

THE EARLY YEARS OF QUANTUM STOCHASTIC CALCULUS

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Dedicated to Professor K. R. Parthasarathy on the occasion of his 75th birthday

ABSTRACT. The origins and early history of quantum stochastic calculus are surveyed, with emphasis on the collaboration between K R Parthasarathy and the author.

1. Introduction

I first met Partha in 1971 when he was at Sheffield University. The occasion was a regional meeting of the UK Royal Statistical Society in Leeds. Partha gave what to me was a brilliantly clear exposition of quantum probability as a new theory of probability in which the σ -field of events was replaced by the non-Boolean lattice of sub-Hilbert spaces of a Hilbert space. Real valued random variables, regarded as lattice homomorphisms from the Borel σ -field to the lattice of events, instead of being the set-mapping inverses of measurable functions as in the classical case, are represented as self-adjoint operators through the spectral theorem. Probability measures are characterised by Gleason's theorem [7] as density operators. At the end of his lecture Partha mentioned that he had learned that a noncommutative central limit theorem had been proved recently in this context, enabling me to introduce myself as the author, with my student C D Cushen, of that theorem [4]. Thereby began the collaboration which has been the most rewarding of my life.

2. The Canonical Central Limit Theorem

In this central limit theorem real-valued random variables are replaced by *canonical pairs*, that is, pairs of self-adjoint operators (p, q) satisfying a mathematically rigorous form of the Heisenberg commutation relation (with Planck's constant set equal to 2π)

$$[p, q] = -i.$$

The fundamental observation leading to the theorem is that if

$$(p_1, q_1), (p_2, q_2), (p_3, q_3), \dots$$

is a sequence of mutually commuting canonical pairs then for each $n = 1, 2, 3, \dots$

$$\left(\frac{1}{\sqrt{n}}(p_1 + p_2 + \dots + p_n), \frac{1}{\sqrt{n}}(q_1 + q_2 + \dots + q_n) \right),$$

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is again a canonical pair, suggesting that if the initial sequence is stochastically independent and identically distributed then this sequence should converge in distribution to a normal limit.

The novelty of the situation arose because canonical pairs do not have a joint probability distribution in the usual sense [35], so that it was not immediately clear how to define convergence in distribution, or indeed stochastic independence or identity of distribution. This situation was remedied using the von Neumann uniqueness theorem [29], that every such pair (p, q) is unitarily equivalent to an ampliation of the Schrödinger pair, essentially $p_0 = -i \frac{d}{dx}$ and $q_0 =$ multiplication by the variable x in the Hilbert space $L^2(\mathbb{R})$. Using this equivalence, given a density operator ρ on the Hilbert space in which p and q act, there is a unique reduced density operator $\rho_{(p,q)}$ acting on $L^2(\mathbb{R})$, called the *distribution operator* of (p, q) , which contains probabilistic information in the state ρ about the pair (p, q) but not about anything else, for example about other canonical pairs which commute with p and q . Convergence in distribution is then defined as convergence of distribution operators in a suitable operator topology, for example one can take the weak* topology of the Banach space pairing of the space of trace-class operators on $L^2(\mathbb{R})$ with its dual, the space of bounded operators. Stochastic independence is defined as the factorisation of the *joint distribution operator* of commuting canonical pairs, which is defined using the analog for several commuting canonical pairs of the von Neumann uniqueness theorem, into the tensor product operator of the individual distribution operators.

To describe the limit distribution, we may assume without loss of generality that the common covariance matrix is of the form

$$\begin{bmatrix} \frac{1}{2}\sigma^2 & 0 \\ 0 & \frac{1}{2}\sigma^2 \end{bmatrix},$$

by applying to each of $(p_1, q_1), (p_2, q_2), (p_3, q_3), \dots$ a common linear canonical transformation of the form

$$(p, q) \mapsto (\alpha p + \beta q, \gamma p + \delta q), \quad \alpha\delta - \beta\gamma = 1.$$

Then the Gaussian limit distribution has distribution operator which is a thermal equilibrium state of the harmonic oscillator Hamiltonian $\frac{1}{2}(p_0^2 + q_0^2)$

$$\rho_\beta = N_\beta e^{-\frac{\beta}{2}(p_0^2 + q_0^2)},$$

where N_β is a normalising constant and the reciprocal temperature β is a function of the variance σ which tends to ∞ as σ approaches the minimum value 1 allowed by the Heisenberg uncertainty principle [27]. In the case $\sigma = 1$ it is the harmonic oscillator ground state.

3. Quantum Brownian Motion

After Partha returned to India, influenced by a succession of classical probabilists who emphasized the power of Donsker's invariance principle [5] or functional central limit theorem, with my student A M Cockroft, I began the search (still not satisfactorily completed notwithstanding [3]) for a functional version of the canonical central limit theorem. It was clear that under the hypotheses of that theorem,

the two sequences (p_1, p_2, p_3, \dots) and (q_1, q_2, q_3, \dots) each consisted of essentially iid classical random variables, and hence by Donsker's theorem one should expect that the sequences

$$P_n(t) = \frac{1}{\sqrt{n}}(p_1 + p_3 + \dots + p_{[nt]-1}) + (nt - [nt])p_{[nt]}, \quad n = 1, 2, \dots \quad (3.1)$$

$$Q_n(t) = \frac{1}{\sqrt{n}}(q_1 + q_2 + \dots + q_{[nt]-1}) + (nt - [nt])q_{[nt]}, \quad n = 1, 2, \dots, \quad (3.2)$$

should each converge in distribution to Brownian motions P and Q respectively of variance σ^2 . Although formulated in terms of self adjoint operators these convergences follow without difficulty from the classical invariance principle since all operators involved commute in each case.

However the Brownian motions P and Q do not commute with each other. From (3.1) and (3.2) it follows that

$$[P_n(s), Q_n(t)] = -is \wedge t + o\left(\frac{1}{n}\right)$$

so one expects the limit Brownian motions to satisfy the commutation relation

$$[P(s), Q(t)] = -is \wedge t. \quad (3.3)$$

In the case $\sigma^2 = 1$ of minimal variance, such a pair is constructed in the *Fock space* $\mathcal{H} = \mathcal{F}(L^2(\mathbb{R}_+))$. This is conveniently defined as the closed linear span of the *exponential vectors* $e(f)$, $f \in L^2(\mathbb{R}_+)$, which satisfy

$$\langle e(f), e(g) \rangle = \exp \langle f, g \rangle, \quad f, g \in L^2(\mathbb{R}_+).$$

The Weyl operators are unitary operators $W(f)$, $f \in L^2(\mathbb{R}_+)$ defined by their actions

$$W(f)e(g) = \exp\left(-\frac{1}{4}\|f\|^2 - \frac{1}{\sqrt{2}}\langle f, g \rangle\right) e\left(\frac{1}{\sqrt{2}}f + g\right); \quad (3.4)$$

they satisfy

$$W(f)W(g) = \exp\left(-\frac{1}{2}i\Im \langle f, g \rangle\right) W(f + g). \quad (3.5)$$

From (3.4) the vacuum expectation is

$$\mathbb{E}[W(f)] = \langle e(0), W(f)e(0) \rangle = \exp\left(-\frac{1}{4}\|f\|^2\right). \quad (3.6)$$

Denoting by $\chi_{[0,t]}$ the indicator function of the interval $[0, t]$, for each $t \in \mathbb{R}_+$, we see from (3.5) that the families of Weyl operators $(W(x\chi_{[0,t]})_{x \in \mathbb{R}}$, $(W(ix\chi_{[0,t]})_{x \in \mathbb{R}}$ are continuous unitary representations of the additive group \mathbb{R} and hence by Stone's theorem there are self adjoint operators $P(t)$ and $Q(t)$ such that each

$$W(x\chi_{[0,t]}) = \exp(ixP(t)), \quad W(ix\chi_{[0,t]}) = \exp(ixQ(t)). \quad (3.7)$$

Again in view of (3.5) these satisfy

$$\exp(ixP(s)) \exp(iyQ(t)) = \exp(i(s \wedge t)xy) \exp(iyQ(t)) \exp(ixP(s))$$

which is the mathematically rigorous Weyl form of (3.3). From (3.6) it follows, firstly, that each $P(t)$ and each $Q(t)$ is normally distributed with zero mean and

variance $\frac{1}{2}$, and, secondly, that the processes P and Q begin anew independently of their pasts at each fixed time s , since, for example for arbitrary $r < s$ and $t > 0$,

$$\begin{aligned}\mathbb{E}[\exp(ix(P(t+s) - P(s)))] &= \exp(-\frac{1}{4}tx^2), \\ \mathbb{E}[\exp(ixP(r)) \exp(iy(P(t+s) - P(s)))] \\ &= \mathbb{E}[\exp(ixP(r))]\mathbb{E}[\exp(iy(P(t+s) - P(s)))].\end{aligned}$$

In this sense P and Q are Brownian motions. In fact variants of the Wiener-Segal isomorphism give diagonalising Hilbert space isomorphisms D_P and $D_Q \neq D_P$ from \mathcal{H} to the complex L^2 -space of Wiener measure which map the vacuum $e(0)$ to the function identically 1, and under conjugation by which P and Q become multiplication by the canonical Brownian motion.

If $\sigma^2 > 1$ we write $\sigma^2 = \alpha^2 + \beta^2$ where α and β are positive real numbers. On the Hilbert space tensor product $\mathcal{H} \otimes \bar{\mathcal{H}}$ of \mathcal{H} with its dual Hilbert space, we define unitary operators $W_\sigma(f)$, $f \in L^2(\mathbb{R}_+)$ by

$$W_\sigma(f) = W(\alpha f) \otimes (W(\beta f))^-.$$

These satisfy the Weyl relation (3.5), but instead of (3.6) we find the expectation in the double vacuum state is

$$\mathbb{E}[W_\sigma(f)] = \langle (e(0) \otimes \bar{e}(0)), W_\sigma(f)(e(0) \otimes \bar{e}(0)) \rangle = \exp\left(-\frac{\sigma^2}{4} \|f\|^2\right).$$

Defining the processes P_σ and Q_σ by replacing W by W_σ in (3.7) we construct Brownian motions of variance σ^2 still satisfying the commutation relations (3.3).

It can be argued that, despite its subsequent popularity and domination of quantum stochastic calculus, the Fock pair (P, Q) of quantum Brownian motions is a degenerate limiting case from the point of view of the functional central limit theorem discussed here. It arises only when the input sequence of iid canonical pairs is of the minimal variance $\sigma^2 = 1$ compatible with and achieving equality in the Heisenberg uncertainty principle. But it is well known in quantum mechanics ([27]) that such pairs must already be in the limiting Gaussian state which in this case is the ground state of the harmonic oscillator Hamiltonian $\frac{1}{2}(p_0^2 + q_0^2)$. The subsequent relative complexity (and richness!) of quantum stochastic calculus and spin-offs such as martingale representation theorems in the Fock case ([33],[19]) compared with the non-Fock case ([17]) may be partly attributed to this degeneracy.

4. The Strong Markov Property and the Genesis of Quantum Stochastic Calculus

Partha visited Britain in the bitterly cold winter of 1979 on an academic tour which included some time spent with me in Nottingham and at home in Southwell. I had two preoccupations which interested him. The first of these was with the quantum strong Markov property for Fock or non-Fock quantum Brownian motions [10] which was subsequently greatly clarified in the Fock case by Partha

and Kalyan Sinha [34]. For its formulation this required the notion of *Markov* or *stop time* essentially a nonnegative self adjoint operator

$$T = \int_0^\infty \lambda dE(\lambda),$$

each of whose spectral projections $E(\lambda)$ belongs to the von Neumann algebra generated by the Weyl operators $W(f)$ for which f vanishes outside the interval $[0, \lambda]$. Given such a stop time T the "Brownian motions starting anew at time T " can be defined [10] as the spectral integrals with operator-valued integrands

$$P(T + t) - P(T) = \int_{\mathbb{R}} (P(\lambda + t) - P(\lambda)) dE(\lambda), \quad t \in \mathbb{R}_+$$

$$Q(T + t) - Q(T) = \int_{\mathbb{R}} (Q(\lambda + t) - Q(\lambda)) dE(\lambda), \quad t \in \mathbb{R}_+$$

which make unambiguous sense because in each case the integrand commutes with the integrator, so that it is not necessary to make an arbitrary choice between "right stopping", integrator on the right of integrand as here, and "left stopping", integrator on the left of integrand, or even "double stopping", idempotent integrator on both sides of integrand. Partha's reaction was to observe that the possibility of defining these integrals and also their approximation by discrete sums, was analogous to a theory of Itô integrals in classical stochastic calculus, with the enabling commutativity of integrand and integrator translating into the independence of adapted integrands and increments in the integrator. Thus perhaps we began to think seriously about quantum stochastic integrals, though in my case I had first been made aware of the possibility by the suggestion of Nelson [28] that "smeared fields" should be expressed as stochastic integrals..

My other preoccupation was with work begun with Patrick Ion in Heidelberg [13] on a non-commutative Feynman-Kac formula. Here the idea was to construct a perturbed semigroup, by premultiplication by a cocycle, of an unperturbed semigroup which we thought of as the vacuum conditional expectation of a "stochastic product integral" such as, in the Fock case,

$$\prod_{0 < x < t} (1 + (p \otimes dQ - q \otimes dP))$$

(a more correct notation would have been

$$\prod_{0 < x < t} (1 + (p \otimes dQ - q \otimes dP - \frac{1}{2}(p^2 + q^2 - 1) \otimes dT)), \quad (4.1)$$

where T is time and (p, q) is the standard canonical pair realised in the Fock space $\mathcal{F}(\mathbb{C})$, so that the product integral is realised in the Fock space $\mathcal{F}(\mathbb{C}) \otimes \mathcal{F}(L^2(\mathbb{R}_+)) = \mathcal{F}(\mathbb{C} \oplus L^2(\mathbb{R}_+))$ and its vacuum expectation is an operator in $\mathcal{F}(\mathbb{C})$, in fact the operator $\exp(-\frac{t}{2}(p^2 + q^2 - 1))$. Thus the harmonic oscillator Hamiltonian replaces the Laplacian of the classical Feynman-Kac formula and the product integral is revealed as a stochastic unitary dilation of the contraction semigroup generated by this Hamiltonian. Untutored in stochastic analysis, we did not think of the product integral as most people would now as the solution $X(t)$ at time t

of the quantum stochastic differential equation

$$dX = X(p \otimes dQ - q \otimes dP - \frac{1}{2}(p^2 + q^2 - 1) \otimes dT), X(0) = 1$$

but we were able to construct an explicit form of it by approximating it by a discrete product of second-quantised small rotations in different planes. This construction interested Partha and led to a more general technique for dilating contraction semigroups, using both Fock [14] and non-Fock [15] quantum Brownian motion.

The technique of constructing product integrals explicitly as limits of second quantisations of discrete products of rotations has recently been extended to double products [11],[12].

5. Evolutions Versus Flows

I visited Bangalore and ISI New Delhi in January 1981. Partha and I struggled towards a satisfactory theory of Fock space stochastic integration and quantum stochastic differential equations. I think there was a creative tension between us. Partha thought that we should be trying to construct unitary valued processes as solutions to quantum stochastic differential equations, whereas I favoured what were later called flows (though at the time we misnamed them "quantum diffusions") of endomorphisms of operator algebras, especially those generated by representations of the canonical commutation relations. Partha's view eventually prevailed, at least in the short term. Partha was also more insightful than I in emphasising the product form of Itô's formula based on the mnemonic for classical Brownian motion B ,

$$(dB)^2 = dT \tag{5.1}$$

[26], whereas I was initially more familiar with the functional form $df(B, T) = f'(B, T)dB + (\dot{f}(B, T) + \frac{1}{2}f''(B, T))dT$ [25].

We were both attracted by the heuristic Fock space eigen-relation

$$dA(t)e(f) = f(t)e(f)dt, \tag{5.2}$$

which gave a nice formula for a stochastic integral of an operator-valued process E against the *annihilation process* $A = 2^{-\frac{1}{2}}(Q + iP)$, namely

$$\left(\int_0^t E(s)dA(s) \right) e(f) = \int_0^t f(s) (E(s)e(f)) ds,$$

or equivalently

$$\left\langle e(f), \left(\int_0^t E(s)dA(s) \right) e(g) \right\rangle = \int_0^t g(s) \langle e(f), E(s)e(g) \rangle ds. \tag{5.3}$$

applicable even to non-adapted operator valued processes E . A corresponding formula for integrals against the *creation process* $A^\dagger = 2^{-\frac{1}{2}}(Q - iP)$, namely

$$\left\langle e(f), \left(\int_0^t E(s)dA^\dagger(s) \right) e(g) \right\rangle = \int_0^t \bar{f}(s) \langle e(f), E(s)e(g) \rangle ds \tag{5.4}$$

required that the process E be adapted, so that when moved to the left hand side of the inner product, the adjoint process E^\dagger could be commuted with the

infinitesimal increment dA . These formulae eventually were subsumed into the "first fundamental formula" of the mature quantum stochastic calculus [20]. In groping our way towards the expected quantum Itô table in the form

$$\begin{array}{c|ccc} & dA^\dagger & dA & dT \\ \hline dA^\dagger & 0 & 0 & 0 \\ dA & dT & 0 & 0 \\ dT & 0 & 0 & 0 \end{array} \quad (5.5)$$

(the number process Λ had yet to appear on the scene), we realised that similar formulas involving two stochastic integrals of the form, $M_E = \int E dA$, $M_E^\dagger = \int E dA^\dagger$, namely

$$\begin{aligned} & \langle M_E(t)e(f), M_F(t)e(g) \rangle \\ &= \int_0^t (g(s) \langle M_E(s)e(f), F(s)e(g) \rangle + \bar{f}(s) \langle E(s)e(f), M_F(s)e(g) \rangle) ds \\ &= \int_0^t \int_0^t \bar{f}(s_1)g(s_2) \langle E(s_1)e(f), F(s_2)e(g) \rangle ds_1 ds_2, \end{aligned}$$

$$\begin{aligned} & \langle M_E^\dagger(t)e(f), M_F(t)e(g) \rangle \\ &= \int_0^t g(s) \left(\langle M_E^\dagger(s)e(f), F(s)e(g) \rangle + \langle E(s)e(f), M_F(s)e(g) \rangle \right) ds \end{aligned}$$

$$\begin{aligned} & \langle M_E(t)e(f), M_F^\dagger(t)e(g) \rangle \\ &= \int_0^t \bar{f}(s) \left(\langle M_g(s)e(f), F(s)e(g) \rangle + \langle E(s)e(f), M_F^\dagger(s)e(g) \rangle \right) ds \end{aligned}$$

could be derived from (5.2) assuming adaptedness, thus accounting for the zero entries in the table (5.5). But the crucial non-zero entry $dA dA^\dagger = dT$ does not succumb to such intuitive arguments.

Partha eventually saw a way of breaking the deadlock, essentially by applying the classical product rule (5.1) to the classical unit-variance Brownian motion $\sqrt{2}Q = A^\dagger + A$ together with the three known zero entries to write

$$dA dA^\dagger = (dA^\dagger + dA)(dA^\dagger + dA) = (\sqrt{2}dQ)^2 = dT$$

which resulted in a tentative and rather complicated first rigorous approach to quantum stochastic calculus [16].

Meanwhile the theory of "quantum diffusions" had at least produced some interesting examples in [20] where, among other things, generalising (4.1), unitary evolutions generated up to a unitarity correction by bilinear forms in (p, q) and (P, Q) were classified up to linear canonical transformation on (p, q) and gauge transformation on (P, Q) and explicit forms were found for all three canonical forms. It appeared to us also that the "flow" approach was equivalent to the "evolution" approach, in so far as every flow appeared to be given by conjugation by an evolution.

6. The Breakthrough [21]

The breakthrough came in 1982 when Partha spent two months in Britain at a Warwick symposium in which I was also able to participate on a part-time basis. I was convinced that the rather cumbersome trick used in [16] could be circumvented using the canonical commutation relations, in the use of which I perhaps had more experience to set against Partha's much more profound understanding of classical probability and stochastic calculus. The natural thing to do was to start, as in classical stochastic calculus, with stochastic integrals of simple, in the sense of piecewise constant, processes whose stochastic integrals were discrete sums of products of their values with increments of the integrator processes, to which the commutation relations were applicable. The difficulty with this approach as I saw it was that the product rule must involve integrals whose integrands were themselves integrals, and these were no longer simple processes so that the integrals were not well defined. How could I find a class of processes whose integrals were well defined and still belonged to the same class so that iterated integrals could be defined? When I showed some very tentative calculations of this kind to Partha, he saw almost immediately what I did not, that the commutation relations gave rise to estimates, such as

$$\left\| \int_0^t E(s) dA^\dagger(s) e(f) \right\|^2 \leq \int_0^t (1 + |f(s)|^2) \|E(s)e(f)\|^2 ds$$

for a simple process E , which made possible the extension by continuity of the integral beyond simple processes in the same way as, in the classical theory of Itô integrals, the Itô isometry allows extension by isometry beyond simple integrands. Such estimates also made possible the solution by the Picard iterative technique of quantum stochastic differential equations.

Progress was now rapid. A paper, essentially giving rigorous meaning to the quantum Itô table (5.5), was quickly written. but it was overtaken by events and never published.

The continuous tensor product structure of the Fock space makes it natural to define a quantum *martingale* for the filtration generated by P and Q in the Fock case as an adapted process M for which whenever $0 < s < t$,

$$\langle e(f), M(s)e(g) \rangle = \langle e(f), M(t)e(g) \rangle$$

for arbitrary $f, g \in L^2(\mathbb{R}_+)$ supported by the interval $[0, s]$. A similar definition can be used in the non-Fock case, though it turns out to be more natural then to use a different definition of a square-integrable martingale [17]. Then P, Q, A and A^\dagger are martingales and every stochastic integral process is a martingale. A natural question is: does every martingale have a stochastic integral representation? Partha's knowledge of classical stochastic analysis and his earlier work with Klaus Schmidt [30],[31],[32], and awareness that infinitely divisible distributions and independent increment processes could be represented naturally in Fock space, led him to answer this question negatively in the Fock case. The *number* or *gauge process* Λ , conveniently defined by

$$\langle e(f), \Lambda(t)e(g) \rangle = \int_0^t \bar{f}(s)g(s) ds \langle e(f), e(g) \rangle,$$

is a martingale which cannot be represented in this way. Stochastic integrals against the new martingale as integrator satisfied

$$\left\langle e(f), \left(\int_0^t E(s) d\Lambda(s) \right) e(g) \right\rangle = \int_0^t \bar{f}(s) g(s) \langle e(f), E(s) e(g) \rangle ds.$$

They are incorporated with the corresponding formulae (5.4) and (5.3) for the creation and annihilation martingales into the first fundamental formula, and also into a new second formula embodying the full one dimensional quantum Itô table

	dA^\dagger	$d\Lambda$	dA	dT
dA^\dagger	0	0	0	0
$d\Lambda$	dA^\dagger	$d\Lambda$	0	0
dA	dT	dA	0	0
dT	0	0	0	0

The new process enabled the Poisson process N and its associated stochastic calculus to be combined in a single unified theory with Brownian motion through the formula

$$N = A^\dagger + \Lambda + A$$

which in the form $N = Q + \Lambda$ may be the classical probabilist's *pons asinorum*, in so far as no two of the three processes N , Q and Λ commute with each other.

A new paper, also never published, incorporating the number process was hastily written. When both were submitted simultaneously the inevitable response was that they must be combined; the eventual result was a much more general multidimensional combined paper [21].

7. Boson-Fermion Unification [23]

The gauge process Λ was crucial to the surprising stochastic differential formula

$$dB^\dagger = (-1)^\Lambda dA^\dagger, dB = (-1)^\Lambda dA \tag{7.1}$$

relating *Boson* fields, regarded as stochastic integrals

$$a^\dagger(f) = \int f dA^\dagger, a(g) = \int \bar{g} dA$$

of scalar field strengths against dA^\dagger and dA , and satisfying the commutation relation

$$[a(f), a^\dagger(g)] = \langle f, g \rangle, [a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0$$

and corresponding *Fermion* fields

$$b^\dagger(f) = \int f dB^\dagger, b(g) = \int \bar{g} dB$$

which obey the *anticommutation relations*

$$[b(f), b^\dagger(g)]_+ = \langle f, g \rangle, [b(f), b(g)]_+ = [b^\dagger(f), b^\dagger(g)]_+ = 0$$

where $[X, Y]_+$ denotes the anticommutator $XY + YX$. This we discovered during a second summer visit by Partha to Warwick in 1984. Previously a Fermionic analog [1] of [21], complete with Fermionic creation and annihilation processes B^\dagger and B , had been constructed by a rather cumbersome use of multiparticle

states in Fermionic Fock space. We were led to the unification (7.1) by a search for analogs of exponential vectors in Fermionic Fock spaces in order to simplify this construction, starting with the formal Bosonic formula $e(f) = \exp(a^\dagger(f))e(0)$. Our success was not without disadvantageous side effects, since it effectively closed down the "Fermion analog" industry as a reliable source of problems for PhD students.

A word about the rival team of Chris Barnett, Ray Streater and Ivan Wilde, of which I was at one time a semi-detached member [25], and who, after my departure, preceded us into publication with a theory of noncommutative stochastic integration which for sheer elegance surely surpasses the Hudson Parthasarathy theory, but which has perhaps proved to be less durable. Of course they didn't have Partha! But another possible reason for our eventual overtaking of them was that, while they produced a lengthy series of analogs in noncommutative probability of the standard classical theory (for example the beautiful first paper [2] is a direct \mathbb{Z}_2 -graded version of the fundamentals of Itô calculus, building in particular on the work of Gross [9] on the \mathbb{Z}_2 -graded analog of the Wiener Segal isomorphism between the L^2 -space of Wiener measure and the Fock space over $L^2(\mathbb{R}_+)$), quantum stochastic calculus in the Hudson-Parthasarathy sense is a noncommutative *extension* of classical calculus and as such seems to have opened more avenues of application.

8. Evolutions Versus Flows Again

The first application was to construct solutions of stochastic differential equations of the form

$$dU = U (L_1 \otimes dA^\dagger + L_2 \otimes d\Lambda + L_3 \otimes dA + L_4 \otimes dT), \quad U(0) = 1, \quad (8.1)$$

for a process living in the tensor product $\mathcal{H}_0 \otimes \mathcal{H}$ where L_1, L_2, L_3 and L_4 are bounded operators in some *initial* Hilbert space \mathcal{H}_0 . Existence and uniqueness of the solution were established [21] by the Picard iterative method. Using the quantum Itô formula in the form

$$d(U^\dagger U) = (dU^\dagger)U + U^\dagger dU + (dU^\dagger)dU$$

it can be seen that a necessary condition for the solution to be unitary is that the 4-tuple (L_1, L_2, L_3, L_4) is of the form $(-WL^\dagger, W - 1, L, iH - \frac{1}{2}L^\dagger L)$ where W, L and H are respectively unitary, arbitrary and self adjoint elements of $B(\mathcal{H}_0)$. In fact this condition is sufficient [21]. Furthermore every shift-covariant adapted unitary evolution satisfying certain regularity conditions is of this form [18].

The unitary process U provides a *stochastic dilation* of the semigroup of contraction operators on \mathcal{H}_0 generated by $iH - \frac{1}{2}L^\dagger L$, in the sense that the vacuum conditional expectation of each $U(t)$ is $\exp(t(iH - \frac{1}{2}L^\dagger L))$. Note that the role of the number process is inessential to this dilation; we can set $W = 1$ thereby eliminating the corresponding term in (8.1). Note also that $U(s)U(t) \neq U(s+t)$; instead the cocycle relation

$$U(s)(\Gamma^\dagger(s)U(t)\Gamma(s)) = U(s+t) \quad (8.2)$$

holds, where $\Gamma(s)$ is the second quantisation of the isometric forward shift on $L^2(\mathbb{R}_+)$. By extending this shift in the natural way to a unitary operator on all of

$L^2(\mathbb{R})$ (8.2) becomes equivalent to the group relation

$$V(s)V(t) = V(s+t)$$

for the operators $V(s) = U(s)\Gamma^\dagger(s)$ on a similarly extended Fock space. The rather singular generator of this one-parameter group has been an intermittent object of study in the intervening years [8].

Of perhaps greater interest than this dilation is the the process $j = (j_t)_{t \in \mathbb{R}_+}$ of C^* -algebra homomorphisms from $B(\mathcal{H}_0)$ to $B(\mathcal{H}_0 \otimes \mathcal{H})$ given by

$$j_t(X) = U(t)(X \otimes 1)U^\dagger(t).$$

This satisfies the system of quantum stochastic differential equations

$$dj(X) = j(\alpha(X))dA^\dagger + j(\lambda(X))d\Lambda + j(\alpha^\dagger(X))dA + j(\tau(X))dT \quad (8.3)$$

with $j_0(X) = X \otimes 1$, where $\alpha, \lambda, \alpha^\dagger$ and τ are the maps from $B(\mathcal{H}_0)$ to itself:

$$\alpha(X) = -W[L^\dagger, X], \quad \lambda(x) = WXW^\dagger - X, \quad \alpha^\dagger(X) = [L, X]W^\dagger, \quad (8.4)$$

$$\tau(X) = i[H, X] - \frac{1}{2}(L^\dagger LX - 2L^\dagger XL + XL^\dagger L). \quad (8.5)$$

The vacuum conditional expectation of each $j_t(X)$ is $\exp(t\tau)(X)$, so that the flow j of operators provides a stochastic dilation of the quantum dynamical semigroup of completely positive maps on $B(\mathcal{H}_0)$ generated by the Lindbladian τ . Note that, once again, the role of the number process is inessential. By using N -dimensional stochastic calculus [21] a similar stochastic dilation can be constructed for a quantum dynamical semigroup whose Lindbladian contains N dissipative terms. The general uniformly continuous quantum dynamical semigroup, in which the Lindbladian can contain infinitely many dissipative terms

$$\tau(X) = i[H, X] - \frac{1}{2} \sum_{j=1}^{\infty} (L_j^\dagger L_j X - 2L_j^\dagger X L_j + X L_j^\dagger L_j)$$

was dealt with subsequently using a perhaps rather clumsy infinite-dimensional version of quantum stochastic calculus [22].

Given arbitrary norm-bounded linear maps $\alpha, \lambda, \alpha^\dagger$ and τ from $B(\mathcal{H}_0)$ to itself, the sytem (8.3) has a unique solution whose existence is established iteratively in same way as that of the equation (8.1). Each j_t commutes with adjunction if and only if

$$\alpha(X^\dagger) = \alpha^\dagger(X), \quad \lambda(X^\dagger) = (\lambda(X))^\dagger, \quad \tau(X^\dagger) = ((X))^\dagger, \quad X \in B(\mathcal{H}_0)$$

Necessary conditions for the resulting flow j to be multiplicative derived from the quantum Itô formula are that, for arbitrary $X, Y \in B(\mathcal{H}_0)$,

$$\alpha(XY) = \alpha(X)Y + X\alpha(Y) + \lambda(X)Y, \quad (8.6)$$

$$\lambda(XY) = \lambda(X)Y + X\lambda(Y) + \lambda(X)\lambda(Y), \quad (8.7)$$

$$\tau(XY) = \tau(X)Y + X\tau(Y) + \alpha(X)\alpha^\dagger(Y). \quad (8.8)$$

These conditions were eventually proved to be sufficient by my student M P Evans. [6].

For a time I believed that the only solutions of the structure equations (8.6), (8.7) and (8.8) were of form (8.4), (8.5); equivalently that every multiplicative flow

with bounded structure maps was inner, in analogy with our heuristic discovery [14] that this was the case for flows on the Weyl algebra generated algebraically by a canonical pair (p, q) . If the initial space is finite-dimensional this is so. But if \mathcal{H}_0 is infinite dimensional, a counterexample is provided by the flow $j_t(X) = \sigma^{\Lambda(t)}(X)$ where σ is an outer endomorphism of $B(\mathcal{H}_0)$, so that

$$dj(X) = j(\sigma(X) - X)d\Lambda, j_0(X) = X \otimes 1.$$

Such outer flows provide a way of dilating the general uniformly continuous quantum dynamical semigroup using only one-dimensional quantum stochastic calculus [24]. This is achieved by expressing the general Lindbladian in the form

$$\tau(X) = i[H, X] - \frac{1}{2}(L^\dagger LX - 2L^\dagger \sigma(X)L + XL^\dagger L),$$

where H and L are elements of $B(\mathcal{H}_0)$ with H self adjoint, and σ is an endomorphism of \mathcal{H}_0 . The dilation is then given by

$$\begin{aligned} dj(X) &= j(LX - \sigma(X)L)dA^\dagger + j(X - \sigma(X))d\Lambda \\ &\quad + j(XL^\dagger - L^\dagger \sigma(X))dA + j(\tau(X))dT \end{aligned}$$

with $j_0(X) = X \otimes 1$.

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