, b) \to (c, d)$ is a homeomorphism and X has the CFS property in $C_x([0, T], (a, b))$ then the process $F(X)$ has the CFS property in $C_{F(x)}([0, T], (c, d))$.

In this paper, we show that if a continuous process X that takes values in the interval (a, b) has the CFS property then for any $\epsilon > 0$ there exists a martingale M_t (under an equivalent change of measure) such that $\sup_{t\in[0,T]} |X_t - M_t| \leq \epsilon$. By abuse of language we will call such M an $\epsilon-$CPS for X. As in [9], the proof of this result will be based on an approximation of X with a discrete process (random walk with retirement; see Definition 2.3 of [9]). We should mention that in the case the process X_t is strictly positive, the proofs of the current paper can be adapted to show the existence of a strictly positive martingale M (under an equivalent change of measure) such that

$$\frac{1}{1+\epsilon} \leq \frac{M_t}{X_t} \leq 1+\epsilon \ \text{ a.s.}$$

In the rest of the paper we will assume that $X_0 = 0$ and X_t takes values in an open interval (a, b). The general case follows from this one by translation. We should mention that our approach in this paper works also for the cases $a = -\infty$ and/or $b = +\infty$.

We need the following definition. The condition in this definition is an equivalent formulation of the CFS property. The proof for this fact will be given in Section 5.

Definition 1.1. We say that the continuous adapted process X_t is *weak f–sticky in* (a,b), for $f \in C_0([0,T])$, if

$$P\left(\left\{\sup_{t\in[s,T]} |X_t - X_s - f(t-s)| < \epsilon\right\} |\mathcal{F}_s\right) > 0 \text{ a.s.}$$

for any $0 \le s \le T$, $\epsilon > 0$, on the set $\left\{a - \inf_{[0,T-s]} f < X_s < b - \sup_{[0,T-s]} f\right\}$.

Here we state the main result of this paper.

Theorem 1.2. *Let X_t be an adapted continuous process that takes values in (a,b). If X_t is weak f-sticky in (a,b) for all $f \in C_0([0,T])$, then X_t admits an $\epsilon-$CPS for all $\epsilon > 0$.*

So far a large class of models are shown to have the CFS property in $C_0([0,T], (-\infty,+\infty))$, see [7, 16, 3] for example. If we compose models with CFS in $C_0([0,T],(-\infty,+\infty))$ with any homeomorphism $h : (-\infty,+\infty) \to (a,b)$, we get models that still satisfy the conditions in Definition 1.1. Therefore we can give the following example.

Example 1.3. Let B_t^H be a fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then the process $e^{\arctan(B_t^H)}$ admits $\epsilon-$CPS for all $\epsilon > 0$.

We should mention that the recent paper [2] introduced a weaker sufficient condition for the existence of CPSs than the CFS property: see Theorem 1 in [2]. By using this weaker condition, [2] gave examples of processes that do not have the CFS property but admit CPSs. The criterion introduced in [2] is for continuous processes with state space $(-\infty,+\infty)$. Our current paper deals with CPSs for general price processes with the CFS property.

2. Sticky Processes

The stickiness condition is reported in the recent paper [8]. Following the definition of [8], an adapted process X_t is sticky if

$$P\left(\sup_{t\in[\tau,T]} |X_t - X_\tau| < \epsilon, \tau < T\right) > 0 \tag{2.1}$$

for any stopping time τ with $P(\tau < T) > 0$ and any $\epsilon > 0$. See also [1] for equivalent definitions of stickiness and some related results. Next we introduce a condition which is an equivalent formulation of the strong CFS property in [9]. The stickiness condition corresponds to $0-$stickiness in this definition.

Definition 2.1. We say that X is *f–sticky in* (a,b), for $f \in C_0([0,T])$, if for any $\epsilon > 0$ and any stopping time τ

$$P\left(\sup_{t\in[\tau,T]} |X_t - X_\tau - f(t-\tau)| < \epsilon |\mathcal{F}_\tau\right) > 0$$

on the set $\left\{a - \inf_{[0,T-\tau]} f < X_\tau < b - \sup_{[0,T-\tau]} f\right\}$.

The following lemma establishes the equivalence of $f-$stickiness with weak $f-$stickiness for each f. In particular, this lemma gives an alternative proof for the equivalence of CFS and the strong CFS conditions as discussed in [9].

Lemma 2.2. *(CFS $\Leftrightarrow$ Strong CFS) For each $f \in C_0([0,T])$, weak $f-$stickiness of X_t in (a,b) is equivalent to $f-$stickiness of X_t in (a,b).*

Proof. It is clear that $f-$stickiness implies weak $f-$stickiness. In the following, we will show that weak $f-$stickiness implies $f-$stickiness. Suppose for a contradiction that X_t is weak f–sticky but not f–sticky. Then there exists a stopping time τ with $P(\tau < T) > 0$, and an $\epsilon > 0$ such that

$$P\left(\tau < T, a - \inf_{[0,T-\tau]} f < X_\tau < b - \sup_{[0,T-\tau]} f\right) > 0$$

and

$$P\left(\sup_{t\in[\tau,T]} |X_t - X_\tau - f(t-\tau)| < \epsilon, \tau < T\right) = 0. \tag{2.2}$$

Since $f \in C_0[0,T]$, there exists a $\delta > 0$ such that for all $t, s \in [0,T]$, $|t-s| < \delta$ implies $|f(t) - f(s)| < \epsilon/3$. In addition, we can find $t_1, t_2 \in [0,T)$, $0 < t_2 - t_1 < \delta$, and $0 < \zeta \le \epsilon/3$ such that

$$P\left(t_1 \le \tau < t_2, a + \zeta - \inf_{[0,T-\tau]} f < X_\tau < b - \zeta - \sup_{[0,T-\tau]} f\right) > 0.$$

Set $I := \mathbb{Q} \cap [t_1, t_2)$ and

$$A := \left\{t_1 \le \tau < t_2, a + \zeta - \inf_{[0,T-\tau]} f < X_\tau < b - \zeta - \sup_{[0,T-\tau]} f\right\}.$$

For each $q \in I$, let $A_q := A \cap \big\{t_1 \le \tau < q\big\} \cap \big\{\sup_{t\in[\tau,q]} |X_t - X_\tau| < \zeta\big\}$.

It is clear that $A_q \in \mathcal{F}_q$ and $A = \cup_{q\in I} A_q$ (X_t has continuous paths). Since $P(A) > 0$, there exists a $q^* \in I$ such that $P(A_{q^*}) > 0$. Note that $A_{q^*} \subset \big\{a - \inf_{[0,T-q^*]} f < X_{q^*} < b - \sup_{[0,T-q^*]} f\big\}$. Hence

$$P\left(A_{q^*} \cap \left\{a - \inf_{[0,T-q^*]} f < X_{q^*} < b - \sup_{[0,T-q^*]} f\right\}\right) > 0.$$

Since X_t is weak f–sticky, we obtain

$$P\left(A_{q^*} \cap \left\{\sup_{t\in[q^*,T]} |X_t - X_{q^*} - f(t-q^*)| < \epsilon/3\right\}\right) > 0.$$

Set $C_{q^*} = A_{q^*} \cap \big\{\sup_{t\in[q^*,T]} |X_t - X_{q^*} - f(t-q^*)| < \epsilon/3\big\}$. We claim that

$$C_{q^*} \subset \left\{\sup_{t\in[\tau,T]} |X_t - X_\tau - f(t-\tau)| < \epsilon\right\} \cap \{\tau < T\}.$$

This contradicts equation (2.2). Indeed, if $\omega \in C_{q^*}$, then for $t \in [\tau, q^*]$ we have

$$|X_t - X_\tau - f(t-\tau)| < |X_t - X_\tau| + |f(t-\tau)| < \epsilon/3 + \epsilon/3 < \epsilon$$

by the definition of A_{q^*} and the uniform continuity of f. Next we show that $|X_t - X_\tau - f(t-\tau)| < \epsilon$ on C_{q^*} whenever $t \in [q^*, T]$. This can easily be obtained as follows:

$$\begin{aligned}|X_t - X_\tau - f(t-\tau)| &\leq |X_t - X_\tau - f(t-\tau) - X_t + X_{q^*} + f(t-q^*)| \\ &\quad + |X_t - X_{q^*} - f(t-q^*)| \\ &\leq |X_\tau - X_{q^*}| + |f(t-q^*) - f(t-\tau)| \\ &\quad + |X_t - X_{q^*} - f(t-q^*)| \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon,\end{aligned}$$

which completes the proof. □

We remark that, while the f-stickiness is equivalent to the weak f-stickiness as the above lemma shows, the weak f-stickiness is easier to check on the concrete processes while the f-stickiness is convenient for the proof of existence of CPSs.

This lemma allows us to state the following corollary about the stickiness property of continuous processes.

Corollary 2.3. *(Stickiness) A continuous process Z_t is sticky as in* [8] *if and only if Z_t satisfies*

$$P\left(\sup_{t\in[0,T-s]} |Z_{s+t} - Z_s| < \epsilon | \mathcal{F}_s\right) > 0, \quad a.s.$$

for all deterministic s.

3. Random Walk With Retirement

Following the definition in [9], a random walk with retirement on the geometric grid is a process $(X_n)_{n\geq 0}$ adapted to a filtration $(\mathcal{G}_n)_{n\geq 0}$ and of the form $X_n = X_0(1+\epsilon)^{\sum_{i=1}^n R_i}$, where $\epsilon > 0, X_0 > 0$ are constants and the process $(R_n)_{n\geq 1}$ takes value in $\{-1, 0, +1\}$ and satisfies

$$\begin{aligned} &(i) && P(R_m = 0 \text{ for all } m \geq n | R_n = 0) = 1, \\ &(ii) && P(R_n = x|\mathcal{G}_{n-1}) > 0 \text{ on } \{R_{n-1} \neq 0\} \text{ for all } x \in \{0, \pm 1\} \text{ and } n \geq 1, \\ &(iii) && P(R_n \neq 0 \text{ for all } n \geq 1) = 0. \end{aligned} \quad (3.1)$$

Similarly, one can define random walk with retirement on the arithmetic grid to be a process of the form $X_n = X_0 + \sum_{i=1}^n \epsilon R_n$. In the paper [9], random walks with retirement play a crucial role in the construction of CPSs. In this paper, we also use random walk with retirement to construct CPSs. However, since we consider general processes that can be bounded, we need to revise the corresponding definition of a random walk with retirement accordingly. More specifically, the random walk with retirement defined in this section involves random jumps $\{\epsilon_i, i \geq 0\}$ (see below) instead of the deterministic jumps ϵ used in [9]. The random jumps in our setting are necessary since we are dealing with possibly bounded processes.

For fixed $\epsilon > 0$, we define the following sequence of stopping times associated with the process X:

$$\tau_0 = 0, \quad \tau_{n+1} = \inf_{t \geq \tau_n} \{|X_t - X_{\tau_n}| \geq \epsilon_{n+1}\} \wedge T, \quad \text{for all } n \geq 0,$$

where $\epsilon_{n+1} := \epsilon \wedge \frac{|b-X_{\tau_n}|}{2} \wedge \frac{|X_{\tau_n}-a|}{2}$.

In addition, for each $n \geq 1$ we define

$$R_n = \begin{cases} \operatorname{sign}(X_{\tau_n} - X_{\tau_{n-1}}) & \text{when } \tau_n < T, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Let $\mathcal{G}_n = \mathcal{F}_{\tau_n}$ for every $n \geq 0$. Note that ϵ_n is bounded and $\mathcal{G}_{n-1}$ measurable.

Lemma 3.1. *If X_t is weak f–sticky in (a,b) for all $f \in C_0([0,T])$, then the discrete process $\{R_n\}$ in (3.2) satisfies (3.1).*

Proof. Property (i) is clear. Property (iii) follows from the fact that $\min_n \epsilon_n(\omega) > 0$ for almost all ω. Note that this is true since almost surely each path of X_t is contained in a compact set in (a,b). To prove property (ii), let us assume that $P(\tau_{n-1} < T) > 0$. Let G_{n-1} be a $\mathcal{G}_{n-1}$ measurable set such that $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$. Then there exist $T' < T$ and $a < a' < b' < b$ such that $b' - a' < \epsilon \wedge \frac{b-b'}{2} \wedge \frac{a'-a}{2}$ and $P\left(G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in [a',b']\}\right) > 0$. It can be easily shown that on the set $G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in [a',b']\}$, the inequality $\epsilon \wedge \frac{b-b'}{2} \wedge \frac{a'-a}{2} \leq \epsilon_n < \frac{3}{4}(b-b')$ holds. Define the stopping time

$$\tau = \begin{cases} \tau_{n-1} & \text{on } \; G_{n-1} \cap \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in [a',b']\}, \\ T & \text{otherwise.} \end{cases}$$

Since X is 0–sticky, for $0 < \epsilon' < (b'-a')/4$ we have

$$P\left(\sup_{t \in [\tau,T]} |X_t - X_\tau| < \epsilon', \tau < T\right) > 0.$$

Observe that $\left\{\sup_{t\in[\tau,T]} |X_t - X_\tau| < \epsilon', \tau < T\right\} \subset \{R_n = 0\}$. Since G_{n-1} was an arbitrary $\mathcal{G}_{n-1}$ measurable set with $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$, we obtain

$$P(R_n = 0|\mathcal{G}_{n-1}) > 0 \text{ on} \{R_{n-1} \neq 0\}.$$

Define

$$f(t) = \begin{cases} \dfrac{7(b-b')}{4(T-T')}t, & \text{if } 0 \leq t \leq \frac{T-T'}{2}, \\ \dfrac{7(b-b')}{8}, & \text{otherwise,} \end{cases}$$

and note that

$$P\left(\tau < T, a - \inf_{[0,T-\tau]} f < X_\tau < b - \sup_{[0,T-\tau]} f\right) > 0.$$

By f–stickiness of X we obtain

$$P\left(\sup_{t\in[\tau,T]} |X_t - X_\tau - f(t-\tau)| < \epsilon', \tau < T\right) > 0,$$

or equivalently $P(A \cap G_{n-1}) > 0$, where

$$A = \begin{array}{l} \left\{\sup_{t\in[\tau_{n-1},T]} |X_t - X_{\tau_{n-1}} - f(t-\tau_{n-1})| < \epsilon'\right\} \cap \\ \{\tau_{n-1} < T'\} \cap \{X_{\tau_{n-1}} \in [a', b']\}. \end{array}$$

We claim that $A \subset \{R_n = 1\}$. Indeed, if $\omega \in A$, we get

$$\begin{aligned} \left|X_{\tau_{n-1}+\frac{T-T'}{2}}(\omega) - X_{\tau_{n-1}}(\omega)\right| \geq f\left(\frac{T-T'}{2}\right) - \\ \left|X_{\tau_{n-1}+\frac{T-T'}{2}}(\omega) - X_{\tau_{n-1}}(\omega) - f\left(\frac{T-T'}{2}\right)\right| \\ \geq \frac{7}{8}(b-b') - \epsilon' \\ > \frac{3}{4}(b-b') > \epsilon_n(\omega). \end{aligned}$$

Hence by continuity of the sample paths we conclude that $\{\tau_n < T\}$ on A. Also, for $\omega \in A$ and for all $t \in [\tau_{n-1}(\omega), T]$ we have

$$\begin{aligned} X_t(\omega) &\geq |X_{\tau_{n-1}}(\omega) + f(t-\tau_{n-1}(\omega))| - |X_t(\omega) - X_{\tau_{n-1}}(\omega) - f(t-\tau_{n-1}(\omega))| \\ &> X_{\tau_{n-1}}(\omega) - \epsilon' > X_{\tau_{n-1}}(\omega) - \epsilon_n(\omega). \end{aligned}$$

The last inequality yields

$$X_{\tau_n}(\omega) > X_{\tau_{n-1}}(\omega) - \epsilon_n(\omega)$$

for all $\omega \in A$, thus $A \subset \{R_n = 1\}$.

Since G_{n-1} was an arbitrary $\mathcal{G}_{n-1}$ measurable set with $P(G_{n-1} \cap \{\tau_{n-1} < T\}) > 0$, we conclude that

$$P(R_n = 1|\mathcal{G}_{n-1}) > 0 \text{ on } \{R_{n-1} \neq 0\}.$$

By an argument similar to the one outlined above we obtain

$$P(R_n = -1|\mathcal{G}_{n-1}) > 0$$

on the set $\{R_{n-1} \neq 0\}$. □

4. Consistent Price Systems

In this section, we construct CPSs for X. Define $\epsilon_n, R_n, n \geq 0$ as in the previous section and let $M_n = X_0 + \sum_{i=1}^n \epsilon_i R_i, n \geq 0$. First we show that $\{M_n\}$ admits an equivalent measure that makes it a uniformly integrable martingale and then we use it to construct CPSs. As stated in the previous section, our setting is complicated by the fact that we might deal with random jumps. However, the proofs of the following lemmas are similar to the corresponding ones in [9]. We include them below for the sake of completeness.

Lemma 4.1. *Suppose X_t satisfies the conditions of Theorem 1.2. Then there exists a measure Q equivalent to P under which the discrete process $\{(M_n, \mathcal{G}_n)\}_{n=0}^{+\infty}$ is a martingale.*

Proof. Define

$$Z_n = \frac{\chi_{\{R_n=0\cap R_{n-1}\neq 0\}}}{3P(R_n=0|\mathcal{G}_{n-1})} + \frac{\chi_{\{R_n=-1\cap R_{n-1}\neq 0\}}}{3P(R_n=-1|\mathcal{G}_{n-1})} + \frac{\chi_{\{R_n=1\cap R_{n-1}\neq 0\}}}{3P(R_n=1|\mathcal{G}_{n-1})} + \chi_{\{R_{n-1}=0\}}.$$

It is easy to check that Z_n satisfies

$$E(Z_n|\mathcal{G}_{n-1}) = 1, \tag{4.1}$$

and

$$E(Z_nR_n|\mathcal{G}_{n-1}) = 0. \tag{4.2}$$

Let $L_n = \prod_{i=1}^n Z_i$ and $L = \lim_{n\to\infty} L_n$. Note that this limit exists almost surely since $L_{n+1} = L_n$ a.s. on $\{R_n = 0\}$ and $\{R_n = 0\} \nearrow \Omega$. From (4.1) and (4.2), we get

$$E(L_n|\mathcal{G}_{n-1}) = L_{n-1}$$
$$E(L_nM_n|\mathcal{G}_{n-1}) = L_{n-1}M_{n-1},$$

which shows that $(L_n)_{n\geq 1}$ and $(M_nL_n)_{n\geq 1}$ are martingales under P. We thus get $E(L_n) = E(Z_1) = 1$, and Fatou's lemma gives $E(L) \leq 1$. We now show that $E(L) \geq 1$, which combined with the previous inequality yields $E(L) = 1$. We have

$$\begin{aligned} E(L) &= E\left(\lim_{n\to\infty} L\chi_{\{R_n=0\}}\right) = \lim_{n\to\infty} E\left(L\chi_{\{R_n=0\}}\right) = \lim_{n\to\infty} E\left(L_n\chi_{\{R_n=0\}}\right) \\ &= 1 - \lim_{n\to\infty} E\left(L_n\chi_{\{R_n\neq 0\}}\right) = 1 - \lim_{n\to\infty} E\left(E\left(L_n\chi_{\{R_n\neq 0\}}|\mathcal{G}_{n-1}\right)\right) \\ &= 1 - \frac{2}{3}\lim_{n\to\infty} E\left(L_{n-1}\chi_{\{R_{n-1}\neq 0\}}\right) \geq 1 - \lim_{n\to\infty}\left(\frac{2}{3}\right)^n = 1. \end{aligned}$$

Combining Fatou's lemma with the equation $E(L_n) = E(L) = 1$ we obtain $E(L|\mathcal{G}_n) = L_n$. Also,

$$\begin{aligned} E(M_nL|\mathcal{G}_{n-1}) &= E(E(M_nL|\mathcal{G}_n)|\mathcal{G}_{n-1}) = E(M_nL_n|\mathcal{G}_{n-1}) \\ &= M_{n-1}L_{n-1} = E(M_{n-1}L|\mathcal{G}_{n-1}). \end{aligned}$$

Hence L is the density of a measure Q under which our discrete process M_n is a martingale. And since $L > 0$ ($L_n > 0$ for all n), Q is equivalent to P. □

Lemma 4.2. *Under the measure Q of Lemma 4.1 the process M_n is uniformly integrable. In particular, $E_Q\big(\sup_{n\geq 0}|M_n|\big) < \infty$.*

Proof. Set $M^* = \sup_{n\geq 0}|M_n|$ and observe that on $\{R_k \neq 0, R_{k+1} = 0\}$ we have

$$M^* \leq |X_0| + |\epsilon_1| + |\epsilon_2| + ... + |\epsilon_k|.$$

Since $\Omega = \bigcup_{k=0}^\infty \{R_k \neq 0, R_{k+1} = 0\}$ (disjoint union), we have

$$\begin{aligned} E_Q(M^*) &= \textstyle\sum_{k=0}^\infty E_Q\left(M^*1_{\{R_k\neq 0\}\cap\{R_{k+1}=0\}}\right) \\ &\leq \textstyle\sum_{k=0}^\infty \left(|X_0| + |\epsilon_1| + |\epsilon_2| + ... + |\epsilon_k|\right) Q(\{R_k \neq 0, R_{k+1} = 0\}). \end{aligned} \tag{4.3}$$

Set $s_k = |X_0| + |\epsilon_1| + |\epsilon_2| + ... + |\epsilon_k|$ and observe that $s_k \leq C(k+1)$ where $C = \max\{|X_0|, |\epsilon|\}$. Inequality (4.3) becomes

$$E_Q(M^*) \leq C\sum_{k=0}^\infty (k+1)Q(\{R_k \neq 0\}) \leq C\sum_{k=1}^\infty (k+1)\left(\frac{2}{3}\right)^k < \infty,$$

where the last inequality is obtained by observing that

$$Q(R_k \neq 0) = Q(R_k \neq 0 | R_{k-1} \neq 0) ... Q(R_1 \neq 0 | R_0 \neq 0) Q(R_0 \neq 0)$$

and $Q(R_k \neq 0 | R_{k-1} \neq 0) = \frac{2}{3}$. □

Now we prove Theorem 1.2.

Proof of Theorem 1.2: By Lemmas 4.1 and 4.2, there exists an equivalent probability measure $Q \sim P$ such that $(M_n, \mathcal{G}_n)_{n \geq 0}$ is a uniformly integrable martingale. Let $M_\infty = \lim_{n\to\infty} M_n$. For each $t \in [0, T]$, set $\tilde{M}_t = E_Q[M_\infty | \mathcal{F}_t]$. Observe that $\tilde{M}_{\tau_n} = E_Q[M_\infty | \mathcal{F}_{\tau_n}] = M_n$, and $\tilde{M}_t = E_Q[\tilde{M}_{\tau_n} | \mathcal{F}_t]$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$ for all $n \geq 0$. Thus the following equation holds:

$$(\tilde{M}_t - X_t) 1_{\{\tau_{n-1} \leq t \leq \tau_n\}} = E_Q \left[(M_n - X_t) 1_{\{\tau_{n-1} \leq t \leq \tau_n\}} \middle| \mathcal{F}_t \right], \quad n \geq 1.$$

We write $M_n - X_t = (M_n - X_{\tau_n}) + (X_{\tau_{n-1}} - X_t) + (X_{\tau_n} - X_{\tau_{n-1}})$. Note that each of $M_n - X_{\tau_n}$, $X_{\tau_{n-1}} - X_t$, and $X_{\tau_n} - X_{\tau_{n-1}}$ takes values in $(-\epsilon, \epsilon)$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$. Therefore we have $-3\epsilon \leq \tilde{M}_t - X_t \leq 3\epsilon$ on the set $\{\tau_{n-1} \leq t \leq \tau_n\}$. Since $\bigcup_{n=1}^\infty \{\tau_{n-1} \leq t \leq \tau_n\} = \Omega$, we conclude that

$$-3\epsilon \leq \tilde{M}_t - X_t \leq 3\epsilon.$$

Since $\epsilon > 0$ is arbitrary, the claim follows. □

5. CFS and f−stickiness

In this paper we used the equivalent condition f–stickiness instead of CFS to prove the existence of CPSs. The advantage is that with our approach some proofs in [9] become more transparent. For example, lemma 2.2 gives a more transparent proof for the equivalence of CFS and strong CFS properties than the original paper [9]. In this section we show that the f–stickiness is actually equivalent to the CFS property.

Let X_t be an adapted continuous process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ and takes values in an open interval (a, b) (including the cases $a = -\infty$ and/or $b = +\infty$). Let $C([t, T], (a, b))$ be the space of continuous functions $f : [t, T] \to (a, b)$ equipped with the usual uniform topology induced by the sup norm. For $x \in (a, b)$, let $C_x([t, T], (a, b))$ be the space of functions $f \in C([t, T], (a, b))$ such that $f(t) = x$. Let "Supp" denote the support, i.e., the smallest closed subset of probability one.

Definition 5.1. We say that X_t has the CFS property in the interval (a, b) if

$$SuppLawP(X|_{[t,T]} | \mathcal{F}_t) = C_{X_t}([t, T], (a, b)), \quad \text{a.s.}$$

In the following proposition we show that the CFS property is equivalent to the weak f-stickiness.

Proposition 5.2. *Let X_t be a continuous process that takes values in (a, b). Then the CFS property of X_t in (a, b) is equivalent to the weak f–stickiness of X in (a, b) for all $f \in C_0([0, T])$.*

Proof. We first show CFS implies weak f–stickiness. To see this observe that

$$\begin{aligned} &P\left(\left\{\sup_{t\in[s,T]}|X_t - X_s - f(t-s)| < \epsilon\right\}|\mathcal{F}_s\right)(\omega) \\ &= Q_s\left(\omega, \left\{g : \sup_{t\in[s,T]}|f_\omega(t) - g(t)| < \epsilon\right\}\right), \end{aligned} \tag{5.1}$$

where Q_s is the $\mathcal{F}_s$−conditional regular law of $X|_{[s,T]}$ and $f_\omega(t) = X_s(\omega)+f(t-s)$. Note that on the set $\{a - \inf_{[0,T-s]} f < X_s < b - \sup_{[0,T-s]} f\}$, we have $f_\omega(\cdot) \in C_{X_s(\omega)}([s,T],(a,b))$. Thus by the CFS property the right hand side of (5.1) is strictly positive on $\{a - \inf_{[0,T-s]} f < X_s < b - \sup_{[0,T-s]} f\}$.

Now we show that f–stickiness implies CFS. It is sufficient to show for any $s \in [0,T)$ the set

$$A := \{\omega \in \Omega \mid SuppLaw\,(X_\theta; s \le \theta \le T|\mathcal{F}_s) \neq C_{X_s}([s,T],(a,b))\}$$

has probability zero.

Let $D = \{f_n\}_{n=1}^{\infty}$ be a countable dense subset of $C([s,T],(a,b))$ and for each $n \ge 1$ set

$$\zeta_n = \left(b - \max_{t\in[s,T]} f_n(t)\right) \wedge \left(\min_{t\in[s,T]} f_n(t) - a\right) \tag{5.2}$$

and note that for $\epsilon > 0$, $B(f_n,\epsilon) \subset C([s,T],(a,b))$ if and only if $\epsilon \le \zeta_n$. Here,

$$B(f,\epsilon) := \left\{g \in C([s,T]) \mid \sup_{t\in[s,T]}|f(t) - g(t)| < \epsilon\right\}$$

for $f \in C([s,T])$. It can easily be shown that the set where the CFS property fails can be written as follows:

$$A = \bigcup_{n\ge1}\bigcup_{\frac{1}{m}\le\zeta_n} D_{n,m}$$

where $D_{n,m} := \{\omega \in \Omega \mid Q_s(\omega, B(f_n, \frac{1}{m})) = 0\} \cap \{\omega \in \Omega \mid |X_s(\omega) - f_n(s)| < \frac{1}{2m}\}$. Note that each $D_{n,m}$ is $\mathcal{F}_s$−measurable and it is sufficient to show that each of them has probability 0.

For $D_{n,m}$ define the function $h \in C_0([0,T])$ as follows: $h(t) = f_n(s+t) - f_n(s)$ for $t \in [0,T-s]$ and $h(t) = h(T-s)$ for $t \in (T-s,T]$. In what follows we want to use the weak h–stickiness property of the process X_t.

We have

$$D_{n,m} \subset \left\{a - \min_{[0,T-s]} h < X_s < b - \max_{[0,T-s]} h\right\}$$

in virtue of (5.2) and the definition of $D_{n,m}$. In addition, for all $\omega \in D_{n,m}$ we have $Q_s(\omega, B(X_s(\omega)+f_n - f_n(s), \frac{1}{2m})) = 0$ since $B(X_s(\omega)+f_n-f_n(s), \frac{1}{2m}) \subset B(f_n, \frac{1}{m})$ and $Q_s(\omega, B(f_n, \frac{1}{m})) = 0$. Hence we conclude that

$$\begin{aligned} &P\left(\left\{\omega \mid \sup_{[s,T]}|X_t(\omega) - X_s(\omega) - h(t-s)| < \tfrac{1}{2m}\right\}|\mathcal{F}_s\right) \\ &= Q_s\left(\omega, B\left(X_s(\omega) + f_n - f_n(s), \tfrac{1}{2m}\right)\right) = 0 \end{aligned}$$

almost surely on the set $D_{n,m} \subset \{a - \min_{[0,T-s]} h < X_s < b - \max_{[0,T-s]} h\}$. Weak h–stickiness will imply that $P(D_{n,m}) = 0$. □

The proposition 5.2 shows that CFS is equivalent to f–stickiness for all $f \in C_0([0,T])$.

In the following proposition, we show that in the case $(a,b) = (-\infty, +\infty)$, the f–stickiness of X for the subclass of linear functions is sufficient to establish the CFS property.

Definition 5.3. We say that X_t is *linear sticky* if it is weak αt–sticky for all $\alpha \in \mathbb{R}$.

Proposition 5.4. *A continuous process X_t has CFS in $C_0([0,T],(-\infty,+\infty))$ if and only if it is linear sticky.*

Proof. It is clear that CFS implies linear stickiness. To show the other direction, we only need to show that linear stickiness implies weak f-stickiness of X_t for each $f \in C_0[0,T]$. Fix a $f \in C_0[0,T]$ and any $\epsilon > 0$. For any $s \in [0,T)$ and $A \in \mathcal{F}_s$ with $P(A) > 0$, we need to show

$$P\left(A \cap \left\{\sup_{t\in[s,T]} |X_t - X_s - f(t-s)| < \epsilon\right\}\right) > 0. \tag{5.3}$$

Set $t_0 = 0$ and define

$$t_n = \inf\left\{t \geq t_{n-1} : |f(t) - f(t_{n-1})| \geq \frac{\epsilon}{4}\right\} \wedge (T-s).$$

Let m be the smallest positive integer such that $t_m = T - s$. It exists since f is uniformly continuous. Define the piecewise linear function $g \in C_0[0,T]$ as follows:

$$g(t) = f(t_i) + \frac{f(t_{i+1}) - f(t_i)}{t_{i+1} - t_i}(t - t_i), \quad t \in [t_i, t_{i+1}].$$

for each $i \in \{1, 2, \cdots, m-1\}$ and $g(t) = g(T-s)$ if $t \in [T-s, T]$. It is clear that

$$\sup_{t\in[0,T]} |f(t) - g(t)| \leq \frac{\epsilon}{2}.$$

Therefore to show (5.3), it is sufficient to show

$$P\left(A \cap \left\{\sup_{t\in[s,T]} |X_t - X_s - g(t-s)| < \frac{\epsilon}{2}\right\}\right) > 0. \tag{5.4}$$

For convenience, we write (5.4) in the following form

$$P\left(A \cap \left\{\sup_{\theta\in[0,T-s]} |X_{s+\theta} - X_s - g(\theta)| < \frac{\epsilon}{2}\right\}\right) > 0. \tag{5.5}$$

Set $A_0 = A$, and let

$$A_{i+1} = A_i \cap \left\{\sup_{\theta\in[t_i,t_{i+1}]} |X_{s+\theta} - X_s - g(\theta)| < \frac{\epsilon}{2^{m-i+1}}\right\}, \quad 0 \leq i \leq m-1.$$

It is clear that $A_m \subset A \cap \left\{ \sup_{\theta \in [0, T-s]} |X_{s+\theta} - X_s - g(\theta)| < \frac{\epsilon}{2} \right\}$. Therefore, it is sufficient to show $P(A_m) > 0$. We have for each $i \in \{1, 2, \cdots, m\}$:

$$\begin{aligned} A_i &= A_{i-1} \cap \left\{ \sup_{\theta \in [t_{i-1}, t_i]} |X_{s+\theta} - X_s - g(\theta)| < \frac{\epsilon}{2^{m-i+2}} \right\} \\ &= A_{i-1} \cap \left\{ \sup_{\tilde{\theta} \in [0, t_i - t_{i-1}]} |X_{s+t_{i-1}+\tilde{\theta}} - g(t_{i-1} + \tilde{\theta}) - X_s| < \frac{\epsilon}{2^{m-i+2}} \right\}. \end{aligned}$$

Denote

$$\begin{aligned} B_i &= A_{i-1} \\ &\cap \left\{ \sup_{\tilde{\theta} \in [0, t_i - t_{i-1}]} |X_{s+t_{i-1}+\tilde{\theta}} - X_{s+t_{i-1}} - [g(t_{i-1} + \tilde{\theta}) - g(t_{i-1})] < \frac{\epsilon}{2^{m-i+3}} \right\} \end{aligned}$$

and

$$C_i = A_{i-1} \cap \left\{ X_{s+t_{i-1}} - X_s - g(t_{i-1}) < \frac{\epsilon}{2^{m-i+3}} \right\}.$$

It is clear that $C_i \cap B_i \subset A_i$ and $C_i \subset A_i$. We conclude that $B_i \subset A_i$. Note that $g(t_{i-1} + \tilde{\theta}) - g(t_{i-1}) = \frac{f(t_i) - f(t_{i-1})}{t_i - t_{i-1}} \tilde{\theta}$, which is a linear function. By the linear stickiness assumption and the fact that $A_{i-1} \in \mathcal{F}_{t_{i-1}}$, we get $P(B_i) > 0$ as long as $P(A_{i-1}) > 0$. Therefore $P(A_i) > 0$ as long as $P(A_{i-1}) > 0$. Now, by induction and the fact that the event $A_0 = A$ has positive probability, we conclude $P(A_m) > 0$. This completes the proof. □

Acknowledgment. We would like to express our gratitude to Mikko Pakkanen and the two anonymous referees for their valuable suggestions and comments.

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