

## AN INTERPOLATING FAMILY OF MEANS

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*Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday*

ABSTRACT. This paper is concerned with a new family of binary symmetric means  $M_p$  of two positive numbers  $a$  and  $b$ :

$$\frac{1}{M_p(a, b)} = c_p \int_0^\infty \frac{dx}{[(x^p + a^p)(x^p + b^p)]^{1/p}}, \quad 0 < p < \infty,$$

where the constant  $c_p$ , depending on  $p$ , is chosen to have  $M_p(a, a) = a$ . Two distinctive members in the family are the well-known logarithmic mean ( $p = 1$ ) and arithmetic-geometric mean ( $p = 2$ ). Different expressions for  $M_p$  are obtained to establish its other properties, including  $M_2(a, b) \leq M_\infty(a, b)$  and the relation between  $M_p$  and the power difference mean. Through investigating the induced operator norm of the integral operator with  $M_p^{-1}$  as its kernel, a generalization of the Hilbert inequality is obtained. Finally positive definiteness of certain matrices as implications of inequalities between two means is also investigated.

### 1. Introduction

Let  $a$  and  $b$  be positive numbers. The *logarithmic mean*  $L(a, b)$  of  $a$  and  $b$  defined as

$$L(a, b) := \frac{a - b}{\ln a - \ln b} \quad (1.1)$$

has long been used in problems related to heat flow [16] and electrical conduction [17]. More recently it has been employed in differential geometry [2, 5]. The well-known *arithmetic-geometric mean*  $AG(a, b)$  of Gauss is defined as follows: the sequences  $\{a_n\}$  and  $\{b_n\}$  defined inductively as

$$\begin{aligned} a_0 &= a, & b_0 &= b, \\ a_{n+1} &= \frac{a_n + b_n}{2}, & b_{n+1} &= \sqrt{a_n b_n}, \end{aligned}$$

have a common limit, and

$$AG(a, b) := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n. \quad (1.2)$$

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This mean, introduced by Legendre and then by Gauss, is related to the evaluation of elliptic integrals, and several other problems in analysis [11, 8].

The expressions (1.1) and (1.2) do not carry any hint that these two means could belong to a common family. There are alternative descriptions for both. It can be seen that

$$\frac{1}{L(a, b)} = \int_0^\infty \frac{dx}{(x+a)(x+b)}, \quad (1.3)$$

and an ingenious calculation, due to Gauss [12], is used to show

$$\frac{1}{AG(a, b)} = \frac{2}{\pi} \int_0^\infty \frac{dx}{\sqrt{(x^2+a^2)(x^2+b^2)}}. \quad (1.4)$$

This similarity between the expressions (1.3) and (1.4) is the motivation for us to introduce a family of means  $M_p(a, b)$ ,  $0 \leq p \leq \infty$ , defined by the relation

$$\frac{1}{M_p(a, b)} := c_p \int_0^\infty \frac{dx}{[(x^p+a^p)(x^p+b^p)]^{1/p}}, \quad 0 < p < \infty, \quad (1.5)$$

where the constant  $c_p$ , depending on  $p$ , will be chosen to have

$$M_p(a, a) = a.$$

Thus

$$\frac{1}{c_p} = a \int_0^\infty \frac{dx}{(x^p+a^p)^{2/p}} = \int_0^\infty \frac{dy}{(y^p+1)^{2/p}}. \quad (1.6)$$

The means  $M_0$  and  $M_\infty$  are defined by taking limits:

$$M_0(a, b) := \lim_{p \rightarrow 0^+} M_p(a, b), \quad M_\infty(a, b) := \lim_{p \rightarrow \infty} M_p(a, b). \quad (1.7)$$

A *binary symmetric mean*  $M(a, b)$  of positive numbers  $a$  and  $b$  is a function that satisfies the following properties:

- (i)  $\min\{a, b\} \leq M(a, b) \leq \max\{a, b\}$  (In particular,  $M(a, a) = a$ );
- (ii)  $M(a, b) = M(b, a)$ ;
- (iii)  $M(\alpha a, \alpha b) = \alpha M(a, b)$  for all  $\alpha > 0$ ;
- (iv)  $M(a, b)$  is non-decreasing in  $a$  and  $b$ .

It is obvious from the definition that the mean  $M_p$  satisfies the properties (ii) – (iv). We will give different expressions for  $M_p$  from which other properties, including (i) above, become apparent. In particular, we will show that

$$M_0(a, b) = \sqrt{ab}, \quad (1.8)$$

$$M_\infty(a, b) = \frac{2 \max\{a, b\}}{2 + \ln \frac{\max\{a, b\}}{\min\{a, b\}}} \leq \frac{a+b}{2}. \quad (1.9)$$

We conjecture that *for fixed  $a$  and  $b$ ,  $M_p(a, b)$  is an increasing function of  $p$* . At this time, we can prove that

$$M_2(a, b) \leq M_\infty(a, b). \quad (1.10)$$

Inequalities already known then lead to the chain

$$M_0(a, b) \leq M_1(a, b) \leq M_2(a, b) \leq M_\infty(a, b). \quad (1.11)$$

The first of these inequalities is well-known and easy to prove; the second has been given different proofs in [9, 10, 19].

The rest of this paper is organized as follows. Section 2 investigates various properties of  $M_p$  in detail, including different expressions for  $M_p$  and the inequality (1.10). Section 3 gives a relation between  $M_p$  and the *power difference mean*  $K_p$ . In section 4, we evaluate the norm of the integral operator induced on the space  $L_2(\mathbb{R}_+)$  by the kernel  $1/M_p(x, y)$ . This gives an extension of the famous Hilbert inequality. In section 5, we discuss positive definiteness of certain matrices as implications of some relations between  $M_p$  and  $K_p$  for which another conjecture is also proposed.

## 2. Mean $M_p$

Expressions (1.8) for  $M_0$  and (1.9) for  $M_\infty$  will be proved after a detailed investigation on  $M_p$  for  $0 < p < \infty$  is completed. Then we will prove the inequality (1.10).

### 2.1. $0 < p < \infty$ .

**Theorem 2.1.**  $\min\{a, b\} \leq \sqrt{ab} \leq M_p(a, b) \leq \left(\frac{a^p + b^p}{2}\right)^{1/p} \leq \max\{a, b\}$ .

*Proof.* The first inequality is easy to see, and the last one is easy to see too by replacing both  $a$  and  $b$  in  $\left(\frac{a^p + b^p}{2}\right)^{1/p}$  with  $\max\{a, b\}$ . We now prove the second and the third inequalities. Since  $(x^p + a^p)(x^p + b^p) = (x^p)^2 + (a^p + b^p)x^p + a^p b^p$ , we have

$$[x^p + (\sqrt{ab})^p]^2 \leq (x^p + a^p)(x^p + b^p) \leq \left[x^p + \frac{a^p + b^p}{2}\right]^2.$$

Therefore by (1.5),

$$\frac{1}{M_p\left(\left[\frac{a^p + b^p}{2}\right]^{1/p}, \left[\frac{a^p + b^p}{2}\right]^{1/p}\right)} \leq \frac{1}{M_p(a, b)} \leq \frac{1}{M_p(\sqrt{ab}, \sqrt{ab})}$$

which, together with the condition  $M_p(z, z) = z$  for any  $z > 0$ , leads to the desired inequalities.  $\square$

In the last integral in (1.6), substitute  $t = (y^p + 1)^{-1}$  to get

$$y^p = \frac{1}{t} - 1 = \frac{1-t}{t}, \quad (2.1)$$

$$p y^{p-1} dy = -\frac{1}{t^2} dt, \quad (2.2)$$

$$\begin{aligned} dy &= -\frac{1}{p} \left(\frac{t}{1-t}\right)^{\frac{p-1}{p}} \frac{1}{t^2} dt \\ &= -\frac{1}{p} t^{-1-1/p} (1-t)^{1/p-1} dt, \end{aligned} \quad (2.3)$$

$$\frac{1}{c_p} = \frac{1}{p} \int_0^1 t^{1/p-1} (1-t)^{1/p-1} dt \quad (2.4)$$

$$= \frac{B\left(\frac{1}{p}, \frac{1}{p}\right)}{p}, \quad (2.5)$$

where  $B(\cdot, \cdot)$  is the Beta-function [1]. In the integral in (1.5), substitute  $x^p + a^p = a^p t^{-1}$  to get

$$\begin{aligned}
x^p &= a^p \left( \frac{1}{t} - 1 \right) = a^p \frac{1-t}{t}, \\
px^{p-1} dx &= -a^p \frac{1}{t^2} dt, \\
dx &= -\frac{a}{p} \left( \frac{t}{1-t} \right)^{\frac{p-1}{p}} \frac{1}{t^2} dt \\
&= -\frac{a}{p} t^{-1-1/p} (1-t)^{1/p-1} dt, \\
\frac{1}{M_p(a, b)} &= c_p \frac{a}{p} \int_0^1 \frac{t^{-1/p-1} (1-t)^{1/p-1}}{[(a^p t^{-1})(a^p \frac{1-t}{t} + b^p)]^{1/p}} dt \\
&= \frac{c_p}{p} \int_0^1 \frac{t^{1/p-1} (1-t)^{1/p-1}}{[a^p(1-t) + b^p t]^{1/p}} dt. \tag{2.6}
\end{aligned}$$

Combine (2.4) and (2.6) to get

$$\frac{1}{M_p(a, b)} = \frac{\int_0^1 \frac{t^{1/p-1} (1-t)^{1/p-1}}{[a^p(1-t) + b^p t]^{1/p}} dt}{\int_0^1 t^{1/p-1} (1-t)^{1/p-1} dt}. \tag{2.7}$$

**Theorem 2.2.** *Given  $a, b > 0$  and  $0 < p < \infty$ , we have*

$$\frac{1}{M_p(a, b)} = (\max\{a, b\})^{-1} \sum_{k=0}^{\infty} \prod_{i=0}^{k-1} \frac{\left(\frac{1}{p} + i\right)^2}{\frac{2}{p} + i} \frac{1}{k!} \left[ 1 - \left( \frac{\min\{a, b\}}{\max\{a, b\}} \right)^p \right]^k, \tag{2.8}$$

where, by convention,  $\prod_{i=0}^{-1}(\dots) \equiv 1$  and  $0! = 1$ .

*Proof.* Both sides of (2.8) are equal to  $a$  if  $a = b$ . Assume without loss of generality that  $a > b > 0$ . Let  $\alpha = 1 - (b/a)^p$  and then  $0 < \alpha < 1$ . We have

$$\begin{aligned}
a^p(1-t) + b^p t &= a^p[1-t + (b/a)^p t] = a^p(1-\alpha t), \\
[a^p(1-t) + b^p t]^{-1/p} &= a^{-1}(1-\alpha t)^{-1/p} \\
&= a^{-1} \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right)}{k!} \alpha^k t^k. \tag{2.9}
\end{aligned}$$

The series in (2.9) converges for  $\alpha < 1$  which justifies the term-by-term integration below. Equation (2.9), together with (2.7), yields

$$\frac{1}{M_p(a, b)} = \frac{a^{-1} \int_0^1 \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right)}{k!} \alpha^k t^{k+1/p-1} (1-t)^{1/p-1} dt}{\int_0^1 t^{1/p-1} (1-t)^{1/p-1} dt}$$

$$\begin{aligned}
& a^{-1} \sum_{k=0}^{\infty} \int_0^1 \frac{\prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right)}{k!} \alpha^k t^{k+1/p-1} (1-t)^{1/p-1} dt \\
&= \frac{\int_0^1 t^{1/p-1} (1-t)^{1/p-1} dt}{\int_0^1 t^{1/p-1} (1-t)^{1/p-1} dt} \\
&= a^{-1} \sum_{k=0}^{\infty} \frac{\prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right)}{k!} \alpha^k \cdot \frac{B(k + \frac{1}{p}, \frac{1}{p})}{B(\frac{1}{p}, \frac{1}{p})}. \tag{2.10}
\end{aligned}$$

Using the well-known properties of the Beta and Gamma functions [1]

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s+t)}, \quad \Gamma(z) = (z-1)\Gamma(z-1),$$

we have

$$\begin{aligned}
\frac{B(k + \frac{1}{p}, \frac{1}{p})}{B(\frac{1}{p}, \frac{1}{p})} &= \frac{\Gamma(k + \frac{1}{p})\Gamma(\frac{1}{p})}{\Gamma(k + \frac{2}{p})} \frac{\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})\Gamma(\frac{1}{p})} \\
&= \frac{\Gamma(k + \frac{1}{p})}{\Gamma(\frac{1}{p})} \frac{\Gamma(\frac{2}{p})}{\Gamma(k + \frac{2}{p})} \\
&= \prod_{i=0}^{k-1} \frac{\frac{1}{p} + i}{\frac{2}{p} + i}.
\end{aligned}$$

Substituting this into (2.10) gives (2.8).  $\square$

**Theorem 2.3.** *Given  $a, b > 0$  and  $0 < p < \infty$ , we have*

$$\frac{1}{M_p(a, b)} = \left(\frac{a^p + b^p}{2}\right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right) \right] \left[ \prod_{i=0}^{2k-1} \frac{\frac{1}{p} + i}{\frac{2}{p} + i} \right] \left(\frac{a^p - b^p}{a^p + b^p}\right)^{2k}. \tag{2.11}$$

*Proof.* We have

$$\begin{aligned}
(x^p + a^p)(x^p + b^p) &= \left(x^p + \frac{a^p + b^p}{2}\right)^2 - \left(\frac{a^p - b^p}{2}\right)^2 \\
&= \left(x^p + \frac{a^p + b^p}{2}\right)^2 (1 - r^2),
\end{aligned}$$

where  $r = \frac{a^p - b^p}{2} / \left(x^p + \frac{a^p + b^p}{2}\right)$ . Therefore  $|r| < 1$  and

$$\begin{aligned}
[(x^p + a^p)(x^p + b^p)]^{-1/p} &= \left(x^p + \frac{a^p + b^p}{2}\right)^{-2/p} (1 - r^2)^{-1/p}, \\
&= \left(x^p + \frac{a^p + b^p}{2}\right)^{-2/p} \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right) \right] \frac{r^{2k}}{k!}, \\
\frac{1}{M_p(a, b)} &= c_p \int_0^{\infty} \frac{1}{\left(x^p + \frac{a^p + b^p}{2}\right)^{2/p}} \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right) \right] \frac{r^{2k}}{k!} dx
\end{aligned}$$

$$\begin{aligned}
&= c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}} \\
&\quad + c_p \sum_{k=1}^\infty \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right) \right] \int_0^\infty \frac{r^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2/p}} dx \\
&= \left(\frac{a^p + b^p}{2}\right)^{-1/p} \\
&\quad + \sum_{k=1}^\infty \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right) \right] c_p \int_0^\infty \frac{\left(\frac{a^p-b^p}{2}\right)^{2k}}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} dx.
\end{aligned} \tag{2.12}$$

Substitute  $x = \left(\frac{a^p+b^p}{2}\right)^{1/p} y$  and  $y^p + 1 = t^{-1}$  as in (2.1) – (2.3) to get

$$\begin{aligned}
\int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p + b^p}{2}\right)^{-2k-1/p} \int_0^\infty \frac{dy}{(y^p + 1)^{2k+2/p}} \\
&= \left(\frac{a^p + b^p}{2}\right)^{-2k-1/p} \int_0^1 \frac{1}{p} t^{2k+1/p-1} (1-t)^{1/p-1} dt \\
&= \left(\frac{a^p + b^p}{2}\right)^{-2k-1/p} \frac{B\left(2k + \frac{1}{p}, \frac{1}{p}\right)}{p}.
\end{aligned}$$

This together with (2.5) leads to

$$\begin{aligned}
c_p \int_0^\infty \frac{dx}{\left(x^p + \frac{a^p+b^p}{2}\right)^{2k+2/p}} &= \left(\frac{a^p + b^p}{2}\right)^{-2k-1/p} \frac{B\left(2k + \frac{1}{p}, \frac{1}{p}\right)}{B\left(\frac{1}{p}, \frac{1}{p}\right)} \\
&= \left(\frac{a^p + b^p}{2}\right)^{-2k-1/p} \prod_{i=0}^{2k-1} \frac{\frac{1}{p} + i}{\frac{2}{p} + i}.
\end{aligned} \tag{2.13}$$

Now (2.11) is a consequence of (2.12) and (2.13).  $\square$

## 2.2. $p = 0$ .

**Theorem 2.4.** *Given  $a, b > 0$ , we have*

$$M_0(a, b) = \sqrt{ab}. \tag{2.14}$$

*Proof.* It can be verified that  $\lim_{p \rightarrow 0^+} \left(\frac{a^p+b^p}{2}\right)^{1/p} = \sqrt{ab}$ . The equality  $M_0(a, b) = \sqrt{ab}$  is then a consequence of Theorem 2.1.  $\square$

## 2.3. $p = \infty$ .

**Theorem 2.5.** *Given  $a, b > 0$ , we have*

$$M_\infty(a, b) = \frac{2 \max\{a, b\}}{2 + \ln \frac{\max\{a, b\}}{\min\{a, b\}}}, \tag{2.15}$$

and

$$M_2(a, b) \leq M_\infty(a, b) \leq (a + b)/2. \tag{2.16}$$

*Proof.* Both (2.15) and (2.16) are obvious if  $a = b$ . Assume that  $a > b > 0$ . Then

$$\begin{aligned} \lim_{p \rightarrow \infty} \int_0^\infty \frac{dy}{(y^p + 1)^{2/p}} &= \int_0^1 dy + \int_1^\infty \frac{dy}{y^2} = 2, \\ \lim_{p \rightarrow \infty} \int_0^\infty \frac{dx}{[(x^p + a^p)(x^p + b^p)]^{1/p}} &= \int_0^b \frac{dx}{ab} + \int_b^a \frac{dx}{xa} + \int_a^\infty \frac{dx}{x^2} \\ &= \frac{1}{a} + \frac{\ln a - \ln b}{a} + \frac{1}{a} \\ &= \frac{2 + (\ln a - \ln b)}{a}. \end{aligned}$$

Therefore  $c_\infty = 1/2$ , and (2.15) holds by definition. The change of order of taking limits and the integrals above is justified by Lebesgue's Dominated Convergence Theorem [21, p.76] because

$$\begin{aligned} \frac{1}{(y^p + 1)^{2/p}} &\leq \begin{cases} 1 & \text{for } 0 \leq y \leq 1, \\ y^{-2} & \text{for } 1 < y, \end{cases} \\ \frac{1}{[(x^p + a^p)(x^p + b^p)]^{1/p}} &\leq \begin{cases} (ab)^{-1} & \text{for } 0 \leq x \leq a, \\ x^{-2} & \text{for } a < x. \end{cases} \end{aligned}$$

The second inequality in (2.16) is relatively easy to show. It goes as follows. Since

$$M_1(a, b) = \frac{a - b}{\ln a - \ln b} \leq \frac{a + b}{2},$$

we have successively

$$\begin{aligned} 2(a - b) &\leq (a + b)(\ln a - \ln b), \\ 2a &\leq 2b + (a + b)(\ln a - \ln b), \\ 4a &= 2(a + b) + (a + b)(\ln a - \ln b), \\ \frac{2a}{2 + \ln a - \ln b} &\leq \frac{a + b}{2}, \end{aligned}$$

as expected.

Let us focus on the first inequality in (2.16) now. As in the proof by John Todd for *Problem 19-17* in [10], let  $r = (a - b)/(a + b)$ . Then  $0 < r < 1$  and  $a = \frac{a+b}{2}(1+r)$  and  $b = \frac{a+b}{2}(1-r)$ . It suffices to show that  $M_2(1+r, 1-r) \leq M_\infty(1+r, 1-r)$ . It was shown by Gauss [12] (see also [8, p.7]) that for  $|r| < 1$

$$\frac{1}{M_2(1+r, 1-r)} = 1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \dots + \left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 r^{2n} + \dots$$

On the other hand, by (1.9),

$$\begin{aligned} \frac{1}{M_\infty(1+r, 1-r)} &= \frac{2 + \ln \frac{1+r}{1-r}}{2(1+r)} \\ &= \frac{2 + 2r \left( 1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \dots + \frac{1}{2n+1}r^{2n} + \dots \right)}{2(1+r)} \end{aligned}$$

$$= \frac{1+r \left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right)}{1+r}.$$

So for  $M_2(1+r, 1-r) \leq M_\infty(1+r, 1-r)$  to hold, it suffices to have

$$(1+r) \left[1 + \frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 r^{2n} + \cdots\right] \geq 1+r \left(1 + \frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right), \quad (2.17)$$

or, equivalently,

$$(1+r) \left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 r^{2n} + \cdots\right] \geq r \left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right). \quad (2.18)$$

Since  $1+r > 2r$ , (2.18) holds if

$$2r \left[\frac{1}{4}r^2 + \frac{9}{64}r^4 + \cdots + \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 r^{2n} + \cdots\right] \geq r \left(\frac{1}{3}r^2 + \frac{1}{5}r^4 + \cdots + \frac{1}{2n+1}r^{2n} + \cdots\right) \quad (2.19)$$

which is guaranteed if the corresponding coefficients of  $r^{2n+1}$  from both sides satisfy

$$2 \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 \geq \frac{1}{2n+1} \quad (2.20)$$

for  $n \geq 1$ . This is what we shall prove now. To this end, we shall use the following estimate for factorial  $n!$  [18, 20]

$$\sqrt{2\pi n}^{n+1/2} e^{-n+1/(12n+1)} < n! < \sqrt{2\pi n}^{n+1/2} e^{-n+1/(12n)}. \quad (2.21)$$

We have

$$\begin{aligned} \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} &= \frac{(2n)!}{2^{2n}(n!)^2} \\ &> \frac{\sqrt{2\pi}(2n)^{2n+1/2} e^{-2n+1/(24n+1)}}{2^{2n}[\sqrt{2\pi n}^{n+1/2} e^{-n+1/(12n)}]^2} \\ &= \frac{e^{1/(24n+1)}}{\sqrt{\pi n} e^{1/(6n)}}, \\ 2(2n+1) \left(\frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)}\right)^2 &> 4n \left(\frac{e^{1/(24n+1)}}{\sqrt{\pi n} e^{1/(6n)}}\right)^2 \\ &= \frac{4}{\pi} e^{2/(24n+1)-1/(3n)} \\ &= \frac{4}{\pi} e^{-(18n+1)/[3n(24n+1)]} \end{aligned}$$



$$\geq 1.12 \quad \text{for } n \geq 2.$$

This proves that (2.19) holds for  $n \geq 2$ . It can be verified that (2.19) holds for  $n = 1$  also. The proof is completed.  $\square$

*Remark 2.6.* One can use (2.21) to also show that

$$\left( \frac{1 \cdot 3 \cdots (2n-1)}{2 \cdot 4 \cdots (2n)} \right)^2 < \frac{1}{\pi n} < \frac{1}{2n+1}.$$

This is used by John Todd [10] to show  $M_1(1+r, 1-r) \leq M_2(1+r, 1-r)$ .

### 3. Relation to the Power Difference Mean

The *power difference mean*  $K_p(a, b)$  is defined for any  $p$  and  $a, b > 0$  as follows [14, 6].

$$K_p(a, b) := \frac{p-1}{p} \frac{a^p - b^p}{a^{p-1} - b^{p-1}}, \quad (3.1)$$

where it is understood that

$$K_p(a, a) := a, \quad K_1(a, b) := \lim_{p \rightarrow 1} K_p(a, b) = L(a, b). \quad (3.2)$$

Alternatively,  $K_p(a, b)$  admits the following integral expression:

$$\frac{1}{K_p(a, b)} = \int_0^1 \frac{dt}{[(1-t)a^p + tb^p]^{1/p}}. \quad (3.3)$$

By (1.1), (1.3), and (3.2), we have  $M_1(a, b) = L(a, b) = K_1(a, b)$ . It makes us wonder what kind of relations are between  $M_p(a, b)$  and  $K_p(a, b)$  for  $p \neq 1$ . Theorem 3.2 below provides an answer. But first we establish an expansion formula for  $K_p(a, b)$ .

**Lemma 3.1.** *Given  $a, b > 0$ , we have*

$$\frac{1}{K_p(a, b)} = \left( \frac{a^p + b^p}{2} \right)^{-1/p} \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} \left[ \prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right) \right] \left( \frac{a^p - b^p}{a^p + b^p} \right)^{2k}. \quad (3.4)$$

*Proof.*  $K_p$  as defined by (3.1) has a removable singularity at  $p = 1$ . In this case, equation (3.4) can be verified either by using  $K_1(a, b) = L(a, b)$  or by taking the limit as  $p$  goes to 1. In what follows, we shall assume  $p \neq 1$ . It suffices to show (3.4) for  $a = 1$  and  $0 < b \neq 1$ . It follows from (3.1) that

$$\begin{aligned} \frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} &= \frac{p}{p-1} \frac{a^{p-1} - b^{p-1}}{a^p - b^p} \left( \frac{a^p + b^p}{2} \right)^{1/p} \\ &= \frac{p}{p-1} \frac{1 - b^{p-1}}{1 - b^p} \left( \frac{1 + b^p}{2} \right)^{1/p}. \end{aligned} \quad (3.5)$$

Let  $r = (1 - b^p)/(1 + b^p)$ . Then  $|r| < 1$ , and

$$b^p = \frac{1-r}{1+r}, \quad \frac{1+b^p}{2} = \frac{1}{1+r}, \quad 1-b^p = \frac{2r}{1+r}, \quad b^{p-1} = (b^p)^{(p-1)/p} = \left( \frac{1-r}{1+r} \right)^{1-1/p}.$$

Therefore by (3.5)

$$\begin{aligned} \frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} &= \frac{p}{p-1} \frac{1 - \left( \frac{1-r}{1+r} \right)^{1-1/p}}{\frac{2r}{1+r}} (1+r)^{-1/p} \\ &= \frac{p}{p-1} \frac{(1+r)^{1-1/p} - (1-r)^{1-1/p}}{2r}. \end{aligned} \quad (3.6)$$

Use the binomial series expansion to get

$$\begin{aligned} (1+r)^{1-1/p} &= \sum_{k=0}^{\infty} \left[ \prod_{i=0}^{k-1} \left( 1 - \frac{1}{p} - i \right) \right] \frac{r^k}{k!} \\ &= 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \left( 1 - \frac{1}{p} \right) \left[ \prod_{i=0}^{k-2} \left( \frac{1}{p} + i \right) \right] \frac{r^k}{k!}, \\ (1-r)^{1-1/p} &= 1 + \sum_{k=1}^{\infty} (-1)^{k-1} \left( 1 - \frac{1}{p} \right) \left[ \prod_{i=0}^{k-2} \left( \frac{1}{p} + i \right) \right] \frac{(-r)^k}{k!} \\ &= 1 - \sum_{k=1}^{\infty} \left( 1 - \frac{1}{p} \right) \left[ \prod_{i=0}^{k-2} \left( \frac{1}{p} + i \right) \right] \frac{r^k}{k!}, \\ (1+r)^{1-1/p} - (1-r)^{1-1/p} &= 2 \left( 1 - \frac{1}{p} \right) \sum_{\ell=0}^{\infty} \left[ \prod_{i=0}^{2\ell-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2\ell+1}}{(2\ell+1)!}, \end{aligned}$$

and from (3.6)

$$\frac{1}{K_p(a, b)} \left( \frac{a^p + b^p}{2} \right)^{1/p} = \sum_{\ell=0}^{\infty} \left[ \prod_{i=0}^{2\ell-1} \left( \frac{1}{p} + i \right) \right] \frac{r^{2\ell}}{(2\ell+1)!},$$

as was to be shown.  $\square$

**Theorem 3.2.** *Given  $a, b > 0$  and  $a \neq b$ , we have*

- (1)  $M_p(a, b) > K_p(a, b)$  for  $0 \leq p < 1$ ,
- (2)  $M_1(a, b) = K_1(a, b)$ ,
- (3)  $M_p(a, b) < K_p(a, b)$  for  $p > 1$ .

*Proof.* We compare the right hand side of (2.11) and that of (3.4). For the purpose here, we may ignore the factor  $\left( \frac{a^p + b^p}{2} \right)^{-1/p}$  in both expressions and compare the two series. Let

$$\alpha_k = \frac{1}{k!} \left[ \prod_{i=0}^{k-1} \left( \frac{1}{p} + i \right) \right] \left[ \prod_{i=0}^{2k-1} \frac{\frac{1}{p} + i}{\frac{2}{p} + i} \right], \quad \beta_k = \frac{1}{(2k+1)!} \left[ \prod_{i=0}^{2k-1} \left( \frac{1}{p} + i \right) \right]$$

which are the coefficients of  $\left( \frac{a^p - b^p}{a^p + b^p} \right)^{2k}$  in the two series, respectively. Since  $\alpha_0 = 1 = \beta_0$ , it suffices to show that for  $k \geq 1$ ,

$$\alpha_k < \beta_k \text{ for } 0 < p < 1; \quad \alpha_k = \beta_k \text{ for } p = 1; \quad \text{and } \alpha_k > \beta_k \text{ for } p > 1.$$

Comparing  $\alpha_k$  and  $\beta_k$ , after canceling the common factor  $\prod_{i=0}^{2k-1} \left(\frac{1}{p} + i\right)$  in  $\alpha_k$  and  $\beta_k$ , is equivalent to comparing the two quantities

$$\frac{\prod_{i=0}^{k-1} \left(\frac{1}{p} + i\right)}{\prod_{i=0}^{2k-1} \left(\frac{2}{p} + i\right)} = \frac{p^k \prod_{i=0}^{k-1} (1 + pi)}{\prod_{i=0}^{2k-1} (2 + pi)} = \frac{p^k}{2^k \prod_{i=1}^k [2 + p(2i - 1)]}$$

and  $k!/(2k + 1)!$ . We have

$$\begin{aligned} \frac{p^k}{2^k \prod_{i=1}^k [2 + p(2i - 1)]} - \frac{k!}{(2k + 1)!} \\ = \frac{p^k (2k + 1)! - k! 2^k (2 + p)(2 + 3p) \cdots [2 + (2k - 1)p]}{2^k (2k + 1)! \prod_{i=1}^k [2 + p(2i - 1)]} \end{aligned}$$

whose numerator denoted by  $g(p)$  is a polynomial of degree  $k$  in  $p$  with the leading coefficient (of  $p^k$ )

$$\begin{aligned} (2k + 1)! - k! 2^k (2k - 1)!! &= (2k + 1)! - k! 2^k \frac{(2k + 1)!}{(2k)!! (2k + 1)} \\ &= (2k + 1)! \left(1 - \frac{1}{2k + 1}\right) > 0, \end{aligned}$$

and the rest of the coefficients (of  $p^i$  for  $i < k$ ) are all negative, and

$$g(1) = (2k + 1)! - k! 2^k (2k + 1)!! = 0.$$

Therefore  $g(p) < g(1)p^k = 0$  for  $0 < p < 1$ , and  $g(p) > g(1)p^k = 0$  for  $p > 1$ . This completes the proof.  $\square$

#### 4. Integral Operators Induced by $M_p$

Let  $\phi^{[0]}(x) \geq 0$  be any function on  $\mathbb{R}_+ = \{x : x > 0\}$ . This introduces a function on  $\mathbb{R}_+ \times \mathbb{R}_+$ :

$$\phi^{[1]}(x, y) = \int_0^\infty \phi^{[0]}(tx) \phi^{[0]}(ty) dt, \quad (4.1)$$

and, in turn, another function

$$\phi^{[2]}(x, y) = \int_0^\infty \phi^{[1]}(x, t) \phi^{[1]}(y, t) dt. \quad (4.2)$$

For  $0 < p < \infty$ , let

$$\phi_p^{[0]}(x) = e^{-x^p}. \quad (4.3)$$

Then for  $x, y > 0$

$$\begin{aligned} \phi_p^{[1]}(x, y) &= \int_0^\infty e^{-t^p(x^p + y^p)} dt && \text{(substitute } s = t^p(x^p + y^p)\text{)} \\ &= \int_0^\infty e^{-s} \frac{1}{p} \frac{s^{1/p-1}}{(x^p + y^p)^{1/p}} ds \\ &= \frac{\Gamma(\frac{1}{p})}{p} \frac{1}{(x^p + y^p)^{1/p}}, \end{aligned} \quad (4.4)$$

$$\begin{aligned}\phi_p^{[2]}(x, y) &= \frac{\Gamma(\frac{1}{p})^2}{p^2} \int_0^\infty \frac{dt}{[(x^p + t^p)(y^p + t^p)]^{1/p}} \quad (\text{use (2.5)}) \\ &= \frac{\Gamma(\frac{1}{p})^2}{p^2} \frac{B(\frac{1}{p}, \frac{1}{p})}{p} \frac{1}{M_p(x, y)}.\end{aligned}\tag{4.5}$$

*Remark 4.1.* Instead of (4.3), we could have started with

$$\phi_p^{[0]}(x) = \alpha_p e^{-x^p}, \quad \alpha_p = \frac{\Gamma(\frac{2}{p})^{1/4} p^{3/4}}{\Gamma(\frac{1}{p})}.\tag{4.3'}$$

Then we will get

$$\phi_p^{[2]}(x) = \frac{1}{M_p(x, y)}.\tag{4.5'}$$

This provides another way of looking at the family of means  $M_p(x, y)$ .

Note for  $p = 1$ , (4.5) is

$$\phi_1^{[2]}(x, y) = \frac{1}{M_1(x, y)} = \frac{1}{L(x, y)},\tag{4.6}$$

and for  $p = 2$ , it is

$$\phi_2^{[2]}(x, y) = \frac{\pi^2}{8} \frac{1}{M_2(x, y)} = \frac{\pi^2}{8} \frac{1}{AG(x, y)}.\tag{4.7}$$

We obtain the values of the norms of the integral operators with kernel  $1/M_p(x, y)$ . These results are extensions of the famous Hilbert inequality. We use a familiar technique from Hardy, Littlewood, and Pólya [13].

**Theorem 4.2** ([13, Theorem 319, page 229]). *Let  $\phi : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$  be homogeneous of order  $-1$ , i.e.,  $\phi(\lambda x, \lambda y) \equiv \lambda^{-1} \phi(x, y)$  for  $\lambda > 0$ , and that*

$$\int_0^\infty \frac{\phi(x, 1)}{\sqrt{x}} dx = \int_0^\infty \frac{\phi(1, y)}{\sqrt{y}} dy =: \kappa\tag{4.8}$$

*Then the induced operator on  $L_2(\mathbb{R}_+)$*

$$\Phi f(x) := \int_0^\infty \phi(x, y) f(y) dy$$

*has norm  $\|\Phi\|_{L_2} \leq \kappa$ . If  $\phi(1, y)$  is uniformly bounded in  $y \in \mathbb{R}_+$ , then<sup>1</sup>  $\|\Phi\|_{L_2} = \kappa$ .*

For the kernel (4.4),  $\phi_p^{[1]}(1, y)$  is uniformly bounded in  $y$  and satisfies (4.8). Let  $\Phi_p^{[1]}$  be the integral operator with  $\phi_p^{[1]}(x, y)$  in (4.4) as its kernel. Apply Theorem 4.2 to get

$$\begin{aligned}\|\Phi_p^{[1]}\|_{L_2} = \kappa_p &:= \int_0^\infty \frac{\phi_p^{[1]}(1, x)}{\sqrt{x}} dx \\ &= \frac{\Gamma(\frac{1}{p})}{p} \int_0^\infty \frac{dx}{(1+x^p)^{1/p} x^{1/2}}\end{aligned}$$

---

<sup>1</sup>This is not explicitly asserted in [13], but can be inferred from the discussion there. See, e.g., [15, page 149].

$$\begin{aligned}
&= \frac{\Gamma(\frac{1}{p})}{p} \frac{1}{p} B(\frac{1}{2p}, \frac{1}{2p}) \\
&= \frac{1}{p^2} [\Gamma(\frac{1}{2p})]^2.
\end{aligned} \tag{4.9}$$

Since the operator  $\Phi_p^{[2]}$  induced by the kernel (4.5) is the square of  $\Phi_p^{[1]}$  induced by (4.4) and also  $\Phi_p^{[1]}$  is self-adjoint because  $\phi_p^{[1]}(x, y) = \phi_p^{[1]}(y, x)$ , we have  $\|\Phi_p^{[2]}\|_{L_2} = \|\Phi_p^{[1]}\|_{L_2}^2$ .

**Theorem 4.3.** *Let  $0 < p < \infty$  and let*

$$\mathcal{M}_p f(x) := \int_0^\infty \frac{1}{M_p(x, y)} f(y) dy.$$

Then  $\mathcal{M}_p$  is a bounded linear operator on  $L_2(\mathbb{R}_+)$  with

$$\|\mathcal{M}_p\|_{L_2} = \frac{\Gamma(\frac{2}{p}) \Gamma(\frac{1}{2p})^4}{p \Gamma(\frac{1}{p})^2}. \tag{4.10}$$

*Proof.* Note by (4.5)

$$\Phi_p^{[2]} = \frac{\Gamma(\frac{1}{p})^2}{p^2} \frac{B(\frac{1}{p}, \frac{1}{p})}{p} \mathcal{M}_p.$$

By the consideration above,

$$\|\mathcal{M}_p\|_{L_2} = \kappa_p^2 \frac{p^3 \Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{1}{2p})^4}{p^4} \frac{p^3 \Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4} = \frac{\Gamma(\frac{2}{p}) \Gamma(\frac{1}{2p})^4}{p \Gamma(\frac{1}{p})^2},$$

as expected.  $\square$

Special case  $p = 1$  gives

$$\|\mathcal{M}_1\|_{L_2} = \pi^2. \tag{4.11}$$

This is noted on [13, page 257] (the last statement of §355). Special case  $p = 2$  gives

$$\|\mathcal{M}_2\|_{L_2} = \frac{\Gamma(1/4)^2}{2\pi^2} = 8.753758\dots \tag{4.12}$$

which happens to be  $2\pi/\text{AG}(\sqrt{2}, 1)^2$ .

*Remark 4.4.* Recall the famous Hilbert inequality that says the norm of the operator induced on  $L_2(\mathbb{R}_+)$  by the kernel  $1/(x+y)$  is  $\pi$ . This is  $\kappa_1$  in (4.9).

More generally, consider the space  $L_r(\mathbb{R}_+)$ , where  $r > 1$ . [13, Theorem 319, page 229] says that if  $\phi$  and  $\Phi$  are as in Theorem 4.2 and

$$\kappa(r) := \int_0^\infty \frac{\phi(1, x)}{x^{1/r}} dx = \int_0^\infty \frac{\phi(1, y)}{y^{1/r'}} dx < \infty, \tag{4.13}$$

then  $\Phi$  is a bounded operator on  $L_r(\mathbb{R}_+)$  with norm  $\|\Phi\|_{L_r} \leq \kappa(r)$ .

In our case, for the kernel (4.4), (4.13) gives

$$\begin{aligned}
\kappa_p(r) &:= \int_0^\infty \frac{dx}{(1+x^p)^{1/p} x^{1/r}} \quad (\text{substitute } t = (1+x^p)^{-1}) \\
&= \frac{1}{p} B(\frac{1-1/r}{p}, \frac{1/r}{p})
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{p} \frac{\Gamma(\frac{1-1/r}{p}) \Gamma(\frac{1/r}{p})}{\Gamma(\frac{1}{p})} \\
&= \frac{1}{p} \frac{\Gamma(\frac{1}{r'p}) \Gamma(\frac{1}{rp})}{\Gamma(\frac{1}{p})}, \tag{4.14}
\end{aligned}$$

where  $1/r + 1/r' = 1$ . So we have

$$\|\mathcal{M}_p\|_{L_r} \leq \kappa_p(r)^2 \frac{p^3 \Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})^4}. \tag{4.15}$$

Special case  $p = 1$

$$\kappa_1(r) = \Gamma(\frac{1}{r'}) \Gamma(\frac{1}{r}) = \pi \csc(\pi/r)$$

is given in [13, pages 226 and 255].

### 5. Positive Definiteness of Certain Matrices

An interesting connection between binary means of positive real numbers and positive definite matrices has been developed in the last few years. See [5, Chapters 4 and 5], [7], [14], and references therein.

Let  $M$  and  $\widetilde{M}$  be two binary means. We say that  $M \ll \widetilde{M}$  if for every  $n$  and for every choice of positive real numbers  $\lambda_1, \lambda_2, \dots, \lambda_n$ , the  $n \times n$  matrix

$$\left( \frac{M(\lambda_i, \lambda_j)}{\widetilde{M}(\lambda_i, \lambda_j)} \right)_{n \times n}$$

is positive semidefinite. For many interesting means, it has been found that the inequality  $M \leq \widetilde{M}$  implies the stronger relation  $M \ll \widetilde{M}$ .

We explore this for the two families  $\mathsf{K}_p$  and  $\mathsf{M}_p$ . First we observe that for every  $x \geq 0$ , the matrix

$$\left( \frac{1}{(x^p + \lambda_i^p)^{1/p} (x^p + \lambda_j^p)^{1/p}} \right)_{n \times n}$$

is positive semidefinite, since it is congruent to the *flat* matrix  $E$  (the matrix with all its entries equal to one). It follows from (1.5) that for  $0 < p < \infty$ , the  $n \times n$  matrices with  $(i, j)$  entries

$$\frac{1}{\mathsf{M}_p(\lambda_i, \lambda_j)} = c_p \int_0^\infty \frac{dx}{(x^p + \lambda_i^p)^{1/p} (x^p + \lambda_j^p)^{1/p}}$$

are positive semidefinite. By a limiting argument, we see that the matrices  $\left( \frac{1}{\mathsf{M}_0(\lambda_i, \lambda_j)} \right)_{n \times n} = \left( \frac{1}{\sqrt{\lambda_i \lambda_j}} \right)_{n \times n}$  and

$$\left( \frac{1}{\mathsf{M}_\infty(\lambda_i, \lambda_j)} \right)_{n \times n} = \left( \frac{2 + \ln \frac{\max\{\lambda_i, \lambda_j\}}{\min\{\lambda_i, \lambda_j\}}}{2 \max\{\lambda_i, \lambda_j\}} \right)_{n \times n} \tag{5.1}$$

are also positive semidefinite.

In [14], Hiai and Kosaki have proved that for  $p \leq 1/2$ , the matrices

$$\left( \mathsf{K}_p(\lambda_i, \lambda_j) \right)_{n \times n}$$

are positive semidefinite. Hence, *the matrix*

$$\left( \frac{K_p(\lambda_i, \lambda_j)}{M_p(\lambda_i, \lambda_j)} \right)_{n \times n}$$

is positive semidefinite for  $p \leq 1/2$ , being the Schur product of two such matrices.

The mean  $K_\infty(a, b)$  is equal to  $\max\{a, b\}$ . Hence we have

$$\frac{M_\infty(\lambda_i, \lambda_j)}{K_\infty(\lambda_i, \lambda_j)} = \frac{2}{2 + \ln \frac{\max\{\lambda_i, \lambda_j\}}{\min\{\lambda_i, \lambda_j\}}} = \frac{2}{2 + |\ln \lambda_i - \ln \lambda_j|} = \frac{1}{1 + |\ln \lambda_i^{1/2} - \ln \lambda_j^{1/2}|}.$$

The matrix with this as its  $(i, j)$  entry is positive semidefinite, in fact, infinitely divisible [5, p.153].

We have proved that  $K_p \ll M_p$  for  $0 \leq p \leq 1/2$ , and that  $M_\infty \ll K_\infty$ . We conjecture that

$$K_p \ll M_p \quad \text{for } 1/2 < p < 1, \text{ and } M_p \ll K_p \quad \text{for } 1 < p < \infty. \quad (5.2)$$

*Remark 5.1.* The positive semidefiniteness of the matrices (5.1) can be expressed in another way: the matrix

$$\left( \frac{1 + \frac{1}{2} |\ln \lambda_i - \ln \lambda_j|}{\max\{\lambda_i, \lambda_j\}} \right)_{n \times n} \quad (5.3)$$

is always positive semidefinite. It is interesting to note that the matrix

$$\left( \frac{1 + |\ln \lambda_i - \ln \lambda_j|}{\max\{\lambda_i, \lambda_j\}} \right)_{n \times n}$$

is not necessarily positive semidefinite, as can be seen from the  $2 \times 2$  example in which  $\lambda_1 = 1$  and  $\lambda_2 = e^2$ . In fact more can be said. Let  $r$  be any real nonnegative number. Then *the matrix*

$$W = \left( \frac{1 + r |\ln \lambda_i - \ln \lambda_j|}{\max\{\lambda_i, \lambda_j\}} \right)_{n \times n}$$

is positive semidefinite for  $0 \leq r \leq 1/2$  and not necessarily positive semidefinite for  $r > 1/2$ . This can be seen as follows. For  $0 \leq r \leq 1/2$ , we have

$$w_{ij} = \frac{1 + |\ln \lambda_i^r - \ln \lambda_j^r|}{\max\{\lambda_i, \lambda_j\}} = \frac{1 + |\ln \lambda_i^r - \ln \lambda_j^r|}{\max\{\lambda_i^{2r}, \lambda_j^{2r}\}} \cdot \frac{1}{[\max\{\lambda_i, \lambda_j\}]^{1-2r}} =: u_{ij} v_{ij}.$$

The matrix  $U = (u_{ij})_{n \times n}$  is positive semidefinite (by the case  $r = 1/2$  already proved), and the matrix  $V = (v_{ij})_{n \times n}$  is positive semidefinite since the matrix  $(1/\max\{\lambda_i, \lambda_j\})_{n \times n}$  is infinitely divisible [3, 5]. So  $W = (w_{ij})_{n \times n}$  being the Schur product of  $U$  and  $V$  is positive semidefinite. Now consider the case  $r > 1/2$ . Let  $\tilde{r}$  be any number such that  $r > \tilde{r} > 1/2$ , and  $\alpha$  be the unique positive root of  $x = 2\tilde{r} \ln(1+x)$  (such a root exists because at  $x = 0$ , the derivative of  $x$  is 1 and the derivative of  $2\tilde{r} \ln(1+x)$  is  $2\tilde{r} > 1$ ). With  $\lambda_1 = 1$  and  $\lambda_2 = e^{\alpha/r}$ , the  $2 \times 2$  matrix  $W$  is

$$W = \begin{pmatrix} 1 & (1+\alpha)e^{-\alpha/r} \\ (1+\alpha)e^{-\alpha/r} & e^{-\alpha/r} \end{pmatrix}$$

whose determinant

$$\det W = e^{-\alpha/r} - (1 + \alpha)^2 e^{-2\alpha/r} = e^{-2\alpha/r} \left[ e^{\alpha/r} - (1 + \alpha)^2 \right] < 0$$

since  $\alpha/r < \alpha/\tilde{r} = 2 \ln(1 + \alpha) = \ln(1 + \alpha)^2$ .

*Remark 5.2.* Examples of means for which  $M \leq \tilde{M}$  but the stronger relation  $M \ll \tilde{M}$  is not true were given in [4], and in [14]. To that list, we add another. We have seen that  $M_\infty(a, b) \leq A(a, b) := (a + b)/2$ , where  $A$  stands for the arithmetic mean. But the relation  $M_\infty \ll A$  is not true. For example, with  $\lambda_1 = 17/100$ ,  $\lambda_2 = 18/100$ , and  $\lambda_3 = 72/100$ , the  $3 \times 3$  matrix with its  $(i, j)$  entry being  $M_\infty(\lambda_i, \lambda_j)/A(\lambda_i, \lambda_j)$  has a negative eigenvalue  $-0.00011509756859$  computed by MATLAB.

## References

1. Andrews, George E., Askey, Richard, and Roy, Ranjan: *Special Functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, UK, 1999.
2. Bhatia, Rajendra: *On the exponential metric increasing property*, Linear Algebra Appl. **375** (2003) 211–220.
3. Bhatia, Rajendra: *Infinitely divisible matrices*, Amer. Math. Monthly **113** (2006) 221–235.
4. Bhatia, Rajendra: *Interpolating the arithmetic-geometric mean inequality and its operator version*, Linear Algebra Appl. **413** (2006) 355–363.
5. Bhatia, Rajendra: *Positive Definite Matrices*, Princeton Series in Applied Mathematics, Princeton University Press, Princeton, New Jersey, 2007.
6. Bhatia, Rajendra and Kosaki, Hideki: *Mean matrices and infinite divisibility*, Linear Algebra Appl. **424** (2007) 36–54.
7. Bhatia, Rajendra and Parthasarathy, K. R.: *Positive definite functions and operator inequalities*, Bull. London Math. Soc. **32** (2000), no. 02, 214–228.
8. Borwein, J. M. and Borwein, P. B.: *Pi and the Agm*, Wiley, 1987.
9. Bracken, Paul: *An arithmetic-geometric mean inequality*, Expo. Math. **19** (2001), no. 3, 273–279.
10. Carlson, B. C. and Vuorinen, M.: *Problem 91-17*, SIAM Rev. **34** (1992) 653.
11. Cox, David A.: *The arithmetic-geometric mean of Gauss*, Enseign. Math., Series II **30** (1984), no. 3-4, 275–330.
12. Gauss, C. F.: *Werke*, vol. 3, Göttingen, 1866–1933.
13. Hardy, G. H., Littlewood, J. E., and Pólya, G.: *Inequalities*, Cambridge University Press, London, 1934.
14. Hiai, F. and Kosaki, H.: *Means of matrices and comparison of their norms*, Indiana Univ. Math. J. **48** (1999) 899–936.
15. Hirsch, F. and Lacombe, G.: *Elements of Functional Analysis*, Graduate Texts in Mathematics, vol. 192, Springer, New York, 1999.
16. McAdams, W. H.: *Heat Transmission*, 3rd ed., McGraw-Hill, 1954.
17. Pólya, G. and Szegő, G.: *Isoperimetric Inequalities in Mathematical Physics*, Princeton University Press, Princeton, 1951.
18. Robbins, H.: *A remark on Stirling's formula*, Amer. Math. Monthly **62** (1955) 26–29.
19. Vamanamurthy, M. K. and Vuorinen, M.: *Inequalities for means*, J. Math. Anal. and Appl. **183** (1994), no. 1, 155–166.
20. Weisstein, Eric W.: *Stirling's approximation*<sup>2</sup>.
21. Wheeden, Richard L. and Zygmund, Antoni.: *Measure and Integral, An Introduction to Real Analysis*, Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1977.

<sup>2</sup><http://mathworld.wolfram.com/stirlingsapproximation.html>.



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