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A CONNECTION BETWEEN THE POISSONIAN WICK PRODUCT AND THE DISCRETE CONVOLUTION

ALBERTO LANCONELLI AND LUIGI SPORTELLI

Abstract. Inspired by Lemma 3.1 in [4], where a connection between the Gaussian Wick product and the classic convolution product is shown, we prove that the Wick product associated to the Poisson distribution is related to the discrete convolution and hence to the law of the sum of discrete independent random variables. The proof of the main result is based on elementary probabilistic tools and on the properties of the Poisson-Charlier polynomials.

1. Introduction

In 1950 the physicist G.C.Wick proposed in the paper [11] a procedure to renormalize certain infinite quantities in quantum field theory. Wick's technique is based on the following definition:

"*An operator* A *acting on the bosonic Fock space is said to be* normally ordered *if in its representation in terms of annihilation and creation operators all the creation operators appear to the left of all annihilation operators.*"

By means of the commutation relations satisfied by annihilation and creation operators, any operator can be reduced to a normally ordered one. In particular the normally ordered powers of the position operator yield in a natural way a new concept of multiplication for the Hermite polynomials and hence for certain classes of functions defined on Gaussian spaces. This is what is known in the mathematical literature as the *Wick product*.

The Wick product plays a crucial role in many important branches of stochastic analysis and of the theory of stochastic partial differential equations. See for instance [6] and the references quoted there for the applications of the so called Wick calculus.

In order to explain the motivation for considering the problem investigated in this paper we briefly recall few basic definitions of the one dimensional Gaussian Wick calculus. In the sequel the symbol $\mathbb{N}^*$ will denote the set of the nonnegative integers, i.e. $\mathbb{N}^* := \mathbb{N} \cup \{0\}$.

Let ν be the standard one dimensional Gaussian measure, that means

$$\nu(A) = \int_A \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx, \quad A \in \mathcal{B}(\mathbb{R}),$$

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and consider the real Hilbert space $\mathcal{L}^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)$ endowed with the standard inner product

$$\langle f,g\rangle := \int_{\mathbb{R}} f(x)g(x)d\nu(x).$$

Let $\{h_n\}_{n\geq 0}$ be the family of Hermite polynomials (h_n is a monic polynomial of exact degree n). This family constitutes an orthogonal basis for $\mathcal{L}^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)$. For $n,m\in\mathbb{N}^*$ define

$$(h_n\diamond h_m)(x) := h_{n+m}(x),\quad x\in\mathbb{R}, \tag{1.1}$$

and extend this bilinear operation to the whole $\mathcal{L}^2(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)$. This yields an unbounded multiplication for functions denoted by $f\diamond g$ and named the *Gaussian Wick product* of f and g.

Moreover for $\alpha\in\mathbb{R}$ with $|\alpha|\leq 1$ and $n\in\mathbb{N}^*$ define the linear bounded operator

$$(\Gamma(\alpha)h_n)(x) = \alpha^n h_n(x),\quad x\in\mathbb{R}.$$

In [4], Lemma 3.1, the authors proved the multidimensional analogue of the following result that provides a connection between the Gaussian Wick product, the operator $\Gamma(\alpha)$ and the classic convolution product.

Lemma 1.1. *Let $\alpha,\beta\in\mathbb{R}$ be such that $\alpha^2+\beta^2=1$. Then for any $f,g\in\mathcal{L}^1(\mathbb{R},\mathcal{B}(\mathbb{R}),\nu)$ we have*

$$(\Gamma(\alpha)f\diamond\Gamma(\beta)g)(x) = \frac{\left[\Big(f(\beta\cdot)\mathcal{N}(\beta\cdot)\Big)*\Big(g(\alpha\cdot)\mathcal{N}(\alpha\cdot)\Big)\right]\left(\frac{x}{\alpha\beta}\right)}{\mathcal{N}(x)},\quad x\in\mathbb{R}, \tag{1.2}$$

where $\mathcal{N}$ stands for the normal density function

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},\quad x\in\mathbb{R},$$

and $*$ denotes the convolution product:

$$(f*g)(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

We now want to give a probabilistic interpretation to equation (1.2). Suppose that f and g are nonnegative functions with

$$\int_{\mathbb{R}} f(x)d\nu(x) = 1 \text{ and } \int_{\mathbb{R}} g(x)d\nu(x) = 1.$$

In this way, f and g can be seen as the Radon-Nikodym derivatives of two probability measures on the real line, say μ_X and μ_Y, with respect to the reference measure ν. We may think that μ_X and μ_Y are the laws of two real valued independent random variables X and Y defined on a common probability space.

From this point of view the right hand side of (1.2) corresponds to the Radon-Nikodym derivative of the law of $\alpha X+\beta Y$ with respect to the measure ν (this follows easily from simple manipulations on the densities). Therefore, equation (1.2) can be rewritten as

$$\left(\Gamma(\alpha)\frac{d\mu_X}{d\nu}\right)\diamond\left(\Gamma(\beta)\frac{d\mu_Y}{d\nu}\right) = \frac{d\mu_{\alpha X+\beta Y}}{d\nu}. \tag{1.3}$$

Actually, going through the proof of the above mentioned Lemma 3.1 from [4], one can easily see that equation (1.3) admits an alternative representation which is free from the weights α and β but contains a correction term; namely

$$\frac{d\mu_X}{d\nu} \diamond \frac{d\mu_Y}{d\nu} \diamond \frac{d\mu_{N(0,2)}}{d\nu} = \frac{d\mu_{X+Y}}{d\nu}, \tag{1.4}$$

where the correction term $\frac{d\mu_{N(0,2)}}{d\nu}$ denotes the density of a normal random variable with mean zero e variance two with respect to the reference measure ν. This quantity can be made explicit as:

$$\frac{d\mu_{N(0,2)}}{d\nu} = \frac{\frac{1}{\sqrt{4\pi}} e^{-\frac{x^2}{4}}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} = \frac{1}{\sqrt{2}} e^{\frac{x^2}{4}}.$$

Now, the Wick product can be formally defined for any probability measure on the real line having finite moments of any order by means of the prescription (1.1) with the polynomials $\{h_n\}_{n\geq 0}$ replaced by the proper family of orthogonal polynomials associated to the given measure. Results in this direction can be found for instance in [9] and [7]. It is therefore natural to ask whether an identity of the type (1.4) holds true also for other measures.

The aim of the present paper is to prove the analogue of (1.4) for the Wick product associated to the Poisson distribution and hence to provide a connection between the Poissonian Wick product and the discrete convolution. Similarly to equation (1.4), we will obtain a formula with a correction term that can be interpreted in the same manner as in the Gaussian case.

The reason for considering the Poisson distribution comes from the theory of orthogonal polynomials; it is in fact well known (see for instance [10] or [3]) that given a probability measure with finite moments of any order, there exists a family of polynomials, which are orthogonal with respect to that measure, satisfying a three-term recursion formula. This recursion formula is specified by two sequences of real numbers, usually denoted by

$$\{\alpha_n\}_{n\geq 0} \quad \text{and} \quad \{\omega_n\}_{n\geq 0}$$

called the Szegö-Jacobi parameters. As it is shown in [9] and [7] the sequence $\{\omega_n\}_{n\geq 0}$ plays the major role when dealing with Wick products associated to the related measure. Now, the Gaussian and the Poisson distributions have the same ω-sequence.

The present paper is a first step in understanding which class of probability measures possess a property of the type (1.4) for the associated Wick product.

We also mention that dealing with the Poisson distribution, which is supported on the set $\mathbb{N}^*$, it is more natural to focus on a formula like (1.4) than a formula like (1.3). In fact, the right hand side of (1.3) is expressed in terms of the random variable $\alpha X + \beta Y$ which is not integer valued for general real numbers α and β.

The paper is organized as follows: Section 2 provides the necessary framework and some useful results of the Poissonian Wick calculus while in Section 3 we state and prove the main result together with some generalizations.

2. Wick Calculus for the Poisson Distribution

We are now going to describe our framework. For more information on the orthogonal polynomials associated to the Poisson distribution we refer the reader to one of the books [3] and [10].

2.1. Poisson-Charlier polynomials and Wick product. Let $(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ be the probability space induced by a *Poisson random variable* with parameter $a \in \mathbb{R}, a > 0$, that means:

$$\mu_a(\{k\}) = \frac{a^k}{k!}e^{-a}, \quad k \in \mathbb{N}^*. \tag{2.1}$$

Let

$$\mathcal{L}^2(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu_a) := \left\{ f : \mathbb{N} \to \mathbb{R} \text{ s. t. } \|f\|^2 := \sum_{k\geq 0} |f(k)|^2 \frac{a^k}{k!} e^{-a} < +\infty \right\}.$$

Since for any $j \in \mathbb{N}$,

$$\sum_{k\geq 0} k^j \mu_a(\{k\}) < +\infty, \tag{2.2}$$

there exists a unique sequence $\{C_n(\cdot, a)\}_{n\geq 0} \subset \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ of monic polynomials satisfying the following orthogonality conditions:

$$\langle C_n(\cdot, a), C_m(\cdot, a)\rangle := \sum_{k\geq 0} C_n(k, a) C_m(k, a) \frac{a^k}{k!} e^{-a} = \delta_{n,m} a^n n!. \tag{2.3}$$

The polynomials $\{C_n(\cdot, a)\}_{n\geq 0}$ are called *Poisson-Charlier polynomials with parameter a* . They can be represented explicitly as (see e.g. [3]):

$$C_n(x, a) = \sum_{i=0}^{n} \binom{n}{i} (x)_i (-a)^{n-i}, \quad x \in \mathbb{N}^*,$$

where $(x)_i := x(x-1)\cdots(x-i+1)$. Moreover since the probability measure μ_a is uniquely determined by its moments (2.2) we deduce that the sequence $\{C_n(\cdot, a)\}_{n\geq 0}$ constitutes an orthogonal basis for $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ (see [1] or [5]). Therefore any $f \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ can be uniquely represented as

$$f(x) = \sum_{n\geq 0} f_n C_n(x, a), \qquad f_n \in \mathbb{R}.$$

In this case one has

$$\|f\|^2 = \sum_{n\geq 0} |f_n|^2 a^n n!.$$

Definition 2.1. For any $n, m \in \mathbb{N}^*$ we define

$$(C_n(\cdot, a) \diamond C_m(\cdot, a))(x) := C_{n+m}(x, a), \quad x \in \mathbb{N}^*. \tag{2.4}$$

(To ease the notation we do not stress the dependence of $\diamond$ on the measure μ_a: the operation $\diamond$ just defined is obviously different from the one in (1.1)).The quantity

$C_n(\cdot,a) \diamond C_m(\cdot,a)$ is called the *(Poissonian) Wick product of* $C_n(\cdot,a)$ *and* $C_m(\cdot,a)$.

More generally, for $f, g \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ with

$$f(x) = \sum_{n\geq 0} f_n C_n(x,a),$$

and

$$g(x) = \sum_{m\geq 0} g_m C_m(x,a),$$

we define

$$\begin{aligned}(f \diamond g)(x) &:= \left(\left[\sum_{n\geq 0} f_n C_n(\cdot,a)\right] \diamond \left[\sum_{m\geq 0} g_m C_m(\cdot,a)\right]\right)(x) \\ &= \sum_{n,m\geq 0} f_n g_m [C_n(\cdot,a) \diamond C_m(\cdot,a)](x) \\ &= \sum_{n,m\geq 0} f_n g_m C_{n+m}(x,a) \\ &= \sum_{n\geq 0}\left(\sum_{k=0}^{n} f_k g_{n-k}\right) C_n(x,a).\end{aligned} \tag{2.5}$$

The quantity $f \diamond g$ is named the *Wick product of* f *and* g.

We would like to mention that one can find in the literature different notions of Wick product for a given probability measure. The idea is in fact to define a product that preserves one of the properties satisfied by the Hermite polynomials, which represent the classical case, and to obtain families of polynomials satisfying the chosen property. Our notion of Wick product preserves the orthogonality of the polynomials that we need for the chaotic representation of the $\mathcal{L}^2$ space.

It is easy to see that the Wick product is a commutative, associative and distributive (w.r.t. the sum) operation between functions and that

$$(f \diamond g)(x) = f(x) \cdot g(x),$$

if g is a constant function. However the Wick product is an unbounded bilinear operator on $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ since in general one has

$$\sum_{n\geq 0}\left|\sum_{k=0}^{n} f_k g_{n-k}\right|^2 a^n n! = +\infty.$$

We are now going to define a pair of spaces which are closed under the Wick product.

2.2. The spaces $\mathcal{G}$ and $\mathcal{G}^*$. The following construction mimics the one in [8] where these spaces are introduced and studied for the first time in the infinite dimensional Gaussian setting.

We begin introducing a family of Hilbert spaces.

For $\lambda \in \mathbb{R}, \lambda \geq 1$, define

$$\mathcal{G}_\lambda := \Bigg\{ f(x) = \sum_{n\geq 0} f_n C_n(x,a) \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a) : \\ \|f\|_\lambda^2 := \sum_{n\geq 0} |f_n|^2 \lambda^{2n} a^n n! < +\infty \Bigg\}.$$

It is straightforward to observe that

$$\mathcal{G}_\lambda \subseteq \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a),$$
$$\mathcal{G}_\lambda \subseteq \mathcal{G}_\mu \quad \text{if} \quad \lambda \geq \mu,$$

and $\|\cdot\|_1 = \|\cdot\|$. Now let

$$\mathcal{G} := \bigcap_{\lambda \geq 1} \mathcal{G}_\lambda, \tag{2.6}$$

endowed with the projective limit topology. Therefore the space $\mathcal{G}$ can be described as

$$\mathcal{G} = \Bigg\{ f(x) = \sum_{n\geq 0} f_n C_n(x,a) \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a) : \\ \sum_{n\geq 0} |f_n|^2 \lambda^{2n} a^n n! < +\infty \text{ for all } \lambda \geq 1 \Bigg\}.$$

The space $\mathcal{G}^*$ is defined as the dual of $\mathcal{G}$ with respect to the inner product of $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ and it is endowed with the inductive limit topology. The space $\mathcal{G}^*$ is a regular generalized function space that can be described formally as

$$\mathcal{G}^* = \Bigg\{ f(x) = \sum_{n\geq 0} f_n C_n(x,a) : \sum_{n\geq 0} |f_n|^2 \lambda^{2n} a^n n! < +\infty \text{ for some } \lambda > 0 \Bigg\}. \tag{2.7}$$

The dual pairing between $\mathcal{G}^*$ and $\mathcal{G}$ will be denoted by $\langle\langle \cdot, \cdot \rangle\rangle$, i.e. if $f(x) = \sum_{n\geq 0} f_n C_n(x,a) \in \mathcal{G}^*$ and $g(x) = \sum_{n\geq 0} g_n C_n(x,a) \in \mathcal{G}$ then

$$\langle\langle f, g \rangle\rangle := \sum_{n\geq 0} f_n g_n a^n n!;$$

in particular for $f \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ and $g \in \mathcal{G}$ we have

$$\langle\langle f, g \rangle\rangle = \langle f, g \rangle.$$

Proposition 2.2. *The spaces $\mathcal{G}$ and $\mathcal{G}^*$ are closed under the Wick product.*

Proof. We will prove the statement only for the space $\mathcal{G}$. The other case can be treated in the same manner. Let

$$f(x) = \sum_{n\geq 0} f_n C_n(x,a) \in \mathcal{G},$$

and

$$g(x) = \sum_{n\geq 0} g_n C_n(x,a) \in \mathcal{G},$$

which means that for any $\lambda \geq 1$ the series

$$\sum_{n\geq 0} |f_n|^2 \lambda^{2n} a^n n! \quad \text{and} \quad \sum_{n\geq 0} |g_n|^2 \lambda^{2n} a^n n!,$$

converge. Recalling that

$$(f \diamond g)(x) = \sum_{n\geq 0} \left(\sum_{k=0}^{n} f_k g_{n-k} \right) C_n(x,a),$$

we get by applying the triangle and the Cauchy-Schwarz inequalities,

$$\begin{aligned}
\|f \diamond g\|_\lambda^2 &= \sum_{n\geq 0} \left| \sum_{k=0}^{n} f_k g_{n-k} \right|^2 \lambda^{2n} a^n n! \\
&\leq \sum_{n\geq 0} \left(\sum_{k=0}^{n} |f_k||g_{n-k}| \right)^2 \lambda^{2n} a^n n! \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{\sqrt{k!}\sqrt{(n-k)!}}{\sqrt{k!}\sqrt{(n-k)!}} |f_k|\,|g_{n-k}| \right)^2 \lambda^{2n} a^n n! \\
&\leq \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{1}{k!(n-k)!} \right) \left(\sum_{k=0}^{n} k!(n-k)!|f_k|^2 |g_{n-k}|^2 \right) \lambda^{2n} a^n n! \\
&= \sum_{n\geq 0} 2^n \left(\sum_{k=0}^{n} k!(n-k)!|f_k|^2 |g_{n-k}|^2 \right) \lambda^{2n} a^n \\
&= \sum_{n\geq 0} \sum_{k=0}^{n} \left[(2\lambda^2 a)^k k! |f_k|^2 \right] \cdot \left[(2\lambda^2 a)^{n-k} (n-k)! |g_{n-k}|^2 \right] \\
&= \left(\sum_{n\geq 0} |f_n|^2 2^n \lambda^{2n} a^n n! \right) \cdot \left(\sum_{n\geq 0} |g_n|^2 2^n \lambda^{2n} a^n n! \right) \\
&= \|f\|_{\sqrt{2}\lambda}^2 \|g\|_{\sqrt{2}\lambda}^2 < +\infty.
\end{aligned}$$

□

Looking through the proof of the previous proposition one gets sufficient conditions for the square integrability of the Wick product of two or more functions.

Corollary 2.3. *If $f, g \in \mathcal{G}_{\sqrt{2}}$, then $f \diamond g \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$. More generally, if for $n \in \mathbb{N}$, $f_1, ..., f_n \in \mathcal{G}_{\sqrt{n}}$, then $f_1 \diamond \cdots \diamond f_n \in \mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$.*

2.3. Stochastic exponentials. For any $t \in \mathbb{R}$ define

$$\mathcal{E}_t(x) := \sum_{n\geq 0} \frac{t^n}{a^n n!} C_n(x,a), \quad x \in \mathbb{N}^*. \tag{2.8}$$

The function $\mathcal{E}_t$ is called *stochastic exponential with parameter* t. Observe that for any $t \in \mathbb{R}$ and $x \in \mathbb{N}^*$ the following identity holds (see e.g. [3]):

$$\mathcal{E}_t(x) = \left(1 + \frac{t}{a}\right)^x e^{-t}. \tag{2.9}$$

One of the most crucial features of the stochastic exponentials is that the linear span of $\{\mathcal{E}_t, t \in \mathbb{R}\}$ is dense in $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ (see [2]) and in $\mathcal{G}_\lambda$ for any $\lambda \geq 1$. Therefore $\mathcal{E}_t \in \mathcal{G}$ and the identity $\langle\langle f, \mathcal{E}_t\rangle\rangle = \langle\langle g, \mathcal{E}_t\rangle\rangle$, for all $t \in \mathbb{R}$, implies that $f = g$ in $\mathcal{G}^*$.

The next proposition collects some useful properties of the stochastic exponentials.

Proposition 2.4. *Let* $t, s \in \mathbb{R}$. *Then*

i) $\langle \mathcal{E}_t, \mathcal{E}_s\rangle = e^{\frac{ts}{a}}$;
ii) $(\mathcal{E}_t \diamond \mathcal{E}_s)(x) = \mathcal{E}_{t+s}(x)$ *for all* $x \in \mathbb{N}^*$;
iii) *For any* $f, g \in \mathcal{G}^*$,

$$\langle\langle f \diamond g, \mathcal{E}_t\rangle\rangle = \langle\langle f, \mathcal{E}_t\rangle\rangle \cdot \langle\langle g, \mathcal{E}_t\rangle\rangle. \tag{2.10}$$

Proof. The proof of *i*) is a simple verification.
Let us prove *ii*). By means of the definition of Wick product we get for $x \in \mathbb{N}^*$,

$$\begin{aligned}
(\mathcal{E}_t \diamond \mathcal{E}_s)(x) &= \sum_{n\geq 0}\left[\sum_{k=0}^{n} \frac{t^k}{a^k k!} \cdot \frac{s^{n-k}}{a^{n-k}(n-k)!}\right] C_n(x,a) \\
&= \sum_{n\geq 0} \frac{1}{n!}\left[\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \cdot \frac{t^k s^{n-k}}{a^n}\right] C_n(x,a) \\
&= \sum_{n\geq 0} \frac{1}{a^n n!}\left[\sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k}\right] C_n(x,a) \\
&= \sum_{n\geq 0} \frac{(t+s)^n}{a^n n!} C_n(x,a) = \mathcal{E}_{t+s}(x).
\end{aligned}$$

We now prove *iii*).

$$\begin{aligned}
\langle\langle f \diamond g, \mathcal{E}_t\rangle\rangle &= \left\langle\!\left\langle \sum_{n\geq 0}\Big(\sum_{k=0}^{n} f_k g_{n-k}\Big) C_n(\cdot,a), \sum_{n\geq 0} \frac{t^n}{a^n n!} C_n(\cdot,a)\right\rangle\!\right\rangle \\
&= \sum_{n\geq 0}\Big(\sum_{k=0}^{n} f_k g_{n-k}\Big)\frac{t^n}{a^n n!} a^n n! \\
&= \sum_{n\geq 0}\Big(\sum_{k=0}^{n} f_k g_{n-k}\Big) t^n \\
&= \Big(\sum_{n\geq 0} f_n t^n\Big) \cdot \Big(\sum_{n\geq 0} g_n t^n\Big) \\
&= \langle\langle f, \mathcal{E}_t\rangle\rangle \cdot \langle\langle g, \mathcal{E}_t\rangle\rangle.
\end{aligned}$$

□

We remark that the application $f \in \mathcal{G}^* \mapsto \{\langle\langle f, \mathcal{E}_t\rangle\rangle\}_{t\in\mathbb{R}}$ considered in (2.10) corresponds to an extension of the so called Segal-Bargamann transform associated to the measure μ_a which has been studied for instance in [2]. From this point of view the Wick product can be defined equivalently as the unique bilinear map on $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$ which is factorized by the Segal-Bargmann transform.

3. Main Results

We are now going to state our main theorem.

Theorem 3.1. *Let X and Y be two independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and taking values on the set of nonnegative integers $\mathbb{N}^*$. Denote by f_X and f_Y the Radon-Nikodym derivatives of the laws of X and Y with respect to the measure μ_a, respectively. Assume that f_X and f_Y belong to $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$.*
Then we have

$$f_X \diamond f_Y \diamond \mathcal{E}_a = f_{X+Y} \tag{3.1}$$

(the equality in Eq.(3.1) holds in the sense of $\mathcal{G}^*$) where f_{X+Y} denotes the Radon-Nikodym derivative of the law of the random variable $X+Y$ with respect to μ_a.

Remark 3.2. We observe that the Radon-Nikodym derivatives f_X and f_Y appearing in the statement of the theorem always exist since the random variables X and Y takes values on $\mathbb{N}^*$; we simply have

$$f_X(k) = \frac{P(X=k)}{\mu_a(\{k\})} \text{ and } f_Y(k) = \frac{P(Y=k)}{\mu_a(\{k\})}, \quad k \in \mathbb{N}^*.$$

Proof. The idea of the proof is to show that for any $t \in \mathbb{R}$,

$$\langle\langle \mathcal{L}, \mathcal{E}_t\rangle\rangle = \langle\langle \mathcal{R}, \mathcal{E}_t\rangle\rangle,$$

where $\mathcal{R}$ and $\mathcal{L}$ denotes the left and the right hand sides of (3.1), respectively. According to our hypothesis and to Proposition 2.2, $\mathcal{L}$ belongs to $\mathcal{G}^*$ and therefore the dual pairing is well defined.(In the calculation below the dot in $\left(1+\frac{t}{a}\right)^{\cdot}$ replaces the variable x which is integrated in the dual pairing. Moreover the square integrability of f_X and f_Y enables us to reduce the dual pairing to an inner product).

We have:

$$\begin{aligned}
\langle\langle f_X \diamond f_Y \diamond \mathcal{E}_a, \mathcal{E}_t\rangle\rangle &= \langle\langle f_X, \mathcal{E}_t\rangle\rangle \cdot \langle\langle f_Y, \mathcal{E}_t\rangle\rangle \cdot \langle\langle \mathcal{E}_a, \mathcal{E}_t\rangle\rangle \\
&= \left\langle\left\langle f_X, \left(1+\frac{t}{a}\right)^{\cdot} e^{-t}\right\rangle\right\rangle \cdot \left\langle\left\langle f_Y, \left(1+\frac{t}{a}\right)^{\cdot} e^{-t}\right\rangle\right\rangle \cdot e^t \\
&= e^{-t}\left\langle f_X, \left(1+\frac{t}{a}\right)^{\cdot}\right\rangle \cdot \left\langle f_Y, \left(1+\frac{t}{a}\right)^{\cdot}\right\rangle \\
&= e^{-t} \cdot \sum_{k\geq 0} f_X(k)\left(1+\frac{t}{a}\right)^k \frac{a^k}{k!} e^{-a} \\
&\quad \cdot \sum_{k\geq 0} f_Y(k)\left(1+\frac{t}{a}\right)^k \frac{a^k}{k!} e^{-a}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-t}\cdot\sum_{k\geq 0}\Big(1+\frac{t}{a}\Big)^k P(X=k)\\
&\quad\cdot\sum_{k\geq 0}\Big(1+\frac{t}{a}\Big)^k P(Y=k)\\
&= e^{-t}\cdot\sum_{k\geq 0}\Big(\sum_{j=0}^{k}P(X=j)P(Y=k-j)\Big)\Big(1+\frac{t}{a}\Big)^k.
\end{aligned}
$$

We now use the independence of X and Y to get

$$
\begin{aligned}
\langle\langle f_X\diamond f_Y\diamond\mathcal{E}_a,\mathcal{E}_t\rangle\rangle &= e^{-t}\cdot\sum_{k\geq 0}\Big(\sum_{j=0}^{k}P(X=j,Y=k-j)\Big)\Big(1+\frac{t}{a}\Big)^k\\
&= e^{-t}\cdot\sum_{k\geq 0}P(X+Y=k)\Big(1+\frac{t}{a}\Big)^k\\
&= \sum_{k\geq 0}P(X+Y=k)\Big(1+\frac{t}{a}\Big)^k e^{-t}\\
&= \sum_{k\geq 0}f_{X+Y}(k)\Big(1+\frac{t}{a}\Big)^k e^{-t}\frac{a^k}{k!}e^{-a}\\
&= \sum_{k\geq 0}f_{X+Y}(k)\mathcal{E}_t(k)\frac{a^k}{k!}e^{-a}\\
&= \langle\langle f_{X+Y},\mathcal{E}_t\rangle\rangle.
\end{aligned}
$$

The proof is complete. □

Remark 3.3. Observe that the correction term $\mathcal{E}_a$ appearing in (3.1) is nothing else than the Radon-Nikodym derivative of a Poisson distribution with parameter $2a$ with respect to the reference measure μ_a; more precisely

$$\mathcal{E}_a=\frac{d\mu_{2a}}{d\mu_a}.$$

In fact for any $k\in\mathbb{N}^*$,

$$\mathcal{E}_a(k)=\Big(1+\frac{a}{a}\Big)^k e^{-a}=2^k e^{-a}=\frac{\frac{(2a)^k}{k!}e^{-2a}}{\frac{a^k}{k!}e^{-a}}.$$

In this way we have a perfect analogy with the Gaussian counterpart of this formula.

Remark 3.4. If we denote by μ_X, μ_Y and μ_{X+Y} the laws of the random variables X, Y and $X+Y$, respectively, then equation (3.1) can be rewritten as

$$\frac{d\mu_X}{d\mu_a}\diamond\frac{d\mu_Y}{d\mu_a}\diamond\frac{d\mu_{2a}}{d\mu_a}=\frac{d\mu_{X+Y}}{d\mu_a}.$$

This equation represents the Poissonian analogue of equation (1.4).

The following generalization is easily obtained.

Theorem 3.5. *Let $X_1, X_2, ..., X_n$ be independent random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and taking values on the set $\mathbb{N}^*$. Denote by $f_1, f_2, ..., f_n$ the Radon-Nikodym derivatives of the laws of $X_1, X_2, ..., X_n$ with respect to the measure μ_a, respectively. Assume that $f_1, f_2, ..., f_n$ belong to the space $\mathcal{L}^2(\mathbb{N}^*, \mathcal{P}(\mathbb{N}^*), \mu_a)$.*
Then we have

$$f_1 \diamond f_2 \diamond \cdots \diamond f_n \diamond \mathcal{E}_{(n-1)a} = f_{X_1+X_2+\cdots+X_n}, \tag{3.2}$$

(the equality in Eq.(3.2) holds in the sense of $\mathcal{G}^*$) where $f_{X_1+X_2+\cdots+X_n}$ denotes the Radon-Nikodym derivative of the law of the random variable $X_1 + X_2 + \cdots + X_n$ with respect to μ_a.

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Alberto Lanconelli: Dipartimento di Matematica, Universita' degli Studi di Bari, Via E. Orabona 4, 70125 Bari, Italia
E-mail address: lanconelli@dm.uniba.it

Luigi Sportelli: Dipartimento di Matematica, Universita' degli Studi di Bari, Via E. Orabona 4, 70125 Bari, Italia
E-mail address: sportelli@dm.uniba.it