

## A VACUUM-ADAPTED APPROACH TO QUANTUM FEYNMAN-KAC FORMULAE

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*Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday*

ABSTRACT. The vacuum-adapted formulation of quantum stochastic calculus is employed to perturb expectation semigroups *via* a Feynman-Kac formula. This gives an alternative perspective on the perturbation theory for quantum stochastic flows that has recently been developed by the authors.

### 1. Introduction

Let  $\alpha = (\alpha_t)_{t \in \mathbb{R}}$  be an ultraweakly continuous group of normal  $*$ -automorphisms of a von Neumann algebra  $\mathbf{A}$  acting faithfully on the Hilbert space  $\mathfrak{h}$ , and let  $\delta$  be its ultraweak generator. Gaussian subordination may be used to construct an ultraweakly continuous semigroup  $\mathcal{P}^0$  on  $\mathbf{A}$  with ultraweak pre-generator  $\frac{1}{2}\delta^2$  [17, Section 1] in the following manner. If  $B = (B_t)_{t \geq 0}$  is standard Brownian motion on Wiener's probability space  $\mathbb{W}$  then, by Itô's formula, the unital  $*$ -homomorphism

$$j_t : \mathbf{A} \rightarrow \mathbf{A} \bar{\otimes} L^\infty(\mathbb{W}) = L^\infty(\mathbb{W}; \mathbf{A}); \quad a \mapsto \alpha_{B_t(\cdot)}(a) \quad (t \geq 0)$$

satisfies the stochastic differential equation

$$j_t(x) = x + \int_0^t j_s(\delta(x)) dB_s + \frac{1}{2} \int_0^t j_s(\delta^2(x)) ds \quad (x \in \text{Dom } \delta^2) \quad (1.1)$$

in the strong sense on  $L^2(\mathbb{W}; \mathfrak{h})$ . Thus

$$\mathcal{P}_t^0(a)u := \mathbb{E}_{\mathbb{W}}[j_t(a)u] \quad (a \in \mathbf{A}, u \in \mathfrak{h} \subset L^2(\mathbb{W}; \mathfrak{h}))$$

defines an ultraweakly continuous semigroup  $(\mathcal{P}_t^0)_{t \geq 0}$  of normal unital completely positive contractions on  $\mathbf{A}$  whose ultraweak generator is as desired.

For the case where  $\alpha$  is unitarily implemented, Lindsay and Sinha obtained an ultraweakly continuous semigroup  $\mathcal{P}^b$  with Feynman-Kac representation

$$\mathcal{P}_t^b(a)u = \mathbb{E}_{\mathbb{W}}[j_t(a)m_t^b u] \quad (t \geq 0, a \in \mathbf{A}, u \in \mathfrak{h}) \quad (1.2)$$

whose ultraweak generator extends  $\frac{1}{2}\delta^2 + \rho_b \delta$ , where  $\rho_b : a \mapsto ab$  is the operator on  $\mathbf{A}$  of right multiplication by  $b$  [17, Theorem 3.2]. Here  $m^b$  is the exponential

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martingale such that

$$m_t^b = I + \int_0^t j_s(b) m_s^b dB_s \quad (t \geq 0),$$

where  $b \in \mathbf{A}$  is self adjoint. For the Laplacian on  $\mathbb{R}^{3d}$  and the commutative von Neumann algebra  $L^\infty(\mathbb{R}^{3d})$ , such vector-field perturbations were studied from this viewpoint by Parthasarathy and Sinha ([20]). Other works on quantum Feynman–Kac formulae include [1], [14], [2] and [5], all of which belong to the pre-quantum stochastic era. The classical Feynman–Kac formula for Schrödinger operators, which is closely related to instances of the Trotter product formula, is well described in the books [21] and [22].

The results of Lindsay and Sinha have been fully generalised in [11]. In that paper a general perturbation theory for quantum stochastic flows is developed, yielding a much wider class of quantum Feynman–Kac formulae. Here we take our inspiration from [6]. The semigroups defined in (1.2) will not, in general, be positive or even real (*i.e.*,  $*$ -preserving). In this light Bahn and Park investigate a more symmetric form of Feynman–Kac perturbation, using instead an operator process  $n^b$  such that

$$n_t^b f = f + \int_0^t j_s(b) \mathbb{E}_{\mathbb{W}}[n_s^b f | \mathcal{B}_s] dB_s - \frac{1}{2} \int_0^t j_s(b^2) \mathbb{E}_{\mathbb{W}}[n_s^b f | \mathcal{B}_s] ds \quad (1.3)$$

for all  $f \in L^2(\mathbb{W}; \mathfrak{h})$ , where  $(\mathcal{B}_t)_{t \geq 0}$  is the canonical filtration of the Brownian motion  $B$ . In this case, letting

$$\mathcal{Q}_t^b(a)u := \mathbb{E}_{\mathbb{W}}[(n_t^b)^* j_t(a) n_t^b u] \quad (a \in \mathbf{A}, u \in \mathfrak{h})$$

gives an ultraweakly continuous completely positive semigroup  $(\mathcal{Q}_t^b)_{t \geq 0}$  on  $\mathbf{A}$ , which is contractive if  $n^b$  is and whose generator extends the map

$$\frac{1}{2} \delta^2 + \lambda_b \delta + \rho_b \delta + \lambda_b \rho_b - \frac{1}{2} \lambda_{b^2} - \frac{1}{2} \rho_{b^2}, \quad (1.4)$$

where  $\lambda_b$  denotes the operator on  $\mathbf{A}$  given by left multiplication by  $b$ .

In this work we are guided by the form of (1.3); the conditional expectations make it reminiscent of a stochastic differential equation used by Alicki and Fannes for dilating quantum dynamical semigroups [4, Equation (12)]. As observed in [7], this type of equation may be profitably interpreted in the vacuum-adapted form of quantum stochastic calculus. In contrast to [11], where the standard identity-adapted (Hudson–Parthasarathy) theory is used, here the analysis is slightly easier although the algebra becomes a bit more complicated.

We describe the contents of the paper next, restricting our description here to the one-dimensional case, for simplicity. The requirement that  $\alpha$  is unitarily implemented is removed; our primary object is a vacuum-adapted quantum stochastic flow. This is an ultraweakly continuous family  $j = (j_t)_{t \geq 0}$  of normal  $*$ -homomorphisms which form a vacuum-adapted quantum stochastic cocycle on Boson Fock space over  $L^2(\mathbb{R}_+)$  and which are as unital as vacuum adaptedness permits. The flow  $j$  is assumed to satisfy the quantum stochastic differential equation

$$dj_t(x) = j_t(\delta_0(x)) dA_t^\dagger + j_t(\pi_0(x)) d\Lambda_t + j_t(\delta_0^\dagger(x)) dA_t + j_t(\tau_0(x)) dt \quad (1.5)$$

for all  $x \in A_0$ , where  $A_0$  is a subset of  $A$ , and the *structure maps*

$$\tau_0, \delta_0, \delta_0^\dagger, \pi_0 : A_0 \rightarrow A$$

must satisfy certain algebraic relations, thanks to the unital and  $*$ -homomorphic properties of  $j$ . Equation (1.5) generalises (1.1), which corresponds to the case where  $A_0 = \text{Dom } \delta^2$ ,  $\pi_0$  is the inclusion map,

$$\delta_0 = \delta_0^\dagger = \delta|_{A_0} \quad \text{and} \quad \tau_0 = \frac{1}{2}\delta^2.$$

The appearance of the non-zero gauge term  $\pi_0$  is due to the fact that we are working in the vacuum-adapted set-up: *cf.* [10, Theorem 7.3]. It follows from (1.5) that the quantum stochastic flow satisfies the equation

$$\langle u\Omega, j_t(x)v\Omega \rangle = \langle u, v \rangle + \int_0^t \langle u\Omega, j_s(\tau_0(x))v\Omega \rangle ds \quad (u, v \in \mathfrak{h}, t \geq 0, x \in A_0),$$

where  $\Omega$  denotes the Fock vacuum vector. The generator of the *vacuum-expectation semigroup*  $\mathcal{P}^0 := (\mathbb{E} \circ j_t)_{t \geq 0}$  therefore extends the map  $\tau_0$ . A natural assumption here is that  $\tau_0$  is a pre-generator of  $\mathcal{P}^0$ , however our results do not require it.

Starting with Evans and Hudson [13], several authors have used conjugation with a unitary process to perturb quantum stochastic flows. These works focused on the case of bounded structure maps, so that the vacuum-expectation semigroup  $\mathcal{P}^0$  is norm continuous, and considered identity-adapted flows and processes. For  $h = h^* \in A$  and  $l \in A$  there exists a unitary process  $U$  such that

$$U_0 = I, \quad dU_t = j_t(l)U_t dA_t^\dagger + j_t(-l^*)U_t dA_t + j_t(-ih - \frac{1}{2}l^*l)U_t dt,$$

and the vacuum-expectation semigroup of the perturbed flow ( $a \mapsto U_t^* j_t(a) U_t$ ) $_{t \geq 0}$  has generator

$$\tau_0 + \lambda_{l^*} \delta_0 + \rho_l \delta_0^\dagger + \lambda_{l^*} \rho_l \pi_0 + i[h, \cdot] - \frac{1}{2}\{l^*l, \cdot\},$$

where  $[\cdot, \cdot]$  and  $\{\cdot, \cdot\}$  denote commutator and anticommutator. The main result obtained here includes this situation as a special case.

For any vacuum-adapted quantum stochastic flow  $j$  and any  $c = \begin{bmatrix} c_0 \\ c_1 \end{bmatrix}$  in  $A \oplus A$ , Theorem 5.3 below gives a process  $M^c$  such that  $M^c - I$  is vacuum adapted and the following quantum stochastic differential equation is satisfied:

$$d(M^c - I)_t = j_t(c_0)M_t^c dt + j_t(c_1)M_t^c dA_t^\dagger.$$

Consequently, for any  $d = \begin{bmatrix} d_0 \\ d_1 \end{bmatrix}$  in  $A \oplus A$ , there is an ultraweakly continuous semigroup  $\mathcal{P}^{c,d}$  on  $A$  with

$$\langle u, \mathcal{P}_t^{c,d}(a)v \rangle = \langle u\Omega, (M_t^c)^* j_t(a) M_t^d v\Omega \rangle \quad (u, v \in \mathfrak{h}, t \geq 0, a \in A).$$

When  $j$  satisfies (1.5), the ultraweak generator of  $\mathcal{P}^{c,d}$  necessarily extends

$$\tau_0 + \lambda_{c_1^*} \delta_0 + \rho_{d_1} \delta_0^\dagger + \lambda_{c_1^*} \rho_{d_1} \pi_0 + \lambda_{c_0^*} + \rho_{d_0}. \quad (1.6)$$

This class of semigroups includes both the Lindsay–Sinha and the Bahn–Park examples, as well as those obtained by unitary conjugation; the generators of the latter correspond to the case

$$c = d = \begin{bmatrix} -ih - \frac{1}{2}l^*l \\ l \end{bmatrix}, \quad \text{where } h = h^*.$$

**1.1. Conventions.** Hilbert spaces are complex with inner products linear in their second argument. The linear, Hilbert-space and ultraweak tensor products are denoted by  $\underline{\otimes}$ ,  $\otimes$  and  $\overline{\otimes}$ , respectively. For a Hilbert space  $H$  we adopt the Dirac-inspired notation  $|H\rangle$  for  $B(\mathbb{C}; H)$  and  $\langle H|$  for the topological dual  $B(H; \mathbb{C})$ , writing  $|u\rangle$  for the operator  $\lambda \mapsto \lambda u$  and  $\langle u|$  for the functional  $v \mapsto \langle u, v \rangle$ , where  $u \in H$ . Recall the *E notation*,

$$E_u := |u\rangle \otimes I \quad \text{and} \quad E^u := (E_u)^* = \langle u| \otimes I \quad (u \in H), \quad (1.7)$$

in which  $I$  denotes the identity operator on a Hilbert space determined by context. The following commutator and anticommutator notation is also used for elements of an algebra:

$$[a, b] := ab - ba \quad \text{and} \quad \{a, b\} := ab + ba. \quad (1.8)$$

## 2. Multipliers for Quantum Stochastic Flows

Fix now, and for the rest of the paper, Hilbert spaces  $\mathfrak{h}$  and  $\mathfrak{k}$ , referred to as the *initial space* and *multiplicity space* or *noise dimension space*, respectively. Fix also a von Neumann algebra  $A$  acting faithfully on  $\mathfrak{h}$ . Set  $\widehat{\mathfrak{k}} := \mathbb{C} \oplus \mathfrak{k}$ ,

$$\widehat{c} := \begin{pmatrix} 1 \\ c \end{pmatrix} \in \widehat{\mathfrak{k}} \quad (c \in \mathfrak{k}) \quad \text{and} \quad \omega := \widehat{0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (2.1)$$

Our basic reference for quantum stochastic calculus is [16].

For a subinterval  $J$  of  $\mathbb{R}_+$ , let  $\mathcal{F}_J$  denote the Boson Fock space over  $L^2(J; \mathfrak{k})$  and let  $\mathbf{N}_J := B(\mathcal{F}_J)$ . For brevity, set  $\mathcal{F} := \mathcal{F}_{\mathbb{R}_+}$ ,  $\mathcal{F}_t := \mathcal{F}_{[0, t]}$  and  $\mathcal{F}_{[t, \infty)} := \mathcal{F}_{[t, \infty)}$ , with corresponding abbreviations for the noise algebra  $\mathbf{N} = B(\mathcal{F})$ . The identifications

$$\mathcal{F} = \mathcal{F}_s \otimes \mathcal{F}_{[s, \infty)} = \mathcal{F}_s \otimes \mathcal{F}_{[s, t]} \otimes \mathcal{F}_t \quad (0 \leq s \leq t < \infty),$$

which arise from the exponential property of Fock space, entail the identifications

$$\mathbf{N} = \mathbf{N}_s \overline{\otimes} \mathbf{N}_{[s, \infty)} = \mathbf{N}_s \overline{\otimes} \mathbf{N}_{[s, t]} \overline{\otimes} \mathbf{N}_t \quad (0 \leq s \leq t < \infty).$$

The notation  $\Omega_J$ ,  $I_J$  and  $\text{id}_J$  for the vacuum vector in  $\mathcal{F}_J$ , the identity operator on  $\mathcal{F}_J$  and the identity map on  $\mathbf{N}_J$ , respectively, is also useful, with corresponding abbreviations for other intervals, such as  $\Omega_{[s, \infty)}$ ,  $I_{[s, \infty)}$  and  $\text{id}_{[s, \infty)}$ , as above.

Denote by  $\Delta$  any of the following projections:

$$P_{\mathfrak{k}} \in B(\widehat{\mathfrak{k}}), \quad P_{\mathfrak{k}} \otimes \text{id}_A \in B(\widehat{\mathfrak{k}}) \overline{\otimes} A, \quad \text{and} \quad P_{\mathfrak{k}} \otimes \text{id}_A \otimes I_{\mathcal{F}} \in B(\widehat{\mathfrak{k}}) \overline{\otimes} A \overline{\otimes} \mathbf{N}, \quad (2.2)$$

where  $P_{\mathfrak{k}} = \begin{bmatrix} 0 & 0 \\ 0 & I_{\mathfrak{k}} \end{bmatrix} \in B(\widehat{\mathfrak{k}})$  is the orthogonal projection onto  $\mathfrak{k}$ .

The right shift

$$s_t : L^2(\mathbb{R}_+; \mathfrak{k}) \rightarrow L^2([t, \infty); \mathfrak{k}); \quad f \mapsto f(\cdot - t) \quad (t \geq 0)$$

has second quantisation

$$S_t : \mathcal{F} \rightarrow \mathcal{F}_{[t, \infty)}; \quad \varepsilon(f) \mapsto \varepsilon(s_t f),$$

where  $\varepsilon(g)$  denotes the exponential vector corresponding to the vector  $g$ , and the map

$$\sigma_t : A \overline{\otimes} \mathbf{N} \rightarrow A \overline{\otimes} \mathbf{N}_{[t, \infty)}; \quad T \mapsto (I_{\mathfrak{h}} \otimes S_t)T(I_{\mathfrak{h}} \otimes S_t)^*$$

is a normal  $*$ -isomorphism for all  $t \geq 0$ .

**Definition 2.1.** A *vacuum-adapted quantum stochastic cocycle*  $k$  on  $\mathbf{A}$  is a family of normal completely bounded maps  $(k_t : \mathbf{A} \rightarrow \mathbf{A} \overline{\otimes} \mathbf{N})_{t \geq 0}$  such that, for all  $a \in \mathbf{A}$  and  $s, t \geq 0$ ,

$$(\Omega\text{-C i}) \quad k_0(a) = a \otimes |\Omega\rangle\langle\Omega|,$$

$$(\Omega\text{-C ii}) \quad k_t(a) = k_t(a) \otimes |\Omega_t\rangle\langle\Omega_t|, \text{ where } k_t(a) \in \mathbf{A} \overline{\otimes} \mathbf{N}_t,$$

$$(\text{C iii}) \quad k_{s+t} = \widehat{k}_s \circ \sigma_s \circ k_t, \text{ where } \widehat{k}_s := k_s \overline{\otimes} \text{id}_s$$

and (C iv)  $r \mapsto k_r(a)$  is ultraweakly continuous.

Such a family is a *flow* on  $\mathbf{A}$  if each  $k_t$  is  $*$ -homomorphic and unital. Following tradition we use the letter  $j$  for quantum stochastic flows.

In the standard theory,  $(\Omega\text{-C i})$  and  $(\Omega\text{-C ii})$  are replaced by their identity-adapted counterparts,

$$(I\text{-C i}) \quad k_0(a) = a \otimes I_{\mathcal{F}}$$

$$\text{and } (I\text{-C ii}) \quad k_t(a) = k_t(a) \otimes I_t, \text{ where } k_t(a) \in \mathbf{A} \overline{\otimes} \mathbf{N}_t.$$

*Remark 2.2.* The prescription

$$k^{(\Omega)} = (k_t)(\cdot) \otimes |\Omega_t\rangle\langle\Omega_t|_{t \geq 0} \mapsto k^{(I)} = (k_t)(\cdot) \otimes I_t_{t \geq 0} \quad (2.3)$$

gives a bijective correspondence between the class of vacuum-adapted quantum stochastic cocycles and the class of identity-adapted quantum stochastic cocycles.

Note that

$$k_t(a) = E^{\Omega_t} k_t(a) E_{\Omega_t} \quad (t \geq 0, a \in \mathbf{A}) \quad (2.4)$$

in both cases.

In terms of the orthogonal projection

$$P_t := I_{\mathfrak{h}} \otimes I_t \otimes |\Omega_t\rangle\langle\Omega_t|, \quad (2.5)$$

condition  $(\Omega\text{-C ii})$  becomes

$$k_t(a) = P_t k_t(a) P_t,$$

whereas  $(I\text{-C ii})$  only implies the weaker commutation relation

$$k_t(a) P_t = P_t k_t(a).$$

Let

$$\mathbb{E} := \text{id}_{\mathbf{A}} \overline{\otimes} \omega_{\Omega} : \mathbf{A} \overline{\otimes} \mathbf{N} \rightarrow \mathbf{A}$$

denote the *vacuum expectation*, where  $\omega_{\Omega}$  is the state on  $\mathbf{N}$  corresponding to the vacuum vector  $\Omega$ .

**Proposition 2.3.** *Let  $k$  be a vacuum-adapted quantum stochastic cocycle on  $\mathbf{A}$ . The ultraweakly continuous family of normal completely bounded maps  $(\mathbb{E} \circ k_t)_{t \geq 0}$  on  $\mathbf{A}$  forms a semigroup, called the vacuum-expectation semigroup of  $k$ .*

*Proof.* For all  $t \geq 0$ , the conditional expectation

$$\mathbb{E}_t^{\Omega} : \mathbf{A} \overline{\otimes} \mathbf{N} \rightarrow \mathbf{A} \overline{\otimes} \mathbf{N}; \quad T \mapsto (\text{id}_{\mathbf{A} \overline{\otimes} \mathbf{N}_t} \overline{\otimes} \omega_{\Omega_t})(T) \otimes |\Omega_t\rangle\langle\Omega_t| = P_t T P_t \quad (2.6)$$

has the tower property  $\mathbb{E} \circ \mathbb{E}_t^{\Omega} = \mathbb{E}$ . The claim follows since any vacuum-adapted quantum stochastic cocycle satisfies the identity

$$\mathbb{E}_t^{\Omega} \circ \widehat{k}_t \circ \sigma_t = k_t \circ \mathbb{E} \quad (t \geq 0). \quad \square$$

Quantum stochastic differential equations of the following form are a basic source of quantum stochastic cocycles.

*Remark 2.4.* Under the correspondence (2.3),  $k^{(\Omega)}$  satisfies a quantum stochastic differential equation of the form

$$k_0(a) = a \otimes |\Omega\rangle\langle\Omega|, \quad dk_t = \tilde{k}_t(\psi(a)) d\Lambda_t \quad (2.7)$$

on a subset  $\mathbf{A}_0$  of  $\mathbf{A}$ , where  $\tilde{k}_t := \text{id}_{B(\widehat{\mathbf{k}})} \overline{\otimes} k_t$ , if and only if  $k^{(I)}$  satisfies a quantum stochastic differential equation of the form

$$k_0(a) = a \otimes I_{\mathcal{F}}, \quad dk_t = \tilde{k}_t(\phi(a)) d\Lambda_t \quad (2.8)$$

on  $\mathbf{A}_0$ , where the maps  $\psi, \phi : \mathbf{A}_0 \rightarrow B(\widehat{\mathbf{k}}) \overline{\otimes} \mathbf{A}$  are related by the following identity:

$$\psi(a) = \phi(a) + \Delta \otimes a \quad (a \in \mathbf{A}_0).$$

This is proved in [10, Theorem 7.3]. Here  $\Lambda$  is the matrix of fundamental quantum stochastic integrators [15]; see [16].

*Remark 2.5* ([18, Section 6]). Let the map  $\phi : \mathbf{A} \rightarrow B(\widehat{\mathbf{k}}) \overline{\otimes} \mathbf{A}$  have the block-matrix form

$$\phi(a) = \begin{bmatrix} i[h, a] - \frac{1}{2}\{r^*r, a\} + r^*\pi(a)r & ar^* - r^*\pi(a) \\ ra - \pi(a)r & \pi(a) - I_{\mathbf{k}} \otimes a \end{bmatrix} \quad (a \in \mathbf{A}), \quad (2.9)$$

where  $h \in \mathbf{A}$  is self adjoint,  $r \in |\mathbf{k}\rangle \overline{\otimes} \mathbf{A}$  and  $\pi : \mathbf{A} \rightarrow B(\mathbf{k}) \overline{\otimes} \mathbf{A}$  is a normal unital  $*$ -homomorphism. Then the quantum stochastic differential equation (2.8) has a unique solution and this is an identity-adapted quantum stochastic flow. Conversely, if an identity-adapted quantum stochastic flow satisfies (2.8) for some normal bounded map  $\phi : \mathbf{A} \rightarrow B(\widehat{\mathbf{k}}) \overline{\otimes} \mathbf{A}$  then  $\phi$  has the form (2.9).

**Definition 2.6.** Let  $j$  be a vacuum-adapted quantum stochastic flow on  $\mathbf{A}$ . A family of operators  $M = (M_t)_{t \geq 0}$  in  $\mathbf{A} \overline{\otimes} \mathbf{N}$  is a *multiplier* for  $j$  if, for all  $s, t \geq 0$ ,

- (M i)  $M_0 = I_{\mathfrak{h} \otimes \mathcal{F}}$ ,
- (M ii)  $M_t P_t = P_t M_t$ ,
- (M iii)  $M_{s+t} = J_s(M_t) M_s$ , where  $J_s := \widehat{j}_s \circ \sigma_s$

and (M iv)  $r \mapsto M_r$  is strongly continuous.

The Banach–Steinhaus Theorem and condition (M iv) imply that  $M$  is locally bounded.

**Theorem 2.7** (Cf. [6, Theorem 2.1]). *Let  $M$  and  $N$  be multipliers for the vacuum-adapted quantum stochastic flow  $j$ . The ultraweakly continuous normal completely bounded family*

$$\mathcal{P} := (a \mapsto \mathbb{E}[M_t^* j_t(a) N_t])_{t \geq 0}$$

*forms a semigroup, which is completely contractive if  $M$  and  $N$  are contractive and is completely positive if  $M = N$ .*

*Proof.* To prove the semigroup property, let  $a \in \mathbf{A}$  and  $s, t \geq 0$ . By the tower property for the conditional expectation  $\mathbb{E}_s^\Omega$  defined in (2.6), it follows that

$$\begin{aligned} \mathcal{P}_{s+t}(a) &= \mathbb{E}[\mathbb{E}_s^\Omega[M_s^* J_s(M_t^*) J_s(j_t(a)) J_s(N_t) N_s]] && \text{by (C iii) and (M iii)} \\ &= \mathbb{E}[M_s^* \mathbb{E}_s^\Omega[J_s(M_t^* j_t(a) N_t)] N_s] && \text{by (M ii)} \\ &= \mathbb{E}[M_s^* j_s(\mathbb{E}[M_t^* j_t(a) N_t]) N_s] && (2.10) \\ &= \mathcal{P}_s(\mathcal{P}_t(a)). \end{aligned}$$

For the equality (2.10), note that if  $a \in \mathbf{A}$  and  $b \in \mathbf{N}$  then

$$\mathbb{E}_s^\Omega[J_s(a \otimes b)] = \langle \Omega, b\Omega \rangle j_s(a) = j_s(\mathbb{E}[a \otimes b]);$$

thus  $\mathbb{E}_s^\Omega \circ J_s = j_s \circ \mathbb{E}$ , by linearity and ultraweak continuity.  $\square$

*Remark 2.8.* Some of the ideas in this section go back to early work of Accardi [1, Sections 2 and 4]; see also [3, Section 2.3].

### 3. A Vacuum-adapted Quantum Stochastic Differential Equation

Let  $(u_t)_{t \in \mathbb{R}}$  be a strongly continuous one-parameter unitary group in  $\mathbf{A}$ , let  $B = (B_t)_{t \geq 0}$  be the canonical Brownian motion on Wiener's probability space  $\mathbb{W}$  and, taking  $\mathfrak{k} = \mathbb{C}$ , identify  $L^2(\mathbb{W})$  with  $\mathcal{F}$  via the Wiener–Itô–Segal isomorphism. If the unitary operator  $U_t \in \mathbf{A} \overline{\otimes} \mathbf{N}$  is such that

$$U_t \xi : \omega \mapsto u_{B_t(\omega)} \xi(\omega) = u_{\omega(t)} \xi(\omega) \quad (\xi \in L^2(\mathbb{W}; \mathfrak{h}))$$

then the family of maps  $(j_t^B : a \mapsto U_t(a \otimes I_{\mathcal{F}}) U_t^*)_{t \geq 0}$  is an identity-adapted quantum stochastic flow on  $\mathbf{A}$  ([6, Lemma 3.1], cf. [16, Section 5]).

Bahn and Park considered the operator stochastic differential equation

$$M_0^a = I_{\mathfrak{h} \otimes \mathcal{F}}, \quad dM_t^a = j_t^B(a) P_t M_t^a dB_t - \frac{1}{2} j_t^B(a^2) P_t M_t^a dt, \quad (3.1)$$

where  $a \in \mathbf{A}$ , and obtained a solution pointwise in  $L^2(\mathbb{W}; \mathfrak{h})$  [6, Proposition 3.2]. They showed that the collection of operators  $(M_t^a)_{t \geq 0}$  forms a multiplier for the quantum stochastic flow  $j^B$  [6, Proposition 3.3].

Fix  $a \in \mathbf{A}$  and set  $N_t := M_t^a - I_{\mathfrak{h} \otimes \mathcal{F}}$  for all  $t \geq 0$ , so that

$$N_t \xi = \int_0^t Q_s \xi dB_s - \frac{1}{2} \int_0^t R_s \xi ds + \int_0^t Q_s N_s \xi dB_s - \frac{1}{2} \int_0^t R_s N_s \xi ds$$

for all  $\xi \in L^2(\mathbb{W}; \mathfrak{h})$ , where

$$Q_t := j_t^B(a) P_t \quad \text{and} \quad R_t := j_t^B(a^2) P_t.$$

As  $(j_t^B(b))_{t \geq 0}$  is identity adapted for all  $b \in \mathbf{A}$ , the processes  $Q$  and  $R$  are vacuum adapted. By [9, Theorem 2.2], the process  $N$  above is the unique vacuum-adapted solution of the quantum stochastic differential equation

$$N_0 = 0, \quad dN_t = Q_t dA_t^\dagger - \frac{1}{2} R_t dt + Q_t N_t dA_t^\dagger - \frac{1}{2} R_t N_t dt. \quad (3.2)$$

To see that (3.2) is the correct quantum stochastic generalisation of (3.1), for simplicity take  $\mathfrak{h} = \mathbb{C}$  and let  $\mathfrak{z}(f)$  denote the Brownian exponential corresponding to  $f \in L^2(\mathbb{R}_+)$ , i.e., the unique element of  $L^2(\mathbb{W})$  such that

$$\mathfrak{z}(f)_t := \mathbb{E}_{\mathbb{W}}[\mathfrak{z}(f)|\mathcal{B}_t] = 1 + \int_0^t f(s)\mathbb{E}_{\mathbb{W}}[\mathfrak{z}(f)|\mathcal{B}_s] dB_s \quad (t \geq 0),$$

where  $(\mathcal{B}_t)_{t \geq 0}$  is the canonical filtration generated by the Brownian motion  $B$ . (Recall that  $\mathfrak{z}(f)$  corresponds to  $\varepsilon(f)$  and  $\mathbb{E}_{\mathbb{W}}[\cdot|\mathcal{B}_t]$  to  $P_t$ .) If  $(X_t)_{t \geq 0}$  is a process of bounded operators on  $\mathcal{F}$  with locally bounded norm and such that  $X_t P_t = P_t X_t$  for all  $t \geq 0$  then, by the (classical) Itô product formula,

$$\begin{aligned} \mathbb{E}_{\mathbb{W}} \left[ \overline{\mathfrak{z}(f)} \int_0^t X_s P_s \mathfrak{z}(g) dB_s \right] &= \mathbb{E}_{\mathbb{W}} \left[ \int_0^t \overline{f(s)\mathfrak{z}(f)}_s X_s \mathfrak{z}(g)_s ds \right] \\ &= \left\langle \varepsilon(f), \int_0^t X_s P_s dA_s^\dagger \varepsilon(g) \right\rangle \quad (f, g \in L^2(\mathbb{R}_+)). \end{aligned}$$

**Definition 3.1.** For a Hilbert space  $\mathbf{H}$ , a *bounded process in  $B(\mathbf{H}) \overline{\otimes} \mathbf{A}$*  is a family of operators  $Z = (Z_t)_{t \geq 0}$  in  $B(\mathbf{H}) \otimes \mathbf{A} \otimes \mathbf{N}$  such that

$$t \mapsto \langle \zeta', Z_t \zeta \rangle \text{ is measurable} \quad (\zeta, \zeta' \in \mathbf{H} \otimes \mathfrak{h} \otimes \mathcal{F});$$

such a process is *vacuum adapted* if

$$Z_t = (I_{\mathbf{H}} \otimes P_t) Z_t (I_{\mathbf{H}} \otimes P_t) \quad (t \geq 0)$$

or, equivalently,

$$Z_t = Z_t \otimes |\Omega_t\rangle\langle\Omega_t| \quad \text{for some } Z_t \in B(\mathbf{H}) \overline{\otimes} \mathbf{A} \overline{\otimes} \mathbf{N}_{[0,t]} \quad (t \geq 0).$$

A vacuum-adapted bounded process  $G$  in  $B(\widehat{\mathbf{k}}) \overline{\otimes} \mathbf{A}$  is an *integrand* process if its block-matrix form  $\begin{bmatrix} k & m \\ l & n \end{bmatrix}$  is such that

$$\|G\|_t := \|k\|_{1,t} + \|l\|_{2,t} + \|m\|_{2,t} + \|n\|_{\infty,t} < \infty \quad (t \geq 0),$$

where, for  $p = 1, 2$  or  $\infty$ ,  $\|f\|_{p,t}$  denotes the  $L^p$  norm of the function  $1_{[0,t]}f$ .

The following result is the coordinate-independent version of [8, Proposition 37], with non-trivial initial space. Recall the notation (1.7) and (2.1).

**Proposition 3.2.** *Let  $G$  be an integrand process. There is a unique bounded vacuum-adapted process  $\int G d\Lambda = \left(\int_0^t G_s d\Lambda_s\right)_{t \geq 0}$  in  $\mathbf{A}$  such that*

$$\langle u\varepsilon(f), \int_0^t G_s d\Lambda_s v\varepsilon(g) \rangle = \int_0^t \langle u\varepsilon(f), E^{\widehat{f}(s)} G_s E_{\widehat{g}(s)} v\varepsilon(g) \rangle ds \quad (t \geq 0)$$

for all  $u, v \in \mathfrak{h}$  and  $f, g \in L^2(\mathbb{R}_+; \mathbf{k})$ . Moreover, the following inequality holds:

$$\left\| \int_0^t G_s d\Lambda_s \right\| \leq \|G\|_t \quad (t \geq 0).$$

We shall need to pass suitably adapted operators inside quantum stochastic integrals. The next lemma takes care of this.



**Lemma 3.3.** *Let  $G$  be an integrand process such that  $G\Delta \equiv 0$  and let  $X$  be a bounded vacuum-adapted process in  $\mathbf{A}$ . Then*

$$\int_s^t G_r d\Lambda_r X_s = \int_s^t G_r (I_{\widehat{\mathfrak{k}}} \otimes X_s) d\Lambda_r \quad (0 \leq s \leq t). \quad (3.3)$$

*Proof.* Let  $u, v \in \mathfrak{h}$  and  $f, g \in L^2(\mathbb{R}_+; \mathfrak{k})$ ; note that

$$\langle u\varepsilon(f), \int_0^t G_r d\Lambda_r v\varepsilon(g) \rangle = \int_0^t \langle u\varepsilon(f), E^{\widehat{f}(r)} G_r E_\omega v\varepsilon(g) \rangle dr,$$

since  $\Delta^\perp E_c = E_\omega$  for all  $c \in \mathfrak{k}$ . If  $A \in B(\mathfrak{h} \otimes \mathcal{F}_s)$  and  $\xi \in \mathfrak{h} \otimes \mathcal{F}$  then, setting  $P_{[s]} := |\Omega_{[s]} \rangle \langle \Omega_{[s]}|$  for brevity, it follows that

$$\begin{aligned} \langle u\varepsilon(f), \int_s^t G_r d\Lambda_r (A \otimes P_{[s]}) \xi \rangle &= \int_s^t \langle u\varepsilon(f), E^{\widehat{f}(r)} G_r E_\omega (A \otimes P_{[s]}) \xi \rangle dr \\ &= \int_s^t \langle u\varepsilon(f), E^{\widehat{f}(r)} G_r (I_{\widehat{\mathfrak{k}}} \otimes A \otimes P_{[s]}) E_\omega \xi \rangle dr \\ &= \langle u\varepsilon(f), \int_s^t G_s (I_{\widehat{\mathfrak{k}}} \otimes A \otimes P_{[s]}) d\Lambda_r \xi \rangle. \quad \square \end{aligned}$$

The following existence and uniqueness theorem is sufficiently general for present purposes.

**Theorem 3.4.** *Let  $G$  and  $X$  be as in Lemma 3.3, with  $X$  locally bounded in norm. Then there is a unique vacuum-adapted process  $Z$  in  $\mathbf{A}$  such that*

$$Z_t = X_t + \int_0^t G_s (I_{\widehat{\mathfrak{k}}} \otimes Z_s) d\Lambda_s \quad (t \geq 0). \quad (3.4)$$

Furthermore,

$$\|Z\|_{\infty, t} \leq \sqrt{2} \|X\|_{\infty, t} \exp(2\|l\|_{2, t}^2 + 2\|k\|_{1, t}^2) \quad (t \geq 0),$$

where  $\begin{bmatrix} k & 0 \\ l & 0 \end{bmatrix}$  is the block-matrix form of  $G$ , and  $Z$  is norm continuous if and only if  $X$  is.

*Proof.* Define a sequence of processes  $(X^{(n)})_{n \geq 0}$  inductively by letting  $X^{(0)} := X$  and

$$X_t^{(n+1)} := \int_0^t G_s (I_{\widehat{\mathfrak{k}}} \otimes X_s^{(n)}) d\Lambda_s \quad (t \geq 0).$$

This process is well defined and such that

$$\|X_t^{(n+1)}\| \leq \|k X^{(n)}\|_{1, t} + \|l X^{(n)}\|_{2, t} \quad (t \geq 0),$$

so, integrating by parts,

$$\|X^{(n+1)}\|_{\infty, t}^2 \leq 2\|k X^{(n)}\|_{1, t}^2 + 2\|l X^{(n)}\|_{2, t}^2 \leq \int_0^t c(s) \|X^{(n)}\|_{\infty, s}^2 ds,$$

where

$$c(s) := 4\|k_s\| \int_0^s \|k_r\| dr + 2\|l_s\|^2.$$

It follows that

$$\|X^{(n+1)}\|_{\infty,t}^2 \leq \frac{1}{n!} \left( \int_0^t c(s) ds \right)^n \|X\|_{\infty,t}^2 \quad (n \geq 0, t \geq 0),$$

so  $Z_t := \sum_{n=0}^{\infty} X_t^{(n)}$  exists for all  $t \geq 0$ , the series being convergent in norm. A dominated-convergence argument shows that  $Z$  satisfies (3.4) and, since

$$\|Z_t\|^2 \leq 2\|X_t\|^2 + 2 \int_0^t c(s) \|Z_s\|^2 ds \quad (t \geq 0),$$

the inequality and so uniqueness follow from Gronwall's lemma. The final claim is immediate.  $\square$

#### 4. Multipliers *via* Quantum Stochastic Differential Equations

Fix a vacuum-adapted quantum stochastic flow  $j$  on  $\mathbf{A}$  and let

$$J_t := \widehat{j}_t \circ \sigma_t : \mathbf{A} \overline{\otimes} \mathbf{N} \rightarrow \mathbf{A} \overline{\otimes} \mathbf{N}_t \overline{\otimes} \mathbf{N}_{|t} = \mathbf{A} \overline{\otimes} \mathbf{N}$$

and  $\widetilde{J}_t := \text{id}_{B(\widehat{\mathbf{k}})} \overline{\otimes} J_t$ , for all  $t \geq 0$ . The ultraweakly continuous family of normal unital  $*$ -homomorphisms  $(J_t)_{t \geq 0}$  form a semigroup (*cf.* [19, Proposition 4.3]).

The following result is a vacuum-adapted version of [11, Lemma 5.1] which suffices here.

**Lemma 4.1.** *If the integrand process  $G$  is norm continuous then the family of operators  $(1_{[s,\infty)}(r) \widetilde{J}_s(G_{r-s}))_{r \geq 0}$ , where  $1_A$  denotes the indicator function of the set  $A$ , defines an integrand process such that*

$$J_s \left( \int_0^t G_r d\Lambda_r \right) = \int_s^{s+t} \widetilde{J}_s(G_{r-s}) d\Lambda_r \quad (t \geq 0).$$

*Sketch proof.* Apply the ampliation of the vector functional  $A \mapsto \langle \varepsilon(f), A\varepsilon(g) \rangle$  to the left-hand side, then consider suitable Riemann sums.  $\square$

With this technical lemma we can construct multipliers of  $j$  by solving quantum stochastic differential equations with coefficients driven by  $j$ .

**Lemma 4.2.** *For all  $c \in \widehat{\mathbf{k}} \overline{\otimes} \mathbf{A}$  there is a unique process  $M^c = (M_t^c)_{t \geq 0}$  in  $\mathbf{A}$  such that  $M^c - I = (M_t^c - I_{\mathfrak{h} \otimes \mathcal{F}})_{t \geq 0}$  is vacuum adapted and*

$$M_t^c = I_{\mathfrak{h} \otimes \mathcal{F}} + \int_0^t \widetilde{j}_s(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_s^c) d\Lambda_s,$$

where  $\widetilde{j}_s := \text{id}_{B(\widehat{\mathbf{k}})} \overline{\otimes} j_s$  and  $\omega := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \widehat{\mathbf{k}}$ , *i.e.*,

$$\langle u\varepsilon(f), (M^c - I)_t v\varepsilon(g) \rangle = \int_0^t \langle u\varepsilon(f), j_s(E^{\widehat{f}(s)}c) M_s^c v\varepsilon(g) \rangle ds \quad (t \geq 0)$$

for all  $u, v \in \mathfrak{h}$  and  $f, g \in L^2(\mathbb{R}_+; \mathbf{k})$ . The process  $M^c$  is norm continuous.

*Proof.* Define an integrand process  $G$  by setting  $G_t := \tilde{j}_t(cE^\omega)$  for all  $t \geq 0$ . In view of the identity  $\tilde{j}(\cdot)\Delta = \tilde{j}(\cdot\Delta)$ , which exploits the abuse of notation (2.2), and the fact that  $E^\omega\Delta = 0$ , Theorem 3.4 gives a vacuum-adapted process  $N$  in  $\mathbf{A}$  which is norm continuous and such that

$$N_t = \int_0^t G_s d\Lambda_s + \int_0^t G_s(I_{\widehat{\mathbf{k}}} \otimes N_s) d\Lambda_s \quad (t \geq 0). \quad (4.1)$$

Hence  $M_t^c := I_{\mathfrak{h} \otimes \mathcal{F}} + N_t$  is a norm-continuous process as required; uniqueness holds because the solution of (4.1) is unique.  $\square$

**Theorem 4.3.** *For all  $c \in |\widehat{\mathbf{k}}\rangle \overline{\otimes} \mathbf{A}$  the process  $M^c$  given by Lemma 4.2 is a multiplier for  $j$ .*

*Proof.* It suffices to verify that condition (M iii) of Definition 2.6 holds. Fix  $s \geq 0$  and let

$$M_t := \begin{cases} M_t^c & \text{if } t \in [0, s), \\ J_s(M_{t-s}^c)M_s^c & \text{if } t \in [s, \infty). \end{cases}$$

Now  $\tilde{J}_s \circ \tilde{j}_{r-s} = \tilde{j}_r$  for all  $r \geq s$ , by (C iii) of Definition 2.1, so Lemma 3.3 and Proposition 4.1 imply that

$$\begin{aligned} M_{s+t} &= J_s \left( I_{\mathfrak{h} \otimes \mathcal{F}} + \int_0^t \tilde{j}_r(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_r^c) d\Lambda_r \right) M_s^c \\ &= M_s^c + \int_s^{s+t} \tilde{J}_s(\tilde{j}_{r-s}(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_{r-s}^c))(I_{\widehat{\mathbf{k}}} \otimes M_s^c) d\Lambda_r \\ &= M_s^c + \int_s^{s+t} \tilde{j}_r(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes (J_s(M_{r-s}^c)M_s^c)) d\Lambda_r \\ &= I_{\mathfrak{h} \otimes \mathcal{F}} + \int_0^s \tilde{j}_r(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_r^c) d\Lambda_r + \int_s^{s+t} \tilde{j}_r(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_r) d\Lambda_r \\ &= I_{\mathfrak{h} \otimes \mathcal{F}} + \int_0^{s+t} \tilde{j}_r(cE^\omega)(I_{\widehat{\mathbf{k}}} \otimes M_r) d\Lambda_r \quad (t \geq 0). \end{aligned}$$

By Lemma 4.2,  $M \equiv M^c$  and  $M_{t+s}^c = M_{t+s} = J_s(M_t^c)M_s^c$ , as required.  $\square$

## 5. Semigroup Perturbation

For vacuum-adapted integrands the quantum Itô product formula takes the following form [8, Section 5.4].

**Lemma 5.1.** *Let  $Z := \int G d\Lambda$  and  $Z' := \int G' d\Lambda$  for integrand processes  $G$  and  $G'$ . Then*

$$H := (I_{\widehat{\mathbf{k}}} \otimes Z)\Delta^\perp G' + G\Delta^\perp(I_{\widehat{\mathbf{k}}} \otimes Z') + G\Delta G'$$

*defines an integrand process such that  $ZZ' = \int H d\Lambda$ .*

The product of three integrals gives the following.

**Corollary 5.2.** *Let  $G$ ,  $G'$  and  $G''$  be integrand processes and let  $Z := \int G \, d\Lambda$ ,  $Z' := \int G' \, d\Lambda$  and  $Z'' := \int G'' \, d\Lambda$ . Then*

$$\begin{aligned} H := & (I_{\widehat{\mathbf{k}}} \otimes ZZ')\Delta^\perp G'' + (I_{\widehat{\mathbf{k}}} \otimes Z)\Delta^\perp G' \Delta^\perp (I_{\widehat{\mathbf{k}}} \otimes Z'') + G\Delta^\perp (I_{\widehat{\mathbf{k}}} \otimes Z'Z'') \\ & + (I_{\widehat{\mathbf{k}}} \otimes Z)\Delta^\perp G' \Delta G'' + G\Delta G' \Delta^\perp (I_{\widehat{\mathbf{k}}} \otimes Z'') + G\Delta G' \Delta G'' \end{aligned}$$

is an integrand process such that  $ZZ'Z'' = \int H \, d\Lambda$ .

We may now give the main result.

**Theorem 5.3.** *Let  $\psi : \mathbf{A}_0 \rightarrow \mathbf{A} \overline{\otimes} B(\widehat{\mathbf{k}})$ , where  $\mathbf{A}_0$  is a subset of  $\mathbf{A}$ , and suppose  $j$  satisfies the vacuum-adapted quantum stochastic differential equation*

$$j_0(x) = x \otimes |\Omega\rangle\langle\Omega|, \quad dj_t(x) = \widetilde{j}_t(\psi(x)) \, d\Lambda_t \quad (x \in \mathbf{A}_0).$$

For each  $c, d \in |\widehat{\mathbf{k}}\rangle \overline{\otimes} \mathbf{A}$ , the generator  $\tau$  of the pointwise ultraweakly continuous semigroup  $\mathcal{P} := (\mathbb{E}[(M_t^c)^* j_t(\cdot) M_t^d])_{t \geq 0}$  satisfies  $\text{Dom } \tau \supset \mathbf{A}_0$  and, for all  $x \in \mathbf{A}_0$ ,

$$\begin{aligned} \tau(x) = & E^\omega \psi(x) E_\omega + c^* \Delta \psi(x) E_\omega + E^\omega \psi(x) \Delta d \\ & + c^* \Delta \psi(x) \Delta d + c^* E_\omega x + x E^\omega d. \end{aligned} \quad (5.1)$$

*Proof.* Let  $x \in \mathbf{A}_0$  and  $t \geq 0$ ; note that  $(M_t^c)^* j_t(x) M_t^d - j_t(x)$  equals

$$\begin{aligned} & (M^c - I)_t^* (j_t - j_0)(x) (M^d - I)_t + (M^c - I)_t^* (j_t - j_0)(x) \\ & + (j_t - j_0)(x) (M^d - I)_t + (M^c - I)_t^* j_0(x) (M^d - I)_t \\ & + (M^c - I)_t^* j_0(x) + j_0(x) (M^d - I)_t. \end{aligned} \quad (5.2)$$

If  $u, v \in \mathfrak{h}$  and  $f, g \in L^2(\mathbb{R}_+; \mathbf{k})$  then, writing  $P_\Omega$  for  $|\Omega\rangle\langle\Omega| \in \mathbf{N}$ ,

$$\begin{aligned} \langle u\varepsilon(f), j_0(x) (M^d - I)_t v\varepsilon(g) \rangle &= \langle (x^* u)\Omega, \int_0^t \widetilde{j}_s(dE^\omega) (I_{\widehat{\mathbf{k}}} \otimes M_s^d) \, d\Lambda_s v\varepsilon(g) \rangle \\ &= \int_0^t \langle (x^* u)\Omega, E^\omega \widetilde{j}_s(dE^\omega) (I_{\widehat{\mathbf{k}}} \otimes M_s^d) E_{g(s)} v\varepsilon(g) \rangle \, ds \\ &= \int_0^t \langle u\varepsilon(f), (x \otimes P_\Omega) j_s(E^\omega d) M_s^d v\varepsilon(g) \rangle \, ds, \end{aligned}$$

therefore

$$j_0(x) (M^d - I)_t = \int_0^t (x \otimes P_\Omega) j_s(E^\omega d) M_s^d \, ds,$$

$$(M^c - I)_t^* j_0(x) = \int_0^t (M_s^c)^* j_s(c^* E_\omega) (x \otimes P_\Omega) \, ds$$

and

$$\begin{aligned} (M^c - I)_t^* j_0(x) (M^d - I)_t &= \int_0^t (M^c - I)_s^* (x \otimes P_\Omega) j_s(E^\omega d) M_s^d \, ds \\ &+ \int_0^t (M_s^c)^* j_s(c^* E_\omega) (x \otimes P_\Omega) (M^d - I)_s \, ds. \end{aligned}$$

This implies that the sum of the last three terms in (5.2) equals

$$\begin{aligned} & \int_0^t (M_s^c)^* ((x \otimes P_\Omega) j_s(E^\omega d) + j_s(c^* E_\omega)(x \otimes P_\Omega)) M_s^d ds \\ &= \int_0^t (\widetilde{M}_s^c)^* ((I_{\widehat{\mathfrak{k}}} \otimes x \otimes P_\Omega) \widetilde{j}_s(\Delta^\perp dE^\omega) + \widetilde{j}_s(E_\omega c^* \Delta^\perp)(I_{\widehat{\mathfrak{k}}} \otimes x \otimes P_\Omega)) \widetilde{M}_s^d d\Lambda_s, \end{aligned}$$

where  $\widetilde{M}_s^e := I_{\widehat{\mathfrak{k}}} \otimes M_s^e$  for  $e = c, d$ .

After some working, with the aid of Lemma 5.1 and Corollary 5.2, it follows that  $(M_t^c)^* j_t(x) M_t^d - j_0(x)$  equals

$$\int_0^t (\widetilde{j}_s(A_1) + \widetilde{j}_s(A_2) \widetilde{M}_s^d + (\widetilde{M}_s^c)^* \widetilde{j}_s(A_3) + (\widetilde{M}_s^c)^* \widetilde{j}_s(A_4) \widetilde{M}_s^d) d\Lambda_s,$$

where

$$A_1 := \Delta\psi(x)\Delta,$$

$$A_2 := \Delta\psi(x)\Delta^\perp + \Delta\psi(x)\Delta dE^\omega,$$

$$A_3 := \Delta^\perp\psi(x)\Delta + E_\omega c^* \Delta\psi(x)\Delta$$

$$\begin{aligned} \text{and } A_4 := & \Delta^\perp\psi(x)\Delta^\perp + E_\omega c^* \Delta\psi(x)\Delta^\perp + \Delta^\perp\psi(x)\Delta dE^\omega \\ & + E_\omega c^* \Delta\psi(x)\Delta dE^\omega + E_\omega c^* \Delta^\perp(I_{\widehat{\mathfrak{k}}} \otimes x) + (I_{\widehat{\mathfrak{k}}} \otimes x)\Delta^\perp dE^\omega. \end{aligned}$$

Hence

$$\begin{aligned} \langle u, (\mathcal{P}_t(x) - x)v \rangle &= \langle u\Omega, ((M_t^c)^* j_t(x) M_t^d - j_0(x))v\Omega \rangle \\ &= \int_0^t \langle u\Omega, (j_s(E^\omega A_1 E_\omega) + j_s(E^\omega A_2 E_\omega) M_s^d \\ &\quad + (M_s^c)^* j_s(E^\omega A_3 E_\omega) + (M_s^c)^* j_s(E^\omega A_4 E_\omega) M_s^d) v\Omega \rangle ds \\ &= \int_0^t \langle u\Omega, (M_s^c)^* j_s(E^\omega A_4 E_\omega) M_s^d v\Omega \rangle ds \\ &= \int_0^t \langle u, \mathcal{P}_s(y)v \rangle ds, \end{aligned}$$

where

$$y = E^\omega \psi(x) E_\omega + c^* \Delta\psi(x) E_\omega + E^\omega \psi(x) \Delta d + c^* \Delta\psi(x) \Delta d + c^* E_\omega x + x E^\omega d,$$

as required.  $\square$

*Remark 5.4.* In terms of the direct-sum decomposition  $\widehat{\mathfrak{k}} = \mathbb{C} \oplus \mathfrak{k}$ , if

$$\psi = \begin{bmatrix} \tau_0 & \delta_0^\dagger \\ \delta_0 & \pi_0 \end{bmatrix}, \quad c = \begin{bmatrix} k_1 \\ l_1 \end{bmatrix} \quad \text{and} \quad d = \begin{bmatrix} k_2 \\ l_2 \end{bmatrix}$$

then (5.1) becomes

$$\tau(x) = \tau_0(x) + l_1^* \delta_0(x) + \delta_0^\dagger(x) l_2 + l_1^* \pi_0(x) l_2 + k_1^* x + x k_2 \quad (x \in \mathbf{A}_0).$$

When  $\psi$  is bounded and  $A_0 = A$ , the map  $\delta_0$  is a bounded  $\pi_0$ -derivation. Since  $\delta_0(A_0) \subset A \overline{\otimes} |\mathbf{k}|$ , it follows that  $\delta_0$  is implemented ([12], see [16, Chapter 6]) and so

$$\begin{aligned} \tau(x) = & i[h, x] - \frac{1}{2}\{r^*r, x\} + r^*\pi_0(x)r + (xr^* - r^*\pi_0(x))l_2 \\ & + l_1^*(rx - \pi_0(x)r) + l_1^*\pi_0(x)l_2 + k_1^*x + xk_2 \end{aligned}$$

for some  $h = h^* \in A$  and  $r \in |\mathbf{k}| \overline{\otimes} A$ . Equivalently,

$$\tau(x) = d_1^*\pi_0(x)d_2 + e_1^*x + xe_2,$$

where  $d_i = l_i - r$  and  $e_i = k_i + r^*l_i - \frac{1}{2}r^*r - ih$  for  $i = 1, 2$ .

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## References

1. Accardi, L.: On the quantum Feynman–Kac formula, *Rend. Sem. Mat. Fis. Milano* **48** (1978) 135–180.
2. Accardi, L.; Frigerio, A.: Markovian cocycles, *Proc. Roy. Irish Acad. Sect. A* **83** (1983), no. 2, 251–263.
3. Accardi, L.; Frigerio, A.; Lewis, J. T.: Quantum stochastic processes, *Publ. Res. Inst. Math. Sci.* **18** (1982), no. 1, 97–133.
4. Alicki, R.; Fannes, M.: Dilations of quantum dynamical semigroups with classical Brownian motion, *Comm. Math. Phys.* **108** (1987), no. 3, 353–361.
5. Arveson, W.: Ten lectures on operator algebras, *CBMS Regional Conference Series in Mathematics* **55**, American Mathematical Society, Providence, 1984.
6. Bahn, C.; Park, Y. M.: Feynman–Kac representation and Markov property of semigroups generated by noncommutative elliptic operators, *Infinite Dim. Anal. Quantum Probab.* **6** (2003), no. 1, 103–121.
7. Belton, A. C. R.: Quantum  $\Omega$ -semimartingales and stochastic evolutions, *J. Funct. Anal.* **187** (2001), no. 1, 94–109.
8. Belton, A. C. R.: An isomorphism of quantum semimartingale algebras, *Q. J. Math.* **55** (2004), no. 2, 135–165.
9. Belton, A. C. R.: Alicki–Fannes and Hudson–Parthasarathy evolution equations, in: *Quantum Probability and Infinite Dimensional Analysis* **20** (2007) 128–133, World Scientific, Singapore.
10. Belton, A. C. R.: Random-walk approximation to vacuum cocycles, *J. London Math. Soc. (2)* **81** (2010), no. 2, 412–434.
11. Belton, A. C. R.; Lindsay, J. M.; Skalski, A. G.: Quantum Feynman–Kac perturbations, *preprint*, 2011.
12. Christensen, E.; Evans, D. E.: Cohomology of operator algebras and quantum dynamical semigroups, *J. London Math. Soc. (2)* **20** (1979), no. 2, 358–368.
13. Evans, M. P.; Hudson, R. L.: Perturbations of quantum diffusions, *J. London Math. Soc. (2)* **41** (1990), no. 2, 373–384.
14. Hudson, R. L.; Ion, P. D. F.; Parthasarathy, K. R.: Time-orthogonal unitary dilations and noncommutative Feynman–Kac formulae, *Comm. Math. Phys.* **83** (1982), no. 2, 261–280; Time-orthogonal unitary dilations and noncommutative Feynman–Kac formulae II, *Publ. Res. Inst. Math. Sci.* **20** (1984), no. 3, 607–633.
15. Hudson, R. L.; Parthasarathy, K. R.: Quantum Itô’s formula and stochastic evolutions, *Comm. Math. Phys.* **93** (1984), no. 3, 301–323.

16. Lindsay, J. M.: Quantum stochastic analysis — an introduction, in: *Quantum Independent Increment Processes I* (2005) 181–271, Lecture Notes in Mathematics **1865**, Springer, Berlin.
17. Lindsay, J. M.; Sinha, K. B.: Feynman–Kac representation of some noncommutative elliptic operators, *J. Funct. Anal.* **147** (1997), no. 2, 400–419.
18. Lindsay, J. M.; Wills, S. J.: Existence, positivity and contractivity for quantum stochastic flows with infinite dimensional noise, *Probab. Theory Related Fields* **116** (2000), no. 4, 505–543.
19. Lindsay, J. M.; Wills, S. J.: Markovian cocycles on operator algebras adapted to a Fock filtration, *J. Funct. Anal.* **178** (2000), no. 2, 269–305.
20. Parthasarathy, K. R.; Sinha, K. B.: A stochastic Dyson series expansion, in: *Theory and Application of Random Fields (Bangalore, 1982)* (1983) 227–232, Lecture Notes in Control and Information Science **49**, Springer, Berlin.
21. Reed, M.; Simon, B.: *Methods of Modern Mathematical Physics II. Fourier Analysis, Self-Adjointness*, Academic Press, New York, 1975.
22. Simon, B.: *Functional Integration and Quantum Physics*, Academic Press, New York, 1979.

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