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LIMITING DISTRIBUTIONS OF GALTON-WATSON BRANCHING PROCESSES WITH IMMIGRATION

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Abstract. In this paper, we introduce a decomposition of the transition probability matrix of Galton-Watson (G-W) branching processes with immigration and applying the decomposition, we give a limiting distribution of the process. Moreover, we extend the definition of this process to that of the multiple G-W branching process, and give an explicit form of a limiting distribution of the multiple G-W branching process.

1. Introduction

The Galton-Watson (G-W) branching process has many applications, and the study of this process has a long history [4]. The G-W branching process with a single immigration component has been considered by Heathcote [5, 6]. It is well known that this process is a Markov chain on $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}$. He has given a necessary and sufficient condition for the existence of the limiting distribution of this process in [5, 6]. Furthermore, this process was studied by E. Seneta [8, 9], A. G. Pakes [7], et al. in the later of these two papers. However, the concrete structure of the transition probability matrix and a formula of the limiting distribution if it exists was not known.

In [10, 11], we found the limiting distribution in the case of the Bernoulli type G-W branching process with immigration. This process is the simplest case of the G-W branching process with immigration. In the case of the Bernoulli type, we can decompose the probability matrix to the upper triangular matrix by using the matrix $\boldsymbol{S} = (\binom{i}{j})_{i,j\in\mathbb{N}_0}$ and its inverse matrix $\boldsymbol{S}^{-1}$, where $\binom{i}{j}$ is the binomial coefficent. In the general case, the probability matrix can be also decomposed by the same matrices, namely, these matrices do not depend on the condition of the process. Therefore, these two matrices $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$ are very important. Moreover, the upper triangular matrix has an interesting characteristic property (Theorem 3.5).

The main purpose of this paper is to give the formula of a limiting distribution and the stationary distribution of the G-W branching process with immigration (Theorem 4.4). Moreover, we give extended definitions of this process (Definition 5.1 and 5.2). We apply the derived theorems related to the G-W branching process with immigration to the extended process.

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This paper is organized as follows. In Section 2, we explain the definition of the G-W branching process with immigration. In Section 3, we decompose the probability matrix of this process (Theorem 3.4). In Section 4, we derive the limiting distribution by using the decompositon of the probability matrix. In section 5, we extend the definition of the G-W branching process with immigration to the multiple G-W branching process and give a limiting distribution of the multiple G-W branching process.

2. Galton-Watson Branching Processes with Immigration

In this section, we explain the definition and notation of Galton-Watson (G-W) branching processes with immigration. Let $\{X_{m,n}\,;\, m \in \mathbb{N},\ n \in \mathbb{N}_0\}$ and $\{I_n\,;\, n \in \mathbb{N}_0\}$ be $\mathbb{N}_0$-valued independent and identically distributed (i.i.d.) random variables, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$. These random variables may be simply written as $\{X_{m,n}\}$ and $\{I_n\}$. Probability distributions of $\{X_{m,n}\}$ and $\{I_n\}$ are denoted by $\boldsymbol{p}_X = (p_i)_{i\in\mathbb{N}_0}$ and $\boldsymbol{p}_I = (q_i)_{i\in\mathbb{N}_0}$, respectively, i.e., for $k, n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, $p_k = P(X_{m,n} = k)$ and $q_k = P(I_n = k)$. For any $n \in \mathbb{N}_0$, Y_n is defined as follows:

$$Y_{n+1} = \sum_{i=1}^{Y_n} X_{i,n} + I_n, \qquad Y_0 = x_0, \tag{2.1}$$

where x_0 is an initial state. The initial state is an $\mathbb{N}_0$-valued random variable which is independent of $\{X_{m,n}\}$ and $\{I_n\}$. Moreover, the probability distribution of x_0 is denoted by $\boldsymbol{\pi}_0$ which is called the initial state probability distribution. Then, the discrete stochastic process $\{Y_n\}_{n\in\mathbb{N}_0}$ is called the G-W branching process with immigration.

This process is applied to the investigation of the fluctuations of population. Then, $X_{m,n}$ represents the number of offsprings of the m-th induvidual in the n-th generation, and I_n represents the number of immigrants. The stochastic process $\{\hat{Y}_n\}_{n\in\mathbb{N}_0}$ is defined by

$$\hat{Y}_{n+1} = \sum_{i=1}^{\hat{Y}_n} X_{i,n}, \qquad \hat{Y}_0 = x_0,$$

and is simply called the G-W branching process. This process corresponds to the G-W branching process with immigration such that $q_0 = 1$. Therefore, this process is included in processes with immigration.

It is well known that the following lemma holds (see e.g. [3]):

Lemma 2.1. *Let A and B denote $\mathbb{N}_0$-valued independent random variables. Then the probability distribution of $S = A + B$ is given by the convolution of the probability distributions of A and B, i.e., $\boldsymbol{p}_S = \boldsymbol{p}_A * \boldsymbol{p}_B$.*

From this result, we see that for any $n \in \mathbb{N}_0$, $P(Y_{n+1} = j | Y_n = i)$ is given by the j-th entry of the vector $\underbrace{\boldsymbol{p}_X * \cdots * \boldsymbol{p}_X}_{i} * \boldsymbol{p}_I$. It means that the process $\{Y_n\}_{n\in\mathbb{N}_0}$ is the Markov chain. In general, the state space of this process is $\mathcal{S} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, i.e., for any $n \in \mathbb{N}_0$, $Y_n \in \mathcal{S}$. Thus the size of probability matrix is ∞

by ∞. To relate numbers of rows and columns of the probability matrix to states, these are numbered from zero:

$$\boldsymbol{P} = (P_{ij})_{i,j\in\mathcal{S}} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ i \\ \vdots \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & \cdots & j & \cdots \end{array} \\ \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0j} & \cdots \\ P_{10} & P_{11} & \cdots & P_{1j} & \cdots \\ \vdots & \vdots & & \vdots & \cdots \\ P_{i0} & P_{i1} & \cdots & P_{ij} & \cdots \\ \vdots & \vdots & & \vdots & \ddots \end{pmatrix} \end{array}, \tag{2.2}$$

where $P_{ij} = P(Y_{n+1} = j | Y_n = i)$. In this paper, boldface capital letters are used to denote matrices. Similarly, boldface small letters are used to denote row vectors. If $q_0 = 1$, then the state 0 becomes the absorbing state, i.e., $P(Y_{n+1} = 0 | Y_n = 0) = 1$. In other cases, this process does not have any absorbing state.

3. Decomposition of Probability Matrices

Now, we describe basic properties of the discrete convolution. Let $\mathbb{R}^{\mathbb{N}_0} = \{(f_i)_{i\in\mathbb{N}_0} |\ f_i \in \mathbb{R},\ i \in \mathbb{N}_0\}$. For any $\boldsymbol{f} = (f_i)_{i\in\mathbb{N}_0}$ and $\boldsymbol{g} = (g_i)_{i\in\mathbb{N}_0}$ in $\mathbb{R}^{\mathbb{N}_0}$, the n-th entry of the vector $\boldsymbol{f} * \boldsymbol{g} \in \mathbb{R}^{\mathbb{N}_0}$ is defined by

$$(f * g)_n = \sum_{k=0}^{n} f_k g_{n-k},\ n = 0, 1, 2, \ldots .$$

Therefore, if we set

$$\boldsymbol{F} = (F_{ij})_{i,j\in\mathbb{N}_0} = \begin{pmatrix} f_0 & f_1 & f_2 & \cdots \\ 0 & f_0 & f_1 & \cdots \\ 0 & 0 & f_0 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}, \qquad F_{ij} = \begin{cases} f_{j-i}, & \text{if } i \le j, \\ 0, & \text{otherwise,} \end{cases}$$

then

$$\boldsymbol{f} * \boldsymbol{g} = \boldsymbol{g}\boldsymbol{F} = \boldsymbol{g} \begin{pmatrix} \boldsymbol{f}\,{}^t\boldsymbol{U}^0 \\ \boldsymbol{f}\,{}^t\boldsymbol{U}^1 \\ \boldsymbol{f}\,{}^t\boldsymbol{U}^2 \\ \vdots \end{pmatrix} = \boldsymbol{g} \begin{pmatrix} \boldsymbol{J}_0{}^t\boldsymbol{f} & \boldsymbol{J}_1{}^t\boldsymbol{f} & \boldsymbol{J}_2{}^t\boldsymbol{f} & \cdots \end{pmatrix},$$

where the matrix $\boldsymbol{U}$ and matrices $\boldsymbol{J}_n,\ n = 0, 1, 2, \ldots$, are given as follows:

$$\boldsymbol{U} = (U_{ij})_{i,j\in\mathbb{N}_0} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix}, \qquad U_{ij} = \begin{cases} 1, & \text{if } i = j-1, \\ 0, & \text{otherwise,} \end{cases}$$

$$\boldsymbol{J}_n = \left(\begin{array}{c|c} \tilde{\boldsymbol{J}}_n & \boldsymbol{O} \\ \hline \boldsymbol{O} & \boldsymbol{O} \end{array}\right), \quad \tilde{\boldsymbol{J}}_0 = (1),\ \tilde{\boldsymbol{J}}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \tilde{\boldsymbol{J}}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ldots .$$

The matrix ${}^t\boldsymbol{U}$ and the vector ${}^t\boldsymbol{f}$ are transposes of the matrix $\boldsymbol{U}$ and vector $\boldsymbol{f}$, respectively. Note that for any matrix $\boldsymbol{X} = (X_{ij})_{i,j\in\mathbb{N}_0}$, rows or columns of $\boldsymbol{X}$ is shifted by $\boldsymbol{U}$:

$$\begin{aligned}
\boldsymbol{UX} &= \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\
{}^t\boldsymbol{UX} &= \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ X_{30} & X_{31} & X_{32} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\
\boldsymbol{XU} &= \begin{pmatrix} X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ \vdots & \ddots & \ddots & \ddots \end{pmatrix} = \begin{pmatrix} X_{01} & X_{02} & X_{03} & \cdots \\ X_{11} & X_{12} & X_{13} & \cdots \\ X_{21} & X_{22} & X_{23} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \\
\boldsymbol{X}{}^t\boldsymbol{U} &= \begin{pmatrix} X_{00} & X_{01} & X_{02} & \cdots \\ X_{10} & X_{11} & X_{12} & \cdots \\ X_{20} & X_{21} & X_{22} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ 0 & 0 & 0 & \ddots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} 0 & X_{00} & X_{01} & \cdots \\ 0 & X_{10} & X_{11} & \cdots \\ 0 & X_{20} & X_{21} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.
\end{aligned}$$

Moreover, we have

$$\boldsymbol{U}{}^t\boldsymbol{U} = \left(\begin{array}{c|ccc} 0 & 0 & 0 & \cdots \\ \hline 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{array}\right) = \left(\begin{array}{c|c} 0 & \boldsymbol{o} \\ \hline {}^t\boldsymbol{o} & \boldsymbol{I} \end{array}\right), \quad {}^t\boldsymbol{UU} = \begin{pmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} = \boldsymbol{I},$$

where $\boldsymbol{o}$ is the zero vector and $\boldsymbol{I}$ is the identity matrix. Then we call $\boldsymbol{F}$ the convolution matrix of $\boldsymbol{f}$. The n-th convolution of $\boldsymbol{f}$ is denoted by $\boldsymbol{f}^{n*}$, i.e.,

$$\boldsymbol{f}^{n*} = \underbrace{\boldsymbol{f} * \cdots * \boldsymbol{f}}_{n},$$

where $\boldsymbol{f}^{0*} \equiv (1\ 0\ 0\ \cdots)$. Thus, if $\boldsymbol{\delta} = \boldsymbol{f}^{0*}$, then it can be expressed by

$$\boldsymbol{f}^{n*} = \boldsymbol{\delta F}^n,\ n = 1, 2, \ldots .$$

The convolution matrix of $\boldsymbol{\delta}$ is the identity matrix $\boldsymbol{I}$. It is known that the convolution has following basic properties:

Lemma 3.1. *Let* $\boldsymbol{f} = (f_i)_{i\in\mathbb{N}_0}$, $\boldsymbol{g} = (g_i)_{i\in\mathbb{N}_0}$, $\boldsymbol{h} = (h_i)_{i\in\mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0}$ *and* $\alpha \in \mathbb{R}$. *Then the following properties hold:*

(1) *Commutativity:* $\boldsymbol{f} * \boldsymbol{g} = \boldsymbol{g} * \boldsymbol{f}$.
(2) *Associativity:* $\boldsymbol{f} * (\boldsymbol{g} * \boldsymbol{h}) = (\boldsymbol{f} * \boldsymbol{g}) * \boldsymbol{h}$.

(3) *Distributivity:* $\boldsymbol{f} * (\boldsymbol{g} + \boldsymbol{h}) = \boldsymbol{f} * \boldsymbol{g} + \boldsymbol{f} * \boldsymbol{h}$.
(4) *Scalar Product:* $\alpha(\boldsymbol{f} * \boldsymbol{g}) = (\alpha\boldsymbol{f}) * \boldsymbol{g} = \boldsymbol{f} * (\alpha\boldsymbol{g})$.

From this lemma, we see that the product of two convolution matrices is commutative. Moreover, we have

$$(\alpha\boldsymbol{f} + \beta\boldsymbol{g})^{n*} = \sum_{k=0}^{n} \tbinom{n}{k} \alpha^k \beta^{n-k} \boldsymbol{f}^{k*} \boldsymbol{g}^{(n-k)*}, \quad \alpha, \beta \in \mathbb{R},$$

where $\binom{a}{b}$ is the binomial coefficient, i.e., for $a, b \in \mathbb{N}_0$,

$$\tbinom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!}, & \text{if } 0 < b < a, \\ 1, & \text{if } 0 \le a = b \text{ or } 0 < a \text{ and } b = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Probability matrices of G-W branching processes with immigration can be expressed as follows:*

$$\boldsymbol{P} = \left(\sum_{k=0}^{\infty} \boldsymbol{E}_{k0} \boldsymbol{P}_X^k \right) \boldsymbol{P}_I, \qquad \boldsymbol{E}_{ij} \equiv \begin{matrix} \\ i \end{matrix} \begin{pmatrix} & \overset{j}{\vdots} & \\ \cdots & 1 & \\ & & \end{pmatrix}, \tag{3.1}$$

where $\boldsymbol{P}_X$ and $\boldsymbol{P}_I$ are convolution matrices of $\boldsymbol{p}_X$ and $\boldsymbol{p}_I$, respectively.

Proof. See [11]. □

Here the limit of a sequence $A_N = (a_{i,j}^{[N]})_{i,j \in \mathbb{N}_0} \in \mathbb{R}^{\mathbb{N}_0} \times \mathbb{R}^{\mathbb{N}_0}$, $N = 1, 2, \dots$ is defined by

$$\lim_{N \to \infty} A_N = (\lim_{N \to \infty} a_{i,j}^{[N]})_{i,j \in \mathbb{N}_0}$$

if the limit exists.

Now, we define two matrices $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$:

$$S_{ij} = \begin{cases} \binom{i}{j}, & \text{if } j \le i, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad S_{ij}^{-1} = \begin{cases} (-1)^{i+j} \binom{i}{j}, & \text{if } j \le i, \\ 0, & \text{otherwise.} \end{cases}$$

The matrix $\boldsymbol{S}^{-1}$ is the inverse matrix of $\boldsymbol{S}$, because if $i \ge j$, the (i, j) entry of the product $\boldsymbol{S}\boldsymbol{S}^{-1}$ is expressed as follows:

$$\begin{aligned} (SS^{-1})_{ij} &= \sum_{k=0}^{i} S_{ik} S_{kj}^{-1} = \sum_{k=j}^{i} \tbinom{i}{k} (-1)^{j+k} \tbinom{k}{j} = \sum_{k=j}^{i} (-1)^{j+k} \frac{i!}{k!(i-k)!} \frac{k!}{j!(k-j)!} \\ &= \sum_{k=j}^{i} (-1)^{j+k} \frac{i!}{(i-k)!} \frac{1}{j!(k-j)!} = \sum_{k=j}^{i} (-1)^{j+k} \frac{i!}{j!(i-j)!} \frac{(i-j)!}{(i-k)!(k-j)!} \\ &= \sum_{k=j}^{i} (-1)^{j+k} \frac{i!}{j!(i-j)!} \frac{(i-j)!}{(i-k)!(k-j)!} = \tbinom{i}{j} \sum_{k=j}^{i} (-1)^{j+k} \tbinom{i-j}{k-j}. \end{aligned}$$

It is clear that if $i < j$ then $(SS^{-1})_{ij} = 0$. Then from (3.1) and the Fubini theorem, we obtain

$$\boldsymbol{S}^{-1}\boldsymbol{P} = \boldsymbol{S}^{-1}\left(\sum_{k=0}^{\infty}\boldsymbol{E}_{k0}\boldsymbol{P}_X^k\right)\boldsymbol{P}_I = \sum_{k=0}^{\infty}\boldsymbol{S}^{-1}\boldsymbol{E}_{k0}\boldsymbol{P}_X^k\boldsymbol{P}_I.$$

In the aboveCfor any $k \in \mathbb{N}_0$, we have

$$\boldsymbol{S}^{-1}\boldsymbol{E}_{k0} = \begin{array}{c} \\ 0 \\ \vdots \\ k-1 \\ k \\ k+1 \\ \vdots \end{array}\begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & \cdots \end{array} \\ \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots \\ S_{kk}^{-1} & 0 & 0 & \cdots \\ S_{k+1k}^{-1} & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}\end{array} = \sum_{\ell=k}^{\infty} S_{\ell k}^{-1}\boldsymbol{E}_{\ell 0}.$$

Thus, the matrix $\boldsymbol{S}^{-1}\boldsymbol{P}$ is expressed by

$$\boldsymbol{S}^{-1}\boldsymbol{P} = \sum_{k=0}^{\infty}\sum_{\ell=k}^{\infty} S_{\ell k}^{-1}\boldsymbol{E}_{\ell 0}\boldsymbol{P}_X^k\boldsymbol{P}_I = \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \\ \vdots \end{array}\begin{pmatrix} \sum_{k=0}^{0} S_{0k}^{-1}\boldsymbol{p}_X^{k*} * \boldsymbol{p}_I \\ \sum_{k=0}^{1} S_{1k}^{-1}\boldsymbol{p}_X^{k*} * \boldsymbol{p}_I \\ \vdots \\ \sum_{k=0}^{i} S_{ik}^{-1}\boldsymbol{p}_X^{k*} * \boldsymbol{p}_I \\ \vdots \end{pmatrix}.$$

The i-th row of this matrix is expressed as follows:

$$\begin{aligned} \sum_{k=0}^{i} S_{ik}^{-1}\boldsymbol{p}_X^{k*} * \boldsymbol{p}_I &= \left(\sum_{k=0}^{i}(-1)^{i+k}\binom{i}{k}\boldsymbol{p}_X^{k*}\right) * \boldsymbol{p}_I = (-1)^i\left(\sum_{k=0}^{i}(-1)^k\binom{i}{k}\boldsymbol{\delta}\boldsymbol{P}_X^k\right) * \boldsymbol{p}_I \\ &= (-1)^i\boldsymbol{\delta}\left(\sum_{k=0}^{i}\binom{i}{k}\boldsymbol{I}^{i-k}(-1)^k\boldsymbol{P}_X^k\right) * \boldsymbol{p}_I = (-1)^i\boldsymbol{\delta}(\boldsymbol{I} - \boldsymbol{P}_X)^i * \boldsymbol{p}_I \\ &= \boldsymbol{\delta}(\boldsymbol{P}_X - \boldsymbol{I})^i * \boldsymbol{p}_I = (\boldsymbol{p}_X - \boldsymbol{\delta})^{i*} * \boldsymbol{p}_I. \end{aligned}$$

Therefore, we have

$$\boldsymbol{S}^{-1}\boldsymbol{P} = \begin{array}{c} 0 \\ 1 \\ \vdots \\ i \\ \vdots \end{array}\begin{pmatrix} (\boldsymbol{p}_X - \boldsymbol{\delta})^{0*} * \boldsymbol{p}_I \\ (\boldsymbol{p}_X - \boldsymbol{\delta})^{1*} * \boldsymbol{p}_I \\ \vdots \\ (\boldsymbol{p}_X - \boldsymbol{\delta})^{i*} * \boldsymbol{p}_I \\ \vdots \end{pmatrix}.$$

Here, we set $\tilde{\boldsymbol{p}} = \boldsymbol{p}_X - \boldsymbol{\delta}$, and the convolution matrix of $\tilde{\boldsymbol{p}}$ is denoted by $\tilde{\boldsymbol{P}}$, i.e.,

$$\tilde{\boldsymbol{p}} = \begin{pmatrix} p_0 - 1 & p_1 & p_2 & \cdots \end{pmatrix}, \qquad \tilde{\boldsymbol{P}} = \begin{pmatrix} p_0 - 1 & p_1 & p_2 & \cdots \\ 0 & p_0 - 1 & p_1 & \cdots \\ 0 & 0 & p_0 - 1 & \ddots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

Then the (i,j) entry λ_{ij} of the matrix $\boldsymbol{\Lambda} = \boldsymbol{S}^{-1}\boldsymbol{P}\boldsymbol{S}$ is given as follows:

$$\lambda_{ij} = (\tilde{\boldsymbol{p}}^{i*} * \boldsymbol{p}_I)\boldsymbol{S}_j^{\text{column}} = \boldsymbol{\delta}\boldsymbol{P}_I\tilde{\boldsymbol{P}}^i\boldsymbol{S}_j^{\text{column}} = \langle {}^t\boldsymbol{\delta}, \boldsymbol{P}_I\tilde{\boldsymbol{P}}^i\boldsymbol{S}_j^{\text{column}}\rangle, \tag{3.2}$$

where $\boldsymbol{S}_j^{\text{column}}$ means the j-th column of the matrix $\boldsymbol{S}$, i.e.,

$$\boldsymbol{S}_j^{\text{column}} = {}^t(S_{0j} \;\; S_{1j} \;\; S_{2j} \;\; \cdots) = {}^t(\tbinom{0}{j} \;\; \tbinom{1}{j} \;\; \tbinom{2}{j} \;\; \cdots).$$

Both $\boldsymbol{S}$ and $\boldsymbol{S}^{-1}$ do not depend on $\boldsymbol{p}_X$ and $\boldsymbol{p}_I$.

For $\boldsymbol{f} = {}^t(f_0\ f_1\ \cdots)$ and $\boldsymbol{g} = {}^t(g_0\ g_1\ \cdots)$ in $\mathbb{R}^{\mathbb{N}_0}$, $\langle \boldsymbol{f}, \boldsymbol{g}\rangle$ is defined as follows:

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle = {}^t\boldsymbol{f}\boldsymbol{g} = \sum_{k=0}^{\infty} f_k g_k$$

if it exists in $\mathbb{R}$.

Lemma 3.3. *The column vector $\tilde{\boldsymbol{P}}\boldsymbol{S}_j^{\text{column}}$ can be expressed by*

$$\tilde{\boldsymbol{P}}\boldsymbol{S}_j^{\text{column}} = \begin{cases} {}^t\boldsymbol{o}, & \text{if } j = 0, \\ \sum_{k=0}^{j-1}\langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_{j-k}^{\text{column}}\rangle \boldsymbol{S}_k^{\text{column}}, & \text{if } j = 1, 2, \ldots, \end{cases} \tag{3.3}$$

if the vector exists in $\mathbb{R}^{\mathbb{N}_0}$.

Proof. We first prove the case of $j = 0$. Then, we have

$$\tilde{\boldsymbol{P}}\boldsymbol{S}_0^{\text{column}} = \begin{pmatrix} \sum_{k=0}^{\infty} p_k - 1 \\ \sum_{k=0}^{\infty} p_k - 1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 1-1 \\ 1-1 \\ \vdots \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \end{pmatrix}.$$

Next, we prove the case of $j \geq 1$ by induction. If $j = 1$, then we have

$$\begin{aligned}
\tilde{\boldsymbol{P}}\boldsymbol{S}_1^{\text{column}} &= \begin{pmatrix} p_0 - 1 & p_1 & p_2 & \cdots \\ 0 & p_0 - 1 & p_1 & \cdots \\ 0 & 0 & p_0 - 1 & \cdots \\ \vdots & \vdots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 0 \\ \binom{1}{1} \\ \binom{2}{1} \\ \vdots \end{pmatrix} \\
&= \begin{matrix} 0 \\ \vdots \\ i \\ \vdots \end{matrix} \begin{pmatrix} \sum_{k=0}^{\infty} kp_k \\ \vdots \\ (\sum_{k=0}^{\infty} kp_k) + i(\sum_{k=0}^{\infty} p_k) - i \\ \vdots \end{pmatrix} = \begin{pmatrix} \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \\ \vdots \\ \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle + i - i \\ \vdots \end{pmatrix} \\
&= \begin{pmatrix} \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \\ \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \\ \vdots \end{pmatrix} = \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix} = \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \boldsymbol{S}_0^{\text{column}}.
\end{aligned}$$

Assume that in the case of $j < n$, (3.3) holds. By the definition of binomial coefficients, it is clear that the column vector $\boldsymbol{S}_j^{\text{column}}$ can be expressed as follows:

$$\boldsymbol{S}_j^{\text{column}} = \sum_{k=1}^{\infty} \boldsymbol{U}^k \boldsymbol{S}_{j-1}^{\text{column}}, \; j = 1, 2, \ldots.$$

From these conditions and the Fubini theorem, we obtain

$$\begin{aligned}
\tilde{\boldsymbol{P}}\boldsymbol{S}_{n+1}^{\text{column}} &= \tilde{\boldsymbol{P}}\sum_{k=1}^{\infty}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}} = \sum_{k=1}^{\infty}\tilde{\boldsymbol{P}}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}} = \sum_{k=1}^{\infty}\begin{pmatrix}\tilde{\boldsymbol{p}}\,{}^t\boldsymbol{U}^0\\ \tilde{\boldsymbol{p}}\,{}^t\boldsymbol{U}^1\\ \tilde{\boldsymbol{p}}\,{}^t\boldsymbol{U}^2\\ \vdots\end{pmatrix}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}}\\
&= \sum_{k=1}^{\infty}\begin{pmatrix}\tilde{\boldsymbol{p}}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}}\\ \tilde{\boldsymbol{p}}\boldsymbol{U}^{k-1}\boldsymbol{S}_n^{\text{column}}\\ \vdots\\ \tilde{\boldsymbol{p}}\boldsymbol{U}\boldsymbol{S}_n^{\text{column}}\\ \hline \tilde{\boldsymbol{P}}\boldsymbol{S}_n^{\text{column}}\end{pmatrix} = \sum_{k=1}^{\infty}\left(\begin{pmatrix}\boldsymbol{o}\\ \boldsymbol{o}\\ \vdots\\ \boldsymbol{o}\\ \hline \tilde{\boldsymbol{P}}\boldsymbol{S}_n^{\text{column}}\end{pmatrix} + \begin{pmatrix}\tilde{\boldsymbol{p}}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}}\\ \tilde{\boldsymbol{p}}\boldsymbol{U}^{k-1}\boldsymbol{S}_n^{\text{column}}\\ \vdots\\ \tilde{\boldsymbol{p}}\boldsymbol{U}\boldsymbol{S}_n^{\text{column}}\\ \hline \boldsymbol{O}\end{pmatrix}\right)\\
&= \sum_{k=1}^{\infty}\left(\boldsymbol{U}^k\tilde{\boldsymbol{P}}\boldsymbol{S}_n^{\text{column}} + (\tilde{\boldsymbol{p}}\boldsymbol{U}^k\boldsymbol{S}_n^{\text{column}})\boldsymbol{S}_0^{\text{column}}\right)\\
&= \sum_{k=1}^{\infty}\left(\boldsymbol{U}^k\tilde{\boldsymbol{P}}\boldsymbol{S}_n^{\text{column}}\right) + \left(\tilde{\boldsymbol{p}}\sum_{\ell=1}^{\infty}\boldsymbol{U}^\ell\boldsymbol{S}_n^{\text{column}}\right)\boldsymbol{S}_0^{\text{column}}\\
&= \sum_{k=1}^{\infty}\left(\boldsymbol{U}^k\sum_{\ell=0}^{n-1}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n-\ell}^{\text{column}}\rangle\boldsymbol{S}_\ell^{\text{column}}\right) + \left(\tilde{\boldsymbol{p}}\boldsymbol{S}_{n+1}^{\text{column}}\right)\boldsymbol{S}_0^{\text{column}}\\
&= \left(\sum_{\ell=0}^{n-1}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n-\ell}^{\text{column}}\rangle\sum_{k=1}^{\infty}\boldsymbol{U}^k\boldsymbol{S}_\ell^{\text{column}}\right) + \langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n+1}^{\text{column}}\rangle\boldsymbol{S}_0^{\text{column}}\\
&= \left(\sum_{\ell=0}^{n-1}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n-\ell}^{\text{column}}\rangle\boldsymbol{S}_{\ell+1}^{\text{column}}\right) + \langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n+1}^{\text{column}}\rangle\boldsymbol{S}_0^{\text{column}}\\
&= \left(\sum_{\ell=1}^{n}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n+1-\ell}^{\text{column}}\rangle\boldsymbol{S}_{\ell}^{\text{column}}\right) + \langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n+1}^{\text{column}}\rangle\boldsymbol{S}_0^{\text{column}}\\
&= \sum_{k=0}^{n}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{n-k}^{\text{column}}\rangle\boldsymbol{S}_k^{\text{column}}.
\end{aligned}$$

Thus the proof is completed. □

Theorem 3.4. *Let $\boldsymbol{P}$ be the transition probability matrix of the G-W branching process with immigration and let $\boldsymbol{\Lambda}$ be the matrix of which the (i,j) entry λ_{ij} is given by*

$$\lambda_{ij} = \begin{cases}\langle{}^t\boldsymbol{p}_I,\boldsymbol{S}_j^{\text{column}}\rangle, & \text{if } i=0,\\ \sum_{k=i-1}^{j-1}\langle{}^t\tilde{\boldsymbol{p}},\boldsymbol{S}_{j-k}^{\text{column}}\rangle\lambda_{i-1k}, & \text{if } i\geq 1 \text{ and } j\geq i,\\ 0, & \text{otherwise,}\end{cases} \tag{3.4}$$

where λ_{ij} exists in $\mathbb{R}$ for each i,j. Then the matrix $\boldsymbol{P}$ can be decomposed as follows:

$$\boldsymbol{P} = \boldsymbol{S}\boldsymbol{\Lambda}\boldsymbol{S}^{-1}. \tag{3.5}$$

Proof. Consider the matrix $\boldsymbol{\Lambda} = \boldsymbol{S}^{-1}\boldsymbol{P}\boldsymbol{S}$. From (3.2), we see that

$$\lambda_{ij} = \langle {}^t\boldsymbol{\delta}, \boldsymbol{P}_I \tilde{\boldsymbol{P}}^i \boldsymbol{S}_j^{\text{column}} \rangle = \langle {}^t\boldsymbol{p}_I, \tilde{\boldsymbol{P}}^i \boldsymbol{S}_j^{\text{column}} \rangle.$$

Thus in the case of $i = 0$, we have

$$\lambda_{0j} = \langle {}^t\boldsymbol{p}_I, \tilde{\boldsymbol{P}}^0 \boldsymbol{S}_j^{\text{column}} \rangle = \langle {}^t\boldsymbol{p}_I, \boldsymbol{S}_j^{\text{column}} \rangle. \tag{3.6}$$

In particular, $\lambda_{00} = \sum_{k=0}^{\infty} q_k = 1$. From Lemma 3.3, we have $\tilde{\boldsymbol{P}}\boldsymbol{S}_0^{\text{column}} = \boldsymbol{o}$. Hence, for any $i \geq 1$, we obtain

$$\lambda_{i0} = \langle {}^t\boldsymbol{p}_I, \tilde{\boldsymbol{P}}^i \boldsymbol{S}_0^{\text{column}} \rangle = \langle {}^t\boldsymbol{p}_I, \boldsymbol{o} \rangle = 0.$$

Moreover, from Lemma 3.3, λ_{ij} can be expressed as follows:

$$\begin{aligned}\lambda_{ij} &= \langle {}^t\boldsymbol{p}_I, \tilde{\boldsymbol{P}}^i \boldsymbol{S}_j^{\text{column}} \rangle = \boldsymbol{p}_I \tilde{\boldsymbol{P}}^i \boldsymbol{S}_j^{\text{column}} = \boldsymbol{p}_I \tilde{\boldsymbol{P}}^{i-1} \sum_{k=0}^{j-1} \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_{j-k}^{\text{column}} \rangle \boldsymbol{S}_k^{\text{column}} \\ &= \sum_{k=0}^{j-1} c_{j-k} \boldsymbol{p}_I \tilde{\boldsymbol{P}}^{i-1} \boldsymbol{S}_k^{\text{column}} = \sum_{k=0}^{j-1} c_{j-k} \lambda_{i-1k}.\end{aligned}$$

In the case of $i \geq 1$, it is proved by induction. From Lemma 3.3, if $i = 1$, then we have

$$\begin{aligned}\lambda_{1j} &= \langle {}^t\boldsymbol{p}_I, \tilde{\boldsymbol{P}} \boldsymbol{S}_j^{\text{column}} \rangle = \langle {}^t\boldsymbol{p}_I, \sum_{k=0}^{j-1} \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_{j-k}^{\text{column}} \rangle \boldsymbol{S}_k^{\text{column}} \rangle \\ &= \sum_{k=0}^{j-1} \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_{j-k}^{\text{column}} \rangle \langle {}^t\boldsymbol{p}_I, \boldsymbol{S}_k^{\text{column}} \rangle.\end{aligned}$$

From (3.6), λ_{1j} can be expressed as follows:

$$\lambda_{1j} = \begin{cases} \sum_{k=0}^{j-1} c_{j-k} \lambda_{0k}, & \text{if } j \geq 1, \\ 0, & \text{otherwise,} \end{cases}$$

with $c_k \equiv \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_k^{\text{column}} \rangle$. Assume that in the case of $i < n-1$, λ_{ij} is given by (3.4). Then if $i = n$, we have

$$\lambda_{n-10} = \lambda_{n-11} = \cdots = \lambda_{n-1n-2} = 0.$$

Thus for any $j \leq n-1$, $\lambda_{nj} = 0$ because

$$\lambda_{nj} = \sum_{k=0}^{j-1} c_{j-k} \lambda_{n-1k} = \sum_{k=0}^{j-1} c_{j-k} 0 = 0.$$

Moreover, for any $j > n-1$, we obtain

$$\begin{aligned}\lambda_{nj} &= \sum_{k=0}^{j-1} c_{j-k} \lambda_{n-1k} = \sum_{\ell=0}^{n-2} c_{j-\ell} \lambda_{n-1\ell} + \sum_{m=n-1}^{j-1} c_{j-m} \lambda_{n-1m} \\ &= \sum_{\ell=0}^{n-2} c_{j-\ell} 0 + \sum_{m=n-1}^{j-1} c_{j-m} \lambda_{n-1m} = \sum_{m=n-1}^{j-1} c_{j-m} \lambda_{n-1m}.\end{aligned}$$

It means that the (i, j) entry of $\boldsymbol{\Lambda}$ is given by (3.4). Thus the proof is completed. □

Theorem 3.5. *Diagonal entries of the upper triangular matrix* $\boldsymbol{\Lambda}$ *are given by*

$$\lambda_{ii} = \mathrm{E}[X]^i, \ i = 0, 1, 2, \ldots, \tag{3.7}$$

where X is a random variable that has the probability distribution $\boldsymbol{p}_X$, i.e., $P(X = i) = p_i$ for every $i \in \mathbb{N}_0$.

Proof. From Theorem 3.4, we have

$$\lambda_{ii} = \sum_{k=i-1}^{i-1} c_{i-k}\lambda_{i-1k} = c_1\lambda_{i-1i-1} = \langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle \lambda_{i-1i-1}, \ i = 1, 2, \cdots .$$

Since $\lambda_{00} = 1 = \mathrm{E}[X]^0$ and $\langle {}^t\tilde{\boldsymbol{p}}, \boldsymbol{S}_1^{\text{column}}\rangle = \mathrm{E}[X]$, we obtain (3.7). □

4. The Limiting Distribution

Let $\boldsymbol{\pi}(n)$ be the probability distribution of Y_n given as in (2.1). If $\lim_{n\to\infty} \boldsymbol{\pi}(n)$ exists, then it is called the limiting distribution. Moreover, the probability distribution which satisfies the condition $\boldsymbol{\pi} = \boldsymbol{\pi}\boldsymbol{P}$ is called the stationary distribution. In this section, we derive the limiting distribution of G-W branching processes with immigration.

First, we introduce a generating function and its properties. Using these properties, we find the limiting distribution of the G-W branching process with immigration.

Definition 4.1. The generating function $\mathcal{G}[x_n]$ of a real sequence $\{x_n\}_{n\in\mathbb{N}_0}$ is defined by

$$\mathcal{G}[x_n](s) \equiv \sum_{n=0}^{\infty} x_n s^n, \quad s \in \mathbb{C}.$$

The generating function of $\{x_n\}_{n\in\mathbb{N}_0}$ is also denoted by $X(s)$. In general, it is assumed that the value s is included in the region of convergence, i.e., $\sum_{n=0}^{\infty} |x_n s^n| < \infty$. The generating function has following properties:

Lemma 4.2. *Basic properties of the generating function:*

1. *Linearity: For any $\alpha, \beta \in \mathbb{R}$, $\mathcal{G}[\alpha x_n + \beta y_n](s) = \alpha X(s) + \beta Y(s)$.*
2. *Left Shift: For any positive integer k, $\mathcal{G}[x_{n+k}](s) = s^{-k}(X(s) - \sum_{\ell=0}^{k-1} x_\ell s^\ell)$.*
3. *Right Shift: For any non-negative integer k, $\mathcal{G}[x_{n-k}](s) = s^k X(s)$.*
4. *Final Value Theorem: if both $\lim_{s\to 1} X(s)$ and $\lim_{n\to\infty} x_n$ exist, then $\lim_{s\to 1} \frac{1-s}{s} X(s) = \lim_{n\to\infty} x_n$.*

The generating function is also called the z-transform. For this theorem, we refer to [2].

From (3.5), the n-th power of the matrix $\boldsymbol{P}$ is expressed in the form

$$\boldsymbol{P}^n = \boldsymbol{S}\boldsymbol{\Lambda}^n\boldsymbol{S}^{-1}.$$

For any $n \in \mathbb{N}_0$, we have

$$\boldsymbol{\Lambda}^{n+1} = \boldsymbol{\Lambda}^n \boldsymbol{\Lambda} = \begin{pmatrix} \lambda_{00}^{(n)} & \lambda_{01}^{(n)} & \cdots & \lambda_{0i}^{(n)} & \cdots \\ & \lambda_{11}^{(n)} & \cdots & \lambda_{1i}^{(n)} & \cdots \\ & & \ddots & \vdots & \\ & & & \lambda_{ii}^{(n)} & \cdots \\ \boldsymbol{O} & & & & \ddots \end{pmatrix} \begin{pmatrix} \lambda_{00} & \lambda_{01} & \cdots & \lambda_{0i} & \cdots \\ & \lambda_{11} & \cdots & \lambda_{1i} & \cdots \\ & & \ddots & \vdots & \\ & & & \lambda_{ii} & \cdots \\ \boldsymbol{O} & & & & \ddots \end{pmatrix},$$

where $\boldsymbol{\Lambda}^0 \equiv \boldsymbol{I}$. Then, we obtain the following recurrence formula with respect to n:

$$\lambda_{ij}^{(n+1)} = \begin{cases} \sum_{\ell=i}^{j} \lambda_{i\ell}^{(n)} \lambda_{\ell j}, & \text{if } i \le j, \\ 0, & \text{if } i > j, \end{cases} \qquad \lambda_{ij}^{(0)} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

For any $i \in \mathbb{N}_0$, the generating function $\mathcal{G}[\lambda_{ii}^{(n)}](s) \equiv \Lambda_{ii}(s)$ of $\{\lambda_{ii}^{(n)}\}_{n\in\mathbb{N}_0}$ is given as follows: From

$$\mathcal{G}[\lambda_{ii}^{(n+1)}](s) = \mathcal{G}[\lambda_{ii}^{(n)} \lambda_{ii}](s),$$

we have

$$s^{-1}(\Lambda_{ii}(s) - \lambda_{ii}^{(0)}) = \lambda_{ii} \Lambda_{ii}(s).$$

Therefore, since

$$s^{-1}(\Lambda_{ii}(s) - 1) = \mathrm{E}[X]^i \Lambda_{ii}(s),$$

we obtain

$$\Lambda_{ii}(s) = \frac{1}{1 - s\mathrm{E}[X]^i}.$$

Similarly, if $i < j$, then the generating function $\mathcal{G}[\lambda_{ij}^{(n)}](s) \equiv \Lambda_{ij}(s)$ of $\{\lambda_{ij}^{(n)}\}_{n\in\mathbb{N}_0}$ is expressed as follows: Since

$$\mathcal{G}[\lambda_{ij}^{(n+1)}](s) = \mathcal{G}\left[\sum_{\ell=i}^{j} \lambda_{i\ell}^{(n)} \lambda_{\ell j}\right](s),$$

we have

$$s^{-1}(\Lambda_{ij}(s) - \lambda_{ij}^{(0)}) = \sum_{\ell=i}^{j} \Lambda_{i\ell}(s) \lambda_{\ell j},$$

and hence

$$s^{-1}(\Lambda_{ij}(s) - 0) - \Lambda_{ij}(s)\lambda_{jj} = \sum_{\ell=i}^{j-1} \Lambda_{i\ell}(s) \lambda_{\ell j}.$$

Consequently we obtain

$$\Lambda_{ij}(s) = \frac{s}{1 - s\mathrm{E}[X]^j} \sum_{\ell=i}^{j-1} \Lambda_{i\ell}(s) \lambda_{\ell j}.$$

It is obvious that the generating function of $\{\lambda_{ij}^{(n)}\}_{n\in\mathbb{N}_0}$ is 0 in the case of $i > j$. Since $\lambda_{ii} = \mathrm{E}[X]^i$ and $\boldsymbol{\Lambda}$ is the upper triangular matrix, $\lambda_{ii}^{(n)} = \mathrm{E}[X]^{i+n}$.

Therefore, we have

$$\lim_{t\to\infty} \lambda_{ii}^{(n)} = \begin{cases} 0, & \text{if } \mathrm{E}[X] < 1 \text{ and } i \neq 0, \\ 1, & \text{if } \mathrm{E}[X] = 1 \text{ or } i = 0, \\ \infty, & \text{if } \mathrm{E}[X] > 1 \text{ and } i \neq 0. \end{cases}$$

In the case of $i < j$, if both $\lim_{s\to 1} \frac{1-s}{s} \Lambda_{ij}(s)$ and $\lim_{n\to\infty} \lambda_{ij}^{(n)}$ exist, then by the final value theorem in Lemma 4.2, $\lambda_{ij}^{(\infty)} \equiv \lim_{n\to\infty} \lambda_{ij}^{(n)}$ is given as follows:

$$\begin{aligned} \lambda_{ij}^{(\infty)} &= \lim_{s\to 1} \frac{1-s}{s} \Lambda_{ij}(s) = \lim_{s\to 1} \frac{1-s}{s} \frac{s}{1-s\mathrm{E}[X]^j} \sum_{\ell=i}^{j-1} \Lambda_{i\ell}(s)\lambda_{\ell j} \\ &= \lim_{s\to 1} \frac{s}{1-s\mathrm{E}[X]^j} \sum_{\ell=i}^{j-1} \frac{1-s}{s} \Lambda_{i\ell}(s)\lambda_{\ell j} = \frac{1}{1-\mathrm{E}[X]^j} \sum_{\ell=i}^{j-1} \lambda_{i\ell}^{(\infty)} \lambda_{\ell j}. \qquad (4.1) \end{aligned}$$

From this, we see that if $\mathrm{E}[X] \geq 1$, then $\lim_{s\to 1} \frac{1-s}{s} \Lambda_{ij}(s)$ does not exist. In this case, the limiting distribution dose not exist.

Lemma 4.3. *If* $\mathrm{E}[X] < 1$, *then the* (i, j) *entry of the matrix* $\boldsymbol{\Lambda}^\infty \equiv \lim_{n\to\infty} \boldsymbol{\Lambda}^n$ *is expressed as follows:*

$$\lambda_{ij}^{(\infty)} = \begin{cases} 1, & \text{if } i = j = 0, \\ \frac{1}{1-\mathrm{E}[X]^j} \sum_{\ell=0}^{j-1} \lambda_{0\ell}^{(\infty)} \lambda_{\ell j}, & \text{if } i = 0 \text{ and } j \geq 1, \\ 0, & \text{otherwise.} \end{cases} \qquad (4.2)$$

Proof. From (4.1), it is obvious in the case of $i = 0$. Therefore, we prove the case of $i \geq 1$. Since $\boldsymbol{\Lambda}$ is the upper triangular matrix, all lower triangular entries are 0. Other cases are proved by induction. For any fixed $i \geq 1$, if $i = j$ then we have

$$\lambda_{ii}^{(\infty)} = \lim_{s\to 1} \frac{1-s}{s} \Lambda_{ii}(s) = \lim_{s\to 1} \frac{1-s}{s} \frac{1}{1-s\mathrm{E}[X]^i} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0. \end{cases}$$

Assume that $\lambda_{ik}^{(\infty)} = 0$ holds in the case of $k = i, i+1, \ldots, j-1$. Then we obtain

$$\lambda_{ij}^{(\infty)} = \lim_{s\to 1} \frac{1-s}{s} \Lambda_{ij}(s) = \frac{1}{1-\mathrm{E}[X]^j} \sum_{\ell=i}^{j-1} \lambda_{i\ell}^{(\infty)} \lambda_{\ell j} = 0.$$

Thus the proof is completed. □

Theorem 4.4. *If the limiting distribution* $\boldsymbol{\pi}$ *of the G-W branching process with immigration exists, then the* j*-th entry of the vector is expressed as follows:*

$$\pi_j = \sum_{k=j}^{\infty} (-1)^{k+j} \binom{k}{j} \lambda_{0k}^{(\infty)}.$$

Moreover, $\boldsymbol{\pi}$ *is the stationary distribution of the process.*

Proof. Let $\boldsymbol{P}$ be the transition probability matrix of the G-W branching process with immigration. If $\boldsymbol{P}^\infty \equiv \lim_{k\to\infty} \boldsymbol{P}^k$ exists, then it can be expressed as follows:

$$\begin{aligned}\boldsymbol{P}^\infty &= \boldsymbol{S}\boldsymbol{\Lambda}^\infty\boldsymbol{S}^{-1} \\ &= \begin{matrix} \\ 0 \\ 1 \\ \vdots \end{matrix}\begin{matrix} 0 & \cdots & j & \cdots \\ \left(\begin{matrix} \sum_{k=0}^\infty (-1)^k \binom{k}{0}\lambda_{0k}^{(\infty)} & \cdots & \sum_{k=j}^\infty (-1)^{k+j}\binom{k}{j}\lambda_{0k}^{(\infty)} & \cdots \\ \sum_{k=0}^\infty (-1)^k \binom{k}{0}\lambda_{0k}^{(\infty)} & \cdots & \sum_{k=j}^\infty (-1)^{k+j}\binom{k}{j}\lambda_{0k}^{(\infty)} & \cdots \\ \vdots & & \vdots & \end{matrix}\right)\end{matrix}.\end{aligned}$$

Note that for all $i, k \in \mathbb{N}_0$ and fixed $j \in \mathbb{N}_0$, $P_{ij}^{(\infty)} = P_{kj}^{(\infty)}$. Therefore, the limiting distribution is given as follows:

$$\begin{aligned}\boldsymbol{\pi} &\equiv \lim_{n\to\infty} \boldsymbol{\pi}(n) = \lim_{n\to\infty} \boldsymbol{\pi}_0 \boldsymbol{P}^n = \boldsymbol{\pi}_0\boldsymbol{P}^\infty \\ &= \left(\sum_{k=0}^\infty (-1)^k \binom{k}{0}\lambda_{0k}^{(\infty)} \quad \cdots \quad \sum_{k=j}^\infty (-1)^{k+j}\binom{k}{j}\lambda_{0k}^{(\infty)} \quad \cdots \right).\end{aligned}$$

Moreover, $\boldsymbol{\pi}\boldsymbol{P} = \boldsymbol{\pi}$ holds since $\boldsymbol{P}^\infty\boldsymbol{P} = \boldsymbol{P}^\infty$. It means that the limiting distribution $\boldsymbol{\pi}$ is also the stationary distribution of the matrix $\boldsymbol{P}$. □

Remark 4.5. If $\mathrm{E}[X] < 1$, the process $\{Y_n\}_{n\in\mathbb{N}_0}$ is called to be sub-critical. The necessary and sufficient condition for the existence of the limiting distribution in the sub-critical case is given by Heathcote [5, 6]. However so far any detail expression of the limiting distribution has not been obtained. Theorem 4.4 gives an explicit expression of the limiting distribution.

Remark 4.6. Let $\boldsymbol{P}$ be a transition probability matrix given as in (2.2), and let ℓ_1 be the Banach space consisting of real sequences $x = (x_n)_{n=0}^\infty$ satisfying $\sum_{n=0}^\infty |x_n| < \infty$ with the norm $\|x\|_1$ given by $\|x\|_1 = \sum_{n=0}^\infty |x_n|$ for $x = (x_n)_{n=0}^\infty \in \ell_1$. Then $\boldsymbol{P}$ is regarded as a continuous linear operator from ℓ_1 into itself:

$$\ell_1 \ni x \mapsto x\boldsymbol{P} \in \ell_1.$$

5. Extended Processes

In this section, we give extended definitions of G-W branching processes with immigration. Previous theorems can be applied to these extended processes.

Let $X_{m,n}, m \in \mathbb{N}, n \in \mathbb{N}_0$ be $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_X$ of some discrete random variable X. Similarly, $I_{k,n},\ k = 1, 2, \ldots, K,\ n \in \mathbb{N}_0$ are $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_{I_k}$ of some discrete random variable I_k. Assume that for any m, n, the random variable $X_{m,n}$ is independent to every $I_{k,n},\ k \in \mathbb{N}, n \in \mathbb{N}_0$. For any $n \in \mathbb{N}_0$, we define a stochastic process $\{\tilde{Y}_n\}$ by

$$\tilde{Y}_{n+1} = \sum_{k=1}^{\tilde{Y}_n} X_{k,n} + \sum_{\ell=1}^{K} I_{\ell,n}, \qquad \tilde{Y}_0 = x_0,$$

where x_0 is the initial state which independent to $\{X_{m,n}\}$ and $\{I_{k,n}\},\ k = 1, 2, \ldots, K$.

Definition 5.1. The stochastic process $\{\tilde{Y}_n\}_{n\in\mathbb{N}_0}$ is called the *G-W branching process* with multiple immigration.

It is clear that usual G-W branching processes with immigration are the special case of this process. From Lemma 2.1, the probability distribution of $\tilde{I} \equiv \sum_{\ell=1}^{K} I_{\ell,n}$ is given as follows:

$$\boldsymbol{p}_{\tilde{I}} = \boldsymbol{p}_{I_1} * \boldsymbol{p}_{I_2} * \cdots * \boldsymbol{p}_{I_K}.$$

Therefore, this process can be considered as the G-W branching process with immigration for which $\boldsymbol{p}_I = \boldsymbol{p}_{\tilde{I}}$. It means that all theorems related to G-W branching processes with immigration are applicable to this process. Moreover, we extend this process as follows:

Let $X_{k,m,n},\ k = 1,2,\dots,M,\ m \in \mathbb{N},\ n \in \mathbb{N}_0$ be $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_{X_k}$ of a discrete random variable $X_k,\ k = 1,2,\dots,M$. Similarly, let $I_{k,\ell,n},\ \ell = 1,2,\dots,N_k,\ n \in \mathbb{N}_0$ be also $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_{I_{k,\ell}}$ of a discrete random variable $I_{k,\ell,n}$. Assume that all $X_{k,m,n}$ are independent of $I_{k,\ell,n}$ for every k and ℓ. For any $n \in \mathbb{N}_0$ and $k \in \mathbb{N}$, we define

$$Y_{k,n+1} = \sum_{a=1}^{Y_{k,n}} X_{k,a,n} + \sum_{b=1}^{N_k} I_{k,b,n}, \qquad Y_{k,0} = x_{k,0}, \tag{5.1}$$

where for each $k \in \{1,2,\dots,M\}$, $x_{k,0}$ is the initial state of the k-th process. If for all $k \in \{1,2,\dots,M\}$, $x_{k,0}$ are mutually independent random variables, and independent of all $X_{k,m,n}$ and $I_{k,\ell,n}$. Then the stochastic processes $\{Y_{k,n}\}_{n\in\mathbb{N}_0},\ k \in \{1,2,\dots,M\}$ become G-W branching processes with multiple immigration. We define a stochastic process $\{Z_n\}_{n\in\mathbb{N}_0}$ by

$$Z_n = \sum_{i=1}^{M} Y_{i,n}. \tag{5.2}$$

Definition 5.2. The stochastic process $\{Z_n\}_{n\in\mathbb{N}_0}$ is called a *multiple G-W branching process.*

Let $\{X_k\}_{k\in\{1,2,\dots,M\}}$ be a finite sequence of independent random variables with probability distributions $\boldsymbol{p}_{X_k}$ for every k. If $\mathrm{E}[X_k] < 1$ for all $k \in \mathbb{N}$, then the limiting distribution of the process $\{Y_{k,n}\}_{n\in\mathbb{N}_0}$ is given by Theorem 4.4 since this process is the G-W branching process with multiple immigration. Therefore, we obtain the following theorem:

Theorem 5.3. *Let $X_{k,m,n},\ m \in \mathbb{N}, n \in \mathbb{N}_0$ be $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_{X_k}$ of a discrete random variable X_k for any $k \in \{1,2,\dots,M\}$. Let $I_{k,\ell,n},\ \ell = 1,2,\dots,N_k,\ n \in \mathbb{N}_0$ be also $\mathbb{N}_0$-valued i.i.d. random variables that have a probability distribution $\boldsymbol{p}_{I_{k,\ell}}$ of a discrete random variable $I_{k,\ell,n}$ for any k,ℓ. If $\mathrm{E}[X_k] < 1$ for all k then the limiting distribution $\boldsymbol{\pi}^{(\infty)}$ of the multiple G-W branching process $\{Z_n\}_{n\in\mathbb{N}_0}$ defined as in (5.2) exists and is given by*

$$\boldsymbol{\pi}^{(\infty)} = \boldsymbol{\pi}_1^{(\infty)} * \cdots * \boldsymbol{\pi}_M^{(\infty)}.$$

Proof. The proof is obvious from Lemma 2.1 and Definition 5.2. □

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