

## A MARTINGALE REPRESENTATION FOR THE MAXIMUM OF A LÉVY PROCESS

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ABSTRACT. By using Malliavin calculus for Lévy processes, we compute an explicit martingale representation for the maximum of a square-integrable Lévy process.

### 1. Introduction and Preliminary Results

The representation of functionals of Brownian motion by stochastic integrals, also known as martingale representation, has been widely studied. The *martingale representation theorem* states that any square-integrable Brownian functional is equal to a stochastic integral with respect to Brownian motion; it plays an important role in mathematical finance, where it appears naturally in optimal portfolio compositions. Martingale representation theorems are also important in stochastic calculus for Lévy processes. For a survey on martingale representation theorems, see [2]. In the Brownian motion case, the Clark-Ocone formula of Malliavin calculus is a powerful tool to get *explicit* martingale representations for path-dependent Brownian functionals; for an example of application, see [11]. Recently, due the development of Malliavin calculus for Lévy processes, many Clark-Ocone formulas have appeared for various sub-classes of this family of processes. For more details, the reader is invited to have a look at [3].

In this note, using a Clark-Ocone formula for Lévy processes, we compute an explicit martingale representation of the maximum of a square-integrable Lévy process, therefore providing a generalization of the well-known explicit martingale representation of the maximum for Brownian motion.

Let  $T$  be a strictly positive real number and let  $X = (X_t)_{t \in [0, T]}$  be a (one-dimensional) Lévy process defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , i.e.,  $X$  is a process with independent and stationary increments, is continuous in probability and starts from 0 almost surely. We assume that  $X$  is the *càdlàg* modification and that the probability space is equipped with the completed filtration  $(\mathcal{F}_t)_{t \in [0, T]}$  generated by  $X$ . We also assume that the  $\sigma$ -field  $\mathcal{F}$  is equal to  $\mathcal{F}_T$ . This filtration satisfies the *usual conditions* and, for any fixed time  $t$ ,  $\mathcal{F}_{t-} = \mathcal{F}_t$ ; see [10].

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When the Lévy process  $X$  is square-integrable, it can be expressed as

$$X_t = \mu t + \sigma W_t + \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz), \quad (1.1)$$

where  $\mu$  is a real number,  $\sigma$  is a strictly positive real number,  $W$  is a standard Brownian motion and  $\tilde{N}$  is the compensated Poisson random measure associated with the Poisson random measure  $N$  of  $X$ . The Poisson random measure  $N$  is independent of the Brownian motion  $W$ . Its compensator measure is denoted by  $\lambda \times \nu$ , where  $\lambda$  is Lebesgue measure on  $[0, T]$  and  $\nu$  is the Lévy measure of  $X$ , i.e.,  $\nu$  is a  $\sigma$ -finite measure on  $\mathbb{R}$  such that  $\nu(\{0\}) = 0$  and

$$\int_{\mathbb{R}} (1 \wedge z^2) \nu(dz) < \infty.$$

Therefore the compensated random measure  $\tilde{N}$  is defined by

$$\tilde{N}([0, t] \times A) = N([0, t] \times A) - t\nu(A).$$

In this setup, let  $\mathcal{P}$  be the predictable  $\sigma$ -field on  $[0, T] \times \Omega$  and  $\mathcal{B}(\mathbb{R})$  the Borel  $\sigma$ -field on  $\mathbb{R}$ . A process  $\psi(t, z, \omega)$  is said to be Borel predictable if it is  $(\mathcal{P} \times \mathcal{B}(\mathbb{R}))$ -measurable. For the rest of the paper, we suppose that  $X$  is a square-integrable Lévy process with a decomposition as in Equation (1.1).

A direct extension of Theorem 9.10 in [3] (see e.g. [12] for more details) yields the following martingale representation theorem for Lévy processes:

**Theorem 1.1.** *Let  $F \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ . There exist a unique Borel predictable process  $\psi \in L^2(\lambda \times \nu \times \mathbb{P})$  and a unique predictable process  $\phi \in L^2(\lambda \times \mathbb{P})$  such that*

$$F = \mathbb{E}[F] + \int_0^T \phi(t) dW_t + \int_0^T \int_{\mathbb{R}} \psi(t, z) \tilde{N}(dt, dz). \quad (1.2)$$

The rest of the paper is organized as follows. In Section 2, we first recall the martingale representation for the maximum of Brownian motion and then state our main result, that is the martingale representation for the maximum of a general square-integrable Lévy process. Then, in Section 3, after a very short presentation of the relevant results from Malliavin calculus for Lévy processes, we proceed to the proof of the main result.

## 2. Martingale Representation for the Running Supremum

For  $0 \leq s < t \leq T$ , define  $M_{s,t} = \sup_{s \leq r \leq t} X_r$  and  $M_t = M_{0,t}$ . Our main result is a generalization of the following one:

**Theorem 2.1.** *If  $X = W$ , i.e. if  $X$  is a standard Brownian motion, then*

$$M_T = \sqrt{\frac{2T}{\pi}} + \int_0^T 2 \left[ 1 - \Phi \left( \frac{M_t - W_t}{\sqrt{T-t}} \right) \right] dW_t,$$

where  $\Phi(x) = \mathbb{P}\{N(0, 1) \leq x\}$ .

This last representation can be found in [13]. Their proof uses Clark's formula (see [1]), which is essentially a Clark-Ocone formula on the canonical space of Brownian motion. It can also be computed using a completely different method

based on Itô’s formula (see [14]). Let’s note that if one extends this by adding a drift to the Brownian motion, the results are similar (see [11]).

Our main result consists in providing explicit expressions for  $\phi$  and  $\psi$  appearing in (1.2) for  $F = M_T$  when  $X$  is an arbitrary square-integrable Lévy process. Set  $\bar{F}_t(y) = \mathbb{P}\{M_t > y\}$ .

**Theorem 2.2.** *If  $X$  a square-integrable Lévy process, then its running maximum can be written as follows:*

$$M_T = \mathbb{E}[M_T] + \int_0^T \phi(t, M_t - X_t) dW_t + \int_0^T \int_{\mathbb{R}} \psi(t, z, M_t - X_t) \tilde{N}(dt, dz),$$

with  $\phi(t, y) = \sigma \bar{F}_{T-t}(y)$  and

$$\psi(t, z, y) = (z - y)^+ + \mathbb{I}_{\{z > y\}} \int_0^y \bar{F}_{T-t}(x) dx + \mathbb{I}_{\{z \leq y\}} \int_{y-z}^y \bar{F}_{T-t}(x) dx.$$

It is easily shown that Theorem 2.2 is an extension of Theorem 2.1; it suffices to notice that when  $X = W$  we have  $\tilde{N}(dt, dz) \equiv 0$  and

$$\bar{F}_t(y) = 2 \left( 1 - \Phi \left( y/\sqrt{t} \right) \right).$$

When  $X$  is a square-integrable subordinator, e.g. a Gamma subordinator, then trivially  $M_T = X_T$  and  $\bar{F}_t(y)$  is the tail of the distribution of  $X_t$ . For many jump-diffusion Lévy processes, i.e., when  $X$  is the sum of a Brownian motion with drift and a compound Poisson jump structure with finite second moment (e.g. Kou’s model [5]), the distribution of  $M_T$  can be obtained up to the inversion of a Laplace transform. The distribution of the running supremum of more sophisticated Lévy processes have been studied by A. Kuznetsov and his co-authors, see [7, 6].

### 3. Proof of Theorem 2.2

Our proof for Theorem 2.2 is based on a Malliavin calculus for square-integrable Lévy processes constructed along the same lines as the *standard* Brownian Malliavin calculus and the Malliavin calculus for pure-jump square-integrable Lévy processes as developed in [8]. See [3] for the main definitions and results, and for more references. Several proofs are provided in [12].

One can define two directional derivative operators, one in the direction of the Brownian motion and one in the direction of the Poisson random measure: these (directional) derivative operators are respectively given by  $D^{(1)}: \mathbb{D}^{(1)} \rightarrow L^2([0, T] \times \Omega)$  and  $D^{(2)}: \mathbb{D}^{(2)} \rightarrow L^2([0, T] \times \mathbb{R} \times \Omega)$ , where  $\mathbb{D}^{(1)}$  and  $\mathbb{D}^{(2)}$  stand for their respective domain. We say that  $F$  is Malliavin differentiable if  $F \in \mathbb{D}^{1,2} := \mathbb{D}^{(1)} \cap \mathbb{D}^{(2)}$ . We use the following norm for the Malliavin derivative  $DF := (D^{(1)}F, D^{(2)}F)$ :

$$\|DF\|^2 = \|D^{(1)}F\|_{L^2(\lambda \times \mathbb{P})}^2 + \|D^{(2)}F\|_{L^2(\lambda \times \nu \times \mathbb{P})}^2.$$

The Malliavin derivative  $D$  is *continuous*:

**Lemma 3.1.** *If  $F$  belongs to  $L^2(\Omega)$ , if  $(F_k)_{k \geq 1}$  is a sequence of elements in  $\mathbb{D}^{1,2}$  converging to  $F$  in the  $L^2(\Omega)$ -norm and if  $\sup_{k \geq 1} \|DF_k\| < \infty$ , then  $F$  belongs to  $\mathbb{D}^{1,2}$  and  $(DF_k)_{k \geq 1}$  converges weakly to  $DF$  in  $L^2(\lambda \times \mathbb{P}) \times L^2(\lambda \times \nu \times \mathbb{P})$ .*

If  $F \in \mathbb{D}^{(1)}$ , all the results about the classical Brownian Malliavin derivative, such as the chain rule for Lipschitz functions, can be applied to  $D^{(1)}F$ ; see Nualart [9] for details. But this is also true for the Poisson random measure Malliavin derivative. For example, if  $F = g(X_{t_1}, \dots, X_{t_n}) \in \mathbb{D}^{(2)}$  and

$$(t, z) \mapsto g(X_{t_1} + z\mathbb{I}_{[0,t_1]}(t), \dots, X_{t_n} + z\mathbb{I}_{[0,t_n]}(t)) - g(X_{t_1}, \dots, X_{t_n})$$

belongs to  $L^2(\lambda \times \nu \times \mathbb{P})$ , then

$$D_{t,z}^{(2)}F = g(X_{t_1} + z\mathbb{I}_{[0,t_1]}(t), \dots, X_{t_n} + z\mathbb{I}_{[0,t_n]}(t)) - g(X_{t_1}, \dots, X_{t_n}).$$

This is the *adding-a-mass* formula; see [3] and the references therein for more details.

Finally, here is the corresponding Clark-Ocone formula:

**Theorem 3.2.** *If  $F$  belongs to  $\mathbb{D}^{1,2}$ , then*

$$F = \mathbb{E}[F] + \int_0^T \mathbb{E}[D_t^{(1)}F | \mathcal{F}_t] dW_t + \int_0^T \int_{\mathbb{R}} \mathbb{E}[D_{t,z}^{(2)}F | \mathcal{F}_t] \tilde{N}(dt, dz).$$

We are now ready to prove Theorem 2.2. First, since  $X$  is a square-integrable martingale with drift, from Doob’s maximal inequality we have that  $M_T$  is a square-integrable random variable. Also, if  $\mathbb{E}[M_T] < \infty$ , then one can show that (see Shiryaev and Yor [14] and Graversen et al. [4])

$$\mathbb{E}[M_T | \mathcal{F}_t] = M_t + \int_{M_t - X_t}^{\infty} \bar{F}_{T-t}(z) dz. \tag{3.1}$$

Now, let  $(t_k)_{k \geq 1}$  be a dense subset of  $[0, T]$ , let  $F = M_T$  and, for each  $n \geq 1$ , define  $F_n = \max\{X_{t_1}, \dots, X_{t_n}\}$ . Clearly,  $(F_n)_{n \geq 1}$  is an increasing sequence bounded by  $F$ . Hence  $F_n$  converges to  $F$  in the  $L^2(\Omega)$ -norm when  $n$  goes to infinity.

We want to prove that each  $F_n$  is Malliavin differentiable, i.e., that each  $F_n$  belongs to  $\mathbb{D}^{1,2} = \mathbb{D}^{(1)} \cap \mathbb{D}^{(2)}$ . This follows from the following two facts. First, since

$$(x_1, \dots, x_n) \mapsto \max\{x_1, \dots, x_n\}$$

is a Lipschitz function on  $\mathbb{R}^n$  and since  $D^{(1)}$  operates like the classical Brownian Malliavin derivative on the Brownian part of  $F_n$ , we have that

$$0 \leq D_t^{(1)}F_n = \sum_{k=1}^n \sigma \mathbb{I}_{\{t \leq t_k\}} \mathbb{I}_{A_k} \leq \sum_{k=1}^n \sigma \mathbb{I}_{A_k} = \sigma,$$

where  $A_1 = \{F_n = X_{t_1}\}$  and  $A_k = \{F_n \neq X_{t_1}, \dots, F_n \neq X_{t_{k-1}}, F_n = X_{t_k}\}$  for  $2 \leq k \leq n$ . This implies that  $\sup_{n \geq 1} \|D^{(1)}F_n\|_{L^2([0,T] \times \Omega)}^2 \leq \sigma^2 T$ . Secondly, since  $D^{(2)}$  operates like the Poisson random measure Malliavin derivative on the Poisson part of  $F_n$ , we have that

$$0 \leq |D_{t,z}^{(2)}F_n| = |\max\{X_{t_1} + z\mathbb{I}_{\{t < t_1\}}, \dots, X_{t_n} + z\mathbb{I}_{\{t < t_n\}}\} - F_n| \leq |z|,$$

where the equality is justified by the following inequality:

$$\|\max\{X_{t_1} + z\mathbb{I}_{\{t < t_1\}}, \dots, X_{t_n} + z\mathbb{I}_{\{t < t_n\}}\} - F_n\|_{L^2([0,T] \times \mathbb{R} \times \Omega)}^2 \leq T \int_{\mathbb{R}} z^2 \nu(dz).$$

Indeed, if  $z \geq 0$ , then  $0 \leq \max \{X_{t_1} + z\mathbb{I}_{\{t < t_1\}}, \dots, X_{t_n} + z\mathbb{I}_{\{t < t_n\}}\} - F_n \leq z$ , and, if  $z < 0$ , then

$$\begin{aligned} 0 &\leq F_n - \max \{X_{t_1} + z\mathbb{I}_{\{t < t_1\}}, \dots, X_{t_n} + z\mathbb{I}_{\{t < t_n\}}\} \\ &= F_n + \min \{-X_{t_1} + |z|\mathbb{I}_{\{t < t_1\}}, \dots, -X_{t_n} + |z|\mathbb{I}_{\{t < t_n\}}\} \\ &= \min \{F_n - X_{t_1} + |z|\mathbb{I}_{\{t < t_1\}}, \dots, F_n - X_{t_n} + |z|\mathbb{I}_{\{t < t_n\}}\} \\ &\leq |z|. \end{aligned}$$

This implies that  $\sup_{n \geq 1} \|D^{(2)}F_n\|_{L^2([0, T] \times \mathbb{R} \times \Omega)}^2 \leq T \int_{\mathbb{R}} z^2 \nu(dz)$ .

Consequently,  $\sup_{n \geq 1} \|DF_n\|^2 \leq T(\sigma^2 + \int_{\mathbb{R}} z^2 \nu(dz))$  and we have that  $F$  is Malliavin differentiable. By the uniqueness of the limit, this means that taking the limit of  $D_t^{(1)}F_n$  when  $n$  goes to infinity yields  $D_t^{(1)}F = \sigma\mathbb{I}_{[0, \tau]}(t)$ , where  $\tau = \inf \{t \in [0, T] : X_t = M_T\}$ , with the convention  $\inf \emptyset = T$ , i.e.  $\tau$  is the first time when the Lévy process  $X$  (not the Brownian motion  $W$ ) reaches its supremum on  $[0, T]$ , and

$$D_{t,z}^{(2)}F = \sup_{0 \leq s \leq T} (X_s + z\mathbb{I}_{\{t < s\}}) - M_T.$$

Hence

$$\mathbb{E} \left[ D_t^{(1)}F \mid \mathcal{F}_t \right] = \sigma\mathbb{P} \{M_t < M_{t,T} \mid \mathcal{F}_t\} = \sigma\mathbb{P} \{M_{T-t} > a\},$$

where  $a = M_t - X_t$ . Since  $M_{t,T} - X_t$  is independent of  $\mathcal{F}_t$  and has the same law as  $M_{T-t}$ , then using Equation (3.1) we get that

$$\begin{aligned} \mathbb{E} \left[ D_{t,z}^{(2)}F \mid \mathcal{F}_t \right] &= \mathbb{E} \left[ \sup_{0 \leq s \leq T} (X_s + z\mathbb{I}_{\{t < s\}}) - M_T \mid \mathcal{F}_t \right] \\ &= \mathbb{E} [\max\{M_t, M_{t,T} + z\} \mid \mathcal{F}_t] - \mathbb{E} [M_T \mid \mathcal{F}_t] \\ &= M_t + \mathbb{E} \left[ (M_{t,T} + z - M_t)^+ \mid \mathcal{F}_t \right] - \mathbb{E} [M_T \mid \mathcal{F}_t] \\ &= \mathbb{E} \left[ (M_{T-t} + z - a)^+ \right] - \int_a^\infty \bar{F}_{T-t}(x) dx, \end{aligned}$$

where  $a = M_t - X_t$ . Since

$$\mathbb{E} \left[ (M_{T-t} + z - a)^+ \right] = \int_{a-z}^\infty \bar{F}_{T-t}(x) dx,$$

we have that

$$\begin{aligned} \psi(t, z, a) &= \int_{a-z}^\infty \bar{F}_{T-t}(x) dx - \int_a^\infty \bar{F}_{T-t}(x) dx \\ &= \mathbb{I}_{\{z \geq a\}}(z - a) + \mathbb{I}_{\{z \geq a\}} \int_0^a \bar{F}_{T-t}(x) dx \\ &\quad + \mathbb{I}_{\{a > z \geq 0\}} \int_{a-z}^a \bar{F}_{T-t}(x) dx - \mathbb{I}_{\{z < 0\}} \int_a^{a-z} \bar{F}_{T-t}(x) dx. \end{aligned}$$

Finally, the martingale representation follows from the Clark-Ocone formula given in Theorem 3.2.

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