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RENORMALIZED SQUARE OF WHITE NOISE QUANTUM TIME SHIFT

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Abstract. Quantum time shifts are a very special class of continuous time random walk type additive processes on Lie algebras whose exponentiation gives rise to quantum Markov processes whose infinitesimal generators possess a total set of unitary eigen–operators: such a class of quantum Markov generators was not known in the literature. This property generalizes the known fact that trigonometric exponentials are eigenvectors of the usual classical Laplacian and was first noticed in connection with the quantum Laplacian in [5]. In that case the eigen–operators turn out to be the Weyl operators, which are known to be non–commutative extensions of the trigonometric exponentials. A generalization of these results, from the Weyl algebra to the harmonic oscillator algebra was obtained in [6]. In the present paper we extend the above results to the renormalized square of white noise algebra. This provides among other things, a quantum extension of the Markov generators of the Meixner processes. To our knowledge such an extension was not previously known.

1. Introduction

The classical heat semigroup

$$P^t = e^{t\Delta}$$

is the Markov semigroup canonically associated to the classical, real valued Brownian motion (W_t) via the formula

$$E_{0]}(v_t^\circ(f(W_0))) = E_{0]}(f(W_t)) = P^t f\,, \quad t \geq 0,$$

where $E_{0]}$ denotes the W-conditional expectation onto the σ–algebra from $-\infty$ to 0 (the past σ–algebra), v_t° is the usual time shift in the Wiener space and f in any Borel measurable function.

In the quantum formulation of the classical, real valued Brownian motion there is also a time shift u_t° and $E_{0]}$ denotes the (restriction of the) vacuum conditional expectation. However u_t° acts trivially on the initial algebra and therefore the generator of the corresponding semigroup is zero.

The root of the problem is that the quantum time shift u_t° does not coincide with the time shift of the classical Brownian motion, but it coincides with the time

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shift of the increment process associated to the classical Brownian motion, i.e., the (integrated) white noise.

Meyer posed the question if there exists a quantum extension of the classical time shift in Wiener space. If such an extension exists, then the generator of the associated quantum Markov semigroup must be a quantum extension of the classical Laplacian.

Meyer's problem was solved in [1] by Accardi and was based on the following idea:

The usual time shift v_t° in Wiener space is the unique endomorphism of the associated algebra of measurable functions given by the map

$$v_t^\circ(W_s) := W_{s+t}.$$

The time shift u_t° on the corresponding increment process is the unique endomorphism of the associated algebra of measurable functions given by the map

$$u_t^\circ\left(W_s - W_r\right) := W_{s+t} - W_{r+t}.$$

Denoting $\hat{j}_t$ the restriction of the Wiener time shift on the time zero algebra, we see that v_t° is uniquely determined by the pair $(\hat{j}_t, u_t^\circ)$ through the identity

$$\begin{aligned} v_t^\circ\left(W_s\right) &:= W_{s+t} \\ &= W_t + \left(W_{s+t} - W_t\right) \\ &= \hat{j}_t\left(W_0\right) + u_t^\circ\left(W_s - W_0\right). \end{aligned}$$

But we know that in the quantum formulation of the classical Wiener process, the initial random variable is identified with the position operator q_0 and the increment (noise) is identified with the momentum process $W_t - W_0 = P_{(0,t]}$.

This leads to the following:

$$W_s = W_0 + W_s - W_0 = q_0 \otimes 1 + 1_0 \otimes P_{(0,s]}.$$

Then

$$\begin{aligned} v_t^\circ(W_s) &= v_t^\circ(q_0 \otimes 1 + 1_0 \otimes P_{(0,s]}) \\ &= \hat{j}_t(q_0 \otimes 1) + 1_0 \otimes u_t^\circ(P_{(0,s]}) \\ &= \hat{j}_t(q_0) + 1_0 \otimes P_{(t,t+s]}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} v_t^\circ(W_s) &= W_{s+t} \\ &= W_0 + (W_{s+t} - W_0) \\ &= q_0 \otimes 1 + 1_0 \otimes P_{(0,s+t]} \\ &= q_0 \otimes 1 + 1_0 \otimes P_{(0,t]} + 1_0 \otimes P_{(t,s+t]}. \end{aligned}$$

From the above, we can deduce that

$$\hat{j}_t(q_0) = q_0 \otimes 1 + 1_0 \otimes P_{(0,t]}$$

is a possible solution to our problem.

Note that this choice is not unique in fact we can choose a process $X_{[0,t]}$ instead of $P_{(0,t]}$.

Papers [5] and [6] began the systematic investigation of all possible solutions of this problem in the cases of the Heisenberg Lie algebra and the oscillator Lie algebra, respectively.

In the present paper, we study the quantum extensions of the time shifts of a class of Lévy process. We consider any unitary irreducible representation of the renormalized square of white noise algebra $\mathcal{L}_2$, which can be obtained as in [7]. This representation produces a new picture of the renormalized square of white noise as unital algebra of the first order annihilation, creation and number operators (i.e., oscillator algebra) acting on the the Fock space $\Gamma(L^2(\mathbb{R}_+, \mathcal{K}))$. Following the above statement, we will consider the Lie algebra time shift

$$\hat{j}_t(x) = x \otimes 1 + 1_0 \otimes X_{[0,t]},$$

where x is an element of the oscillator Lie algebra. (We refer to [6] for the definition of the Lie algebra time shift and related results).

This induces a Lie algebra time shift of the renormalized square of white noise algebra $\hat{j}_t \circ \pi$, which will be also denoted by $\hat{j}_t$.

We can show that the identity

$$\hat{j}_t(x) = x \otimes 1 + 1_0 \otimes X_{[0,t]}$$

holds also when x is an element of the renormalized square of white noise algebra.

Taking the quantum Markov process canonically associated to the class of Lévy processes $X_{[0,t]}$ defined on the quadratic Weyl algebra by

$$j_t(e^{ix}) := e^{\hat{j}_t(x)},$$

one can compute the explicit form action on the quadratic Weyl operators of the associated semigroup.

The generator of this Markov semigroup constitutes a quantum extension of the classical Lévy process induced by the Lie algebra time shift.

2. Notations and Preliminaries

2.1. Boson Fock space. In the following all Hilbert spaces are assumed to be complex and separable with inner product linear in the second variable denoted, $\langle \cdot, \cdot \rangle$. For any Hilbert space $\mathcal{H}$, we denote:

- $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$
- $\Gamma(\mathcal{H})$ the symmetric (boson) Fock space over $\mathcal{H}$
- ψ_u ($u \in \mathcal{H}$) the exponential vector associated with u:

$$\psi_u := \sum_{n\geq 0} \frac{u^{\otimes n}}{\sqrt{n!}} \in \Gamma(\mathcal{H}) \quad ; \quad \psi_0 = \Phi \text{ vacuum vector.}$$

For any dense linear subspace S of $\mathcal{H}$ the family $\{\psi_u, u \in S\}$ is total and linearly independent in $\Gamma(\mathcal{H})$. We denote by $\mathcal{E}(S)$ the vector space algebraically generated by it. If S is as above a linear operator may be defined densely on $\Gamma(\mathcal{H})$ by giving arbitrarily its action on the family $\{\psi_u, u \in S\}$. We simply use the notation $\mathcal{E}$ when $S = \mathcal{H}$.

The annihilation, creation and Weyl operators are defined respectively by:

$$A^-(v)\psi_u := \langle v,u\rangle\psi_u\,; \qquad A^+(v)\psi_u := \frac{d}{ds}\,|_{s=0}\ \psi_{u+sv}\,,$$
$$W(v)\psi_u = e^{-\frac{1}{2}\|v\|^2-\langle v,u\rangle}\psi_{u+v}\,; \qquad \forall v\in\mathcal{H}.$$

The boson creation and annihilation operators satisfy the *canonical commutations relations* (CCR):

$$[A^-(u),A^+(v)] \;=\; <u,v> 1_0 \tag{2.1}$$
$$[A^-(u),A^-(v)] \;=\; [A^+(u),A^+(v)] = 0\,, \tag{2.2}$$

for any $u,v\in\mathcal{H}$, where $[x,y]:=xy-yx$ is the commutator and 1_0 is the identity operator on $\Gamma(\mathcal{H})$.

The second quantized $\Gamma(T)$ of a self-adjoint bounded operator T on $\mathcal{H}$ is given by the relation:

$$\Gamma(T)\psi_u := \psi_{Tu}.$$

The differential second quantization operator $\Lambda(T)$ (or the number operator) of T is defined via the Stone theorem by:

$$\Gamma(e^{itT}) =: e^{it\Lambda(T)}\,, \qquad t\in\mathbb{R}.$$

Its action on $\mathcal{E}(S)$ is given by:

$$\Lambda(T)\psi_u = \frac{1}{i}\frac{d}{ds}|_{s=0}\psi_{e^{isT}u}.$$

If T is a bounded but not necessarily self-adjoint operator on $\mathcal{H}$, then by writing T as "sum" of two self-adjoint operators

$$T = T_1 + iT_2 = \frac{T+T^*}{2} + i\frac{T-T^*}{2i},$$

then we can define $\Lambda(T)$ by:

$$\Lambda(T) := \Lambda(T_1) + i\Lambda(T_2).$$

Hence $\Lambda(T)$ is linear in T and the following canonical commutations relations hold weakly on the set of the exponential vectors

$$[\Lambda(T),A^+(u)] = A^+(Tu),\ \ [\Lambda(T),A(u)] = -A(T^*u) \tag{2.3}$$

and

$$[\Lambda(T_1),\Lambda(T_2)] = \Lambda([T_1,T_2]). \tag{2.4}$$

2.2. Markov flows on white noise spaces. In the notations of section (2.1), if the space $\mathcal{H}$ has the form

$$L^2(\mathbb{R}_+,\mathcal{H}_0) \equiv L^2(\mathbb{R}_+)\otimes\mathcal{H}_0$$

where $\mathcal{H}_0$ is an Hilbert space, the associated Fock space is called a *white noise space* and the space $\mathcal{H}_0$ a *multiplicity (or polarization) space*. Usually an *initial (or system)* space is added to the white noise space and in the following we will fix the choice

$$\mathcal{H}_w := \Gamma(\mathcal{H}_0)\otimes\Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big).$$

Notice that we choose the initial space to be the Fock space over the multiplicity space. This is a common feature in the theory of quantum time shifts (see [5]) whose motivation will be clear from the following development. This space has two natural Hilbert space filtrations (past and future) defined (asymmetrically) by

$$\mathcal{H}_{t]} := \Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2([0,t],\mathcal{H}_0)\big) \quad ; \quad \mathcal{H}_{[t} = \Gamma\big(L^2([t,+\infty[,\mathcal{H}_0)\big) \quad ; \quad t \geq 0$$

where, here and in the following, $\mathcal{H}_{t]}$ (resp. $\mathcal{H}_{[t}$) will be identified to the subspace $\Phi_0 \otimes \mathcal{H}_{t]} \otimes \Phi_{[t}$ (resp. $\Phi_{t]} \otimes \mathcal{H}_{[t}$), $\Phi_{[t}$ (resp. $\Phi_0, \Phi_{t]}$) being the vacuum vector in $\mathcal{H}_{[t}$ (resp. $\mathcal{H}_0, \mathcal{H}_{t]}$). With these notations the following factorization property holds

$$\mathcal{H}_w = \Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big) = \mathcal{H}_{t]} \otimes \mathcal{H}_{[t}.$$

Similarly we define the Von Neumann algebra pure noise filtration:

$$\mathcal{B}_{t]} = \mathcal{B}\big(\Gamma\big(L^2([0,t],\mathcal{H}_0)\big)\big) \equiv \mathcal{B}\big(\Gamma\big(L^2([0,t],\mathcal{H}_0)\big)\big) \otimes 1_{[t},$$

$$\mathcal{B}_{[t} = \mathcal{B}\big(\Gamma\big(L^2([t,+\infty[,\mathcal{H}_0)\big)\big) \equiv 1_{t]} \otimes \mathcal{B}(\mathcal{H}_{[t})$$

and

$$\mathcal{B} = \mathcal{B}\big(\Gamma(L^2(\mathbb{R}_+),\mathcal{H}_0)\big) \equiv \mathcal{B}_{t]} \otimes \mathcal{B}_{[t},$$

where $1_{t]}$ and $1_{[t}$ are, respectively, the identities on the spaces $\Gamma\big(L^2([0,t],\mathcal{H}_0)\big)$ and $\Gamma\big(L^2([t,+\infty[,\mathcal{H}_0)\big)$.

If $\mathcal{A}_0$ is a C^*–subalgebra of $\mathcal{B}(\Gamma(\mathcal{H}_0))$, we define the filtrations:

$$\mathcal{A}_{t]} := \mathcal{A}_0 \otimes \mathcal{B}_{t]},\ \mathcal{A}_{[t} := \mathcal{B}_{[t},\ \mathcal{A} = \mathcal{A}_{t]} \otimes \mathcal{A}_{[t} = \mathcal{A}_0 \otimes \mathcal{B}$$

and denote by 1_0 (resp. 1) the identity on $\Gamma(\mathcal{H}_0)$ (resp. $\Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big)$).

The noise creation, annihilation and conservation increment processes, acting on $\Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big)$, will be denoted respectively

$$A^+_{s,t}(\xi) := A^+(\chi_{[s,t]} \otimes \xi),\quad A^-_{s,t}(\xi) := A^-(\chi_{[s,t]} \otimes \xi),\quad \Lambda_{s,t}(\xi) := \Lambda(M_{\chi_{[s,t]}} \otimes T). \tag{2.5}$$

If $s = 0$, we simply write

$$A^+_t(\xi) := A^+_{0,t}(\xi),\quad A^-_t(\xi) := A^-_{0,t}(\xi),\quad \Lambda_t(T) := \Lambda_{0,t}(T) \tag{2.6}$$

and, when no ambiguity is possible, the same notation will be used for their natural action on $\Gamma(\mathcal{H}_0) \otimes \Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big)$.

Denote by θ_t, the right shift on $L^2(\mathbb{R}_+,\mathcal{H}_0)$, so that $\forall t \geq 0$

$$\theta_t f(s) = \begin{cases} f(s-t), & \text{if } s \geq t \geq 0; \\ 0, & \text{if } 0 \leq s \leq t. \end{cases} \tag{2.7}$$

The operator θ_t is isometric with $\theta_t^* f(s) = f(s+t)$. The white noise time shift is the 1–parameter endomorphism semigroup of $\mathcal{B}\big(\Gamma(\mathcal{H}_0)\big) \otimes \mathcal{B}\big(\Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big)\big)$ characterized by the property that, for all $b_0 \in \mathcal{B}(\Gamma(\mathcal{H}_0))$, $b \in \mathcal{B}\big(\Gamma\big(L^2(\mathbb{R}_+,\mathcal{H}_0)\big)\big)$ and $t \geq 0$ one has:

$$u^\circ_t(b_0 \otimes b) = b_0 \otimes \Gamma(\theta_t) b \Gamma(\theta_t^*). \tag{2.8}$$

The shift u°_t is a normal, injective $*$–endomorphism and satisfies the following:

(1) For $s,t \geq 0$, $u^\circ_s u^\circ_t = u^\circ_{s+t}$.
(2) For $s \geq 0$, $u^\circ_s(\mathcal{A}) = \mathcal{A}_0 \otimes 1_{s]} \otimes \mathcal{B}_{[s}$.

The vacuum conditional expectations, defined on the algebra $\mathcal{A}$ and with values in the past filtration algebra are defined by:

$$E_{t]}(b_{t]} \otimes b_{[t}) := \langle \Phi_{[0}, b_{[t} \Phi_{[0} \rangle b_{t]}, \quad b_{t]} \in \mathcal{A}_{t]}, \quad b_{[t} \in \mathcal{A}_{[t}. \tag{2.9}$$

Definition 2.1. A *stochastic process* on $\mathcal{A}_0$ is a family $\{j_t : \mathcal{A}_0 \longrightarrow \mathcal{A}_{t]}, t \geq 0\}$ of $*$–homomorphisms with $j_0(x) = x \otimes 1$. A stochastic process is called *normal* if for each $t \geq 0$, j_t is σ-weakly continuous.

Let $(j_t)_{t \geq 0}$ be a normal stochastic process on $\mathcal{A}_0$. We define $\tilde{j}_t$ as the unique normal $*$–homomorphism $\tilde{j}_t : \mathcal{A}_0 \otimes 1_{[0,t]} \otimes \mathcal{B}_{[t} \longrightarrow \mathcal{A}$ characterized by

$$\tilde{j}_t(x \otimes 1) = j_t(x), \quad \tilde{j}_t(1 \otimes b_{[t}) = j_t(1) \otimes b_{[t}. \tag{2.10}$$

Each $\tilde{j}_t$ can be extended in an obvious way to the algebraic linear span of the elements of the form $x \otimes 1_{[0,t]} \otimes Y_{[t}$, where $x \in \mathcal{A}_0$, $Y_{[t}$ is an operator on $\mathcal{H}_{[t}$.

Definition 2.2. A stochastic process on $\mathcal{A}_0$ is said to be a *Markov cocycle* if, in the notations of (2.10), it satisfies the cocycle equation: for all $s, t \geq 0$ and $x \in \mathcal{A}_0$

$$j_0(x) = x \otimes 1 \;\; ; \;\; j_{s+t}(x) = \tilde{j}_s \circ u_s^\circ \circ j_t(x). \tag{2.11}$$

It is said to be σ-*weakly continuous* if the map $(t, x) \longmapsto j_t(x)$ is continuous w.r.t the σ-weak topology of the Von Neumann algebras involved.

According to the Feynman-Kac formula to every Markov cocycle (j_t) on $\mathcal{A}_0$, one can associate Markov semigroup (P^t) on $\mathcal{A}_0$, characterized by the identity:

$$P^t(x) = E_{0]} j_t(x) \tag{2.12}$$

for all $t \geq 0$, $x \in \mathcal{A}_0$. Moreover the cocycle identity (2.11) condition (2.12) imply that for each $s, t \geq 0$

$$E_{s]} \circ j_{s+t} = j_s \circ P^t. \tag{2.13}$$

3. The Quadratic Weyl C^*–algebra

In the present section we construct a quantum extension of the time shift of the renormalized square of white noise (RSWN) algebra and we extend it to the associated quadratic Weyl algebra.

Recall that $sl(2, \mathbb{R})$ is the $*$–Lie algebra with generators $\{B^-, B^+, M\}$ satisfying the following commutation relations:

$$[B^-, B^+] = M, \quad [M, B^\pm] = \pm 2B^\pm$$

and involution given by:

$$(B^-)^* = B^+, \quad M^* = M.$$

The space $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ has a natural structure of Hilbert algebra (i.e., it is a pre–Hilbert space with a $*$–algebra structure) induced by the inner product $\langle \cdot, \cdot \rangle$ on $L^2(\mathbb{R})$ and we denote by $\mathcal{T}_S$ (test function algebra) the sub–Hilbert algebra of $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$ consisting of the step functions with a finite range

$$\phi = \sum_{I \in F} \phi_I \chi_I \;, \qquad \varphi = \sum_{I \in F} \varphi_I \chi_I, \tag{3.1}$$

where F is a finite set of mutually disjoint bounded intervals in $\mathbb{R}$ and $\phi_I, \varphi_I \in \mathbb{C}$ (one can always suppose that the family F is the same for both ϕ and φ).

Definition 3.1. The *renormalized square* of white noise algebra $\mathcal{L}_2$ over $\mathcal{T}_S$ is by definition the unital $*$–Lie–algebra

$$\mathcal{L}_2 := \mathcal{L}\left(sl(2,\mathbb{R}); \mathcal{T}_S\right)$$

with linearly independent generators:

$$1_0 \text{ (central element)} \ , \ b_0^-(\phi) \ , \ b_0^+(\phi) \ , \ n_0(\phi)$$

and relations:

$$[b_0^-(\phi), b_0^+(\psi)] = \gamma\langle\phi,\psi\rangle 1_0 + n_0(\overline{\phi}\psi) \tag{3.2}$$

$$[n_0(\phi), b_0^+(\psi)] = 2b_0^+(\phi\psi) \tag{3.3}$$

$$[n_0(\phi), b_0^-(\psi)] = -2b_0^-(\overline{\phi}\psi) \tag{3.4}$$

$$[n_0(\phi), n_0(\psi)] = [b_0^+(\phi), b_0^+(\psi)] = [b_0^-(\phi), b_0^-(\psi)] = 0 \tag{3.5}$$

$$b_0^-(\phi)^* = b_0^+(\phi), \quad n_0(\phi)^* = n_0(\overline{\phi}) \tag{3.6}$$

where γ is a fixed strictly positive real parameter (called the renormalization constant), $\phi, \psi \in \mathcal{T}_S$, and the dependence on test functions in $\mathcal{T}_S$ is linear for b_0^+ and n_0 and anti–linear for b_0^-.

Linear independence, in the context of test functions, is meant in the sense that $\phi_0 1_0 + b_0^+(\phi_1) + b_0^-(\phi_2) + n_0(\phi_3) = 0$, with $\phi_j \in \mathcal{T}_S, \phi_0 \in \mathbb{C}$, if and only if $\phi_0 = 0, \phi_1 = \phi_2 = \phi_3 = 0$.

Let b, b^+ and 1 be the generators of the Schrodinger representation of the 1–mode Heisenberg $*$–Lie–algebra acting on $L^2(\mathbb{R})$. The $*$–Lie algebra isomorphism

$$B^- \mapsto b^2, \quad B^+ \mapsto (b^+)^2, \quad M \mapsto 4b^+b + 2\cdot 1$$

identifies the generators of $sl(2,\mathbb{R})$ with self-adjoint operators acting on $L^2(\mathbb{R})$. This identification extends to the universal enveloping algebra $\mathcal{U}(sl(2,\mathbb{R}))$ on $L^2(\mathbb{R})$.

From the commutation relations of $\mathcal{L}_2$ it is clear that, for any bounded interval $I \subset \mathbb{R}$, the sub–space of $\mathcal{L}_2$ with (linearly independent) generators

$$1_0 \text{ (central element)} \ , \quad b_0^-(\chi_I) \ , \ b_0^+(\chi_I) \ , \ n_0(\chi_I) \qquad \phi \in \mathcal{T}_S$$

is a sub–$*$–Lie algebra of $\mathcal{L}_2$ and that the correspondence

$$b_0^-(\chi_I) \mapsto \sqrt{\gamma|I|}b^2 \ ; \ b_0^+(\chi_I) \mapsto \sqrt{\gamma|I|}(b^+)^2 \ ; \ n_0(\chi_I) \mapsto \gamma|I|(1+4b^+b) \ ; \ 1_0 \mapsto 1$$

extends by linearity to an isomorphism of $*$–Lie algebras.

Using this isomorphism one can identify any element of $\mathcal{L}_2$ of the form

$$z_I b_0^+(\chi_I) + \bar{z}_I b_0^-(\chi_I) + c_I n_0(\chi_I) + d_I 1_0, \qquad z_I \in \mathbb{C} \ , \ c_I, d_I \in \mathbb{R} \tag{3.7}$$

to a self-adjoint operator acting on $L^2(\mathbb{R})$. This allows to identify the operators

$$e^{i(z_I b_0^+(\chi_I) + \bar{z}_I b_0^-(\chi_I) + c_I n_0(\chi_I) + d_I 1_0)}, \qquad z_I \in \mathbb{C} \ , \ c_I, d_I \in \mathbb{R}$$

to unitary operators acting on $L^2(\mathbb{R})$.

Finally, using the factorization property of $\Gamma(L^2(\mathbb{R}))$, one concludes that, for any simple test function of the form (3.1), the exponential

$$e^{i(b_0^+(\varphi)+b_0^-(\varphi)+n_0(\phi)+c1_0)} =: e^{ic}W_2(\varphi,\phi), \qquad \varphi,\phi,\in \mathcal{T}_S \ , \ c\in\mathbb{R} \tag{3.8}$$

is well defined on $\Gamma(L^2(\mathbb{R}))$ and in fact is a unitary operator.

Definition 3.2. Let $\varphi,\phi\in\mathcal{T}_S$ be test functions with ϕ real valued. The unitary operator $W_2(\varphi,\phi)$ defined in (3.8) is called the $\mathcal{L}_2$*–Weyl operator* (or simply the *quadratic Weyl operator*) with test functions $\varphi,\phi\in\mathcal{T}_S$. The C^*–algebra, with minimal C^*–norm, generated by the quadratic Weyl operators will be denoted $\mathcal{W}_2(\mathcal{T}_S)$ and called the *quadratic Weyl* C^**–algebra over* $\mathcal{T}_S$.

4. Representations of $\mathcal{L}_2$

Following [7] a class of representations of $\mathcal{L}_2$ can be obtained by choosing:

- an irreducible representation ρ_0 of $sl(2,\mathbb{R})$, canonically extended to the universal enveloping algebra $\mathcal{U}(sl(2,\mathbb{R}))$ and acting on a Hilbert space $\mathcal{H}_0$
- a ρ_0-1-cocycle $\eta_0:\mathcal{U}(sl(2,\mathbb{R}))\to\mathcal{H}_0$, i.e., a linear functional with dense range $D\subseteq\mathcal{H}_0$ satisfying the condition

$$\eta_0(uv)=\rho_0(u)\eta_0(v), \qquad \forall u,v\in\mathcal{U}(sl(2,\mathbb{R})).$$

The cocycle condition implies that the linear functional $L:\mathcal{U}(sl(2,\mathbb{R}))\to\mathbb{C}$ defined by

$$L_0(v)=\langle\eta_0(1),\eta_0(v)\rangle$$

(1 denoting the identity in $\mathcal{U}(sl(2,\mathbb{R}))$) is in fact a state on $\mathcal{U}(sl(2,\mathbb{R}))$.

With these notations and in the notation (2.5) the increment process

$$j_{s,t}:\mathcal{U}(sl(2,\mathbb{R}))\longrightarrow\mathcal{L}(\Gamma(L^2(\mathbb{R},\mathcal{H}_0)) \tag{4.1}$$

defined by:

$$j_{s,t}(X):=\Lambda_{s,t}(\rho_0(X))+A^+_{s,t}(\eta_0(X))+A^-_{s,t}(\eta_0(X^*))+L_0(X)(t-s)1 \tag{4.2}$$

is a Lévy process with respect to the vacuum vector Φ and the relations

$$\pi(b_0^+(\chi_{[s,t[})) := j_{s,t}(B^+) \tag{4.3}$$

$$\pi(b_0^-(\chi_{[s,t[})) := j_{s,t}(B^-) \tag{4.4}$$

$$\pi(n_0(\chi_{[s,t[})) := j_{s,t}(M)-\gamma(t-s)1 \tag{4.5}$$

define a unique representation π of the $*$–Lie algebra $\mathcal{L}_2$ on the Hilbert space $\Gamma(L^2(\mathbb{R},\mathcal{H}_0))$. The following lemma shows that π induces a $*$–representation, still denoted by π, of the quadratic Weyl C^*–algebra over $\mathcal{T}_S$.

Following [10], the generalized Weyl operator over a separable Hilbert space $\mathcal{H}$, is given as follow:

Theorem 4.1. *Let $\mathcal{H}$ be a separable Hilbert space, $T\in\mathcal{B}(\mathcal{H})$ be a self-adjoint bounded operator on $\mathcal{H}$ and $\xi\in\mathcal{H}$. Then the operator*

$$W(\xi,T):=e^{iH(\xi,T)}=e^{i(A^+(\xi)+A^-(\xi)+\Lambda(T))} \tag{4.6}$$

where

$$H(\xi,T):=A^+(\xi)+A^-(\xi)+\Lambda(T),$$

is a unitary operator on the Fock space $\Gamma(\mathcal{H})$. It is called the generalized Weyl operator over $\mathcal{H}$.

Definition 4.2. The C^*–subalgebra of $\mathcal{B}\big(\Gamma(\mathcal{H})\big)$, generated by the set of generalized Weyl operators is called *oscillator Weyl algebra* over $\mathcal{H}$, we denote it by $\mathcal{W}_g(\mathcal{H})$.

Lemma 4.3. *Let $W_2(\varphi,\phi) := e^{i\left(b_0^+(\varphi)+b_0^-(\varphi)+n_0(\phi)\right)}$ be the quadratic Weyl operator as in Definition (3.2). Define:*

$$\begin{aligned} \xi_{\varphi,\phi} &:= \varphi\otimes\eta_0(B^+)+\overline{\varphi}\otimes\eta_0(B^-)+\phi\otimes\eta_0(M)\in\mathcal{H}:=L^2(\mathbb{R},\mathcal{H}_0) && (4.7)\\ \varepsilon(f) &:= \int f(x)dx\in\mathbb{C}\qquad \forall f\in\mathcal{T}_S && (4.8)\\ \alpha_{\varphi,\phi} &:= \varepsilon(\varphi)L_0(B^+)+\varepsilon(\overline{\varphi})L_0(B^-)+\varepsilon(\phi)(L_0(M)-\gamma)\in\mathbb{C} && (4.9)\\ T_{\varphi,\phi} &:= \varphi\otimes\rho_0(B^+)+\overline{\varphi}\otimes\rho_0(B^-)+\phi\otimes\rho_0(M). && (4.10)\end{aligned}$$

Then

$$\pi\Big(W_2(\varphi,\phi)\Big)=e^{i\alpha_{\varphi,\phi}}W(\xi_{\varphi,\phi},T_{\varphi,\phi}). \tag{4.11}$$

In particular $T_{\varphi,\phi}\in\mathcal{B}(\mathcal{H})$ is a self-adjoint operator acting on $\mathcal{H}:=L^2(\mathbb{R},\mathcal{H}_0)$.

Proof. First notice that since f is in $L^2(\mathbb{R})\cap L^\infty(\mathbb{R})$, it is also in $L^1(\mathbb{R})$ hence (4.8) is well defined. Let $\phi,\varphi\in\mathcal{T}_S$ be as in Definition (3.2) and suppose that they are step functions with a finite range

$$\phi=\sum_i\phi_i\chi_{[s_i,t_i[}\ ,\qquad \varphi=\sum_i\varphi_i\chi_{[s_i,t_i[}$$

Denote

$$X(\varphi,\phi)=b_0^+(\varphi)+b_0^-(\varphi)+n_0(\phi)\in\mathcal{L}_2.$$

Then

$$\begin{aligned}\pi\big(X(\varphi,\phi)\big) &= \sum_i\varphi_i\pi\big(b_0^+(\chi_{[s_i,t_i[})\big)+\overline{\varphi_i}\pi\big(b_0^-(\chi_{[s_i,t_i[})\big)+\phi_i\pi\big(n_0(\chi_{[s_i,t_i[})\big)\\ &= \sum_i\varphi_i j_{s_i,t_i}(B^+)+\overline{\varphi_i}j_{s_i,_it}(B^-)+\phi_i(j_{s_i,t_i}(M)-\gamma(t_i-s_i))\\ &= \sum_i\varphi_i\Big(\Lambda_{s_i,t_i}(\rho_0(B^+))+A^+_{s_i,t_i}(\eta_0(B^+))+A^-_{s_i,t_i}(\eta_0(B^-))\\ &\quad +L_0(B^+)(t_i-s_i)1\Big)\\ &\quad +\sum_i\overline{\varphi_i}\Big(\Lambda_{s_i,t_i}(\rho_0(B^-))+A^+_{s_i,t_i}(\eta_0(B^-))+A^-_{s_i,t_i}(\eta_0(B^+))\\ &\quad +L_0(B^-)(t_i-s_i)1\Big)\\ &\quad +\sum_i\phi_i\{\Lambda_{s_i,t_i}(\rho_0(M))+A^+_{s_i,t_i}(\eta_0(M))+A^-_{s_i,t_i}(\eta_0(M))\\ &\quad +(L_0(M)-\gamma)(t_i-s_i)1\}\end{aligned}$$

$$
\begin{aligned}
&= \sum_i \varphi_i \Big(\Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(B^+)) + A^+(\chi_{[s_i,t_i[} \otimes \eta_0(B^+)) \\
&\quad + A^-(\chi_{[s_i,t_i[} \otimes \eta_0(B^-)) + L_0(B^+)(t_i - s_i)1 \Big) \\
&\quad + \sum_i \overline{\varphi_i} \Big(\Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(B^-)) + A^+(\chi_{[s_i,t_i[} \otimes \eta_0(B^-)) \\
&\quad + A^-(\chi_{[s_i,t_i[} \otimes \eta_0(B^+)) + L_0(B^-)(t_i - s_i)1 \Big) \\
&\quad + \sum_i \phi_i \Big(\Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(M)) + A^+(\chi_{[s_i,t_i[} \otimes \eta_0(M)) \\
&\quad + A^-(\chi_{[s_i,t_i[} \otimes \eta_0(M)) + (L_0(M) - \gamma)(t_i - s_i)1 \Big) \\
&= \sum_i \varphi_i \Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(B^+)) + \overline{\varphi_i} \Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(B^-)) + \phi_i \Lambda(\chi_{[s_i,t_i[} \otimes \rho_0(M)) \\
&\quad + \sum_i \varphi_i A^+(\chi_{[s_i,t_i[} \otimes \eta_0(B^+)) + \overline{\varphi_i} A^+(\chi_{[s_i,t_i[} \otimes \eta_0(B^-)) \\
&\quad + \phi_i A^+(\chi_{[s_i,t_i[} \otimes \eta_0(M)) \\
&\quad + \sum_i \varphi_i A^-(\chi_{[s_i,t_i[} \otimes \eta_0(B^-)) + \overline{\varphi_i} A^-(\chi_{[s_i,t_i[} \otimes \eta_0(B^+)) \\
&\quad + \phi_i A^-(\chi_{[s_i,t_i[} \otimes \eta_0(M)) \\
&\quad + \sum_i \varphi_i L_0(B^+)(t_i - s_i)1 + \overline{\varphi_i} L_0(B^-)(t_i - s_i)1 + \phi_i (L_0(M) - \gamma)(t_i - s_i)1 \\
&= \Lambda\Big(\varphi \otimes \rho_0(B^+) + \overline{\varphi} \otimes \rho_0(B^-) + \phi \otimes \rho_0(M) \Big) \\
&\quad + A^+\Big(\varphi \otimes \eta_0(B^+) + \overline{\varphi} \otimes \eta_0(B^-) + \phi \otimes \eta_0(M) \Big) \\
&\quad + A^-\Big(\varphi \otimes \eta_0(B^+) + \overline{\varphi} \otimes \eta_0(B^-) + \phi \otimes \eta_0(M) \Big) \\
&\quad + \{ \varepsilon(\varphi) L_0(B^+) + \varepsilon(\overline{\varphi}) L_0(B^-) + \varepsilon(\phi)(L_0(M) - \gamma) \} 1.
\end{aligned}
$$

Using the notations

$$\xi_{\varphi,\phi} = \varphi \otimes \eta_0(B^+) + \overline{\varphi} \otimes \eta_0(B^-) + \phi \otimes \eta_0(M) \in \mathcal{H} = L^2(\mathbb{R}) \otimes \mathcal{H}_0 \equiv L^2(\mathbb{R}, \mathcal{H}_0),$$

$$T_{\varphi,\phi} = \varphi \otimes \rho_0(B^+) + \overline{\varphi} \otimes \rho_0(B^-) + \phi \otimes \rho_0(M) \in \mathcal{B}(\mathcal{H})$$

and

$$\alpha_{\varphi,\phi} = \varepsilon(\varphi) L_0(B^+) + \varepsilon(\overline{\varphi}) L_0(B^-) + \varepsilon(\phi)(L_0(M) - \gamma) \in \mathbb{R},$$

we obtain

$$
\begin{aligned}
\pi\big(X(\varphi,\phi)\big) &= \pi\big(b_0^+(\varphi) + b_0^-(\varphi) + n_0(\phi)\big) \\
&= A^+(\xi_{\varphi,\phi}) + A^-(\xi_{\varphi,\phi}) + \Lambda(T_{\varphi,\phi}) + \alpha_{\varphi,\phi} 1 \\
&= H(\xi_{\varphi,\phi}, T_{\varphi,\phi}) + \alpha_{\varphi,\phi} 1.
\end{aligned}
$$

Hence

$$\begin{aligned}\pi\Big(W_2(\varphi,\phi)\Big) &= e^{i\pi\big(X(\varphi,\phi)\big)}\\ &= e^{iH(\xi_{\varphi,\phi},T_{\varphi,\phi})+i\alpha_{\varphi,\phi}1}\\ &= e^{i\alpha_{\varphi,\phi}}W(\xi_{\varphi,\phi},T_{\varphi,\phi}).\end{aligned} \tag{4.12}$$

□

5. Time Shift of the Renormalized Square of White Noise

5.1. Lie algebra time shift. We refer to papers [6] and [10] for precise formulation and definition of the Lie algebra time shift.

Let us fix:

(1) a continuous $*$–homomorphism $\rho : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$;
(2) a surjective ρ-1-cocycle, i.e., a linear map $\delta : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{H}$, with the property

$$\delta(T'T) = \rho(T')\delta(T) \quad \forall\, T, T' \in \mathcal{B}(\mathcal{H}). \tag{5.1}$$

Property (5.1) implies that the linear functional $L : \mathcal{B}(\mathcal{H}) \longrightarrow \mathbb{C}$, defined by:

$$L(T) := \langle \delta(1), \delta(T)\rangle \quad \forall\, T \in \mathcal{B}(\mathcal{H}) \tag{5.2}$$

is hermitian and satisfies:

$$L(TT') = \langle \delta(T^*), \delta(T')\rangle \quad \forall\, T, T' \in \mathcal{B}(\mathcal{H}) \tag{5.3}$$

In fact we have

$$\begin{aligned}\overline{L(T)} &= \langle \delta(T), \delta(1)\rangle = \langle \rho(T)\delta(1), \delta(1)\rangle = \langle \delta(1), \rho(T^*)\delta(1)\rangle\\ &= \langle \delta(1), \delta(T^*)\rangle = L(T^*)\end{aligned}$$

and

$$\begin{aligned}L(TT') &= \langle \delta(1), \delta(TT')\rangle = \langle \delta(1), \rho(T)\delta(T')\rangle = \langle \rho(T^*)\delta(1), \delta(T')\rangle\\ &= \langle \delta(T^*), \delta(T')\rangle.\end{aligned}$$

A Lie algebra time shift of the oscillator Lie algebra over a Hilbert space $\mathcal{H}$ is given, in [10], by the following proposition:

Proposition 5.1. *Let $\mathcal{L}_{osc}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ be the current algebra over $\mathbb{R}$ of the oscillator algebra $\mathcal{L}_{osc}$ and*

$$\hat{j}_t : \mathcal{L}_{osc} \longrightarrow \mathcal{L}_{osc} \otimes \mathcal{L}_{osc}(\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

be a family of $*$–homomorphisms of Lie algebras with action on the self-adjoint elements given by

$$\hat{j}_t\big(H(\xi,T)+\alpha 1_0\big) = (H(\xi,T)+\alpha 1_0)\otimes 1 + 1_0 \otimes \Big(H_t\big(\delta(T),\rho(T)\big)+tL(T)1\Big), \tag{5.4}$$

where $H_t(\xi, T) := A^+_t(\xi) + A^-_t(\xi) + \Lambda_t(T)$. Then $\hat{j}_t$ is a Lie algebra time shift of the oscillator Lie algebra $\mathcal{L}_{osc}$ (i.e., the associated map defined on $\mathcal{W}_g(\mathcal{H})$ by

$$j_t\big(W(\xi,T)\big) = j_t\big(e^{iH(\xi,T)}\big) := e^{i\hat{j}_t(H(\xi,T))} = W(\xi,T)\otimes e^{iH_t(\delta(T),\rho(T))}e^{itL(T)}$$

extends to a $*$–homomorphism of C^*–algebras).

In the following, the RSWN-algebra will be identified to its representation under π. We have the following theorem:

Theorem 5.2. *The Lie algebra time shift of the RSWN-algebra is given by the identity*

$$\hat{j}_t(X(\varphi,\phi)) = X(\varphi,\phi)\otimes 1 + 1_0\otimes X_{[0,t]}(\varphi,\phi) \tag{5.5}$$

with

$$X_{[0,t]}(\varphi,\phi) = A_t^+(\delta(T_{\varphi,\phi})) + A_t^-(\delta(T_{\varphi,\phi})) + \Lambda_t(\rho(T_{\varphi,\phi})) + tL(T_{\varphi,\phi})1. \tag{5.6}$$

Proof. We have

$$\begin{aligned}
&\hat{j}_t(X(\varphi,\phi))\\
&= \hat{j}_t\big(H(\xi_{\varphi,\phi},T_{\varphi,\phi}) + \alpha_{\varphi,\phi}1_0\big)\\
&= \Big(H(\xi_{\varphi,\phi},T_{\varphi,\phi}) + \alpha_{\varphi,\phi}1_0\Big)\otimes 1\\
&\quad +1_0\otimes\Big(H_t(\delta(T_{\varphi,\phi}),\rho(T_{\varphi,\phi})) + tL(T_{\varphi,\phi})1\Big)\\
&= X(\varphi,\phi)\otimes 1 + 1_0\otimes\Big(A_t^+(\delta(T_{\varphi,\phi})) + A_t^-(\delta(T_{\varphi,\phi})) + \Lambda_t(\rho(T_{\varphi,\phi}))\\
&\quad +tL(T_{\varphi,\phi})1\Big).
\end{aligned}$$

∎

5.2. Semigroup and generator associated to the Lie algebra time shift.

Using the notation j_t for the map $j_t\circ\pi$, one has the following theorem:

Theorem 5.3. *The family*

$$\{j_t : \mathcal{W}_2(\mathcal{T}_S)\longrightarrow \mathcal{A}_{t]},\ t\geq 0\}$$

is a Markov cocycle. Moreover it is given by the $*$–homomorphism

$$j_t\Big(W_2(\varphi,\phi)\Big) = W_2(\varphi,\phi)\otimes e^{iX_{[0,t]}(\varphi,\phi)}. \tag{5.7}$$

Proof. We have

$$\begin{aligned}
j_t\Big(W_2(\varphi,\phi)\Big) &= j_t\Big(e^{i\alpha_{\varphi,\phi}}W(\xi_{\varphi,\phi},T_{\varphi,\phi})\Big)\\
&= e^{i\alpha_{\varphi,\phi}}j_t\Big(W(\xi_{\varphi,\phi},T_{\varphi,\phi})\Big)\\
&= e^{i\alpha_{\varphi,\phi}}W(\xi_{\varphi,\phi},T_{\varphi,\phi})\otimes e^{iX_{[0,t]}(\varphi,\phi)}\\
&= W_2(\varphi,\phi)\otimes e^{iX_{[0,t]}(\varphi,\phi)},
\end{aligned}$$

which proves the first part of theorem. On the other hand, we have:

$$\begin{aligned}
\tilde{j}_s \circ u_s^\circ \circ j_t\Big(W_2(\varphi,\phi)\Big) &= \tilde{j}_s \circ u_s^\circ\Big(W_2(\varphi,\phi)\otimes e^{iX_{[0,t]}(\varphi,\phi)}\Big)\\
&= \tilde{j}_s\Big(W_2(\varphi,\phi)\otimes \Gamma(\theta_s)e^{iX_{[0,t]}(\varphi,\phi)}\Gamma(\theta_s^*)\Big)\\
&= \tilde{j}_s\Big(W_2(\varphi,\phi)\otimes e^{iX_{[s,s+t]}(\varphi,\phi)}\Big)\\
&= \tilde{j}_s\Big(\big(W_2(\varphi,\phi)\otimes 1\big)\big(1\otimes 1_{[0,s]}\otimes e^{iX_{[s,s+t]}(\varphi,\phi)}\big)\Big)\\
&= \tilde{j}_s\Big(W_2(\varphi,\phi)\otimes 1\Big)\tilde{j}_s\Big(1\otimes 1_{[0,s]}\otimes e^{iX_{[s,s+t]}(\varphi,\phi)}\Big)\\
&= j_s\Big(W_2(\varphi,\phi)\Big)\Big(j_s(1)\otimes e^{iX_{[s,s+t]}(\varphi,\phi)}\Big)\\
&= W_2(\varphi,\phi)\otimes e^{iX_{[0,s]}(\varphi,\phi)}\otimes e^{iX_{[s,s+t]}(\varphi,\phi)}\\
&= W_2(\varphi,\phi)\otimes e^{i(X_{[0,s]}(\varphi,\phi)+X_{[s,s+t]}(\varphi,\phi))}\\
&= W_2(\varphi,\phi)\otimes e^{iX_{[0,s+t]}(\varphi,\phi)}\\
&= j_{s+t}\Big(W_2(\varphi,\phi)\Big).
\end{aligned}$$

By the quantum Feynmann-Kac formula [3], the Markovian cocycle j_t defines a Markovian semigroup on the C^*–algebra $\mathcal{W}_2(\mathcal{T}_S)$ given by

$$P^t\Big(W_2(\varphi,\phi)\Big) = E_{0]}j_t\Big(W_2(\varphi,\phi)\Big). \tag{5.8}$$

□

Theorem 5.4. *Let P^t be the Markovian semigroup, defined by the equation (5.8), with generator A. Then the quadratic Weyl operators are eigenvectors of A. More precisely, we have*

$$P^t\big(W_2(\varphi,\phi)\big) = e^{tL\left(e^{iT_{\varphi,\phi}}-1\right)}W_2(\varphi,\phi) \tag{5.9}$$

and then

$$A\big(W_2(\varphi,\phi)\big) = L\big(e^{iT_{\varphi,\phi}}-1\big)W_2(\varphi,\phi). \tag{5.10}$$

To prove this theorem we use the following splitting formula from [6].

Lemma 5.5. *For all bounded operator $T\in\mathcal{B}(\mathcal{H})$ and $\xi,\eta\in\mathcal{H},\alpha\in\mathbb{C}$, we have*

$$e^{A^+(\xi)+\Lambda(T)+A^-(\eta)+\alpha} = e^{A^+(\tilde{\xi})}e^{\Lambda(T)}e^{A^-(\tilde{\eta})}e^{\tilde{\alpha}}, \tag{5.11}$$

where

$$\tilde{\xi} = \sum_{n=1}^{\infty}\frac{1}{n!}T^{n-1}\xi =: e_1(T)\xi,\ \tilde{\eta} = \sum_{n=1}^{\infty}\frac{1}{n!}(T^*)^{n-1}\eta =: e_1(T^*)\eta$$

and

$$\tilde{\alpha} = \alpha + \sum_{n=2}^{\infty}\frac{1}{n!}\langle\eta, T^{n-2}\xi\rangle =: \alpha + \langle\eta, e_2(T)\xi\rangle$$

Proof. (of theorem 5.4) We have

$$\begin{aligned} P^t\big(W_2(\varphi,\phi)\big) &= E_{0]}j_t\big(W_2(\varphi,\phi)\big) \\ &= E_{0]}\big(W_2(\varphi,\phi)\otimes e^{iX_{[0,t]}(\varphi,\phi)}\big) \\ &= \langle\Phi_{[0}, e^{iX_{[0,t]}(\varphi,\phi)}\Phi_{[0}\rangle W_2(\varphi,\phi). \end{aligned}$$

The formulas (5.11) and (5.6) imply

$$\begin{aligned} e^{iX_{[0,t]}(\varphi,\phi)} &= e^{i\left(A_t^+(\delta(T_{\varphi,\phi}))+A_t^-(\delta(T_{\varphi,\phi}))+\Lambda_t(\rho(T_{\varphi,\phi}))+tL(T_{\varphi,\phi})1\right)} \\ &= e^{A^+\left(i\chi_{[0,t]}\otimes\delta(T_{\varphi,\phi})\right)+A^-\left(-i\chi_{[0,t]}\otimes\delta(T_{\varphi,\phi})\right)+\Lambda\left(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi})\right)+itL(T_{\varphi,\phi})} \\ &= e^{A^+(\xi_t)}e^{\Lambda(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi}))}e^{A^-(\eta_t)}e^{\alpha_t+itL(T_{\varphi,\phi})}, \end{aligned}$$

for some vectors $\xi_t,\eta_t\in L^2(\mathbb{R}_+)\otimes\mathcal{H}$ and real number α_t is given by

$$\begin{aligned} \alpha_t &= \langle -i\chi_{[0,t]}\otimes\delta(T_{\varphi,\phi}), e_2\big(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi})\big)(i\chi_{[0,t]}\otimes\delta(T_{\varphi,\phi}))\rangle \\ &= -t\langle\delta(T_{\varphi,\phi}), e_2(i\rho(T_{\varphi,\phi}))\delta(T_{\varphi,\phi})\rangle \\ &= -t\langle\delta(T_{\varphi,\phi}), \rho(e_2(iT_{\varphi,\phi}))\delta(T_{\varphi,\phi})\rangle \\ &= -t\langle\delta(T_{\varphi,\phi}), \delta(e_2(iT_{\varphi,\phi})T_{\varphi,\phi})\rangle \\ &= -tL\big(e_2(iT_{\varphi,\phi})T_{\varphi,\phi}^2\big) \\ &= tL\big(e^{iT_{\varphi,\phi}}-iT_{\varphi,\phi}-1\big). \end{aligned}$$

This gives

$$\begin{aligned} P^t\big(W_2(\varphi,\phi)\big) &= e^{\alpha_t+itL(T_{\varphi,\phi})}\langle\Phi_{[0}, e^{A^+(\xi_t)}e^{\Lambda(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi}))}e^{A^-(\eta_t)}\Phi_{[0}\rangle W_2(\varphi,\phi) \\ &= e^{\alpha_t+itL(T_{\varphi,\phi})}\langle e^{A^-(\xi_t)}\Phi_{[0}, e^{\Lambda(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi}))}e^{A^-(\eta_t)}\Phi_{[0}\rangle W_2(\varphi,\phi) \\ &= e^{itL(T_{\varphi,\phi})+tL(e^{iT_{\varphi,\phi}}-iT_{\varphi,\phi}-1)}\langle\Phi_{[0}, e^{\Lambda(i\chi_{[0,t]}\otimes\rho(T_{\varphi,\phi}))}\Phi_{[0}\rangle W_2(\varphi,\phi) \\ &= e^{tL(e^{iT_{\varphi,\phi}}-1)}W_2(\varphi,\phi). \end{aligned}$$

which proves (5.9). The identity (5.10) is a simple consequence from the above. □

5.3. Conclusion. The representation of the square of white noise algebra introduced in [7], becomes a good way to give an answer to the Meyer's problem in our case and to construct a quantum extension generators of a class of classical Lévy processes. We have seen that it translates all difficulties coming from the "second Weyl canonical commutation relations" to the oscillator one, which are now well-known.

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