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MRM-APPLICABLE MEASURES FOR THE POWER FUNCTION OF THE SECOND ORDER

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ABSTRACT. The authors have studied multiplicative renormalization method (MRM) for generating functions of orthogonal polynomials. They determined all MRM-applicable measures for renormalizing functions $h(x) = \exp[x]$, $h(x) = (1-x)^{-\kappa}$, $\kappa = 1/2$ and 1. For the cases of $h(x) = \exp[x]$ MRM-applicable measures are Meixner class and very rich. For the case $h(x) = (1-x)^{-\kappa}$, $\kappa = 1$, MRM-applicable measures are rather rich. For the case $\kappa = 1/2$, MRM-applicable measures are only uniform measures on intervals. In this paper, they determine all MRM-applicable measures for $h(x) = (1-x)^{-2}$, whose typical cases are three special beta distributions.

1. Multiplicative Renormalization

A probability measure μ on $\mathbb{R}$ with density $f_\mu(x)$ is said to be applicable to *the multiplicative renormalization method for* $h(x)$ if there exists a suitable analytic function $\rho(t)$ around $t = 0$ with $\rho(0) = 0$, $\alpha = \rho'(0) \neq 0$ such that

$$\psi(t,x) = \frac{h(\rho(t)x)}{\theta(\rho(t))} \quad \text{with} \quad \theta(t) = \int h(tx)\, d\mu(x) \tag{1.1}$$

is a generating function of the orthogonal polynomials $\{P_n(x)\}$ in $L^2(\mu)$ with the leading coefficient equal 1. We say that μ is *MRM-applicable for* $h(x)$ *with* $\rho(t)$, simply. Then there exist Jacobi-Szegö parameters $\{\alpha_n, \omega_n\}$ satisfying the recursive relation

$$P_{n+1}(x) = (x - \alpha_n)P_n(x) - \omega_{n-1}P_{n-1}(x) \quad \text{for} \quad n \geq 0 \tag{1.2}$$

with conventions $\omega_{-1} = 1, P_{-1}(x) = 0$. It is known that

$$\|P_n\|^2 = \lambda_n = \omega_{-1}\omega_0\omega_1 \cdots \omega_{n-1} \quad \text{for} \;\; n \geq 0. \tag{1.3}$$

The following proposition is shown in [3].

Proposition 1.1. *The function* $\psi(t,x) = \dfrac{h(\rho(t)x)}{\theta(\rho(t))}$ *is a generating function of orthogonal polynomials if and only if*

$$\Theta_\rho(t,s) = \frac{\widetilde{\theta}(\rho(t),\rho(s))}{\theta(\rho(t))\theta(\rho(s))} \tag{1.4}$$

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is a function $J(ts)$ depending only on ts, where

$$\widetilde{\theta}(t,s) = \int h(tx)h(sx)\,d\mu(x). \tag{1.5}$$

In [3], we showed that Gaussian measure, Poisson measure, Gamma distribution and negative binomial are MRM-applicable for $h(x) = e^x$ and that beta distribution $\widetilde{B}(\kappa+1/2, \kappa+1/2)$, which is defined in (4.10), is MRM-applicable for $h^\kappa(x) = 1/(1-x)^\kappa$. Especially, uniform measure is MRM-applicable for $h^{1/2}(x)$ and the semi-circle distribution is for $h^1(x)$.

The problem to determine all MRM-measures applicable for $h(x) = e^x$ is solved in [6], [12]. The measures coincide with Meixner class. In [8], we showed that typical measures MRM-applicable for $h^1(x)$ are rich and typical cases are given by

$$f_\mu(x) = \frac{b\sqrt{1-x^2}}{\pi\big(a^2+b^2-2a(1-b)x+(1-2b)x^2\big)} \quad \text{for} \quad |x| \le 1,$$

where $b > 0$ and $|a| \le 1-b$. In [9], we showed that only uniform measures are MRM-applicable for $h^{1/2}(x)$.

Our purpose is to determine all MRM-applicable measures for $h^k(x)$. In a forth coming paper, we will discuss for general $h^k(x)$, $\kappa \ne 1/2, 1$. The cases of $\kappa = 1/2$ and $\kappa = 1$ are extremal and must be discussed separately. The case of $\kappa = 2$ can be included in the general case, but it is so special that we can obtain μ from $\theta(s)$ directly applying 'the inverse Hilbert transform' discussed in [8]. In this paper we assume that $h(x) = \dfrac{1}{(1-x)^2}$ and μ is MRM-applicable for $h(x)$. Then

$$\psi(t,x) = \frac{1}{\theta(\rho(t))(1-\rho(t)x)^2} \quad \text{with} \quad \theta(t) = \int \frac{f_\mu(x)}{(1-tx)^2}\,dx. \tag{1.1$'$}$$

Put $\varphi(t) = \theta(\rho(t))$ and

$$\begin{array}{lllllllll}
\alpha &=& \rho'(0), & c_1 &=& \alpha^{-2}\rho''(0), & c_4 &=& \alpha^{-3}\rho'''(0),\\
a &=& \alpha^{-1}\varphi'(0), & c_2 &=& \alpha^{-2}\varphi''(0), & c_5 &=& \alpha^{-3}\varphi'''(0),\\
b &=& \alpha^{-1}J'(0), & c_3 &=& \alpha^{-2}J''(0), & c_6 &=& \alpha^{-3}J'''(0).
\end{array} \tag{1.6}$$

Those parameters are not free. Here, we will see some of constrains. From $h(x) = \sum_{n=0}^{\infty}(n+1)x^n$, we get the expansion

$$\psi(t,x) = \sum_{n=0}^{\infty}(n+1)\alpha^n P_n(x)t^n. \tag{1.7}$$

By $\theta'(0) = \varphi'(0)/\rho'(0) = a$, we see that

$$\alpha_0 = \int x f_\mu(x)\,dx = \frac{1}{2}\theta'(0) = \frac{a}{2}.$$

Since $\varphi''(0) = \rho'(0)^2\theta''(0) + \rho''(0)\theta'(0)$,

$$\omega_0 = \|P_1\|^2 = \int (x-\alpha_0)^2 f_\mu(x)\,dx = \frac{1}{6}\theta''(0) - \alpha_0^2 = \frac{2c_2 - 2ac_1 - 3a^2}{12} > 0$$

by (1.3). Thus we have

$$\alpha_0 = \frac{a}{2} \quad \text{and} \quad \omega_0 = \frac{2c_2 - 2ac_1 - 3a^2}{12} > 0. \tag{1.8}$$

Since $J(t) = \sum_{n=0}^{\infty}(n+1)^2\alpha^{2n}\|P_n\|^2 t^n$ holds by the orthogonality and (1.7),

$$\alpha b = 4\alpha^2\|P_1\|^2 > 0, \quad \alpha^2 c_3 = 18\alpha^4\|P_2\|^2 > 0, \quad \alpha^3 c_6 = 96\alpha^6\|P_3\|^2 > 0. \tag{1.9}$$

By (1.8) and (1.9), we have an equality $\alpha b = 4\alpha^2(2c_2 - 2ac_1 - 3a^2)/12$. Hence

$$\alpha = \frac{3b}{2c_2 - 2ac_1 - 3a^2}. \tag{1.10}$$

For the convenience of discussion we introduce

$$\theta_0(t) = \int \frac{1}{1-tx}\, d\mu(x), \quad \varphi(t) = \theta(\rho(t)), \quad \varphi_0(t) = \theta_0(\rho(t)). \tag{1.11}$$

Since $\psi(t,x)$ is analytic around $t=0$, the support of μ must be bounded. Moreover, $\theta(t)$ cannot be analytic at t with $1/t \in S_\mu$, where S_μ is the support of μ.

Define

$$\begin{aligned} F_1(X,Y,t) &= \frac{bX + 2(a+bt)Y}{2-(a+bt)X}, \\ F_2(X,Y,t) &= \frac{-3b + 2(ab+c_3t)X + 2(c_2 + 2abt + c_3t^2)Y}{2c_1 + 3a + 3bt - (c_2 + 2abt + c_3t^2)X}, \\ F_3(X,Y,t) &= \frac{A(X,Y,t)}{B(X,Y,t)}, \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} A(X,Y,t) &= -8ab - 5bc_1 - 8c_3t + 3(bc_2 + 2ac_3t + c_6t^2)X \\ &\quad +2(c_5 + 3bc_2t + 3ac_3 + c_6t^2)Y, \\ B(X,Y,t) &= 5ac_1 + 4c_2 + 2c_4 + (8ab + 5c_1)t + 4c_3t^2 \\ &\quad -(c_5 + 3bc_2t + 3ac_3t^2 + c_6t^3)X. \end{aligned} \tag{1.13}$$

The following proposition can be proved similarly to Lemma 3.3, 3.4 and 3.5 of [9].

Proposition 1.2. *Assume that $\Theta_\rho(t,s)$ is a function of ts. Then the following conditions are satisfied.*

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = F_2(\rho(t), \rho'(t), t) = F_3(\rho(t), \rho'(t), t). \tag{1.14}$$

It is useful to remark that the equations (1.14) at $t=0$ is considered as

$$\alpha = \varphi'(0) = \lim_{t\to 0} F_1(\rho(t), \rho'(t), t) = \lim_{t\to 0} F_2(\rho(t), \rho'(t), t) = \lim_{t\to 0} F_3(\rho(t), \rho'(t), t) \tag{1.15}$$

and $\rho(t)$ satisfies

$$\rho(t) = \alpha t + \frac{\alpha^2 c_1}{2} t^2 + \frac{\alpha^3 c_4}{6} t^3 + O(t^4),$$

if necessary.

2. Actions of the Motion Group

We now observe the motion group. Let $f_\mu(x)$ be the density of μ ($d\mu(x) = f_\mu(x)\,dx$). Define

$$\tau_{p,q}x = px + q, \quad d\tau_{p,q}\mu(x) = |p| f_\mu(px+q)\,dx \tag{2.1}$$

and the scale modification rt of t by λ_r. Then we can see how functions and parameters are deformed by the actions similarly to §2 in [9].

Proposition 2.1. *For a given probability measure μ, let $\tau_{p,q}\mu$ be given by (2.1). If μ is MRM-applicable for $h(x)$ with $\rho(t)$ and $\varphi(t)$, then $\tau_{p,q}\mu$ is so for $h(x)$ with $\lambda_r\tau_{p,q}\rho(t)$ and $\lambda_r\tau_{p,q}\varphi(t)$, where*

$$\begin{aligned}
\tau_{p,q}\theta(t) &= \left(\frac{p}{p+qt}\right)^2 \theta\left(\frac{t}{p+qt}\right), \\
\tau_{p,q}\widetilde{\theta}(t,s) &= \left(\frac{p}{p+qt}\right)^2 \left(\frac{p}{p+qs}\right)^2 \widetilde{\theta}\left(\frac{t}{p+qt}, \frac{s}{p+qs}\right), \\
\lambda_r\tau_{p,q}\rho(t) &= \frac{p\rho(rt)}{1-q\rho(rt)}, \\
\lambda_r\tau_{p,q}\varphi(t) &= (1-q\rho(rt))^2\,\varphi(rt)
\end{aligned} \tag{2.2}$$

for $p \neq 0, r \neq 0$.

Proposition 2.2. *Parameters are deformed by $\tau_{p,q}$ and λ_r as*

$$\begin{aligned}
\widehat{\alpha} &= pr\alpha, \quad \widehat{a} = \frac{a-2q}{p}, \quad \widehat{b} = \frac{rb}{p}, \quad \widehat{c}_1 = \frac{c_1+2q}{p}, \\
\widehat{c}_2 &= \frac{c_2 - 4qa - 2qc_1 + 2q^2}{p^2}, \quad \widehat{c}_3 = \frac{r^2c_3}{p^2}, \quad \widehat{c}_4 = \frac{c_4 + 6qc_1 + 6q^2}{p^2}, \\
\widehat{c}_5 &= \frac{c_5 - 6qac_1 - 6qc_2 - 2qc_4 + 6q^2a + 6q^2c_1}{p^3}, \quad \widehat{c}_6 = \frac{r^3c_6}{p^3}.
\end{aligned} \tag{2.3}$$

Replacing the parameters $a,\ b,\ c_1,\ c_2,\ c_3,\ c_4,\ c_5$ and c_6 in $F_1(X,Y,t), F_2(X,Y,t), F_3(X,Y,t), A(X,Y,t)$ and $B(X,Y,t)$ by $\widehat{a},\ \widehat{b},\ \widehat{c}_1,\ \widehat{c}_2,\ \widehat{c}_3,\ \widehat{c}_4,\ \widehat{c}_5$ and $\widehat{c}_6$, denote them by $\widehat{F}_1(X,Y,t),\ \widehat{F}_2(X,Y,t),\ \widehat{F}_3(X,Y,t),\ \widehat{A}(X,Y,t)$ and $\widehat{B}(X,Y,t)$, respectively. Then $\rho_{pqr}(t) = \lambda_r\tau_{pq}\rho(t)$ and $\varphi_{pq}(t) = \tau_{pq}\varphi(t)$ satisfy

$$\frac{\varphi'_{p,q}(t)}{\varphi_{pq}(t)} = \widehat{F}_1(\rho_{pqr}(t), \rho'_{pqr}(t), t) = \widehat{F}_2(\rho_{pqr}(t), \rho'_{pqr}(t), t) = \widehat{F}_3(\rho_{pqr}(t), \rho'_{pqr}(t), t). \tag{2.4}$$

Corollary 2.3. *Let us take $q = \dfrac{a}{2}$. Then*

$$\begin{aligned}
\widehat{\alpha} &= pr\alpha, \quad \widehat{a} = 0, \quad \widehat{b} = \frac{rb}{p}, \quad \widehat{c}_1 = \frac{c_1+a}{p}, \\
\widehat{c}_2 &= \frac{2c_2 - 2ac_1 - 3a^2}{2p^2}, \quad \widehat{c}_3 = \frac{r^2c_3}{p^2}, \quad \widehat{c}_4 = \frac{2c_4 + 6ac_1 + 3a^2}{2p^2}, \\
\widehat{c}_5 &= \frac{2c_5 - 3a^2c_1 - 6ac_2 - 2ac_4 + 3a^3}{2p^3}, \quad \widehat{c}_6 = \frac{r^3c_6}{p^3}.
\end{aligned} \tag{2.5}$$

3. Formulas of $\rho(t)$ and $\varphi(t)$

By Proposition 1.2, $\rho(t)$ satisfies the equations

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = F_2(\rho(t), \rho'(t), t) = F_3(\rho(t), \rho'(t), t). \tag{1.14}$$

The parameters must satisfy

$$\begin{aligned} \alpha &= \frac{3b}{2c_2 - 2ac_1 - 3a^2}, \\ \frac{\alpha}{b} &= \frac{3}{2(c_2 - ac_1) - 3a^2} > 0, \quad \frac{c_3}{b^2} > 0, \quad \frac{c_6}{b^3} > 0 \end{aligned} \tag{3.1}$$

by (1.8), (1.9) and (1.10). Since we can show that the case $5ac_1 + 4c_2 + 2c_4 = 0$ is impossible, we have an equation

$$\alpha a = \lim_{t \to 0} F_3(\rho(t), \rho'(t), t) = \frac{-8ab - 5bc_1 + 2\alpha c_5}{5ac_1 + 4c_2 + 2c_4},$$

by (1.13), (1.14) and (1.15). Solving the equation in c_5, we have

$$c_5 = -\frac{1}{3}\Big(12a^3 - 14ac_2 + 8a^2c_1 + 5ac_1^2 - 5c_1c_2 - 3ac_4\Big). \tag{3.2}$$

Theorem 3.1. *For parameters satisfying (3.1), suppose that $\rho(t)$ and $\varphi(t)$ satisfy (1.14). Then there are only three possible cases as follows:*

(i) $b \neq 0,\ a + c_1 = 0,\ c_3 = \frac{45}{32}b^2$

In this case, the parameters and functions satisfy

$$c_1 = -a, \quad c_2 > \frac{1}{2}a^2, \quad c_3 = \frac{45}{32}b^2 > 0,$$

$$c_4 = \frac{3}{4}\big(3a^2 - 2c_2\big), \quad c_5 = \frac{3}{4}\big(2c_2 - a^2\big)a, \quad c_6 = \frac{81}{32}b^3$$

and

$$\begin{aligned} \rho(t) &= \frac{24bt}{8a^2\big(2c_2 - a^2\big) + 12abt + 9b^2t^2}, \\ \varphi(t) &= \left(\frac{8\big(2c_2 - a^2\big) + 12abt + 9b^2t^2}{8\big(2c_2 - a^2\big)}\right)^2. \end{aligned}$$

(ii) $b \neq 0,\ a + c_1 = 0,\ c_3 = \frac{9}{8}b^2$

In this case, the parameters and functions satisfy

$$c_1 = -a, \quad c_2 > \frac{1}{2}a^2, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = 2a^2 - c_2, \quad c_5 = a\big(2c_2 - a^2\big), \quad c_6 = \frac{3}{2}b^3$$

and

$$\rho(t) = \frac{12bt}{4\big(2c_2 - a^2\big) + 6abt + 3b^2t^2},$$

$$\varphi(t) = \frac{4\big(2c_2 - a^2\big)}{4\big(2c_2 - a^2\big) - 3b^2t^2}\left(\frac{4\big(2c_2 - a^2\big) + 6abt + 3b^2t^2}{4\big(2c_2 - a^2\big)}\right)^2.$$

(iii) $b \neq 0,\ a + c_1 \neq 0,\ \ c_3 = \dfrac{4}{3}b^2$

In this case, the parameters and functions satisfy

$$c_2 = \frac{1}{2}(12a^2 + 20ac_1 + 9c_1^2), \quad c_3 = \frac{4}{3}b^2, \quad c_4 = -\frac{3}{2}(2a + c_1)(2a + 3c_1),$$

$$c_5 = \frac{1}{2}\big(36a^3 + 36a^2c_1 - 3ac_1^2 - c_1^3 + 48a^2c_1 + 66ac_1^2 + 16c_1^3\big), \quad c_6 = \frac{20}{9}b^3$$

and

$$\rho(t) = \frac{6bt}{18(a + c_1)^2 - 3bc_1t + 2b^2t^2},$$

$$\varphi(t) = \frac{\big(18(c_1 + a)^2 - 3bc_1t + 2b^2t^2\big)^2}{108(c_1 + a)^3\big(3(c_1 + a) - bt\big)}.$$

Proof. Remark that $a + c_1$ is transformed by $\tau_{p,q}$ as $\widehat{a} + \widehat{c}_1 = (a + c_1)/p$ because of (2.3). Hence the condition $a + c_1 = 0$ is invariant under the motion $\tau_{p,q}$. Since $\widehat{\alpha}\widehat{b} = \alpha b,\ \widehat{\alpha}\widehat{c}_6 = \alpha c_6/p^2,\ \widehat{c}_3 = c_3/p^2$ hold by (2.3), the signs of $\alpha b,\ c_3, \alpha c_6$ are invariant also. By (2.5), we can deform the measure μ by the shift $\tau_{1,a/2}$ as $\widehat{a} = 0$. Hence it is sufficient to prove the following lemma for the case $a = 0$. □

Lemma 3.2. *For parameters satisfying*

$$\begin{aligned} a &= 0, \quad c_5 = 10c_1c_2, \quad \alpha = \frac{3b}{2c_2} \\ c_2 &> 0, \quad \frac{c_3}{b^2} > 0, \quad \frac{c_6}{b^3} > 0, \end{aligned} \tag{3.1$'$}$$

suppose that $\rho(t)$ and $\varphi(t)$ satisfy (1.14). Then there are only three possible cases in the following:

(i) $a = 0,\ a + c_1 = 0,\ \ c_3 = \dfrac{45}{32}b^2$

In this case, $c_2 > 0$,

$$a = 0,\ b \neq 0, \quad c_1 = 0,\ c_3 = \frac{45}{32}b^2,\ c_4 = -\frac{1}{2}c_2,\ c_5 = 0,\ c_6 = \frac{81}{32}b^3,$$

$$\rho(t) = \frac{24bt}{16c_2 + 9b^2t^2} \quad \textit{and} \quad \varphi(t) = \left(\frac{16c_2 + 9b^2t^2}{16c_2}\right)^2.$$

(ii) $a = 0,\ a + c_1 = 0,\ c_3 = \dfrac{9}{8}b^2$

In this case, $c_2 > 0$,

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = -\frac{1}{3}c_2, \quad c_5 = 0, \quad c_6 = \frac{3}{2}b^3,$$

$$\rho(t) = \frac{12bt}{8c_2 + 3b^2t^2} \quad \textit{and} \quad \varphi(t) = \frac{8c_2}{8c_2 - 3b^2t^2}\left(\frac{8c_2 + 3b^2t^2}{8c_2}\right)^2.$$

(iii) $a = 0,\ a + c_1 \neq 0,\ \ c_3 = \dfrac{4}{3}b^2$

In this case,

$$a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 = \frac{9}{2}c_1^2, \quad c_3 = \frac{4}{3}b^2,$$

$$c_4 = -\frac{9}{2}c_1^2, \quad c_5 = \frac{15}{2}c_1^3, \quad c_6 = \frac{20}{9}b^3,$$

$$\rho(t) = \frac{6bt}{18c_1^2 - 3bc_1t + 2b^2t^2} \quad \textit{and} \quad \varphi(t) = \frac{\left(18c_1^2 - 3bc_1t + 2b^2t^2\right)^2}{108c_1^3(3c_1 - bt)}.$$

Proof. Case (1) Now we assume that $a + c_1 = 0$ with $(3.1)'$ and prove (i), (ii) in the lemma. Here conditions on the parameters are

$$\begin{array}{l} a = 0, \quad c_1 = 0, \quad c_5 = 0, \quad \alpha = \dfrac{3b}{2c_2}, \\ c_2 > 0, \quad \dfrac{c_3}{b^2} > 0, \quad \dfrac{c_6}{b^3} > 0 \end{array} \tag{3.3}$$

by $(3.1)'$. The equations (1.12) are

$$\begin{aligned} F_1(X, Y, t) &= \frac{b(X + 2tY)}{(2 - btX)}, \\ F_2(X, Y, t) &= \frac{-3b + 2c_3tX + 2(c_2 + c_3t^2)Y}{3bt - (c_2 + c_3t^2)X}, \\ F_3(X, Y, t) &= \frac{-8c_3t + 3(bc_2 + c_6t^2)X + 2t(3bc_2 + c_6t^2)Y}{4c_2 + 2c_4 + 4c_3t^2 - t(3bc_2 + c_6t^2)X}. \end{aligned}$$

Solve the equations

$$F_1(X, Y, t) = F_2(X, Y, t) \quad \text{and} \quad F_1(X, Y, t) = F_3(X, Y, t)$$

in Y, and denote their solutions $Y_2(X, t)$ and $Y_3(X, t)$, which are

$$\begin{aligned} Y_2(X, t) &= \frac{6b - 4c_3tX - b(c_2 - c_3t^2)X^2}{2\left(2c_2 - \left(3b^2 - 2c_3\right)t^2\right)}, \\ Y_3(X, t) &= \frac{8c_3t - \left(b(c_2 - c_4) - (2bc_3 + 3c_6)t^2\right)X + bc_6t^3X^2}{2t\left(b(c_2 - c_4) - (2bc_3 - c_6)t^2\right)}, \end{aligned}$$

respectively. Now, a candidate of $\rho(t)$ satisfying $\rho(0) = 0$ is given by solving the equation $Y_2(X, t) = Y_3(X, t)$ in X as

$$\rho(t) = \frac{t \times R_{num}(t)}{R_{den}(t)}, \tag{3.4}$$

where

$$\begin{aligned} R_{num}(t) &= 3b^2(c_2 - c_4) - 8c_2c_3 + \left(6b^2c_3 - 8c_3^2 + 3bc_6\right)t^2, \\ R_{den}(t) &= -bc_2(c_2 - c_4) + \left(3bc_2c_3 - bc_3c_4 - 3c_2c_6\right)t^2 - \left(2bc_3^2 - 3b^2c_6 + c_3c_6\right)t^4. \end{aligned}$$

If $c_4 = c_2$, then $R_{num}(0) \neq 0$ is shown and hence $\rho'(0)$ does not exist; that is, $c_4 \neq c_2$. We see

$$\alpha = \rho'(0) = \lim_{t\to 0} \frac{R_{num}(t)}{R_{den}(t)} = \frac{3b^2(c_2 - c_4) - 8c_2c_3}{-bc_2(c_2 - c_4)}.$$

By (3.3), we have an equation

$$\frac{3b^2(c_2 - c_4) - 8c_2c_3}{-bc_2(c_2 - c_4)} = \frac{3b}{2c_2}.$$

Solving it in c_4, we have

$$c_4 = \frac{c_2(9b^2 - 16c_3)}{9b^2}.$$

Then we can see the numerator of $\rho'(t) - Y_2(\rho(t), t)$ is expressed in the form

$$\left(K_2t^2 + K_4t^4 + K_6t^6\right), \tag{3.5}$$

where

$$\begin{aligned} K_2 &= 16c_2^2c_3\Big(72b^2c_3 - 224c_3^2 + 135bc_6\Big), \\ K_4 &= -3c_2\Big(108b^4c_3^2 + 324b^3c_3c_6 - 405b^2c_6^2 - 480b^2c_3^3 + 912bc_3^2c_6 - 512c_3^4\Big), \\ K_6 &= 27b^2c_3\Big(- 36bc_3c_6 - 36b^2c_3^2 + 36b^3c_6 + 3c_6^2 + 32c_3^2\Big). \end{aligned}$$

Since the form (3.5) must be equal to 0 with $b \neq 0$, we get

$$c_6 = \frac{8c_3(-9b^2 + 28c_3)}{135b} \tag{3.6}$$

solving the equation $K_2 = 0$ in c_6. Replacing c_6 in K_4 by the right hand side of (3.6), we have expressions of K_4 and K_6 as

$$\begin{aligned} K_4 &= \frac{4}{3}c_2c_3^2\big(45b^2 - 32c_3\big)\big(9b^2 - 8c_3\big), \\ K_6 &= -\frac{4}{225}c_3^2\big(72b^2 - 49c_3\big)\big(45b^2 - 32c_3\big)\big(9b^2 - 8c_3\big). \end{aligned}$$

The condition for $K_4 = 0$ and $K_6 = 0$ is given by

$$c_3 = \frac{45}{32}b^2 \quad \text{or} \quad c_3 = \frac{9}{8}b^2.$$

Case (1–1) : $c_3 = \dfrac{45}{32}b^2$

In this case, parameters are

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_2 > 0, \quad c_3 = \frac{45}{32}b^2,$$

$$c_4 = -\frac{3}{2}c_2, \quad c_5 = 0, \quad c_6 = \frac{81}{32}b^3$$

and $\rho(t)$ is

$$\rho(t) = \frac{24bt}{16c_2 + 9b^2t^2}, \quad \rho'(0) = \alpha = \frac{3b}{2c_2}$$

by (3.4). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{36b^2t}{16c_2 + 9b^2t^2}, \quad \varphi(0) = 1.$$

Therefore, we have

$$\log \varphi(t) \quad = \quad \int_0^t \frac{36b^2t}{16c_2 + 9b^2t^2}\, dt = 2 \log \frac{16c_2 + 9b^2t^2}{16c_2}.$$

Thus we proved (i). □

Case (1–2) : $c_3 = \dfrac{9}{8}b^2$

In this case, parameters are

$$a = 0, \quad b \neq 0, \quad c_1 = 0, \quad c_2 > 0, \quad c_3 = \frac{9}{8}b^2,$$

$$c_4 = -c_2, \quad c_5 = 0, \quad c_6 = \frac{3}{2}b^3$$

and $\rho(t)$ is

$$\rho(t) = \frac{12bt}{8c_2 + 3b^2t^2}, \quad \rho'(0) = \alpha = \frac{3b}{2c_2}$$

by (3.4). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{18b^2t\big(8c_2 - b^2t^2\big)}{\big(8c_2 - 3b^2t^2\big)\big(8c_2 + 3b^2t^2\big)}.$$

Therefore, we have

$$\begin{aligned} \log \varphi(t) \quad &= \quad \int_0^t \frac{18b^2t(8c_2 - b^2t^2)}{(8c_2 + 3b^2t^2)(8c_2 - 3b^2t^2)}\, dt \\ &= \quad 2 \log \frac{8c_2 + 3b^2t^2}{8c_2} - \log \frac{8c_2 - 3b^2t^2}{8c_2}. \end{aligned}$$

Thus we proved (ii). □

Case (2) Assume $(3.1)'$ and $a + c_1 \neq 0$. Then

$$a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 > 0, \quad c_3 > 0, \quad c_5 = \frac{5}{3}c_1c_2, \quad \frac{c_6}{b^3} > 0 \tag{3.7}$$

by $(3.1)'$. The equations (1.12) are

$$\begin{aligned} F_1(X, Y, t) \quad &= \quad \frac{b(X + 2tY)}{(2 - btX)}, \\ F_2(X, Y, t) \quad &= \quad \frac{-3b + 2c_3tX + 2(c_2 + c_3t^2)Y}{2c_1 + 3bt - (c_2 + c_3t^2)X}, \\ F_3(X, Y, t) \quad &= \quad \frac{-15bc_1 - 24c_3t + 9(bc_2 + c_6t^2\big)X + 2(5c_1c_2 + 9bc_2t + 3c_6t^3)Y}{6\big(2c_2 + c_4\big) + 15bc_1t + 12c_3t^2 - \big(5c_1c_2 + 9bc_2t + 3c_6t^3\big)X}. \end{aligned}$$

Solve equations

$$F_1(X, Y, t) = F_2(X, Y, t) \quad \text{and} \quad F_1(X, Y, t) = F_3(X, Y, t)$$

in Y, and denote their solutions $Y_2(X,t)$ and $Y_3(X,t)$, which are

$$Y_2(X,t) = \frac{6b + 2(bc_1 - 2c_3t)X - b(c_2 - c_3t^2)X^2}{2\left(2c_2 - 2bc_1t - \big(3b^2 - 2c_3)t^2\big)\right)},$$

$$Y_3(X,t) = \frac{30bc_1 + 48c_3t - 6\big(b(c_2 - c_4) + (2bc_3 + 3c_6)t^2\big)X - b(5c_1c_2 - 6c_6t^3)X^2}{2\big(10c_1c_2 + 6b(c_2 - c_4)t - 15b^2c_1t^2 - 6(2bc_3 - c_6)t^3\big)}.$$

Now, the candidate of $\rho(t)$ satisfying $\rho(0) = 0$ is given by solving the equation $Y_2(X,t) = Y_3(X,t)$ in X as

$$\rho(t) = \frac{t \times R_{num}(t)}{R_{den}(t)}, \tag{3.8}$$

where

$$\begin{aligned} R_{num}(t) &= 6\big(3b^2(c_2 - c_4) + 5b^2c_1^2 - 8c_2c_3 + 3bc_1c_3t + \big(6b^2c_3 - 8c_3^2 + 3bc_6\big)t^2\Big), \\ R_{den}(t) &= -2bc_2\big(3(c_2 - c_4) + 5c_1^2\big) + 20c_1c_2c_3t + 6(3bc_2c_3 - bc_3c_4 - 3c_2c_6)t^2 \\ &\quad -3bc_1\big(5bc_3 - 4c_6\big)t^3 + 6\big(3b^2c_6 - 2bc_3^2 - c_3c_6\big)t^4. \end{aligned}$$

If $3c_4 - 5c_1^2 - 3c_2 = 0$, then we have

$$0 = \rho(0) = \lim_{t\to 0} \frac{tR_{num}(t)}{R_{den}(t)} = -\frac{12}{5c_1},$$

which is a contradiction. Therefore, we see that $3c_4 - 5c_1^2 - 3c_2 \neq 0$ and

$$\rho'(0) = \lim_{t\to 0} \frac{R_{num}(t)}{R_{den}(t)} = \frac{3\big(3b^2c_2 + 5b^2c_1^2 - 8c_2c_3 - 3b^2c_4\big)}{-bc_2\big(3(c_2 - c_4) + 5c_1^2\big)}. \tag{3.9}$$

By (3.1)$'$, (3.7) and (3.9) with $\rho'(0) = \alpha$, we have an equation

$$\frac{3\big(3b^2c_2 + 5b^2c_1^2 - 8c_2c_3 - 3b^2c_4\big)}{-bc_2\big(3(c_2 - c_4) + 5c_1^2\big)} = \frac{3b}{2c_2}.$$

Solving it in c_4, we have

$$c_4 = \frac{9b^2c_2 + 15b^2c_1^2 - 16c_2c_3}{9b^2}.$$

Then we can see the numerator of $\rho'(t) - Y_2(\rho(t),t)$ is expressed in the form

$$-3b\left(K_2t^2 + K_3t^3 + K_4t^4 + K_5t^5 + K_6t^6\right), \tag{3.10}$$

where

$$\begin{aligned} K_2 &= 16c_2c_3\Big(144b^2c_2c_3 + 15b^2c_1^2c_3 - 448c_2c_3^2 + 270bc_2c_6\Big), \\ K_3 &= -12bc_1c_3\Big(168b^2c_2c_3 + 15b^2c_1^2c_3 - 736c_2c_3^2 + 486bc_2c_6\Big), \\ K_4 &= 3\Big(32c_2c_3^2(32c_3^2 - 57bc_6) + 30b^2\big(-32c_1^2c_3^3 + 32c_2c_3^3 + 27c_2c_6^2\big) \\ &\qquad +18b^3c_3c_6\big(41c_1^2 - 36c_2\big) + 9b^4c_3^2\big(5c_1^2 - 24c_2\big)\Big), \\ K_5 &= 324b^2c_1\Big(-10bc_3^3 + 9b^2c_3c_6 + 3c_3^2c_6 - 3bc_6^2\Big), \\ K_6 &= 54b^2c_3\Big(32c_3^3 + 3c_6^2 - 36bc_3c_6 - 36b^2c_3^2 + 36b^3c_6\Big). \end{aligned}$$

Since the form (3.10) must be equal to 0, we get

$$c_6 \quad = \quad -\frac{c_3}{270bc_2}\Big(15b^2c_1^2 + 144b^2c_2 - 448c_2c_3\Big)$$

solving $K_2 = 0$, because of $b \neq 0, c_2 > 0, c_3 > 0$. Replacing c_6 in K_3 by the above value and solving $K_3 = 0$ in c_3, we have

$$c_3 = \frac{3b^2\big(38c_2 + 5c_1^2\big)}{88c_2}. \tag{3.11}$$

For these values of c_3 and c_6, K_4 is

$$K_4 \quad = \quad \frac{135b^8}{1874048c_2^3}\big(-2c_2 + 9c_1^2\big)\big(38c_2 + 5c_1^2\big)^2\big(104c_2 + 27c_1^2\big).$$

Since $c_1 \neq 0,\ c_2 > 0$, $K_4 = 0$ implies

$$c_2 = \frac{9}{2}c_1^2.$$

Then the parameters are

$$a = 0, \quad b \neq 0, \quad c_1 \neq 0, \quad c_2 = \frac{9}{2}c_1^2, \quad c_3 = \frac{4}{3}b^2,$$
$$c_4 = -\frac{9}{2}c_1^2, \quad c_5 = \frac{15}{2}c_1^3, \quad c_6 = \frac{20}{9}b^3$$

and $\rho(t)$ is

$$\rho(t) = \frac{6bt}{18c_1^2 - 3bc_1t + 2b^2t^2}$$

by (3.8). Then

$$\frac{\varphi'(t)}{\varphi(t)} = F_1(\rho(t), \rho'(t), t) = \frac{3b^2t\big(9c_1 - 2bt\big)}{(3c_1 - bt)(18c_1^2 - 3bc_1t + 2b^2t^2)}.$$

Integrating it from 0 to t, we have

$$\log\varphi(t) \quad = \quad 2\log\frac{18c_1^2 - 3bc_1t + 2b^2t^2}{18c_1^2} - \log\frac{3c_1 - bt}{3c_1}.$$

Thus we conclude (iii) and all statements in the lemma. □

4. MRM-applicable Measures for $h(x) = \dfrac{1}{(1-x)^2}$

The case $\kappa = 2$ is a very special case for which we can calculate $\Theta_\rho(t,s)$ and *'the Hilbert transform' of the measure* μ

$$\widetilde{\theta}_0(s) = \text{p.v.}\int \frac{f_\mu(x)}{1 - sx}\,dx \quad \text{for } s \in \mathbb{R}. \tag{4.1}$$

Obviously,

$$\theta_0(s) = \widetilde{\theta}_0(s) \quad \text{for} \quad \frac{1}{s} \notin S_\mu$$

holds. Then we can obtain measures μ explicitly by 'the inverse Hilbert transform' by applying the method in [8].

It is easy to see that

$$\theta(t) = \theta_0(t) + t\theta_0'(t) = \{t\theta_0(t)\}'. \tag{4.2}$$

and

$$\tilde{\theta}(t,s) = \int \frac{1}{(1-tx)^2(1-sx)^2}\,d\mu(x) = \frac{1}{(t-s)^3}\big(t^2(t-s)\theta(t) - 2t^2 s\theta_0(t) + 2ts^2\theta_0(s) + s^2(t-s)\theta(s)\big) \tag{4.3}$$

by (1.5). Therefore, we have

$$\tilde{j}(t,s) = \frac{\tilde{\theta}(t,s)}{\theta(t)\theta(s)} = \frac{1}{(t-s)^3}\left(\frac{t^2(t-s)}{\theta(s)} - \frac{2t^2 s\theta_0(t)}{\theta(t)\theta(s)} + \frac{2ts^2\theta_0(s)}{\theta(t)\theta(s)} + \frac{s^2(t-s)}{\theta(t)}\right) \tag{4.4}$$

and hence

$$\frac{\partial \tilde{j}(t,s)}{\partial s}\Big|_{s=0} = \frac{2}{t} + \frac{1}{t^3}\left(-\theta'(0)t^2 - \frac{2t^2\theta_0(t)}{\theta(t)}\right) = \frac{2}{t} - \theta'(0) - \frac{2\theta_0(t)}{t\theta(t)}. \tag{4.5}$$

Suppose that μ is MRM-applicable for $h(x) = 1/(1-x)^2$ with $\rho(t)$. Then $\Theta_\rho(t,s) = \tilde{j}(\rho(t),\rho(s))$ depends only on ts, say $J(ts) = \Theta_\rho(t,s)$. By (4.5), we have

$$\frac{tJ'(0)}{\rho'(0)} = \frac{1}{\rho'(0)}\frac{\partial \tilde{j}(\rho(t),s)}{\partial s}\rho'(s)\Big|_{s=0} = \frac{2}{\rho(t)} - \theta'(0) - \frac{2\theta_0(\rho(t))}{\rho(t)\theta(\rho(t))} \tag{4.6}$$

and equivalently

$$\frac{J'(0)}{\rho'(0)}\rho^{-1}(t) = \frac{2}{t} - \theta'(0) - \frac{2\theta_0(t)}{t\theta(t)}. \tag{4.6$'$}$$

Hence

$$\theta_0(s) = \frac{\theta(s)}{2}\big(2 - \big(a + b\rho^{-1}(s)\big)s\big) \tag{4.7}$$

with $\alpha = \rho'(0),\ a = \theta'(0),\ b = J'(0)/\rho'(0)$. When $\rho^{-1}(s)$ and $\theta(s)$ are given, we can calculate $\theta_0(s)$ and $\tilde{j}(t,s)$ by (4.7) and (4.4). The equalities (4.3) and (4.4) can be rewritten as

$$\widetilde{\theta}(t,s) = \frac{t^2\big(t - 3s + (a + b\rho^{-1}(t))ts\big)\theta(t) - s^2\big(s - 3t + (a + b\rho^{-1}(s))ts\big)\theta(s)}{(t-s)^3}, \tag{4.3$'$}$$

$$\widetilde{j}(t,s) = \frac{1}{(t-s)^3}\left(\frac{t^2\big(t - 3s + (a + b\rho^{-1}(t))ts\big)}{\theta(s)} - \frac{s^2\big(s - 3t + (a + b\rho^{-1}(s))ts\big)}{\theta(t)}\right). \tag{4.4$'$}$$

To calculate $J(ts) = \Theta_\rho(t,s)$ in (1.4), the following expression is convenient:

$$\begin{aligned}\Theta_\rho(t,s) &= \frac{\tilde{\theta}(\rho(t),\rho(s))}{\theta(\rho(t))\theta(\rho(s))} \\ &= \frac{1}{(\rho(t)-\rho(s))^3}\left(\frac{\rho(t)^2(\rho(t)-\rho(s))}{\varphi(s)} - \frac{2\rho(t)^2\rho(s)\varphi_0(t)}{\varphi(t)\varphi(s)} + \frac{2\rho(t)\rho(s)^2\varphi_0(s)}{\varphi(t)\varphi(s)} + \frac{\rho(s)^2(\rho(t)-\rho(s))}{\varphi(t)}\right),\end{aligned} \tag{4.8}$$

where $\varphi_0(t) = \theta_0(\rho(t))$ is calculated by

$$\varphi_0(t) = \frac{\varphi(t)}{2}\big(2 - (a + bt)\rho(t)\big) \tag{4.9}$$

using (4.7). Appealing (4.7), (4.8) and (4.9), we have the following theorems by Theorems 4.1, 4.2 and 4.3. Those cases are all possible ones, which give probability measures MRM-applicable.

Now we see three typical cases and corresponding three general cases. To state assertions, we introduce *beta distributions* $\widetilde{B}(p, q)$ *over* $[-1,\ 1]$ by

$$d\widetilde{\mu}_{p,q}(x) = \widetilde{f}_{p,q}(x)\,dx,$$

$$\widetilde{f}_{p,q}(x) = \begin{cases} \dfrac{\Gamma(p+q)}{2^{p+q-1}\Gamma(p)\Gamma(q)}(1+x)^{p-1}(1-x)^{q-1} & \text{if } \ -1 < x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{4.10}$$

for $p, q > 0$. Let us denote the usual beta distribution over $[0,\ 1]$ by $B(p, q)$;

$$d\mu_{p,q}(x) = f_{p,q}(x)\,dx, \quad f_{p,q}(x) = \begin{cases} \dfrac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)}x^{p-1}(1-x)^{q-1} & \text{if } \ 0 < x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.1. Put $a = 0,\ b = \dfrac{4}{3}$ in Theorem 3.1 (i). Then parameters are $c_1 = 0,\ c_2 = 1,\ c_3 = \dfrac{5}{2},\ c_4 = -\dfrac{3}{2},\ c_5 = 0,\ c_6 = 6$ and functions are

$$\begin{aligned}
\rho(t) &= \frac{2t}{1+t^2}, \quad \varphi(t) = (1+t^2)^2, \quad \varphi_0(t) = \frac{1}{3}(3-t^2)(1+t^2), \\
\rho^{-1}(s) &= \frac{s}{1+\sqrt{1-s^2}}, \quad \theta(s) = \frac{4}{(1+\sqrt{1-s^2})^2}, \\
\theta_0(s) &= \frac{4(3-2s^2+3\sqrt{1-s^2})}{3(1+\sqrt{1-s^2})^3}, \quad J(ts) = \Theta_\rho(t,s) = \frac{3-ts}{3(1-ts)^3}, \\
\widetilde{\theta}(t,s) &= \frac{4}{3(t-s)^3}\left(\frac{s^2(-3s+5t+4t\sqrt{1-s^2})}{(1+\sqrt{1-s^2})^2} + \frac{t^2(-5s+3t-4s\sqrt{1-t^2})}{(1+\sqrt{1-t^2})^2}\right), \\
f_\mu(x) &= \frac{8}{3\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}} \quad (\text{beta distribution } \widetilde{B}(\tfrac{5}{2}, \tfrac{5}{2}) \text{ over } [-1,\ 1]).
\end{aligned}$$

Proof. Applying Lemma 3.2 (i), we see $\rho(t) = \dfrac{2t}{1+t^2}$ and $\varphi(t) = (1+t^2)^2$. Solving $\rho(t) = s$ in t, we see $\rho^{-1}(s) = \dfrac{s}{1+\sqrt{1-s^2}}$. Hence

$$\theta(s) = \varphi(\rho^{-1}(s)) = \big(1 + \rho^{-1}(s)^2\big)^2 = \left(\frac{2}{1+\sqrt{1-s^2}}\right)^2,$$

since

$$1 + \rho^{-1}(s)^2 = 1 + \left(\frac{s}{1+\sqrt{1-s^2}}\right)^2 = \frac{2 - s^2 + 2\sqrt{1-s^2} + s^2}{\big(1+\sqrt{1-s^2}\big)^2} = \frac{2}{1+\sqrt{1-s^2}}.$$

By (4.7), we have

$$\theta_0(s) = \frac{1}{2}\left(\frac{2}{1+\sqrt{1-s^2}}\right)^2 \left(2 - \frac{4}{3}\frac{s^2}{1+\sqrt{1-s^2}}\right) = \frac{4(3-2s^2+3\sqrt{1-s^2})}{3(1+\sqrt{1-s^2})^3},$$

which satisfies (4.2). $\varphi_0(t)$ is obtained by (4.9) as

$$\varphi_0(t) = \frac{1}{2}(1+t^2)^2\left(2 - \frac{4t}{3}\frac{2t}{1+t^2}\right) = \frac{1}{3}(1+t^2)(3-t^2).$$

Replacing as $\rho(t) = 2t/(1+t^2)$ and $\varphi(t) = (1+t^2)^2$ in Equation (4.8), we have

$$\begin{aligned}\Theta_\rho(t,s) &= \frac{1}{3(s-t)^3(1-ts)^3}\Big(4s^2\big(6s(1+t^2) - 18t(1+s^2) + 16s^2t\big) \\ &\qquad -4t^2\big(6t(1+s^2) - 18s(1+t^2) + 16t^2t\big)\Big) \\ &= \frac{8(s-t)^3(3-ts)}{3\times 8(s-t)^3(1-ts)^3} = \frac{(3-ts)}{3(1-ts)^3},\end{aligned}$$

since

$$\begin{aligned}\rho(s)^2\varphi(s) &= 4s^2, \\ 3\rho(s) - 9\rho(t) + 4s\rho(s)\rho(t) &= \frac{2(s-3t)(3-ts)}{(1+t^2)(1+s^2)}, \\ (\rho(s)-\rho(t))^3\varphi(t)\varphi(s) &= \frac{\big(2s(1+t^2) - 2t(1+s^2)\big)^3}{(1+t^2)(1+s^2)} = \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)}.\end{aligned}$$

By $(4.3)'$, $\theta(s) = 4/(1+\sqrt{1-s^2})^2$, $\rho^{-1}(s) = s/(1+\sqrt{1-s^3})$ and $1/(1+\sqrt{1-t^2}) = (1-\sqrt{1-t^2})/t^2$,

$$\begin{aligned}\widetilde{\theta}(t,s) &= \frac{1}{3(t-s)^3}\left(t^2\big(3t - 9s + \frac{4t^2s}{1+\sqrt{1-t^2}}\big)\theta(t)\right. \\ &\qquad \left.-s^2\big(3s - 9t + \frac{4t^2s}{1+\sqrt{1-s^2}}\big)\theta(s)\right) \\ &= \frac{1}{3(t-s)^3}\left(\frac{t^2\big(3t-5s-4s\sqrt{1-t^2}\big)}{(1+\sqrt{1-t^2})^2} - \frac{s^2\big(3s-5t-4t\sqrt{1-s^2})\big)}{(1+\sqrt{1-s^2})^2}\right).\end{aligned}$$

The corresponding measure μ can be obtained by 'the inverse Hilbert transform' of $\theta_0(s)$. However, here we get it more simply by observing the result in Section 4.7 of [4]. There we showed that $\widetilde{B}(\beta+\frac{1}{2}, \beta+\frac{1}{2})$ is MRM-applicable for $h_\beta(x) = \dfrac{1}{(1-x)^\beta}$ with $\theta(s) = \left(\dfrac{2}{1+\sqrt{1-s^2}}\right)^\beta$. Hence we see that μ is $\widetilde{B}(\frac{5}{2}, \frac{5}{2})$. □

Theorem 4.2. *For* $c_1 = -a,\ c_2 > \frac{1}{2}a^2,\ c_3 = \frac{45}{32}b^2,\ c_4 = -\frac{3}{4}(2c_2 - a^2),\ c_5 = \frac{3}{4}a(2c_2 - a^2),\ c_6 = \frac{81}{32}b^3$, *corresponding functions are given as*

$$\begin{aligned}
\rho(t) &= \frac{24bt}{8(2c_2 - a^2) + 12abt + 9b^2t^2},\\
\varphi(t) &= \frac{\big(8(2c_2 - a^2) + 12abt + 9b^2t^2\big)^2}{64(2c_2 - a^2)^2},\\
\varphi_0(t) &= \frac{\big(8(2c_2 - a^2) + 12abt + 9b^2t^2\big)\big(8(2c_2 - a^2) - 3b^2t^2\big)}{64(2c_2 - a^2)^2},\\
J(ts) &= \Theta_\rho(t,s) = \frac{64(2c_2 - a^2)^2\big(8(2c_2 - a^2) - 3b^2st\big)}{\big(8(2c_2 - a^2) - 9b^2st\big)^3},
\end{aligned}$$

$$\begin{aligned}
\rho^{-1}(s) &= \frac{4(2c_2 - a^2)s}{3b\big(2 - as + \sqrt{4 - 4as + (3a^2 - 4c_2)s^2}\big)},\\
\theta(s) &= \frac{16}{3b\big(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2}\big)},\\
\theta_0(s) &= \frac{8\big(2 - as + 2\sqrt{4 - 4as - (4c_2 - 3a^2)s^2}\big)}{3\big(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2}\big)^2},
\end{aligned}$$

$$\begin{aligned}
\widetilde{\theta}(t,s) &= \frac{16}{3(t-s)^3)}\Bigg(\frac{s^2\big(-3s + 5t - ast + 2\sqrt{4 - 4as - (4c_2 - 3a^2)s^2}t\big)}{\big(2 - as + \sqrt{4 - 4as - (4c_2 - 3a^2)s^2}\big)^2}\\
&\qquad + \frac{t^2\big(-5s + 3t + ast - 2s\sqrt{4 - 4at - (4c_2 - 3a^2)t^2}\big)}{\big(2 - at + \sqrt{4 - 4at - (4c_2 - 3a^2)t^2}\big)^2}\Bigg),\\
f_\mu(x) &= \frac{4}{3\pi(2c_2 - a^2)^2}\big(2(2c_2 - a^2) - (2x - a)^2\big)^{\frac{3}{2}}.
\end{aligned}$$

Proof. Let us apply Proposition 2.1 to Example 4.1 with parameters

$$p = \frac{2}{\sqrt{2}\sqrt{2c_2 - a^2}}, \quad q = -\frac{a}{\sqrt{2}\sqrt{2c_2 - a^2}}, \quad r = \frac{3b}{\sqrt{2}\sqrt{2c_2 - a^2}}.$$

Then we have the conclusion. □

Example 4.3. Put $a = 0,\ b = 2$ in Theorem 3.1 (ii). Then parameters are $c_1 = 0,\ c_2 = \frac{3}{2},\ c_3 = \frac{9}{2},\ c_4 = -\frac{3}{2},\ c_5 = 0,\ c_6 = 12$ and functions are

$$\begin{aligned}
\rho(t) &= \frac{2t}{1+t^2}, \quad \varphi(t) = \frac{(1+t^2)^2}{1-t^2}, \quad \varphi_0(t) = 1 + t^2,\\
\rho^{-1}(s) &= \frac{s}{1+\sqrt{1-s^2}}, \quad \theta(s) = \frac{2}{\sqrt{1-s^2}\big(1+\sqrt{1-s^2}\big)}, \quad \theta_0(s) = \frac{2}{1+\sqrt{1-s^2}},\\
J(ts) &= \Theta_\rho(t,s) = \frac{1+ts}{(1-ts)^3},\\
\widetilde{\theta}(t,s) &= \frac{2}{(t-s)^3}\left(\frac{t + s - 2t^2s}{\sqrt{1-t^2}} - \frac{t + s - 2ts^2}{\sqrt{1-s^2}}\right),\\
f_\mu(x) &= \frac{2}{\pi}\sqrt{1-x^2} \quad \text{(semi-circle law).}
\end{aligned}$$

Proof. Applying Lemma 3.2 (ii), we see $\rho(t) = \dfrac{2t}{1+t^2}$ and $\varphi(t) = \dfrac{(1+t^2)^2}{1-t^2}$. We see $\rho^{-1}(s) = \dfrac{s}{1+\sqrt{1-s^2}}$ as in Example 4.1. Hence

$$\theta(s) = \frac{\big(1+\rho^{-1}(s)^2\big)^2}{1-\rho^{-1}(s)^2} = \left(\frac{2}{1+\sqrt{1-s^2}}\right)^2 \frac{1+\sqrt{1-s^2}}{2\sqrt{1-s^2}} = \frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})},$$

since

$$1+\rho^{-1}(s)^2 = \frac{2}{1+\sqrt{1-s^2}},\ 1-\rho^{-1}(s)^2 = \frac{2-s^2+2\sqrt{1-s^2}-s^2}{\big(1+\sqrt{1-s^2}\big)^2} = \frac{2\sqrt{1-s^2}}{1+\sqrt{1-s^2}}.$$

By (4.7), we obtain

$$\theta_0(s) = \frac{1}{2}\frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})}\left(2-2\frac{s^2}{1+\sqrt{1-s^2}}\right) = \frac{2}{1+\sqrt{1-s^2}},$$

which satisfies (4.2). $\varphi_0(t)$ is obtained by (4.9) as

$$\varphi_0(t) = \frac{1}{2}\frac{(1+t^2)^2}{1-t^2}\left(2-2\frac{2t^2}{1+t^2}\right) = \frac{(1+t^2)^2(1-t^2)}{(1+t^2)(1-t^2)} = 1+t^2.$$

By (4.8) with $\rho(t) = 2t/(1+t^2)$ and $\varphi(t) = (1+t^2)^2/(1-t^2)$,

$$\begin{aligned}\Theta_\rho(t,s) &= \frac{(1+t^2)(1+s^2)(1-t^2)(1-s^2)}{8(s-t)^3(1-ts)^3}\\ &\quad\times\left(\frac{8s^2\big(s-3t+s^2t+st^2\big)}{(1+t^2)(1+s^2)(1-s^2)} - \frac{8t^2\big(t-3s+t^2s+ts^2\big)}{(1+t^2)(1+s^2)(1-t^2)}\right)\\ &= \frac{8(s-t)^3(3-ts)}{3\times 8(s-t)^3(1-ts)^3} = \frac{(1+ts)}{(1-ts)^3},\end{aligned}$$

since

$$\begin{aligned}\rho(t)^2\varphi(t) &= \frac{4t^2}{1-t^2},\\ \rho(s)-3\rho(t)+2s\rho(s)\rho(t) &= \frac{2(s-3t+s^2+st^2)}{(1+t^2)(1+s^2)},\\ (\rho(s)-\rho(t))^3\varphi(t)\varphi(s) &= \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)(1-t^2)(1-s^2)}.\end{aligned}$$

By (4.3)′ with $\rho^{-1}(s) = s/(1+\sqrt{1-s^2})$ and $\theta(s) = 2/\Big(\sqrt{1-s^2}\big(1+\sqrt{1-s^2}\big)\Big)$,

$$\begin{aligned}\widetilde{\theta}(t,s) &= \frac{1}{(t-s)^3}\Bigg(t^2\big(t-3s+\frac{2t^2s}{1+\sqrt{1-t^2}}\big)\theta(t)\\ &\qquad\qquad -s^2\big(s-3t+\frac{2t^2s}{1+\sqrt{1-s^2}}\big)\theta(s)\Bigg)\\ &= \frac{2}{(t-s)^3}\left(\frac{t^2\big(t-s-2s\sqrt{1-t^2}\big)}{\sqrt{1-t^2}(1+\sqrt{1-t^2})} - \frac{s^2\big(s-t-2t\sqrt{1-s^2}\big)}{\sqrt{1-s^2}(1+\sqrt{1-s^2})}\right)\\ &= \frac{2}{(t-s)^3}\left(\frac{t+s-2t^2s}{\sqrt{1-t^2}} - \frac{t+s-2ts^2}{\sqrt{1-s^2}}\right).\end{aligned}$$

The corresponding measure μ can be obtained by 'the inverse Hilbert transform' of $\theta_0(s) = \dfrac{2}{1+\sqrt{1-s^2}}$. We showed that it is the semi-circle distribution in §4.6 of [4]. □

Theorem 4.4. *For* $c_1 = -a,\ c_2 > \frac{1}{2}a^2,\ c_3 = \frac{9}{8}b^2,\ c_4 = -c_2 + 2a^2,\ c_5 = a(2c_2 - a^2),\ c_6 = \frac{3}{2}b^3$, *corresponding functions are given as*

$$\begin{aligned}
\rho(t) &= \frac{12bt}{4(2c_2 - a^2) + 6abt + 3b^2t^2}, \\
\varphi(t) &= \frac{\big(4(2c_2 - a^2) + 6abt + 3b^2t^2\big)^2}{4(2c_2 - a^2)\big(4(2c_2 - a^2) - 3b^2t^2\big)}, \\
\varphi_0(t) &= \frac{4(2c_2 - a^2) + 6abt + 3b^2t^2}{4(2c_2 - a^2)},
\end{aligned}$$

$$\begin{aligned}
J(ts) &= \Theta_\rho(t,s) = \frac{16(2c_2 - a^2)^2\big(4(2_c - a^2) + 3b^2st\big)}{\big(4(2c_2 - a^2) - 3b^2st\big)^3}, \\
\rho^{-1}(s) &= \frac{4(2c_2 - a^2)s}{b\big(6 - 3as + \sqrt{3}\sqrt{12 - 12as - (8c_2 - 7a^2)s^2}\big)}, \\
\theta(s) &= \frac{24}{12 - 12as - (8c_2 - 7a^2)s^2 + \sqrt{3}(2 - as)\sqrt{12 - 12as - (8c_2 - 7a^2)s^2}}, \\
\theta_0(s) &= \frac{12}{(3(2 - as) + \sqrt{3}\sqrt{12(1 - as) - (8c_2 - 7a^2)s^2}}, \\
\widetilde{\theta}(t,s) &= \frac{2\sqrt{3}}{(2c_2 - a^2)(s - t)^3} \\
&\quad \times \Bigg(\frac{(6s + 6t - 9ast - 8c_2st^2 - 3at^2 + 7a^2st^2)}{\sqrt{12 - 12at - (8c_2 - 7a^2)t^2}} \\
&\qquad - \frac{(6s + 6t - 9ast - 8c_2s^2t - 3as^2 + 7a^2ts^2)}{\sqrt{12 - 12as - (8c_2 - 7a^2)s^2}} \Bigg) \\
f_\mu(x) &= \frac{\sqrt{3}}{\pi(2c_2 - a^2)}\sqrt{4(2c_2 - a^2) - 3(2x - a)^2}.
\end{aligned}$$

Proof. Let us apply Proposition 2.1 to Example 4.3 with parameters

$$p = \frac{\sqrt{3}}{\sqrt{2c_2 - a^2}}, \quad q = -\frac{\sqrt{3}a}{2\sqrt{2c_2 - a^2}}, \quad r = \frac{\sqrt{3}b}{2\sqrt{2c_2 - a^2}}.$$

Then we have the conclusion. □

Let us observe the last case.

Example 4.5. Take $a = \frac{1}{2},\ b = \frac{3}{2},\ c_1 = 0,\ c_2 = \frac{3}{2},\ c_3 = 3,\ c_4 = -\frac{3}{2},\ c_5 = \frac{9}{4},\ c_6 = \frac{15}{2}$. Then

$$\begin{aligned}
\rho(t) &= \frac{2t}{1+t^2}, \quad \varphi(t) = \frac{(1+t^2)^2}{1-t}, \quad \varphi_0(t) = \frac{1}{2}(2+t)(1+t^2)^2,\\
\rho^{-1}(s) &= \frac{s}{1+\sqrt{1-s^2}}, \quad \theta(s) = \frac{4}{2-s-s^2+(2-s)\sqrt{1-s^2}},\\
\theta_0(s) &= \frac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}},\\
J(ts) &= \Theta_\rho(t,s) = \frac{1}{(1-ts)^3},\\
\widetilde{\theta}(t,s) &= \frac{2(1-s)(2+s-t-2ts)\sqrt{1-t^2} - 2(1-t)(2-s+t-2ts)\sqrt{1-s^2}}{(1-t)(1-s)(t-s)^3},\\
f_\mu(x) &= \frac{2}{\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{1}{2}} \quad \text{(beta distribution } \widetilde{B}(\tfrac{5}{2},\tfrac{3}{2}) \text{ over } [-1,\ 1]).
\end{aligned}$$

Proof. In Lemma 3.2 (iii), put $b = \frac{3}{2},\ c_1 = \frac{1}{2}$. Then

$$a = 0, \quad b = \frac{3}{2}, \quad c_1 = \frac{1}{2}, \quad c_2 = \frac{9}{8}, \quad c_3 = 3, \quad c_4 = -\frac{9}{8}, \quad c_5 = \frac{15}{16}, \quad c_6 = \frac{15}{2},$$

$$\rho(t) = \frac{4t}{2-t+2t^2}, \quad \text{and} \quad \varphi(t) = \frac{\left(2-t+2t^2\right)^2}{4(1-t)}.$$

Deform those parameters and functions by $\tau_{1,-\frac{1}{4}}$, and assigned them by " $\widehat{\ }$ ". Then we have

$$\widehat{a} = \frac{1}{2}, \quad \widehat{b} = \frac{3}{2}, \quad \widehat{c}_1 = 0, \quad \widehat{c}_2 = \frac{3}{2}, \quad \widehat{c}_3 = 3, \quad \widehat{c}_4 = -\frac{3}{2}, \quad \widehat{c}_5 = \frac{9}{4}, \quad \widehat{c}_6 = \frac{15}{2},$$

$$\widehat{\rho}(t) = \tau_{1,-\frac{1}{4}}\rho(t) = \frac{2t}{1+t^2} \quad \text{and} \quad \widehat{\varphi}(t) = \tau_{1,-\frac{1}{4}}\varphi(t) = \frac{(1+t^2)^2}{1-t}$$

by Proposition 2.1. Obviously, $\widehat{\rho}^{-1}(s) = \dfrac{s}{1+\sqrt{1-s^2}}$. Hence

$$\begin{aligned}
\widehat{\theta}(s) &= \widehat{\varphi}(\widehat{\rho}^{-1}(s)) = \left(\frac{2}{1+\sqrt{1-s^2}}\right)^2 \frac{1+\sqrt{1-s^2}}{1-s+\sqrt{1-s^2}}\\
&= \frac{4}{2-s-s^2+(2-s)\sqrt{1-s^2}},
\end{aligned}$$

$$\begin{aligned}
\widehat{\theta}_0(s) &= \frac{2}{(1-s)(2+s)+(2-s)\sqrt{1-s^2}}\left(2 - \left(\frac{1}{2} + \frac{3}{2}\frac{2}{1+\sqrt{1-s^2}}\right)\right)\\
&= \frac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}}
\end{aligned}$$

by (4.7). $\widehat{\theta}_0(s)$ satisfies $\widehat{\theta}(s) = \widehat{\theta}_0(s) + s\widehat{\theta}_0'(s)$. By (4.9), we see

$$\widehat{\varphi}_0(t) = \frac{(1+t^2)^2}{2(1-t)}\left(2 - \left(\frac{1}{2} + \frac{3}{2}t\right)\frac{2t}{1+t^2}\right) = \frac{1}{2}(2+t)(1+t^2).$$

Now, we obtain the corresponding measure $\widehat{\mu}$. 'The Hilbert transform' $\widetilde{\theta}_0(s)$ of $\widehat{\mu}$ is given in §5 of [8] as follows. Firstly, let us denote the analytic continuation of $\widehat{\theta}(s)$ for s with $s^{-1} \in \mathbb{C}\backslash S_{\widehat{\mu}}$ by the same symbol $\widehat{\theta}(s)$.

$$\widetilde{\theta}_0(s) = \lim_{\epsilon \to +0} \frac{\widehat{\theta}_0(s+i\epsilon) + \widehat{\theta}_0(s-i\epsilon)}{2} \quad \text{for} \quad s \in \mathbb{R}. \tag{4.11}$$

Actually,

$$\widetilde{\theta}_0(s) = \text{p.v.} \int_{-\infty}^{\infty} \frac{f_{\widehat{\mu}}(x)}{1-sx}\, dx.$$

holds. Then the density $f_{\widehat{\mu}}(x)$ is given by 'the inverse Hilbert transform'

$$f_{\widehat{\mu}}(x) = \text{p.v.} \int_{-\infty}^{\infty} \frac{1}{\pi^2} \frac{\widetilde{\theta}_0(s)}{1-xs}\, ds. \tag{4.12}$$

By (4.11)

$$\widetilde{\theta}_0(s) = \begin{cases} \dfrac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}} & \text{if } |s|<1, \\ \dfrac{2+2s-s^2}{s^3} & \text{otherwise,} \end{cases}$$

since

$$\begin{aligned} \widehat{\theta}_0(s) &= \frac{4+5s}{2+2s-s^2+2(1+s)\sqrt{1-s^2}} \\ &= \frac{(4+5s)(2+2s-s^2-2(1+s)\sqrt{1-s^2})}{(2+2s-s^2)^2-4(1+s)^2(1-s^2)} \\ &= \frac{2+2s-s^2-2(1+s)\sqrt{1-s^2}}{s^3}. \end{aligned}$$

Then for $|x|>1$,

$$\begin{aligned} &\text{p.v.} \int_{|s|>1} \frac{\widetilde{\theta}_0(s)}{1-xs}\, ds = \text{p.v.} \int_{|s|>1} \frac{2+2s-s^2}{s^3(1-xs)}\, ds \\ &= \lim_{A\to\infty} \int_1^A \left(\frac{4(1+x)}{s^2} - \frac{x(1-2x-2x^2)}{1-xs} - \frac{x(1-2x-2x^2)}{1+xs} \right) ds \\ &= 4(1+x) + (1-2x-2x^2)\log\frac{|1+x|}{|1-x|}. \end{aligned}$$

Similarly, for $0 \le x < 1$,

$$\begin{aligned} &\text{p.v.} \int_{|s|>1} \frac{\widetilde{\theta}_0(s)}{1-xs}\, ds = \text{p.v.} \int_{|s|>1} \frac{2+2s-s^2}{s^3(1-xs)}\, ds \\ &= \text{p.v.} \int_{|s|>1} \left(\frac{-1+2x+2x^2}{s} + \frac{2(1+x)}{s^2} + \frac{2}{s^3} - \frac{x(1-2x-2x^2)}{1-xs} \right) ds \\ &= \lim_{A\to\infty, \epsilon\to 0} \left(\int_0^{\frac{1}{x}-\epsilon} + \int_{\frac{1}{x}+\epsilon}^{A} \right) \\ &\qquad \times \left(\frac{4(1+x)}{s^2} - \frac{x(1-2x-2x^2)}{1-xs} - \frac{x(1-2x-2x^2)}{1+xs} \right) ds \\ &= 4(1+x) + (1-2x-2x^2)\log\frac{|1+x|}{|1-x|}. \end{aligned}$$

For $|x| < 1$, we have

$$\begin{aligned}
&\text{p.v.} \int_{|s|<1} \frac{\widetilde{\theta}_0(s)}{1-xs}\, ds \\
= \;& \text{p.v.} \int_{|s|<1} \frac{4+5s}{\left(2+2s-s^2+2(1+s)\sqrt{1-s^2}\right)(1-xs)}\, ds \\
= \;& \text{p.v.} \int_0^\infty \frac{4+5\frac{u^2-1}{u^2+1}}{\left(2+2\frac{u^2-1}{u^2+1}-\frac{(u^2-1)^2}{(u^2+1)^2}+2(1+\frac{u^2-1}{u^2+1})\frac{2u}{u^2+1}\right)} \\
&\qquad \times \frac{1}{\left(1-x\frac{u^2-1}{u^2+1}\right)} \frac{4u}{(u^2+1)^2}\, du \\
= \;& \text{p.v.} \int_0^\infty \frac{4u(3u+1)}{(u+1)^3\left(1+x+(1-x)u^2\right)}\, du \\
= \;& \lim_{A\to\infty} \int_0^A \Bigg(-\frac{2(1-2x-2x^2)}{u+1} - \frac{2(2x+3)}{(u+1)^2} + \frac{4}{(u+1)^3} \\
&\qquad +2(1-x)\frac{(1-2x-2x^2)u+2(1+x)^2}{1+x+(1-x)u^2} \Bigg)\, du \\
= \;& (1-2x-2x^2)\log\frac{1-x}{1+x} - 4(x+1) + 2\pi(1+x)\sqrt{1-x^2},
\end{aligned}$$

replacing $u = \sqrt{\dfrac{1+x}{1-x}},\ s = \dfrac{u^2-1}{u^2+1},\ ds = \dfrac{4u}{(u^2+1)^2}$. For $x > 1$, we have similarly

$$\begin{aligned}
&\text{p.v.} \int_{|s|<1} \frac{\widetilde{\theta}_0(s)}{1-xs}\, ds \\
= \;& \lim_{A\to\infty} \int_0^A \Bigg(-\frac{2(1-2x-2x^2)}{u+1} - \frac{2(2x+3)}{(u+1)^2} \\
&\qquad + \frac{4}{(u+1)^3} + (1-x)\frac{2(1-2x-2x^2)u}{1+x+(1-x)u^2} \Bigg)\, du \\
&\qquad - \lim_{A\to\infty,\epsilon\to+0} \left(\int_0^{\sqrt{\frac{x+1}{x-1}}-\epsilon} + \int_{\sqrt{\frac{x+1}{x-1}}+\epsilon}^A \right) \frac{4(1+x)^2(x-1)}{x+1-(x-1)u^2}\, du \\
= \;& (1-2x-2x^2)\log\frac{1-x}{1+x} - 4(x+1).
\end{aligned}$$

Thus we obtain

$$f_{\widehat{\mu}}(x) = \text{p.v.} \int \frac{\widetilde{\theta}_0(s)}{\pi^2(1-xs)}\, ds = \begin{cases} \dfrac{2}{\pi}(1+x)\sqrt{1-x^2} & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 1 \end{cases}$$

using (4.12). By (4.8) with $\widehat{\rho}(t) = 2t/(1+t^2)$ and $\widehat{\varphi}(t) = (1+t^2)^2/(1-t)$,

$$\begin{aligned}
\widehat{\Theta}_{\widehat{\rho}}(t,s) \;=\;& \frac{(1+t^2)(1+s^2)(1-t)(1-s)}{8(s-t)^3(1-ts)^3} \\
&\times \left(\frac{8s^2\left(s-3t+ts+st^2\right)}{(1+t^2)(1+s^2)(1-s)} - \frac{8t^2\left(t-3s+ts+ts^2\right)}{(1+t^2)(1+s^2)(1-t)} \right)
\end{aligned}$$

$$= \frac{1}{(1-ts)^3},$$

since

$$\begin{aligned}
\widehat{\rho}(t)^2\widehat{\varphi}(t) &= \frac{4t^2}{1-t}, \\
2\widehat{\rho}(s) - 6\widehat{\rho}(t) + (1+3s)\widehat{\rho}(s)\widehat{\rho}(t) &= \frac{4(s-3t+ts+st^2)}{(1+t^2)(1+s^2)}, \\
(\widehat{\rho}(s) - \widehat{\rho}(t))^3\widehat{\varphi}(t)\widehat{\varphi}(s) &= \frac{8(s-t)^3(1-ts)^3}{(1+t^2)(1+s^2)(1-t)(1-s)}.
\end{aligned}$$

$\widetilde{\widetilde{\theta}}(t,s)$ can be calculated by $(4.3)'$. □

Theorem 4.6. *For* $c_1 \neq -a$, $c_2 = \frac{1}{2}(12a^2 + 20ac_1 + 9c_1^2)$, $c_3 = \frac{4}{3}b^2$, $c_4 = -\frac{3}{2}(2a+c_1)(2a+3c_1)$, $c_5 = \frac{3}{2}(a+c_1)(2a+c_1)(6a+5c_1)$, $c_6 = \frac{20}{9}b^3$, *corresponding functions are given as*

$$\begin{aligned}
\rho(t) &= \frac{6bt}{18(a+c_1)^2 - 3bc_1t + 2b^2t^2}, \\
\varphi(t) &= \frac{\big(18(a+c_1)^2 - 3bc_1t + 2b^2t^2\big)^2}{108(a+c_1)^3\big(3(a+c_1) - bt\big)}, \\
\varphi_0(t) &= \frac{\big(6(a+c_1) + bt\big)\big(18(a+c_1)^2 - 3bc_1t + 2b^2t^2\big)}{108(a+c_1)^3}, \\
J(ts) &= \Theta_\rho(t,s) = \frac{729(a+c_1)^6}{(9(a+c_1)^2 - b^2st)^3}, \\
\rho^{-1}(s) &= \frac{12(a+c_1)^2 s}{b\left(2 + c_1 s + \sqrt{\big(2-(4a+3c_1)s\big)\big(2+(4a+5c_1)s\big)}\right)}, \\
\theta(s) &= \frac{8}{\Big(\big(2+(4a+5c_1)s\big)\big(2-(4a+3c_1)s\big) + \big(2-(2a+c_1)s\big)\sqrt{\big(2-(4a+3c_1)s\big)\big(2+(4a+5c_1)s\big)}\Big)}, \\
\theta_0(s) &= \frac{4\big(2+(5a+6c_1)s\big)}{\Big(4 + 4(2a+3c_1)s - (8a^2+12ac_1+3c_1^2)s^2 + \big(2+(4a+5c_1)s\big)\sqrt{\big(2-(4a+3c_1)s\big)\big(2+(4a+5c_1)s\big)}\Big)}, \\
f_\mu(x) &= \frac{\sqrt{(4a+5c_1+2x)^3(4a+3c_1-2x)}}{16\pi|a+c_1|^3}.
\end{aligned}$$

Proof. Let us apply Proposition 2.1 to Example 4.5 with parameters

$$p = \frac{1}{2(a+c_1)}, \quad q = \frac{c_1}{4(a+c_1)}, \quad r = \frac{b}{3(a+c_1)}.$$

Then we have the conclusion. □

5. Tables of Typical MRM-applicable Measures

We have seen that for the renormalizing function $h(x) = (1-x)^{-2}$, MRM-applicable measures are classified into three classes. Their typical cases are beta distributions over $[-1,\ 1]$ (or over $[0,\ 1]$). They are shown in the following Table 1 and Table 2. Here, we say that a measure μ is *typical* in a class, if other measures in the class can be obtained by an action of motion group, i.e. an affine transform, of the measure μ.

Table 1 shows the examples in §4 together with generating functions, orthogonal polynomials and Jacobi-Szegö parameters. Generating function $\psi(t,x)$ is easily obtained by using the equality $\psi(t,x) = h(\rho(t)x)/\varphi(t)$. From the generating function, we can calculate orthogonal polynomials $\{P_n(x)\}$ and Jacobi-Szegö parameters applying the method in [3]. In the table, $\{C_n^\kappa(x)\}$ are Gegenbauer polynomials.

Beta Distributions over $[-1,\ 1]$ as Typical Examples for $h(x) = \frac{1}{(1-x)^2}$			
	$a=0, b=\frac{4}{3}$, $c_1=0, c_2=1$, $c_3=\frac{5}{2}, c_4=-\frac{3}{2}$, $c_5=0, c_6=6$	$a=0, b=2$, $c_1=0, c_2=\frac{3}{2}$, $c_3=\frac{9}{2}, c_4=-\frac{3}{2}$, $c_5=0, c_6=12$	$a=\frac{1}{2}, b=\frac{3}{2}$, $c_1=0, c_2=\frac{3}{2}$, $c_3=3, c_4=-\frac{3}{2}$, $c_5=\frac{9}{4}, c_6=\frac{15}{2}$
$\rho(t)$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$
$\varphi(t)$	$(1+t^2)^2$	$\frac{(1+t^2)^2}{1-t^2}$	$\frac{(1+t^2)^2}{1-t}$
$\varphi_0(t)$	$\frac{1}{3}(3-t^2)(1+t^2)$	$1+t^2$	$\frac{1}{2}(2+t)(1+t^2)$
$\psi(t,x)$	$\frac{1}{(1-2xt+t^2)^2}$	$\frac{1-t^2}{(1-2xt+t^2)^2}$	$\frac{1-t}{(1-2xt+t^2)^2}$
$\Theta_\rho(t,s)$	$\frac{3-ts}{3(1-ts)^3}$	$\frac{1+ts}{(1-ts)^3}$	$\frac{1}{(1-ts)^3}$
$\rho^{-1}(s)$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$
$\theta(s)$	$\frac{4}{(1+\sqrt{1-s^2})^2}$	$\frac{2}{\sqrt{1-s^2}(1+\sqrt{1-s^2})}$	$\frac{4}{(2-s-s^2+(2-s)\sqrt{1-s^2})}$
$\theta_0(t)$	$\frac{4\left(1+2\sqrt{1-s^2}\right)}{3\left(1+\sqrt{1-s^2}\right)^2}$	$\frac{2}{1+\sqrt{1-s^2}}$	$\frac{4+5s}{(2+2s-s^2+2(1+s)\sqrt{1-s^2})}$
μ	$\widetilde{B}(\frac{5}{2},\frac{5}{2})$	$\widetilde{B}(\frac{3}{2},\frac{3}{2})$	$\widetilde{B}(\frac{5}{2},\frac{3}{2})$
$f_\mu(x)$	$\frac{8}{3\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{3}{2}}$	$\frac{2}{\pi}\sqrt{1-x^2}$	$\frac{2}{\pi}(1+x)^{\frac{3}{2}}(1-x)^{\frac{1}{2}}$
$P_n(x)$	$\frac{1}{2^n(n+1)}C_n^2(x)$	$\frac{1}{2^n}C_n^1(x)$	$\frac{1}{2^n(n+1)}\left(C_n^2(x)-C_{n-1}^1(x)\right)$
α_n	0	0	$\frac{1}{2(n+1)(n+2)}$
ω_n	$\frac{(n+1)(n+4)}{4(n+2)(n+3)}$	$\frac{1}{4}$	$\frac{(n+1)(n+3)}{4(n+2)^2}$

(The numbering of $\{\omega_n\}$ is different from some of our papers (e.g. [4]).

Table 1

Beta Distributions over [0, 1] as Typical Examples for $h(x) = \frac{1}{(1-x)^2}$			
	$a=1, b=\frac{2}{3}$, $c_1=-1, c_2=\frac{3}{4}, c_3=\frac{5}{8}$, $c_4=\frac{9}{8}$, $c_5=\frac{3}{8}$, $c_6=\frac{3}{4}$	$a=1, b=1$, $c_1=-1, c_2=\frac{7}{8}, c_3=\frac{9}{8}$, $c_4=\frac{9}{8}, c_5=\frac{3}{4}, c_6=\frac{3}{2}$	$a=\frac{5}{4}, b=\frac{3}{4}$, $c_1=-1, c_2=\frac{11}{8}, c_3=\frac{3}{4}$, $c_4=\frac{9}{8}, c_5=\frac{45}{32}, c_6=\frac{15}{16}$
$\rho(t)$	$\frac{4t}{(1+t)^2}$	$\frac{4t}{(1+t)^2}$	$\frac{4t}{(1+t)^2}$
$\varphi(t)$	$(1+t)^4$	$\frac{(1+t)^4}{1-t^2}$	$\frac{(1+t)^4}{1-t}$
$\varphi_0(t)$	$\frac{1}{3}(1+t)^2(3-t^2)$	$(1+t)^2$	$\frac{1}{2}(2+t)(1+t)^2$
$\psi(t,x)$	$\frac{1}{(1+2(1-2x)t+t^2)^2}$	$\frac{1-t^2}{(1+2(1-2x)t+t^2)^2}$	$\frac{1-t}{(1+2(1-2x)t+t^2)^2}$
$\Theta_\rho(t,s)$	$\frac{3-ts}{3(1-ts)^3}$	$\frac{1+ts}{(1-ts)^3}$	$\frac{1}{(1-ts)^3}$
$\rho^{-1}(s)$	$\frac{s}{(1+\sqrt{1-s})^2}$	$\frac{s}{(1+\sqrt{1-s})^2}$	$\frac{s}{(1+\sqrt{1-s})^2}$
$\theta(s)$	$\frac{16}{(1+\sqrt{1-s})^4}$	$\frac{4}{\sqrt{1-s}\,(1+\sqrt{1-s})^2}$	$\frac{8}{\sqrt{1-s}\,(1+\sqrt{1-s})^3}$
$\theta_0(t)$	$\frac{8\left(2-s+4\sqrt{1-s}\right)}{3(1+\sqrt{1-s})^4}$	$\frac{4}{(1+\sqrt{1-s})^2}$	$\frac{2\left(3+\sqrt{1-s}\right)}{(1+\sqrt{1-s})^3}$
μ	$B(\frac{5}{2}, \frac{5}{2})$	$B(\frac{3}{2}, \frac{3}{2})$	$B(\frac{5}{2}, \frac{3}{2})$
$f_\mu(x)$	$\frac{128}{3\pi}x^{\frac{3}{2}}(1-x)^{\frac{3}{2}}$	$\frac{8}{\pi}\sqrt{x(1-x)}$	$\frac{16}{\pi}x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}$

Table 2

The following Table 3 shows all MRM-applicable measures for the general case, which will be proved separately. A similar discussion is given by Demni [5].

Beta Distributions over [−1, 1] as Typical Examples for $h(x) = \frac{1}{(1-x)^\kappa}$ $(\kappa \neq 0,\ 1)$			
κ	$\kappa > -\frac{1}{2}$	$\kappa > \frac{1}{2}$	$\kappa > \frac{1}{2}$
$\rho(t)$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$	$\frac{2t}{1+t^2}$
$\varphi(t)$	$(1+t^2)^\kappa$	$\frac{(1+t^2)^\kappa}{1-t^2}$	$\frac{(1+t^2)^\kappa}{1-t}$
$\psi(t,x)$	$\frac{1}{(1-2xt+t^2)^\kappa}$	$\frac{1-t^2}{(1-2xt+t^2)^\kappa}$	$\frac{1-t}{(1-2xt+t^2)^\kappa}$
$\rho^{-1}(s)$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$	$\frac{s}{1+\sqrt{1-s^2}}$
$\theta(s)$	$\frac{2^\kappa}{(1+\sqrt{1-s^2})^\kappa}$	$\frac{1}{\sqrt{1-s^2}}\frac{2^{\kappa-1}}{\left(1+\sqrt{1-s^2}\right)^{\kappa-1}}$	$\frac{2}{1-s+\sqrt{1-s^2}} \times \frac{2^{\kappa-1}}{\left(1+\sqrt{1-s^2}\right)^{\kappa-1}}$
μ	$\widetilde{B}(\kappa+\frac{1}{2}, \kappa+\frac{1}{2})$	$\widetilde{B}(\kappa-\frac{1}{2}, \kappa-\frac{1}{2})$	$\widetilde{B}(\kappa+\frac{1}{2}, \kappa-\frac{1}{2})$

Table 3

Obviously, the case $\kappa = 0$ is not appropriate for multiplicative renormalization method. The case $\kappa = 1/2$ is described in the table also. Actually $\widetilde{B}(1,1)$ is the uniform measure on $[-1,\ 1]$. The table of typical measures on $[0,\ 1]$ obtained by the motion group similarly to Table 2 is omitted here. One mode interacting Fock spaces related to MRM-applicable measures and the case of finitely supported measures will be studied too.

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