

## THE MINIMAL MARTINGALE MEASURE FOR THE PRICE PROCESS WITH POISSON SHOT NOISE JUMPS

JUN YAN\*

ABSTRACT. In this article, we consider the problem of the minimal martingale measures for the price process with Poisson shot noise jumps. We give out the minimal martingale measure for this price process by the exponential martingale method.

### 1. Introduction

Let  $S(t), 0 \leq t \leq T$  be a price process which is defined on probability space  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ , where  $\mathcal{F}_t$  denotes the completion of  $\sigma(S(u), 0 \leq u \leq t)$ ,  $\mathbb{F} = \{\mathcal{F}_t, 0 \leq t \leq T\}$ , and  $S(t)$  evolves as

$$S(t) := \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t \right\} \prod_{k=1}^{N(t)} (1 + aH(t - T_k, Y_k)) \quad (1.1)$$

$b, \sigma$  are positive constants,  $a$  is a constant,  $\{B(t), 0 \leq t \leq T\}$  is a standard Brown motion,  $Y_k, k \geq 1$  are a sequence of i.i.d random variables, their common distribution is  $F(\cdot)$ ,  $T_k, k \geq 1$  are the jump times of the Poisson process  $\{N(t), 0 \leq t \leq T\}$  with non random intensity process  $\lambda(t)$ , and we assume  $\int_0^T \lambda(u) du < +\infty$ , the bivariate function  $H(t, y)$  is nonnegative, and  $H(t, y) = 0$  for  $t < 0$ , and we make an assumption that for any  $y$ ,  $H(\cdot, y)$  does not increase. We also assume that  $aH(t - T_k, Y_k) > -1$ , otherwise  $S(t)$  may take negative value, which conflicts with the reality. By employing the random measure, we can rewrite  $S(t)$  by

$$\begin{aligned} S(t) &= \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t + \sum_{k=1}^{N(t)} \log(1 + aH(t - T_k, Y_k)) \right\} \\ &= \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t + \int_0^t \int_0^\infty \log(1 + aH(t - u, y)) N(dy, du) \right\} \end{aligned} \quad (1.2)$$

where  $N(dy, du)$  is a Poisson random measure with intensity  $\lambda(u)F(dy)du$ , and we denote  $\tilde{N}(dy, du) = N(dy, du) - \lambda(u)F(dy)du$ ,  $\tilde{N}(dy, du)$  is usually called martingale measure. In financial theory, we usually need to consider the discounted

---

Received 2011-6-19; Communicated by J. Xiong.

2000 *Mathematics Subject Classification.* Primary 60H30; Secondary 60J75, 91B24.

*Key words and phrases.* Minimal martingale measure, Poisson shot noise process, exponential martingale.

\* Research supported by Youth Foundation of Tianyuan Mathematics(No.11026114).

price process which is defined by  $\widehat{S}(t) = S(t)e^{-rt}$  with respect to price process  $S$ , where  $r$  denotes the constant interest rate.

If  $a = 0$ , the price process (1.1) decreases into the following form

$$S_B(t) := \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t \right\} \quad (1.3)$$

which is usually called geometric Brown motion, and we know  $S_B(t)$  is a continuous stochastic process, however, in financial market, the price process may take jumps, so we can extend  $S_B(t)$  into  $S_J(t)$  ([12]) which evolves as

$$S_J(t) := \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t \right\} \prod_{k=1}^{N(t)} (1 + Y_k) \quad (1.4)$$

where  $Y_j$ ,  $j \geq 1$  denote the size of jumps, and  $N(t)$  denotes the time of jump before  $t$ .

In reality, the influence of the jumps should fade away as time goes, in order to reflect this effect, we use  $H(t - T_k, Y_k)$  to replace  $Y_k$ , indeed, due to the fact that  $H(t, y)$  does not increase with respect to the first variable, the influence of  $Y_k$  fades away as time goes. The part  $\sum_{k=1}^{N(t)} \log(1 + aH(t - T_k, Y_k))$  is usually called Poisson shot noise process.

Recall that if any probability measure  $Q \sim P$ , and satisfies that  $\widehat{S}$  is a martingale under  $Q$ , then we usually call  $Q$  the equivalent martingale measure. It is obvious that  $S$  is a price process with jumps, according to the theory of mathematical finance, the market should be incomplete, and the contingent claim is not always attainable, while the equivalent martingale measure is not unique, how to select an equivalent martingale measure as the pricing measure is a problem. Many principles such as minimal entropy martingale measure ([4], [5], [3]), minimal Hellinger martingale measure ([18]), q-optimal martingale measure ([8], [11]) are employed to choose a equivalent martingale measure.

Recently, many literature is interested in minimal martingale measures of all kinds of processes. Schweizer ([17]) studies the minimal martingale measure for the semimartingale, he provides several characterizations of the minimal martingale measure. Takuji ([19]) studies the minimal martingale measure for the jump diffusion process and Paola ([15]) discusses the pure jump case. Altmann, Schmidt and Winfried ([1]) study the minimal martingale measure of shot noise model, for detail, they discuss the following price process

$$S_h(t) := \exp \left\{ bt + \sigma B(t) - \frac{1}{2} \sigma^2 t \right\} \prod_{k=1}^{N(t)} (1 + Y_k h(t - T_k)) \quad (1.5)$$

which is a particular case of our model (1.1) by taking  $H(t, y) = yh(t)$ , where the function  $h(\cdot)$  is nonnegative and non increasing on the positive real line.

Motivated by above studies, in this article, we establish the minimal martingale measure for the price process  $S$  with Poisson shot noise jumps by constructing an exponential martingale, and the definition of the minimal martingale measure is as follows.

**Definition 1.1.** An equivalent martingale measure  $Q$  is called *minimal* if any square integrable  $P$  martingale which is orthogonal to the martingale part of  $\widehat{S}$  under  $P$  remains a martingale under  $Q$ , and  $Q$  is usually called *minimal martingale measure*.

Now let us define the probability measure  $Q$  by

$$\begin{aligned} \frac{dQ}{dP} \Big|_{\mathcal{F}_t} &= Z(t) \\ &:= \frac{\exp \left\{ \int_0^t \sigma G(u) dB(u) + \int_0^t \int_0^\infty c_t(u, y) H(t-u, y) N(dy, du) \right\}}{E \left( \exp \left\{ \int_0^t \sigma G(u) dB(u) + \int_0^t \int_0^\infty c_t(u, y) H(t-u, y) N(dy, du) \right\} \right)} \end{aligned} \quad (1.6)$$

where  $G(\cdot)$  which is nonnegative and  $c_t(\cdot, \cdot)$  satisfy the following condition:

$$(H) : \int_0^T G^2(u) du + E \left( \exp \{ 2c_T(Y_1) H(0, Y_1) \} \right) < +\infty$$

where  $c_T(y) = \sup_{u \in [0, T]} c_T(u, y)$ .

As follows, we introduce a lemma in which we construct an exponential martingale, the lemma plays a key role in this paper, and the proof will be postponed for conciseness. The process  $Z(t)$  in (1.6) is not explicit, with the help of the following key lemma, we can give a clear expression for  $Z(t)$ .

**Lemma 1.2.** *If there exists a  $\delta$  such that  $\int_0^t \int_0^\infty e^{\delta g_t(u, y) H(0, y)} \lambda(u) F(dy) du < +\infty$ , for all  $t \in [0, T]$ , where  $g_t(u, y)$  is a non random function. Then for any Borel measurable function  $\theta(\cdot)$  satisfying  $\sup_{0 \leq t \leq T} \theta(t) < \delta$ , we have*

$$\mathbb{E} \left( e^{\int_0^t \theta(u) dL(u)} \right) = \exp \left\{ \int_0^t \int_0^\infty \left( e^{\theta(u) g_t(u, y) H(t-u, y)} - 1 \right) \lambda(u) F(dy) du \right\} \quad (1.7)$$

where  $L(t) = \sum_{k=1}^{N(t)} g_t(T_k, Y_k) H(t - T_k, Y_k) = \int_0^t \int_0^\infty g_t(u, y) H(t-u, y) N(dy, du)$ .

Take  $\theta(u) = 1$ ,  $g_t(\cdot, \cdot) = c_t(\cdot, \cdot)$  in above lemma, then we can have

$$\begin{aligned} &E \left( \exp \left\{ \sigma \int_0^t G(u) dB(u) + \int_0^t \int_0^\infty c_t(u, y) H(t-u, y) N(dy, du) \right\} \right) \\ &= \exp \left\{ \frac{\sigma^2}{2} \int_0^t G^2(u) du + \int_0^t \int_0^\infty \left( e^{c_t(u, y) H(t-u, y)} - 1 \right) \lambda(u) F(dy) du \right\} \end{aligned} \quad (1.8)$$

by noting the fact that

$$E \left( \exp \left\{ \sigma \int_0^t G(u) dB(u) \right\} \right) = \exp \left\{ \frac{\sigma^2}{2} \int_0^t G^2(u) du \right\} \quad (1.9)$$

so we can give a explicit expression for  $Z(t)$ , that is,

$$\begin{aligned} Z(t) &= \exp \left\{ \sigma \int_0^t G(u) dB(u) + \int_0^t \int_0^\infty c_t(u, y) H(t-u, y) N(dy, du) \right. \\ &\quad \left. - \frac{\sigma^2}{2} \int_0^t G^2(u) du - \int_0^t \int_0^\infty \left( e^{c_t(u, y) H(t-u, y)} - 1 \right) \lambda(u) F(dy) du \right\} \end{aligned} \quad (1.10)$$

by Itô formula ([16]), we can achieve that

$$Z(t) = 1 + \sigma \int_0^t Z(u-)G(u)dB(u) + \int_0^t \int_0^\infty Z(u-) \left( e^{c_t(u,y)H(t-u,y)} - 1 \right) \tilde{N}(dy, du) \quad (1.11)$$

Recall that, in the case of Brown motion, that is, for the price process  $S_B(t)$ , we can define the equivalent probability measure by

$$\frac{dQ_B}{dP} \Big|_{\mathcal{F}_t} = Z_B(t) \quad (1.12)$$

with  $Z_B(t)$  satisfying the stochastic differential equation

$$dZ_B(t) = \sigma a(t)Z_B(t)dB(t) \quad (1.13)$$

where  $a(t)$  makes the corresponding discounted price process

$$\widehat{S}_B(t) := e^{-rt}S_B(t) = \exp \left\{ bt + \sigma B(t) - \frac{1}{2}\sigma^2 t - rt \right\} \quad (1.14)$$

a martingale under  $Q_B$ , for detail,  $a(t)$  satisfies

$$(b-r)t + \sigma^2 \int_0^t a(u)du = 0, \quad 0 \leq t \leq T \quad (1.15)$$

indeed, by Itô formula, we have

$$d\widehat{S}_B(t) = (b-r)\widehat{S}_B(t)dt + \sigma\widehat{S}_B(t)dB(t) \quad (1.16)$$

by Girsanov-Meyer theorem([16])

$$\begin{aligned} dL_B(t) &:= \sigma\widehat{S}_B(t)dB(t) - (Z_B(t))^{-1}d \left[ Z_B, \sigma \int_0^t \widehat{S}_B(u)dB(u) \right]_t \\ &= \sigma\widehat{S}_B(t)dB(t) - \sigma^2 a(t)\widehat{S}_B(t)dt \end{aligned} \quad (1.17)$$

is a  $Q_B$  local martingale, where  $[\cdot, \cdot]$  denotes the quadratic variation([16]), and we can rewrite  $d\widehat{S}_B(t)$  by

$$d\widehat{S}_B(t) = (b-r)\widehat{S}_B(t)dt + \sigma^2 a(t)\widehat{S}_B(t)dt + dL_B(t) \quad (1.18)$$

so under the condition (1.15),  $\widehat{S}_B(t)$  is a  $Q_B$  local martingale, and under some slight condition, we can get  $\widehat{S}_B(t)$  is truly a  $Q_B$  martingale, motivated by above discussion, we assume, in our model (1.1),  $Z(t)$  satisfies the follow stochastic differential equation

$$dZ(t) = \gamma(t)Z(t-)(\sigma dB(t) + dH(t)) \quad (1.19)$$

where  $\gamma(\cdot)$  is an undetermined function, and  $H(t) = \int_0^t \int_0^\infty H(t-u, y)\tilde{N}(dy, du)$ .

Compare (1.11) and (1.19), we know that

$$\begin{cases} G(u) = \gamma(u) \\ e^{c_t(u,y)H(t-u,y)} - 1 = \gamma(u)H(t-u, y) \end{cases} \quad (1.20)$$

that is

$$\begin{cases} G(u) = \gamma(u) \\ c_t(u, y) = \frac{\log(1 + \gamma(u)H(t-u, y))}{H(t-u, y)} \end{cases} \quad (1.21)$$

and we can claim that  $\gamma(\cdot)$  satisfies the follow condition which is usually called martingale condition:

$$(b-r)t + \sigma^2 \int_0^t \gamma(u)du + a \int_0^t \int_0^\infty (H(t-u, y) + \gamma(u)H^2(t-u, y))\lambda(u)F(dy)du = 0 \tag{1.22}$$

with  $0 \leq t \leq T$ , we denote  $Q$  in (1.6) which is determined by  $\gamma(u)$  satisfying (1.22) by  $Q^*$ , that is

$$\begin{aligned} \frac{dQ^*}{dP} \Big|_{\mathcal{F}_t} = \exp \left\{ \sigma \int_0^t \gamma(u)dB(u) + \int_0^t \int_0^\infty \log(1 + \gamma(u)H(t-u, y))N(dy, du) \right. \\ \left. - \frac{\sigma^2}{2} \int_0^t \gamma^2(u)du - \int_0^t \int_0^\infty \gamma(u)H(t-u, y)\lambda(u)F(dy)du \right\} \end{aligned} \tag{1.23}$$

where  $\gamma(\cdot)$  is determined by (1.22). For  $Q^*$ , we have the following lemma.

**Lemma 1.3.** *Under the conditions (H) and (1.22), the discounted price process  $\widehat{S}(t)$  is a martingale under the measure  $Q^*$ .*

The proof of Lemma 1.3 is postponed to the next section.

## 2. Main Result

In this section, we give out our main result, and the proof of Lemma 1.2 and Lemma 1.3 are also presented. Here is our main result.

**Theorem 2.1.** *Under the condition (H), the equivalent martingale measure  $Q^*$  is the minimal martingale measure with respect to the price process  $S$ .*

*Proof.* By Itô formula

$$\begin{aligned} \widehat{S}(t) &= 1 + \sigma \int_0^t \widehat{S}(u-)dB(u) + a \int_0^t \widehat{S}(u-)H(t-u, y)N(dy, du) \\ &\quad + (b-r) \int_0^t \widehat{S}(u-)du \\ &= 1 + \sigma \int_0^t \widehat{S}(u-)dB(u) + a \int_0^t \widehat{S}(u-)H(t-u, y)\widetilde{N}(dy, du) \\ &\quad + (b-r) \int_0^t \widehat{S}(u-)du + a \int_0^t \widehat{S}(u-)H(t-u, y)\lambda(u)F(dy)du \end{aligned} \tag{2.1}$$

obviously, the part  $\sigma \int_0^t \widehat{S}(u-)dB(u) + a \int_0^t \widehat{S}(u-)H(t-u, y)\widetilde{N}(dy, du)$  is a  $P$  martingale. Let  $L$  be any square integrable  $P$  martingale satisfying

$$E \left( L(t) \left( \sigma \int_0^t \widehat{S}(u-)dB(u) + a \int_0^t \widehat{S}(u-)H(t-u, y)\widetilde{N}(dy, du) \right) \right) = 0 \tag{2.2}$$

i.e.,  $L$  is orthogonal to  $\sigma \int_0^t \widehat{S}(u-)dB(u) + a \int_0^t \widehat{S}(u-)H(t-u, y)\widetilde{N}(dy, du)$ , then we have

$$E(L(t)Z(t)) = 0 \tag{2.3}$$

that is

$$E^{Q^*}(L(t)) = 0 \tag{2.4}$$

i.e.,  $L$  is a  $Q^*$  local martingale, and furthermore  $L$  is truly a martingale, indeed we know

$$\begin{aligned} & \sup_{0 \leq t \leq T} E(Z^2(t)) \\ &= \exp \left\{ \sigma^2 \int_0^t G^2(u) du + \int_0^t \int_0^\infty \left( e^{c_t(u,y)H(t-u,y)} - 1 \right)^2 \lambda(u) F(dy) du \right\} \\ &\leq \exp \left\{ \int_0^t (\sigma^2 G^2(u) + \lambda(u)) du + 2 \int_0^t \int_0^\infty e^{2c_t(u,y)H(0,y)} \lambda(u) F(dy) du \right\} \end{aligned} \quad (2.5)$$

therefore, by the condition **(H)**

$$\sup_{0 \leq t \leq T} E(Z^2(t)) < +\infty \quad (2.6)$$

furthermore,  $L$  is a square integrable martingale, i.e.

$$\sup_{0 \leq t \leq T} E(L^2(t)) < +\infty \quad (2.7)$$

by Hölder inequality, we can conclude that

$$\sup_{0 \leq t \leq T} E(L(t)Z(t)) \leq \sqrt{\sup_{0 \leq t \leq T} E(L^2(t))} \sqrt{\sup_{0 \leq t \leq T} E(Z^2(t))} < +\infty \quad (2.8)$$

i.e.,  $L(t)Z(t)$  is bounded under  $P$ , equivalently,  $L(t)$  is bounded under  $Q^*$ , so we have proved that  $L$  is truly a martingale under  $Q^*$ .  $\square$

The proof of Lemma 1.2.

*Proof.* Firstly, we can infer  $\{L(t) - \int_0^t \int_0^\infty g_t(u,y)H(t-u,y)\lambda(u)F(dy)du, \mathcal{G}_t, P\}$  is a martingale, where  $\mathcal{G}_t = \sigma\{N(s), s \leq t\} \vee \sigma\{Y_k, k \leq N(t)\}$ . Indeed

$$\begin{aligned} & \mathbb{E}(L(t)|\mathcal{G}_s) \\ &= L(s) + \mathbb{E} \left( \sum_{k=1}^{N(t)-N(s)} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \end{aligned} \quad (2.9)$$

and by total probability formula

$$\begin{aligned} & \mathbb{E} \left( \sum_{k=1}^{N(t)-N(s)} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \\ &= \sum_{m=1}^{\infty} \sum_{k=1}^m \mathbb{E} \left( I_{N(t)-N(s)=m} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \end{aligned} \quad (2.10)$$

then by Fubini theorem

$$\begin{aligned}
 & \sum_{m=1}^{\infty} \sum_{k=1}^m \mathbb{E} \left( I_{N(t)-N(s)=m} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \\
 &= \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} \mathbb{E} \left( I_{N(t)-N(s)=m} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \\
 &= \sum_{k=1}^{\infty} \mathbb{E} \left( I_{N(t)-N(s) \geq k} g_t(T_{k+N(s)}, Y_{k+N(s)}) H(t - T_{k+N(s)}, Y_{k+N(s)}) \middle| \mathcal{G}_s \right) \\
 &= \sum_{k=1}^{\infty} \mathbb{E} \left( I_{\tilde{T}_k \leq t-s} \int_0^{\infty} g_{t-s}(\tilde{T}_k, y) H(t - s - \tilde{T}_k, y) F(dy) \middle| \mathcal{G}_s \right) \\
 &= \mathbb{E} \left( \int_0^{\infty} I_{(0, t-s]}(u) \int_0^{\infty} g_{t-s}(u, y) H(t - s - u, y) F(dy) \lambda(u + s) du \middle| \mathcal{G}_s \right) \\
 &= \int_s^t \int_0^{\infty} g_t(u, y) H(t - u, y) \lambda(u) F(dy) du
 \end{aligned} \tag{2.11}$$

which implies  $\{L(t) - \int_0^t \int_0^{\infty} g_t(u, y) H(t - u, y) F(dy) du, \mathcal{G}_t, P\}$  is a martingale, where  $\tilde{T}_k = \inf\{u \geq 0 : N(s + u) - N(s) = k\}$ . Let  $D(t) = \int_0^t \theta(u) dL(u)$ , then by Itô formula and martingale properties of  $\{L(t) - \int_0^t \int_0^{\infty} g_t(u, y) H(t - u, y) \lambda(u) F(dy) du, \mathcal{G}_t, P\}$ , we can get that

$$\begin{aligned}
 \mathbb{E} \left( e^{D(t)} \right) &= 1 + \int_0^t \int_0^{\infty} \mathbb{E} \left( e^{D(u^-)} \right) \theta(u) g_t(u, y) H(t - u, y) \lambda(u) F(dy) du \\
 &\quad + \mathbb{E} \left( \left\{ \sum_{k=1}^{\infty} e^{D(T_k-1)} p(\theta(T_k) g_t(T_k, Y_k) H(t - T_k, Y_k)) \right\} I_{T_k \leq t} \right) \\
 &= 1 + \int_0^t \int_0^{\infty} \mathbb{E} \left( e^{D(u^-)} \right) \left( e^{\theta(u) g_t(u, y) H(t - u, y)} - 1 \right) \lambda(u) F(dy) du
 \end{aligned} \tag{2.12}$$

where  $p(x) = e^x - 1 - x$ , so we have

$$\mathbb{E} \left( e^{\int_0^t \theta(u) dL(u)} \right) = \exp \left\{ \int_0^t \int_0^{\infty} \left( e^{\theta(u) g_t(u, y) H(t - u, y)} - 1 \right) \lambda(u) F(dy) du \right\}. \tag{2.13}$$

□

*Remark 2.2.* From the proof of Lemma 1.2, we know

$$\begin{aligned}
 & \int_0^t \int_0^{\infty} g_t(u, y) H(t - u, y) (N(dy, du) - \lambda(u) F(dy) du) \\
 &= \int_0^t \int_0^{\infty} g_t(u, y) H(t - u, y) \tilde{N}(dy, du)
 \end{aligned} \tag{2.14}$$

is a  $P$  martingale, that is why we call  $\tilde{N}(dy, du)$  martingale measure.

The proof of Lemma 1.3.

*Proof.* We know from the proof of Theorem 2.1

$$\widehat{S}(t) = 1 + M^B(t) + M^H(t) + C(t) \quad (2.15)$$

where

$$M_B(t) = \sigma \int_0^t \widehat{S}(u-) (dB(u) - \sigma G(u) du) \quad (2.16)$$

$$\begin{aligned} M_H(t) = & a \int_0^t \widehat{S}(u-) H(t-u, y) \widetilde{N}(dy, du) \\ & - a \int_0^t \int_0^\infty \widehat{S}(u-) H(t-u, y) \left( e^{c_t(u, y) H(t-u, y)} - 1 \right) \lambda(u) F(dy) du \end{aligned} \quad (2.17)$$

$$\begin{aligned} C(t) = & (b-r) \int_0^t \widehat{S}(u-) du + \sigma^2 \int_0^t \widehat{S}(u-) G(u) du \\ & + a \int_0^t \int_0^\infty \widehat{S}(u-) H(t-u, y) e^{c_t(u, y) H(t-u, y)} \lambda(u) F(dy) du \end{aligned} \quad (2.18)$$

by Girsanov-Meyer theorem([16]), we know  $M_B(t)$  is a  $Q^*$  local martingales, for  $M_H(t)$ , according to the formula of integration by parts([16]), we have

$$\begin{aligned} Z(t)M_H(t) &= \int_0^t Z(u-) dM_H(u) + \int_0^t M_H(u-) dZ(u) + [Z(\cdot), M_H(\cdot)]_t \\ &= \sigma \int_0^t M_H(u-) Z(u-) G(u) dB(u) \\ &\quad + a \int_0^t \int_0^\infty Z(u-) \widehat{S}(u-) H(t-u, y) e^{c_t(u, y) H(t-u, y)} \widetilde{N}(dy, du) \\ &\quad + a \int_0^t \int_0^\infty M_H(u-) Z(u-) \left( e^{c_t(u, y) H(t-u, y)} - 1 \right) \widetilde{N}(dy, du) \end{aligned} \quad (2.19)$$

which implies that  $Z(t)M_H(t)$  is a  $P$  local martingale, equivalently,  $M_H(t)$  is a  $Q^*$  local martingale, for detail see [10], so if we need to make  $\widehat{S}$  a local martingale under  $Q^*$ , we should insure that

$$C(t) = 0, \quad 0 \leq t \leq T \quad (2.20)$$

take (1.21) in mind, we know the above equation is equivalent to (1.22). In order to confirm that  $\widehat{S}(t)$  is a martingale, we need to check

$$E^{Q^*} \left( \sup_{0 \leq t \leq T} \widehat{S}(t) \right) < +\infty, \quad 0 \leq t \leq T \quad (2.21)$$

according to Theorem 51 in Chapter I of [16], indeed

$$\begin{aligned} & E^{Q^*} \left( \sup_{0 \leq t \leq T} \widehat{S}(t) \right) \\ & \leq e^{|b-r|T} E^{Q^*} \left( \sup_{0 \leq t \leq T} e^{\sigma B(t)} \right) E^{Q^*} \left( \exp \left\{ \sum_{k=1}^{N(T)} \log(1 + |a|H(0, Y_k)) \right\} \right) \end{aligned} \quad (2.22)$$



by maximal inequality for martingales and noting that  $\sigma B(t) - \frac{\sigma^2}{2} \int_0^t G^2(u)du$  is a  $Q^*$  martingale, we have

$$\begin{aligned} & E^{Q^*} \left( \sup_{0 \leq t \leq T} \exp \left\{ \sigma B(t) - \frac{\sigma^2}{2} \int_0^t G^2(u)du \right\} \right) \\ & \leq \frac{e}{e-1} \left( 1 + \sup_{0 \leq t \leq T} E^{Q^*} \left( q \left( \sigma B(t) - \frac{\sigma^2}{2} \int_0^t G^2(u)du \right) \right) \right) \\ & \leq \frac{e}{e-1} \left( 1 + \sup_{0 \leq t \leq T} E^{Q^*} \left( \exp \left\{ 2\sigma B(t) - \sigma^2 \int_0^t G^2(u)du \right\} \right) \right) \\ & = \frac{e}{e-1} \left( 1 + \exp \left\{ \sigma^2 \int_0^T G^2(u)du \right\} \right) < +\infty \end{aligned} \tag{2.23}$$

where  $q(x) = x^+ e^x$  and  $x^+ := \max\{x, 0\}$ , so we can obtain

$$\begin{aligned} & E^{Q^*} \left( \sup_{0 \leq t \leq T} e^{\sigma B(t)} \right) \\ & \leq \exp \left\{ \frac{\sigma^2}{2} \int_0^T G^2(u)du \right\} E^{Q^*} \left( \sup_{0 \leq t \leq T} \exp \left\{ \sigma B(t) - \frac{\sigma^2}{2} \int_0^t G^2(u)du \right\} \right) \\ & \leq \frac{e}{e-1} \left( \exp \left\{ \frac{\sigma^2}{2} \int_0^T G^2(u)du \right\} + \exp \left\{ \frac{3\sigma^2}{2} \int_0^T G^2(u)du \right\} \right) < +\infty \end{aligned} \tag{2.24}$$

furthermore, by Hölder inequality

$$\begin{aligned} & E^{Q^*} \left( \exp \left\{ \sum_{k=1}^{N(T)} \log(1 + |a|H(0, Y_k)) \right\} \right) \\ & = E \left( Z(T) \exp \left\{ \sum_{k=1}^{N(T)} \log(1 + |a|H(0, Y_k)) \right\} \right) \\ & \leq \sqrt{E(Z^2(T))} \sqrt{E \left( \exp \left\{ \sum_{k=1}^{N(T)} 2 \log(1 + |a|H(0, Y_k)) \right\} \right)} \end{aligned} \tag{2.25}$$

by Lemma 1.2 and noting the condition **(H)**, we know

$$\begin{aligned} & E(Z^2(T)) \\ & = \exp \left\{ \sigma^2 \int_0^T G^2(u)du + \int_0^T \int_0^\infty \left( e^{c_T(u,y)H(T-u,y)} - 1 \right)^2 \lambda(u)F(dy)du \right\} \\ & \leq \exp \left\{ \int_0^T (\sigma^2 G^2(u) + \lambda(u))du + 2 \int_0^T \int_0^\infty e^{2c_T(u,y)H(0,y)} \lambda(u)F(dy)du \right\} < +\infty \end{aligned} \tag{2.26}$$

and by Lemma 1.2 again, we achieve that

$$\begin{aligned} & E \left( \exp \left\{ \sum_{k=1}^{N(T)} 2 \log(1 + |a|H(0, Y_k)) \right\} \right) \\ &= \exp \left\{ \int_0^T \lambda(u) du \int_0^\infty (a^2 H^2(0, y) + 2|a|H(0, y)) F(dy) \right\} < +\infty \end{aligned} \quad (2.27)$$

so we have

$$E^{\mathcal{Q}^*} \left( \exp \left\{ \sum_{k=1}^{N(T)} \log(1 + |a|H(0, Y_k)) \right\} \right) < +\infty \quad (2.28)$$

finally, (2.24) combining with (2.28) implies (2.21), and we complete the proof.  $\square$

**Acknowledgment.** The author would like to thank the referees for their helpful suggestions and comments. Further, we thank the support of the Youth Foundation of Tianyuan Mathematics of China(No. 11026114).

### References

1. Altmann, T., Schmidt, T., and Winfried, S.: A Shot Noise Model for Financial Assets, *International Journal of Theoretical and Applied Finance* (2010).
2. Embrechts, P., Kluppelberg, C., and Mikosch, T.: *Modeling Extremal Events for Insurance And Finance*, Springer Verlag, 2003.
3. Esche, F., and Schweizer, M.: Minimal entropy preserves the Lévy property: how and why, *Stoch. Proc. Appl.* **115** (2005) 299–327.
4. Frittelli, M.: The minimal entropy martingale measure and the valuation problem in incomplete markets, *Math. Finance* **10** (2000) 39–52.
5. Fujiwara, T. and Miyahara, Y.: The minimal entropy martingale measure for geometric Lévy processes, *Finance Stoch.* **7** (2003) 509–531.
6. Fuqing G. and Jun, Y.: Functional large deviations and moderate deviations for Markov-modulated risk models with reinsurance, *J. Appl. Probab.* **45** (2008) 800–817.
7. Goll, T. and Rüschemdorf, L.: Minimax and minimal distance martingale measures and their relationship to portfolio optimization, *Finance Stoch.* **5** (2001) 557–581.
8. Grandits, P.: The p-optimal martingale measure and its asymptotic relation with the minimal entropy martingale measure, *Bernoulli* **5** (1999) 225–247.
9. He, S. W., Wang, J. G., and Yan, J. A.: *Semimartingale Theory and Stochastic Calculus*, Science Press & CRC Press Inc., Beijing & Boca Raton, 1992.
10. Jacod, J. and Shiryaev, A. N.: *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin, 2003.
11. Jeanblanc, M., Klöppel, S., and Miyahara, Y.: Minimal  $f^q$  martingale measures for exponential Lévy process, *Ann. Appl. Prob.* **17** (2007) 1615–1638.
12. Merton, R. C.: Option pricing when underlying stock return are discontinuous, *J. Financial. Econ.* **3** (1976) 125–144.
13. Miyahara, Y.: Pricing model and related estimation problems, *Asia-Pacific Financial Markets* **8** (2001) 45–60.
14. Möller, M.: Pricing PCS options with use of Esscher transforms, in: *Afir Colloquium Nürnberg, Oct. 1-3.* (1996).
15. Paola, T.: Minimal martingale measure: Pricing and hedging in a pure jump model under restricted information, *Nonlinear Anal.* **71** (2009) e1771–e1787.
16. Protter, P. E., *Stochastic Integration and Differential Equation*, Springer-Verlag, Berlin, 2005.

17. Schweizer, M.: On the minimal martingale measure and the Föllmer-Schweizer decomposition, *Stoch. Ana. Appl.* **13** (1995) 573–599.
18. Tahir, C. and Christophe, S.: Comparing the minimal Hellinger martingale measure of order  $q$  to the  $q$ -optimal martingale measure, *Stoch. Ana. Appl.* **119** (2009) 1368–1385.
19. Takuji, A.: Minimal martingale measures for jump diffusion processes, *J. Appl. Prob.* **41** (2004) 263–270.
20. Terence, C.: Pricing contingent claims on stocks driven by Lévy processes, *Ann. Appl. Prob.* **9** (1999) 504–528.
21. Thorsten, S. and Winfried, S.: Shot-noise processes and the minimal martingale measure, *Stat. Proba. Lett.* **9** (2007) 504–528.

JUN YAN: SCHOOL OF MATHEMATICAL SCIENCES, YANGZHOU UNIVERSITY, YANGZHOU 225002, P.R.CHINA

*E-mail address:* [yanjunwh@gmail.com](mailto:yanjunwh@gmail.com)