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WICK PRODUCT AND CLARK-OCONE FORMULA IN ABSTRACT WIENER SPACE

FARIBORZ ASADIAN

Abstract. We explore properties of the Wick inner product of vector valued functions in the context of an abstract Wiener space $(\mathbb{H}, \mathbb{B})$. We investigate convergence of chaos expansion of the Wick product of test functions and obtain a series expansion of the Wick (ordinary) inner product of test Wiener-functions in terms of the ordinary (Wick) inner products of their derivatives. Next, we use an extension of stochastic integral defined on the space of generalized processes to obtain a generalization of the Clark-Ocone formula involving Wiener functions that are not necessarily differentiable in the sense of Malliavin. As a framework for our investigation, we utilize the spaces of distributions and test functions introduced by Körzlioglu and Üstünel and constructed via the second quantization of an (unbounded and positive) operator on $\mathbb{H}$.

1. Introduction

Let $(\mathbb{H}, \mathbb{B})$ be an abstract Wiener space, and let μ be the corresponding standard Wiener measure in $\mathbb{B}$. The space, $\mathcal{L}^2$, of μ-square integrable functions defined on $\mathbb{B}$ admits an orthogonal decomposition that is known as the Wiener-chaos decomposition. This decomposition provides a framework in which analytical notions are tied to the Hilbert space structure of (tensor products of) $\mathbb{H}$. In this abstract setting, the Wick product appears naturally and quite frequently as the binary operation, on Wiener functions, associated with the adjoint of the Malliavin derivative. (For an investigation of Wick product in the context of white noise analysis, see Kuo [4] and Holden, et al. [1].) Naturally, an exploration of the properties of the Wick product in the abstract setting must utilize the chaos decomposition of $\mathcal{L}^2$. However, $\mathcal{L}^2$ is neither closed with respect to the ordinary product nor closed with respect to the Wick product. Therefore, such an investigation must be carried out within the framework of appropriately defined spaces of test functions and distributions.

The prototype of the abstract Wiener space is the classical Wiener space, and on this space, the adjoint of the Malliavin derivative is the well known Skorokhod integral. In [6], Potthof and Timple used the Wiener-chaos decompositions of Wiener functions defined on the classical Wiener space to construct spaces of test

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functions and distributions. Lanconelli [5] chose this construction as a framework to investigate the connection between Wick and ordinary products and to prove formulas relating these two operations. Subsequently, these formulas were used in [5] to explore projection properties of Wiener functions (associated with second quantization operators) analogous to the defining property of conditional expectation in the theory of probability.

From the point of view of infinite dimensional stochastic analysis, it is desirable to develop these ideas in an abstract setting, so that the results can be applied to functions defined on the Wiener space consisting of paths in an infinite dimensional separable Hilbert space. One aim of this article is to explore the connection between the ordinary and Wick inner products of *vector-valued* functions in this setting. This investigation is motivated by the expectation that the results obtained will find applications in problems of interest in stochastic analysis where the objects of investigation are infinite dimensional. In the setting of an abstract Wiener space, the structure provided by Körezlioglu and Üstünel [2] appears to be a suitable framework for investigation. This structure is built via the second quantization of an unbounded positive operator on $\mathbb{H}$. Our approach will not necessarily require the unboundedness property; instead, the required property will be that the inverse of the operator has (operator) norm less than one. The space of test functions defined in this way is closed with respect to both the ordinary and Wick products.

In section 2 we introduce notation and review chaos decomposition of $\mathcal{L}^2$. Moreover, adapting the scheme of Körezlioglu and Üstünel, we use the second quantization of a positive linear operator on $\mathbb{H}$ to construct spaces of test functions and distributions. The duality relationship between these spaces will be used to extend the definition of Skorokhod integral. The material covered in section 2 will serve as background information for sections 3 and 4. The main results relating Wick and ordinary inner products of infinite dimensional Wiener functions will be proved in section 3. In section 4, we will use the generalization of the Skorokhod integral to extend the Clark-Ocone formula in abstract Wiener space to functions that are not necessarily differentiable in the sense of Malliavin.

2. Stochastic Analysis in Abstract Wiener Space

2.1. Preliminaries. Let $\mathbb{H}$ be a fixed (finite dimensional or separable and infinite dimensional) Hilbert space with norm $|\cdot|_{\mathbb{H}} = \sqrt{\langle\cdot,\cdot\rangle_{\mathbb{H}}}$. Throughout, $\mathbb{K}$ will be an arbitrary separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathbb{K}}$. The collection of Hilbert Schmidt operators from a separable Hilbert space $\mathbb{K}_1$ to a separable Hilbert space $\mathbb{K}_2$ will be denoted by $L_2(\mathbb{K}_1, \mathbb{K}_2)$. This collection itself is a separable Hilbert space and is identified with the tensor product, $\mathbb{K}_1 \otimes \mathbb{K}_2$, of $\mathbb{K}_1$ and $\mathbb{K}_2$. The tensor product of elements $k_1, \cdots, k_n$ in $\mathbb{K}$ will be denoted by $k_1 \otimes \cdots \otimes k_n$ or by $\otimes_{i=1}^n k_i$. For each $k \in \mathbb{K}$, the tensor product $k \otimes \cdots \otimes k$ in the tensor product space of n copies of $\mathbb{K}$ will be denoted by k^n.

Let $\{\mathfrak{e}_\alpha\,;\,\alpha \in \mathcal{A}\}$ be an orthonormal basis of $\mathbb{H}$, and let $\{\lambda_\alpha\,;\,\alpha \in \mathcal{A}\}$ be a collection of real numbers such that $\lambda_\alpha > 1$ for every $\alpha \in \mathcal{A}$. We define the linear operator $\mathrm{Q} : \textit{Domain}\ \mathrm{Q}(\subset \mathbb{H}) \to \mathbb{H}$ by $\mathrm{Q}(\mathfrak{e}_\alpha) = \lambda_\alpha\,\mathfrak{e}_\alpha$ for every $\alpha \in \mathcal{A}$. Throughout the article, $\sup_{\alpha\in\mathcal{A}} \frac{1}{\lambda_\alpha}$ will be denoted by $\mathbb{M}$.

For each $\beta \geq 0$, we let $\mathbb{H}_\beta = \{f \in \mathbb{H}\,;\, \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^\beta \langle f, \mathfrak{e}_\alpha\rangle^2 < \infty\}$. In this space we define the inner product

$$\langle f, g\rangle_{\mathbb{H}_\beta} = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^\beta \langle f, \mathfrak{e}_\alpha\rangle \langle g, \mathfrak{e}_\alpha\rangle$$

and denote the induced norm by $|\cdot|_{\mathbb{H}_\beta}$. The space $\mathbb{H}_\beta$ with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}_\beta}$ is a Hilbert space, and $\{\lambda_\alpha^{-\beta/2}\mathfrak{e}_\alpha\,;\,\alpha \in \mathcal{A}\}$ is an orthonormal basis of this space.

We let $\mathbb{H}_\infty = \bigcap_{\beta>0} \mathbb{H}_\beta$ and endow this space with the topology generated by the norms $|\cdot|_{\mathbb{H}_\beta}, \beta \geq 0$.

Next, for a given $\beta \geq 0$, we define in $\mathbb{H}$ the inner product

$$\langle f, g\rangle_{\mathbb{H}_{-\beta}} = \sum_{\alpha\in\mathcal{A}} \lambda_\alpha^{-\beta} \langle f, \mathfrak{e}_\alpha\rangle \langle g, \mathfrak{e}_\alpha\rangle$$

and note that this inner product induces a norm $|\cdot|_{\mathbb{H}_{-\beta}}$ that is weaker than the (original) norm of $\mathbb{H}$. We complete $\mathbb{H}$ with respect to $|\cdot|_{\mathbb{H}_{-\beta}}$ and obtain a space that we denote by $\mathbb{H}_{-\beta}$. This space together with the inner product $\langle\cdot,\cdot\rangle_{\mathbb{H}_{-\beta}}$ is a Hilbert space, and in it, the set $\{\lambda_\alpha^{\beta/2}\mathfrak{e}_\alpha\,;\,\alpha \in \mathcal{A}\}$ is an orthonormal basis.

We let $\mathbb{H}_{-\infty} = \bigcup_{\beta>0} \mathbb{H}_{-\beta}$ and endow this space with the inductive topology. Thus a sequence $(x_n\,;\, n = 1, 2, \cdots)$ in this space converges if there is a $\beta > 0$ such that $x_n \in \mathbb{H}_{-\beta}$ for every n and the sequence converges in the topology of $\mathbb{H}_{-\beta}$. Clearly for every $\beta > 0$ we have

$$\mathbb{H}_\infty \subseteq \mathbb{H}_\beta \subseteq \mathbb{H} \subseteq \mathbb{H}_{-\beta} \subseteq \mathbb{H}_{-\infty}.$$

The following two well known concrete examples of our abstract structure are frequently used in applications:

Example 2.1. Let $\mathbb{H} = L^2(\mathbb{R}^d)$ equipped with its natural inner product. For each integer $n \geq 0$ we let $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}\left[e^{-x^2}\right]$ and $e_n = \frac{1}{\sqrt{\sqrt{\pi}2^n n!}} H_n(x) e^{-x^2/2}$; i.e., H_n is the Hermite polynomial of degree n and e_n is the corresponding Hermite function. For each $(i_1, \cdots, i_d) \in \mathbb{N}^d$, we denote the tensor product $e_{i_1} \otimes \cdots \otimes e_{i_d}$ by $\mathfrak{e}_{(i_1,\cdots,i_d)}$. The collection $\{\,\mathfrak{e}_{(i_1,\cdots,i_d)}\,;\,(i_1, \cdots, i_d) \in \mathbb{N}^d\,\}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$. Let A be the densely defined differential operator $A = -\frac{d^2}{dx^2} + x^2 + 1$ on $L^2(\mathbb{R})$, and let $Q = A^{\otimes d}$. This operator has spectral decomposition $Q(\cdot) = \sum_{(i_1,\cdots,i_d)\in\mathbb{N}^d} 2^d (i_1 + 1) \cdots (i_d + 1) \langle\cdot, \mathfrak{e}_{(i_1,\cdots,i_d)}\rangle_{L^2(\mathbb{R}^d)} \mathfrak{e}_{(i_1,\cdots,i_d)}$. In this example, the space $\mathbb{H}_\infty$ is the well known Schwartz space of rapidly decreasing functions and $\mathbb{H}_{-\infty}$ is the space of tempered distributions.

Example 2.2. We let $[a, b]$ be a closed and bounded interval in $\mathbb{R}$. Let $\mathbb{H} = L^2([a, b]^d)$, the space of Lebesgue square-integrable functions with support in $[a, b]^d$. For each positive integer n, let $e_n(x) = \sqrt{\frac{2}{b-a}} \sin \frac{n\pi(x-a)}{b-a}$. For each $(i_1, \cdots, i_d) \in \mathbb{N}^d$ we denote the tensor product $e_{i_1} \otimes \cdots \otimes e_{i_d}$ by $\mathfrak{e}_{(i_1,\cdots,i_d)}$. The collection $\{\mathfrak{e}_{(i_1,\cdots,i_d)}\,;\,(i_1, \cdots, i_d) \in \mathbb{N}^d\}$ is an orthonormal basis of $L^2([a, b]^d)$. Let $Q = A^{\otimes d}$, where A is the densely defined operator $A = -\frac{d^2}{dx^2}+1$ on $L^2([a, b])$. Then $Q(\cdot) = \sum_{(i_1,\cdots,i_d)\in\mathbb{N}^d} (c^2 i_1^2 + 1) \cdots (c^2 i_d^2 + 1) \langle\cdot, \mathfrak{e}_{(i_1,\cdots,i_d)}\rangle_{L^2([a,b]^d)} \mathfrak{e}_{(i_1,\cdots,i_d)}$, where

$c = \frac{\pi}{b-a}$. In this example, the space $\mathbb{H}_\infty$ can be identified with the collection of C^∞-functions on $\mathbb{R}^d$ that vanish outside $[a,b]^d$, and $\mathbb{H}_{-\infty}$ is identified with the space of (Schwartz) distributions that have support in $[a,b]^d$.

For every $\beta \in \mathbb{R}$, the tensor product of n copies of $\mathbb{H}_\beta$ will be denoted by $\mathbb{H}_\beta^{\otimes n}$. The set

$$\Big\{ \otimes_{i=1}^n \left(\lambda_{\alpha_i}^{-\beta/2} \mathfrak{e}_{\alpha_i} \right) \; ; \; \alpha_1, \cdots, \alpha_n \in \mathcal{A} \Big\}$$

is an orthonormal basis of $\mathbb{H}_\beta^{\otimes n}$. We denote the inner product of elements $\phi, \psi \in \mathbb{H}_\beta^{\otimes n}$ by $\langle \phi, \psi \rangle_{\mathbb{H}_\beta^{\otimes n}}$ and the (induced) norm of an element $\phi \in \mathbb{H}_\beta^{\otimes n}$ by $|\phi|_{\mathbb{H}_\beta^{\otimes n}}$.

Note that if $\beta \geq 0$ and $f, g \in \mathbb{H}_\beta^{\otimes n}$, or $\beta < 0$ and $f, g \in \mathbb{H}^{\otimes n}$, then

$$\langle f, g \rangle_{\mathbb{H}_\beta^{\otimes n}} = \sum_{\alpha_1, \cdots, \alpha_n \in \mathcal{A}} (\lambda_{\alpha_1} \cdots \lambda_{\alpha_n})^\beta \, \langle f, \otimes_{i=1}^n \mathfrak{e}_{\alpha_i} \rangle_{\mathbb{H}^{\otimes n}} \langle g, \otimes_{i=1}^n \mathfrak{e}_{\alpha_i} \rangle_{\mathbb{H}^{\otimes n}}.$$

For each $\beta \geq 0$, there is a natural pairing of each element $f \in \mathbb{H}_\beta^{\otimes n}$ and each $g \in \mathbb{H}_{-\beta}^{\otimes n}$ denoted by $\langle g, f \rangle_{\mathbb{H}_{\mp\beta}^{\otimes n}}$ and defined by

$$\langle g, f \rangle_{\mathbb{H}_{\mp\beta}^{\otimes n}} = \sum_{\alpha_1, \cdots, \alpha_n \in \mathcal{A}} \langle g, \otimes_{i=1}^n \mathfrak{e}_{\alpha_i} \rangle_{\mathbb{H}_{-\beta}^{\otimes n}} \langle f, \otimes_{i=1}^n \mathfrak{e}_{\alpha_i} \rangle_{\mathbb{H}_\beta^{\otimes n}}.$$

Through this natural pairing, $\mathbb{H}_{-\beta}^{\otimes n}$ is identified with the dual space of $\mathbb{H}_\beta^{\otimes n}$.

Remark 2.3. If $\mathbb{H}$ is finite dimensional, all norms introduced above are equivalent. In fact, if $\lambda_\alpha, \alpha \in \mathcal{A}$ are the eigenvalues of Q and β and β' are real numbers, then for every element $f \in \mathbb{H}^{\otimes n}$ we have $|f|_{\mathbb{H}_\beta^{\otimes n}} \leq \sqrt{\max_{\alpha \in \mathcal{A}} \left(\lambda_\alpha^{(\beta-\beta')n} \right)} |f|_{\mathbb{H}_{\beta'}^{\otimes n}}$. Therefore, for every $\beta > 0$, $\mathbb{H}_\beta = \mathbb{H}_{-\beta} = \mathbb{H}_\infty = \mathbb{H}_{-\infty} = \mathbb{H}$.

2.1.1. *Symmetrization Operator.* Let $\beta \in \mathbb{R}$. We define the symmetric tensor product of elements $f_1, \cdots, f_n \in \mathbb{H}_\beta$ by

$$f_1 \widehat{\otimes} \cdots \widehat{\otimes} f_n = \frac{1}{n!} \sum_\tau f_{\tau(1)} \otimes \cdots \otimes f_{\tau(n)},$$

where the summation is over all permutations of the set $\{1, \cdots, n\}$. (Throughout, the symbol $\widehat{\otimes}$ will be used to denote symmetric tensor product.) We denote by S_n the linear operator that sends each simple element $f_1 \otimes \cdots \otimes f_n \in \mathbb{H}_\beta^{\otimes n}$ to $f_1 \widehat{\otimes} \cdots \widehat{\otimes} f_n$. This operator is a projection and can be extended to the entirety of $\mathbb{H}_\beta^{\otimes n}$. We denote the range of S_n, which is a closed subspace of $\mathbb{H}_\beta^{\otimes n}$, by $\mathbb{H}_\beta^{\widehat{\otimes} n}$ and call the elements in this space *symmetric*. An element in $\mathbb{H}_\beta^{\otimes n}$ is called *antisymmetric* if it belongs to the kernel of S_n. The space $\mathbb{H}_\beta^{\widehat{\otimes} n}$ with the inner product it inherits from $\mathbb{H}_\beta^{\otimes n}$ is a Hilbert space.

The projection S_n defines an equivalence relation on $\mathbb{H}_\beta^{\otimes n}$. Thus, two elements $f, g \in \mathbb{H}_\beta^{\otimes n}$ are related if and only if $f - g$ is antisymmetric. In the obvious way, the equivalence classes (mod S_n) can be identified with elements in the space $\mathbb{H}_\beta^{\widehat{\otimes} n}$.

Every simple symmetric element in $\mathbb{H}_\beta^{\widehat{\otimes}n}$ can be written, via ***Polarization Identity***, as a finite sum of elements of the type h^n. In fact, if $h_1, \cdots, h_n \in \mathbb{H}_\beta$, then

$$h_1 \widehat{\otimes} \cdots \widehat{\otimes} h_n = \frac{1}{2^n n!} \sum_{\sigma_1=\pm1,\cdots,\sigma_n=\pm1} (\sigma_1 h_1 + \cdots + \sigma_n h_n)^n.$$

Remark 2.4. If $\mathbb{H}$ is finite dimensional with dimension n, then the space $\mathbb{H}^{\widehat{\otimes}i}$ has dimension $\binom{n+i-1}{n-1}$.

We need a suitable notation to identify an orthonormal basis for $\mathbb{H}_\beta^{\widehat{\otimes}n}$. We let $\mathbf{\Gamma}$ be the collection of all indexed sets $\gamma = \{\gamma_\alpha\,;\,\alpha \in \mathcal{A}\}$ whose terms are nonnegative integers and $|\gamma| = \sum_{\alpha\in\mathcal{A}} \gamma_\alpha$ is finite. For each $\gamma = \{\gamma_\alpha\,;\,\alpha \in \mathcal{A}\}$ we set $\gamma! = \prod_{\alpha\in\mathcal{A}} \gamma_\alpha!$.

For $\alpha_1 \cdots \alpha_n \in \mathcal{A}$, the product $\mathfrak{e}_{\alpha_1} \otimes \cdots \otimes \mathfrak{e}_{\alpha_n}$ is clearly in equivalence relation with a product $\mathfrak{e}_{\alpha_{i_1}}^{j_1} \otimes \cdots \otimes \mathfrak{e}_{\alpha_{i_k}}^{j_k}$, $j_1 + \cdots + j_k = n$, in which the indices $\alpha_{i_1}, \cdots, \alpha_{i_k}$ are distinct. Since symmetrization of the product $\mathfrak{e}_{\alpha_{i_1}}^{j_1} \otimes \cdots \otimes \mathfrak{e}_{\alpha_{i_k}}^{j_k}$ produces a sum of $n!$ terms of which $\frac{n!}{j_1!\cdots j_k!}$ are distinct and pairwise orthogonal, we clearly see that

$$\left(\lambda_{\alpha_{i_1}} \cdots \lambda_{\alpha_{i_k}}\right)^{-\beta/2} \left| \widehat{\otimes}_{m=1}^k \mathfrak{e}_{\alpha_{i_m}}^{j_m} \right|_{n,\mathbb{H}_\beta} = \sqrt{\frac{j_1! \cdots j_k!}{n!}}.$$

Hence we have the following proposition:

Proposition 2.5. *For each $n \in \mathbb{N}$, the collection*

$$\left\{ \sqrt{\frac{n!}{\gamma!}}\, \widehat{\otimes}_{\alpha\in\mathcal{A}} \left(\lambda_\alpha^{-\beta\gamma(\alpha)/2} \mathfrak{e}_\alpha^{\gamma(\alpha)} \right) \;:\; \gamma \in \Gamma, |\gamma| = n \right\}$$

is an orthonormal basis of $\mathbb{H}_\beta^{\widehat{\otimes}n}$.

If $\beta \geq 0$ and $f \in \mathbb{H}_\beta^{\widehat{\otimes}n}$, or if $\beta < 0$ and $f \in \mathbb{H}^{\widehat{\otimes}n}$, then

$$\begin{aligned} |f|^2_{\mathbb{H}_\beta^{\otimes n}} &= \sum_{\gamma\in\Gamma,|\gamma|=n} \frac{n!}{\gamma!} \prod_{\alpha\in\mathcal{A}} \lambda_\alpha^{-\beta\gamma(\alpha)} \langle f, \widehat{\otimes}_{\alpha\in\mathcal{A}} \mathfrak{e}_\alpha^{\gamma(\alpha)} \rangle^2_{\mathbb{H}_\beta^{\otimes n}} \\ &= \sum_{\gamma\in\Gamma,|\gamma|=n} \frac{n!}{\gamma!} \prod_{\alpha\in\mathcal{A}} \lambda_\alpha^{\beta\gamma(\alpha)} \langle f, \widehat{\otimes}_{\alpha\in\mathcal{A}} \mathfrak{e}_\alpha^{\gamma(\alpha)} \rangle^2_{\mathbb{H}^{\otimes n}}. \end{aligned}$$

From this (and the fact that $\mathbb{H}^{\widehat{\otimes}n}$ is dense in $\mathbb{H}_\beta^{\widehat{\otimes}n}$ if $\beta < 0$) we see that if $\beta_1 \geq \beta_2$, then $\mathbb{H}_{\beta_1} \subseteq \mathbb{H}_{\beta_2}$, and in this case we have

$$|f|_{\mathbb{H}_{\beta_2}^{\otimes n}} \leq \mathbb{M}^{n(\beta_1-\beta_2)/2} |f|_{\mathbb{H}_{\beta_1}^{\otimes n}} \tag{2.1}$$

for every $f \in \mathbb{H}_{\beta_1}^{\widehat{\otimes}n}$. (Recall that $\mathbb{M} = \sup_{\alpha\in\mathcal{A}} \frac{1}{\lambda(\alpha)}$.)

Definition 2.6. Let i and j be positive integers. Two elements $f \in \mathbb{H}^{\widehat{\otimes}i}$ and $g \in \mathbb{H}^{\widehat{\otimes}j}$ are said to be *mutually* orthogonal if there are subspaces $\mathbb{V}_1$ and $\mathbb{V}_2$ of $\mathbb{H}$ such that $\mathbb{V}_1 \perp \mathbb{V}_2$, $f \in \mathbb{V}_1^{\widehat{\otimes}i}$, and $g \in \mathbb{V}_2^{\widehat{\otimes}j}$.

2.2. Structure of the space of square integrable Wiener functions. Let μ be the standard Gauss measure defined on the cylindrical subsets of $\mathbb{H}$. It is well known that μ is not σ-additive. Let $\mathbb{B}$ be a Banach space obtained by completing $\mathbb{H}$ with respect to a μ-measurable norm. (For the definition of μ-measurable norm, see Kuo [3].) Then μ, defined on the cylindrical subsets of $\mathbb{B}$, has a σ-additive extension to the Borel σ-field of $\mathbb{B}$. The pair $(\mathbb{H}, \mathbb{B})$ is called an abstract Wiener space. Identifying $\mathbb{H}$ with its dual space $\mathbb{H}^*$ via Riesz Representation Theorem, we obtain the inclusions

$$\mathbb{B}^* \subseteq \mathbb{H}^* \equiv \mathbb{H} \subseteq \mathbb{B}.$$

In this way the dual space, $\mathbb{B}^*$, of $\mathbb{B}$ is identified with a dense subset of $\mathbb{H}$.

We denote the collection of all μ-square integrable random variables by $\mathcal{L}^2$. The inner product and the norm in this space will be denoted by $\ll \cdot, \cdot \gg$ and $\| \cdot \|$, respectively. Integration of an element $F \in \mathcal{L}^2$ with respect to μ will be denoted by $\mathbb{E}F$.

Let $h \in \mathbb{H}$, and let $h_n, n = 1, 2, \cdots$, be a sequence in $\mathbb{B}^*$ that converges (in $\mathbb{H}$) to h. For each n, the map $x \mapsto h_n(x)$ is Gaussian with mean zero and variance $|h_n|^2_{\mathbb{H}}$. The sequence of random variables $h_n(x)$ is therefore Cauchy in $\mathcal{L}^2$ and converges to an almost surely defined random variable, denoted by $\delta[h]$, that is normally distributed with mean zero and variance $|h|^2_{\mathbb{H}}$. The map δ is unitary from $\mathbb{H}$ to $\mathcal{L}^2$. It is easy to verify that for each positive integer n, $\mathbb{E}\delta[h]^{2n} = \frac{(2n)!}{2^n n!}|h|^{2n}_{\mathbb{H}}$.

Let $\widehat{\mathcal{H}}_0$ be the collection of all real numbers. For each $n = 1, 2, \cdots$, let $\widehat{\mathcal{H}}_n$ be the closure, in $\mathcal{L}^2$, of the collection consisting of finite linear combinations of constants and random variables of the type $\prod_{j=1}^{i} \delta[h_j]$, where $i = 0, 1, \cdots, n$ and $h_j \in \mathbb{H}$. Now let $\mathcal{H}_0 = \widehat{\mathcal{H}}_0$, and for each $n = 1, 2, \cdots$, let $\mathcal{H}_n = \widehat{\mathcal{H}}_n \ominus \widehat{\mathcal{H}}_{n-1}$; i.e., $\mathcal{H}_n$ is the orthogonal complement of $\widehat{\mathcal{H}}_{n-1}$ in $\widehat{\mathcal{H}}_n$.

For each $\beta \in R$ and index $\gamma \in \Gamma$ we let

$$\Psi_{\beta;\gamma} = \frac{1}{\sqrt{\gamma!}} \prod_{\alpha \in \mathcal{A}} \lambda_\alpha^{-\beta\gamma(\alpha)/2} H_{\gamma(\alpha)}\left(\delta[\mathfrak{e}_\alpha]\right),$$

where $H_{\gamma(\alpha)}$ is the Hermite polynomial of degree $\gamma(\alpha)$.

Proposition 2.7. *For each positive integer n, the set $\{\Psi_{0;\gamma} : \gamma \in \Gamma, |\gamma| = n\}$ is an orthonormal basis for $\mathcal{H}_n$.*

The following well known proposition follows from the fact that the collection of elements of the type $P(\delta[h_1], \cdots, \delta[h_m])$, where m is a nonnegative integer, P is a polynomial function of m variables and $h_1, \cdots h_m \in \mathbb{H}$, is dense in $\mathcal{L}^2$, and that each such element can be written as a linear combination of elements of the type $H_i(\delta(h))$, where H_i is a Hermite polynomial and $h \in \mathbb{H}$.

Proposition 2.8. *$\mathcal{L}^2$ is the orthogonal sum of its subspaces $\mathcal{H}_n, n = 0, 1, 2, \cdots$.*

The orthogonal projection of an element in $\mathcal{L}^2$ onto the space $\mathcal{H}_n$ will be denoted by $\mathbb{J}_n$. For an elementary object $h_1 \otimes \cdots \otimes h_n$ in the space $\mathbb{H}^{\otimes n}$ we define

$$\delta^{(n)}[h_1 \otimes \cdots \otimes h_n] = \mathbb{J}_n[\delta[h_1] \cdots \delta[h_n]].$$

We enlarge the domain of $\delta^{(n)}$, by linearity, to include finite linear combinations of elementary objects (simple elements) in $\mathbb{H}^{\otimes n}$. Clearly

$$\delta^{(n)}[h_1 \widehat{\otimes} \cdots \widehat{\otimes} h_n] = \delta^{(n)}[h_1 \otimes \cdots \otimes h_n],$$

and this is true for all simple elements as well.

Example 2.9. It is easy to see that if $h_1, h_2, h_3 \in \mathbb{H}$, then $\delta^{(2)}[h_1 \otimes h_2] = \delta[h_1]\delta[h_2] - \langle h_1, h_2\rangle_{\mathbb{H}}$, and $\delta^{(3)}[h_1 \otimes h_2 \otimes h_3] = \delta[h_1]\delta[h_2]\delta[h_3] - \langle h_1, h_2\rangle_{\mathbb{H}}\delta[h_3] - \langle h_1, h_3\rangle_{\mathbb{H}}\delta[h_2] - \langle h_2, h_3\rangle_{\mathbb{H}}\delta[h_1]$.

Remark 2.10. For every $h \in \mathbb{H}$ and positive integer n, $\delta^{(n)}[h^n] = H_n(\delta[h])$, where H_n is the Hermite polynomial of degree n. The following property follows from a well known property of Hermite polynomials.

$$\delta^{(n)}[h^n]\delta^{(m)}[h^m] = \sum_{k=0}^{n\wedge m} k!\binom{n}{k}\binom{m}{k}\delta^{(n+m-2k)}[h^{n+m-2k}].$$

Proposition 2.11. *For every positive integer n, the linear map $\delta^{(n)} : \mathbb{H}^{\otimes n} \to \mathcal{H}_n$ defined above is bounded, and its restriction to $\mathbb{H}^{\widehat{\otimes} n}$ is unitary up to multiplication by $\sqrt{n!}$. The range of this map is the entire space $\mathcal{H}_n$ and its kernel is $[\mathbb{H}^{\widehat{\otimes} n}]^{\perp}$.*

Proof. For every index γ with $|\gamma| = n$, the element $\otimes_{\alpha\in\mathcal{A}}\left(\mathfrak{e}_\alpha^{\gamma(\alpha)}\right)$ in $\mathbb{H}^{\otimes n}$ is mapped by $\delta^{(n)}$ to the element $\prod_{\alpha\in\mathcal{A}} H_{\gamma(\alpha)}(\delta[\mathfrak{e}_\alpha])$ in $\mathcal{H}_n$. The assertions of this theorem follow from Proposition 2.7 and the fact that $\delta^{(n)}[\mathcal{S}_n(f)] = \delta^{(n)}[f]$ for every simple element $f \in \mathbb{H}^{\otimes n}$. □

2.3. Generalization of multiple stochastic integral. For each $\beta \in \mathbb{R}$ and positive integer n we define in $\mathcal{H}_n$ the inner product

$$\ll F, G \gg_\beta = \sum_{\gamma\in\Gamma, |\gamma|=n} \ll F, \Psi_{-\beta;\gamma} \gg \ll G, \Psi_{-\beta;\gamma} \gg,$$

if the sum converges. We denote the induced norm by $\| \cdot \|_\beta$. Now, if $\beta > 0$, then $\| \cdot \|_\beta$ is stronger than $\| \cdot \|$. In this case we let $\mathcal{H}_{n,\beta} = \{F \in \mathcal{H}_n : \| F \|_\beta < \infty\}$. If $\beta < 0$, then $\| \cdot \|$ is stronger than $\| \cdot \|_\beta$ and we define $\mathcal{H}_{n,\beta}$ to be the completion of $\mathcal{H}_n$ with respect to $\| \cdot \|_\beta$. In either case $\mathcal{H}_{n,\beta}$ is a Hilbert space with the inner product $\ll \cdot, \cdot \gg_\beta$, and one of its orthonormal bases is the set $\{\Psi_{\beta;\gamma} : \gamma \in \Gamma, |\gamma| = n\}$.

For each $\beta > 0$, there is a natural pairing of elements of $\mathcal{H}_{n,-\beta}$ with elements of $\mathcal{H}_{n,\beta}$, through which the space $\mathcal{H}_{n,-\beta}$ is identified with the dual space of $\mathcal{H}_{n,\beta}$. This pairing of an element $G \in \mathcal{H}_{n,-\beta}$ and an element $F \in \mathcal{H}_{n,\beta}$ will be denoted by $\ll G, F \gg_{\mp\beta}$ and is defined by

$$\ll G, F \gg_{\mp\beta} = \sum_{\gamma\in\Gamma, |\gamma|=n} \ll G, \Psi_{-\beta;\gamma} \gg_{-\beta} \ll F, \Psi_{\beta;\gamma} \gg_\beta . \tag{2.2}$$

Evidently this pairing corresponds to the natural paring of elements of $\widehat{\otimes}^n\mathbb{H}_{-\beta}$ with elements of $\widehat{\otimes}^n\mathbb{H}_\beta$ defined in section 2.1.

For every $\beta > 0$, we can use the natural pairing of $\mathcal{H}_{n,\beta}$ and $\mathcal{H}_{n,-\beta}$ to expand the domain of $\delta^{(n)}$ to include the entire space $\mathbb{H}_{-\beta}^{\otimes n}$. For each $g \in \mathbb{H}_{-\beta}^{\otimes n}$ we define the map $\eta_g : \mathcal{H}_{n,\beta} \to \mathbb{R}$ by $\eta_g(F) = n!\langle g, f\rangle_{\mathbb{H}^{\otimes n}_{\mp\beta}}$, where $F = \delta^{(n)}[f]$. This map is a bounded linear functional on $\mathcal{H}_{n,\beta}$ and hence can be represented by a unique element $\delta^{(n)}[g]$ in $\mathcal{H}_{n,-\beta}$.

Now let $\beta > 0$ and $\gamma \in \Gamma, |\gamma| = n$. Since $\Psi_{-\beta,\gamma} \in \mathcal{H}_n$ for every β, it follows from (2.2) that for every $f \in \mathbb{H}^{\widehat{\otimes} n}_{\beta}$,

$$\begin{aligned}\ll \Psi_{-\beta,\gamma}, \delta^{(n)}[f] \gg_{\mp\beta} &= \ll \Psi_{-\beta,\gamma}, \delta^{(n)}[f] \gg \\ &= n!\langle \sqrt{\frac{n!}{\gamma!}} \widehat{\otimes}_{\alpha\in\mathcal{A}} \left(\lambda_\alpha^{\beta\gamma_\alpha/2}\mathfrak{e}_\alpha^{\gamma_\alpha}\right), f\rangle_{\mathbb{H}^{\otimes n}_{\mp\beta}}.\end{aligned}$$

Therefore, $\delta^{(n)}$ maps each element $\sqrt{\frac{n!}{\gamma!}}\widehat{\otimes}_{\alpha\in\mathcal{A}}\left(\lambda_\alpha^{\beta\gamma_\alpha/2}\mathfrak{e}_\alpha^{\gamma_\alpha}\right)$ in $\mathbb{H}^{\otimes n}_{-\beta}$ to $\Psi_{-\beta,\gamma}$ in $\mathcal{H}_{n,-\beta}$. Furthermore, we note that if $g \in \left[\mathbb{H}^{\widehat{\otimes} n}_{-\beta}\right]^{\perp}$, then η_g is the zero map. From these observations we get the following proposition.

Proposition 2.12. *For every positive integer n and nonnegative number β, the linear map $\delta^{(n)} : \mathbb{H}^{\otimes n}_{-\beta} \to \mathcal{H}_{n,\beta}$ defined above is bounded, and its restriction to $\mathbb{H}^{\widehat{\otimes} n}_{-\beta}$ is unitary up to multiplication by $\sqrt{n!}$. The range of this map is the entire space $\mathcal{H}_{n,-\beta}$ and its kernel is $\left[\mathbb{H}^{\widehat{\otimes} n}_{-\beta}\right]^{\perp}$.*

For each $\beta \in \mathbb{R}$, we define $\mathcal{L}^{2,\beta} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{n,\beta}$. Therefore, each $F \in \mathcal{L}^{2,\beta}$ has a chaos decomposition $\sum_{n=0}^{\infty} \delta^{(n)}[f_n]$, where each f_n belongs to the space $\mathbb{H}^{\widehat{\otimes} n}_{\beta}$. We denote the inner product in this space by $\ll \cdot, \cdot \gg_\beta$ and the corresponding norm by $\|\cdot\|_\beta$.

For every $\beta > 0$, we use the same notation in (2.2) and denote the natural pairing of elements in $\mathcal{L}^{2,-\beta}$ with those in $\mathcal{L}^{2,\beta}$ by $\ll \cdot, \cdot \gg_{\mp\beta}$.

Definition 2.13. Let $\beta \in \mathbb{R}$, and let $F, G \in \mathcal{L}^{2,\beta}$ with Wiener-chaos decompositions $F = \sum_{n=0}^{\infty} \delta^{(n)}[f_n]$ and $G = \sum_{n=0}^{\infty} \delta^{(n)}[g_n]$. The *Wick product* of F and G is denoted by $F \diamond G$ and defined by

$$F \diamond G = \sum_{i=0}^{\infty} \sum_{n+m=i} \delta^{(i)}[f_n \widehat{\otimes} g_m],$$

provided the sum converges in $\mathcal{L}^{2,\beta'}$ for some $\beta' \in \mathbb{R}$.

2.4. Vector-valued Wiener maps. Let $\beta \in \mathbb{R}$. For each simple element $\sum_{i=1}^{m} f_i \otimes k_i$ in $\mathbb{H}^{\otimes n}_{\beta} \otimes \mathbb{K}$, we define

$$\begin{aligned}\delta^{(n)}\left[\sum_{i=1}^{m} f_i \otimes k_i\right] &= \sum_{i=1}^{m} \delta^{(n)}[f_i] \otimes k_i \\ &\in \mathcal{H}_{n,\beta} \otimes \mathbb{K}.\end{aligned}$$

We note that $\delta^{(n)}\left[\sum_{i=1}^{m} \mathcal{S}_n(f_i) \otimes k_i\right] = \sum_{i=1}^{m} \delta^{(n)}[f_i] \otimes k_i$. Moreover, if $\{k_i\, ;\, i \in \mathbb{N}\}$ is an orthonormal basis in $\mathbb{K}$, then $\delta^{(n)}$ is a linear densely defined function that maps (for every index γ with $|\gamma| = n$) the element $\otimes_{\alpha\in\mathcal{A}}\left(\mathfrak{e}_\alpha^{\gamma(\alpha)}\right) \otimes k_i$ in $\mathbb{H}^{\otimes n}_{\beta} \otimes \mathbb{K}$ to the element $\prod_{\alpha\in\mathcal{A}} H_{\gamma(\alpha)}(\delta[\mathfrak{e}_\alpha])\otimes k_i$ in $\mathcal{H}_{n,\beta}\otimes\mathbb{K}$. Therefore, $\delta^{(n)}$ can be extended to a bounded linear surjection from $\mathbb{H}_\beta^{\otimes n}\otimes\mathbb{K}$ to $\mathcal{H}_{n,\beta}\otimes\mathbb{K}$, whose restriction to $\mathbb{H}_\beta^{\widehat{\otimes}n}\otimes\mathbb{K}$ is unitary up to the constant $\sqrt{n!}$.

The inner product in the tensor product space $\mathcal{L}^{2,\beta}\otimes\mathbb{K}$ will be denoted by $\ll\cdot,\cdot\gg_{\beta,\mathbb{K}}$ and the corresponding norm by $\|\cdot\|_{\beta,\mathbb{K}}$. For $\beta>0$, the natural pairing of elements in $\mathcal{L}^{2,-\beta}\otimes\mathbb{H}_{-\beta}$ and $\mathcal{L}^{2,\beta}\otimes\mathbb{H}_\beta$ will be denoted by $\ll\cdot,\cdot\gg_{\mp,\mathbb{H}_{\mp\beta}}$. We have the following proposition.

Proposition 2.14. *For every real number β, $\mathcal{L}^{2,\beta}\otimes\mathbb{K}=\bigoplus_{n=0}^\infty(\mathcal{H}_{n,\beta}\otimes\mathbb{K})$ (orthogonal sum). Therefore, each $F\in\mathcal{L}^{2,\beta}\otimes\mathbb{K}$ has a Wiener-chaos decomposition $\sum_{n=0}^\infty\delta^{(n)}[f_n]$, where each f_n belongs to the space $\mathbb{H}_\beta^{\widehat{\otimes}n}\otimes\mathbb{K}$. Furthermore, $\| F\|_{\beta,\mathbb{K}}^2=\sum_{n=0}^\infty n!|f_n|^2_{\mathbb{H}_\beta^{\otimes n}\otimes\mathbb{K}}$.*

Definition 2.15. Let $F,G\in\mathcal{L}^{2,\beta}\otimes\mathbb{K}$ for some $\beta\in\mathbb{R}$. Let $\sum_{n=0}^\infty\delta^{(n)}[f_n]$ and $\sum_{n=0}^\infty\delta^{(n)}[g_n]$ be the Wiener-chaos decompositions of F and G, respectively. For each n, let $f_n=\sum_{j=1}^\infty f_{n,j}\otimes k_j$ and $g_n=\sum_{j=1}^\infty g_{n,j}\otimes k_j$, where each $f_{n,j}$ and each $g_{n,j}$ belongs to $\mathbb{H}_\beta^{\widehat{\otimes}n}$, and the set $\{k_i\,;\,i\in\mathbb{N}\}$ is an orthonormal basis for $\mathbb{K}$. The Wick inner product of F and G is denoted by $\langle F\diamond G\rangle_K$ and defined by

$$\langle F\diamond G\rangle_{\mathbb{K}}=\sum_{i=0}^\infty\sum_{n+m=i}\delta^{(i)}\left[\sum_{j=1}^\infty f_{n,j}\widehat{\otimes}g_{m,j}\right],$$

provided the sum converges in $\mathcal{L}^{2,\beta'}$ for some $\beta'\in\mathbb{R}$.

Lemma 2.16. *Let $\beta\in\mathbb{R}$, and let $f\in\mathbb{H}_\beta^{\widehat{\otimes}n}\otimes\mathbb{K}$ and $g\in\mathbb{H}_\beta^{\widehat{\otimes}m}\otimes\mathbb{K}$. Then*

$$\|\langle\delta^{(n)}[f]\diamond\delta^{(m)}[g]\rangle_{\mathbb{K}}\|_\beta\le\sqrt{\binom{n+m}{n}}\,\|\delta^{(n)}[f]\|_{\beta,\mathbb{K}}\|\delta^{(m)}[g]\|_{\beta,\mathbb{K}}\,. \tag{2.3}$$

Proof. Let $\{k_i\,;\,i\in\mathbb{N}\}$ be an orthonormal basis of $\mathbb{K}$. Then there are elements $f_i\in\mathbb{H}_\beta^{\widehat{\otimes}n}$ and $g_i\in\mathbb{H}_\beta^{\widehat{\otimes}m}$, $i\in\mathbb{N}$, such that $f=\sum_i f_i\otimes k_i$ and $g=\sum_i g_i\otimes k_i$. Then we have $\langle\delta^{(n)}[f]\diamond\delta^{(m)}[g]\rangle_{\mathbb{K}}=\sum_i\delta^{(n+m)}[f_i\widehat{\otimes}g_i]$. Therefore,

$$\begin{aligned}\|\langle\delta^{(n)}[f]\diamond\delta^{(m)}[g]\rangle_{\mathbb{K}}\|_\beta^2&=(n+m)!\Big|\sum_{i=1}^\infty f_i\widehat{\otimes}g_i\Big|^2_{\mathbb{H}_\beta^{\otimes(n+m)}}\\&\le(n+m)!\Big\langle\sum_{i=1}^\infty f_i\otimes g_i,\sum_{j=1}^\infty f_j\otimes g_j\Big\rangle_{\mathbb{H}_\beta^{\otimes(n+m)}}\\&\le(n+m)!\sum_{i=1}^\infty\sum_{j=1}^\infty|f_i|_{\mathbb{H}_\beta^{\otimes n}}|f_j|_{\mathbb{H}_\beta^{\otimes n}}|g_i|_{\mathbb{H}_\beta^{\otimes m}}|g_j|_{\mathbb{H}_\beta^{\otimes m}}\\&\le(n+m)!\sum_{i=1}^\infty|f_i|_{\mathbb{H}_\beta^{\otimes n}}|g_i|_{\mathbb{H}_\beta^{\otimes m}}\sum_{j=1}^\infty|f_j|_{\mathbb{H}_\beta^{\otimes n}}|g_j|_{\mathbb{H}_\beta^{\otimes m}}\\&\le(n+m)!\sum_{i=1}^\infty|f_i|^2_{\mathbb{H}_\beta^{\otimes n}}\sum_{i=1}^\infty|g_i|^2_{\mathbb{H}_\beta^{\otimes m}}\end{aligned}$$

$$= \binom{n+m}{n} \parallel \delta^{(n)}[f] \parallel^2_{\beta,\mathbb{K}} \parallel \delta^{(m)}[g] \parallel^2_{\beta,\mathbb{K}} .$$

□

2.4.1. *Symmetric Processes.* Let n and i be nonnegative integers, and let $\beta \in \mathbb{R}$. Elements in $\mathcal{H}_{n,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ are i-parameter stochastic processes. An element $F = \delta^{(n)}[f]$ in this space, where $f \in \left(\mathbb{H}_\beta^{\otimes n}\right) \otimes \left(\mathbb{H}_\beta^{\otimes i}\right)$, is called a *totally symmetric* process if $f \in \mathbb{H}_\beta^{\widehat{\otimes} n+i}$. In this case we clearly have $\delta^{(n)}[f] = \delta^{(n)}[\mathcal{S}_{n+i} f]$.

Example 2.17. If f_1, f_2, and f_3 are elements in $\mathbb{H}_\beta$, then $\delta^{(2)}[f_1 \widehat{\otimes} f_2] \otimes f_3 + \delta^{(2)}[f_1 \widehat{\otimes} f_3] \otimes f_2 + \delta^{(2)}[f_2 \widehat{\otimes} f_3] \otimes f_1$ is a totally symmetric one-parameter process in $\mathcal{H}_{2,\beta} \otimes \mathbb{H}_\beta$.

We denote the (closed) subspace of $\mathcal{H}_{n,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ consisting of totally symmetric processes by $\mathcal{H}_{n,\beta} \widehat{\widehat{\otimes}} (\mathbb{H}_\beta^{\otimes i})$. Elements that belong to the orthogonal complement of $\mathcal{H}_{n,\beta} \widehat{\widehat{\otimes}} (\mathbb{H}_\beta^{\otimes i})$ are called *totally antisymmetric*.

An element $F \in \mathcal{L}^{2,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ is called a totally symmetric process if $\mathbb{J}_n F \in \mathcal{H}_{n,\beta} \widehat{\widehat{\otimes}} (\mathbb{H}_\beta^{\otimes i})$ for every n. The (closed) subspace of $\mathcal{L}^{2,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ consisting of totally symmetric processes will be denoted by $\mathcal{L}^{2,\beta} \widehat{\widehat{\otimes}} (\mathbb{H}_\beta^{\otimes i})$. Processes in the orthogonal complement of $\mathcal{L}^{2,\beta} \widehat{\widehat{\otimes}} (\mathbb{H}_\beta^{\otimes i})$ will be called totally antisymmetric.

2.5. Malliavin differentiation. Let i and n be nonnegative integers, and let $f \in \mathbb{H}^{\otimes n}$. We denote the ith order Malliavin derivative of $\delta^{(n)}[f]$ by $\mathcal{D}^i(\delta^{(n)}[f])$ and define it to be the zero vector in the space $\mathbb{H}^{\otimes i}$ if $n < i$, and the element $n \cdots (n-i+1) \delta^{(n-i)}[f]$ in $\mathcal{H}_{n-i} \widehat{\widehat{\otimes}} \mathbb{H}^{\otimes i}$ if $n \geq i$. An element $F = \sum_{n=0}^\infty \delta^{(n)}[f_n] \in \mathcal{L}^2$ is called i-times differentiable in the sense of Malliavin if applying $\mathcal{D}^i$ to each term of its chaos decomposition produces a convergent series in $\mathcal{L}^2 \otimes (\mathbb{H}^{\otimes i})$. Clearly, in this case, $\mathcal{D}^i F$ belongs to $\mathcal{L}^2 \widehat{\widehat{\otimes}} (\mathbb{H}^{\otimes i})$ and $\parallel \mathcal{D}^i F \parallel^2_{\mathbb{H}^{\otimes i}} = \sum_{n=i}^\infty n \cdots (n-i+1) n! |f_n|^2_{\mathbb{H}^{\otimes n}}$. We shall denote the subcollection of $\mathcal{L}^2$ consisting of elements that are i-times differentiable in the sense of Malliavin by $\mathcal{D}^{(i)}$.

Proposition 2.18. *Let i be a positive integer, and let $\beta \geq \log_{\frac{1}{\mathbb{M}}} 2^{i+1}$. Then each element F in $\mathcal{L}^2$ has ith order Malliavin derivative in $\mathcal{L}^{2,-\beta} \otimes \mathbb{H}^{\otimes i}$. Moreover, $\parallel \mathcal{D}^i F \parallel_{-\beta,\mathbb{H}^{\otimes i}} \leq \sqrt{i!} \parallel F \parallel$.*

Proof. First we note that if $\beta \geq \log_{\frac{1}{\mathbb{M}}} 2^{i+1}$, then for every $n \geq i+1$ we have

$$\begin{aligned} \mathbb{M}^{(n-i)\beta} &\leq 2^{-(i+1)(n-i)} \\ &\leq 2^{-n} \\ &\leq \binom{n}{i}^{-1}. \end{aligned}$$

Now let $F = \sum_{n=0}^\infty \delta^{(n)}[f_n] \in \mathcal{L}^2$. Then

$$\parallel \mathcal{D}^i F \parallel^2_{-\beta,\mathbb{H}^{\otimes i}} = \sum_{n=i}^\infty n \cdots (n-i+1) n! |f_n|^2_{\mathbb{H}_{-\beta}^{\otimes n-i} \otimes \mathbb{H}^{\otimes i}}$$

$$\leq i! \sum_{n=i}^{\infty} \binom{n}{i} \mathbb{M}^{(n-i)\beta} n! |f_n|^2_{\mathbb{H}^{\otimes n}} \qquad \text{(by inequality 2.1)}$$
$$\leq i! \sum_{n=i}^{\infty} n! |f_n|^2_{\mathbb{H}^{\otimes n}}$$
$$\leq i! \parallel F \parallel^2 .$$

□

We saw above that if $F \in \mathcal{D}^{(i)}$, then $\mathcal{D}^i F$ belongs to the space $\mathcal{L}^2 \widehat{\widehat{\otimes}}(\mathbb{H}^{\otimes i})$. Now we show that the range of $\mathcal{D}^i$ is in fact the entire space $\mathcal{L}^2 \widehat{\widehat{\otimes}}(\mathbb{H}^{\otimes i})$. Therefore, an i-parameter stochastic process with values in $\mathbb{H}^{\otimes i}$ is the ith derivative of a Wiener map if and only if it is *completely* symmetric and hence, necessarily, $\mathbb{H}^{\widehat{\otimes} i}$-valued. To show this suppose $G \in \mathcal{L}^2 \widehat{\widehat{\otimes}}(\mathbb{H}^{\otimes i})$ has chaos decomposition $\sum_{n=0}^{\infty} \delta^{(n)}[g_n]$. Let $F \in \mathcal{L}^2$ be such that $\mathbb{J}_0 F, \cdots, \mathbb{J}_{i-1} F$ are zero vectors (each in its own space), and $\mathbb{J}_m(F) = \frac{(m-i)!}{m!} \delta^{(m)}[g_{m-i}]$ for $m = i, i+1, \cdots$. It is easy to verify that $F \in \mathcal{D}^{(i)}$ and $\mathcal{D}^i F = G$.

Remark 2.19. Let $f \in \mathbb{H}^{\otimes n}$ and $G = \sum_{j=0}^{\infty} \delta^{(j)}[g_j] \in \mathcal{L}^2$. Then

$$\begin{aligned} \ll \delta^{(n)}[f], G \gg &= \mathbb{E} \delta^{(n)}[f] \delta^{(n)}[g_n] \\ &= n! \langle f, g_n \rangle_{\mathbb{H}^{\otimes n}} \\ &= \sum_{j=n}^{\infty} j \cdots (j-n+1) \mathbb{E} \langle f, \delta^{(j-n)}[g_j] \rangle_{\mathbb{H}^{\otimes n}} . \end{aligned}$$

Therefore, we have

$$\ll \delta^{(n)}[f], G \gg = \ll f, \mathcal{D}^n G \gg_{\mathbb{H}^{\otimes n}} . \tag{2.4}$$

Lemma 2.20. *Let $i_1, \cdots, i_n$ be nonnegative integers and let $f_{i_1} \in \mathbb{H}^{\widehat{\otimes} i_1}, \cdots, f_{i_n} \in \mathbb{H}^{\widehat{\otimes} i_n}$ be pairwise mutually orthogonal (Definition 2.6). Then*

$$\delta^{(i_1)}[f_{i_1}] \diamond \cdots \diamond \delta^{(i_n)}[f_{i_n}] = \delta^{(i_1)}[f_{i_1}] \cdots \delta^{(i_n)}[f_{i_n}].$$

Proof. We begin by proving the lemma for the case $n = 2$. In light of the Polarization Identity and continuity of the maps $\delta^{(i_1)}$ and $\delta^{(i_2)}$, it suffices to prove the assertion in this case for elements of the type $f_1 = h_1^{i_1}$ and $f_2 = h_2^{i_2}$, where $h_1, h_2 \in \mathbb{H}$ and $h_1 \perp h_2$. In this case, the assertion is clearly true if $i_1 = 0$ or $i_2 = 0$. Suppose i_1 and i_2 are both positive integers. Let $y \in \mathbb{H}$. Then, using (2.4) we have

$$\begin{aligned} &\ll \delta^{(i_1)}[h_1^{i_1}] \delta^{(i_2)}[h_2^{i_2}], \delta^{(i_1+i_2)}[y^{i_1+i_2}] \gg \\ &\qquad = \mathbb{E} \left[\delta^{(i_1)}[h_1^{i_1}] \delta^{(i_2)}[h_2^{i_2}] \delta^{(i_1+i_2)}[y^{i_1+i_2}] \right] \\ &\qquad = \mathbb{E} \left[\langle h_1^{i_1}, \mathcal{D}^{(i_1)} \left[\delta^{(i_2)}[h_2^{i_2}] \delta^{(i_1+i_2)}[y^{i_1+i_2}] \right] \rangle_{\mathbb{H}^{\otimes i_1}} \right] \\ &\qquad = \sum_{k=0}^{i_1} \binom{i_1}{k} \frac{i_2!}{(i_2-k)!} \frac{(i_1+i_2)!}{(i_2+k)!} \langle h_1^{i_1}, h_2^k \otimes y^{i_1-k} \rangle_{\mathbb{H}^{\otimes i_1}} \end{aligned}$$

$$\times\, \mathbb{E}\left[\delta^{(i_2-k)}[h_2^{i_2-k}]\delta^{(i_2+k)}[y^{i_2+k}]\right]$$

We note that if k is positive, then the k^{th} term of the sum is zero. Therefore,

$$\begin{aligned}\ll \delta^{(i_1)}[h_1^{i_1}]\delta^{(i_2)}[h_2^{i_2}], \delta^{(i_1+i_2)}[y^{i_1+i_2}] \gg &= (i_1+i_2)!\langle h_1^{i_1}\widehat{\otimes} h_2^{i_2}, y^{i_1+i_2}\rangle_{\mathbb{H}^{\otimes i_1+i_2}} \\ &= \ll \delta^{(i_1+i_2)}[h_1^{i_1}\widehat{\otimes} h_2^{i_2}], \delta^{(i_1+i_2)}[y^{i_1+i_2}] \gg .\end{aligned}$$

Invoking the linearity of $\delta^{(i_1+i_2)}$ and the Polarization Identity, we infer the truth of the assertion if y is replaced by a simple element in $\mathbb{H}^{\widehat{\otimes}(i_1+i_2)}$. By a passage to limit, the equality holds for an arbitrarily selected element in $\mathbb{H}^{\widehat{\otimes}(i_1+i_2)}$. The proof for the case $n = 2$ is complete.

Now suppose the assertion of the Lamma is true for $n = k \geq 2$, and let $f_{i_1} \in \mathbb{H}^{\widehat{\otimes} i_1}, \cdots, f_{i_{k+1}} \in \mathbb{H}^{\widehat{\otimes} i_{k+1}}$ be pairwise mutually orthogonal. Then

$$\begin{aligned}\delta^{(i_1)}[f_{i_1}]\cdots\delta^{(i_{k+1})}[f_{i_{k+1}}] &= \left(\delta^{(i_1)}[f_{i_1}]\diamond\cdots\diamond\delta^{(i_k)}[f_{i_k}]\right)\delta^{(i_{k+1})}[f_{i_{k+1}}] \\ &= \delta^{(i_1+\cdots+i_k)}[f_{i_1}\widehat{\otimes}\cdots\widehat{\otimes} f_{i_k}]\delta^{(i_{k+1})}[f_{i_{k+1}}] \\ &= \delta^{(i_1+\cdots i_{k+1})}[f_{i_1}\widehat{\otimes}\cdots\widehat{\otimes} f_{i_{k+1}}] \\ &= \delta^{(i_1)}[f_{i_1}]\diamond\cdots\diamond\delta^{(i_{k+1})}[f_{i_{k+1}}],\end{aligned}$$

since the elements $f_1\widehat{\otimes}\cdots\widehat{\otimes} f_{i_k}$ and $f_{i_{k+1}}$ are mutually orthogonal. □

2.6. Adjoint of the Malliavin derivative. Let $\beta \geq 0$. We know from the preceding subsections that for fixed nonnegative integers n and i, $i \leq n$, the map $\mathcal{D}^i : \mathcal{H}_{n+i,\beta} \to \mathcal{H}_{n,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ is bounded and linear and its range is $\mathcal{H}_{n,\beta}\widehat{\widehat{\otimes}}(\mathbb{H}_\beta^{\otimes i})$. We denote the adjoint of this map by $\delta^{(i)}$. We choose this notation since in the case $n = 0$ and $\beta = 0$, this definition coincides with the definition of $\delta^{(i)}$ given in subsection 2.2. We identify the adjoint of $\mathcal{H}_{n,\beta} \otimes (\mathbb{H}_\beta^{\otimes i})$ with $\mathcal{H}_{n,-\beta} \otimes (\mathbb{H}_{-\beta}^{\otimes i})$ and the adjoint of $\mathcal{H}_{n+i,\beta}$ with $\mathcal{H}_{n+i,-\beta}$. Hence, if $F \in \mathcal{H}_{n,-\beta} \otimes (\mathbb{H}_{-\beta}^{\otimes i})$, then $\delta^{(i)}F$ is the unique element in $\mathcal{H}_{n+i,-\beta}$ such that

$$\ll \delta^{(i)}F, G \gg_{\mp\beta} = \ll F, \mathcal{D}^i G \gg_{\mp\beta,\otimes^i\mathbb{H}_{\mp\beta}} .$$

for every $G \in \mathcal{H}_{n+i,\beta}$. The following proposition clarifies the action of $\delta^{(i)}$ on elements of the space $\mathcal{H}_{n,-\beta} \otimes (\mathbb{H}_{-\beta}^{\otimes i})$.

Proposition 2.21. *Let $\beta \geq 0$, and let $f \in \mathbb{H}_{-\beta}^{\widehat{\otimes} n}$ and $g \in \mathbb{H}_{-\beta}^{\widehat{\otimes} i}$. Then*

$$\delta^{(i)}\left[\delta^{(n)}[f]\otimes g\right] = \delta^{(n)}[f]\diamond\delta^{(i)}[g]. \tag{2.5}$$

Proof. Clearly $\delta^{(i)}\left[\delta^{(n)}[f]\otimes g\right]$ is an element in $\mathcal{H}_{n+i,-\beta}$. We prove this proposition by showing that $\ll \delta^{(i)}\left[\delta^{(n)}[f]\otimes g\right], Y \gg_{\mp\beta} = \ll \delta^{(n+i)}[f\widehat{\otimes} g], Y \gg_{\mp\beta}$ for every $Y \in \mathcal{H}_{n+i,\beta}$. It suffices to prove the proposition for the case in which f and g are elementary objects. So let $f = \widehat{\otimes}_{j=1}^n f_j$ and $g = \otimes_{j=1}^i f_{j+n}$, where $f_1, \cdots, f_{i+n} \in \mathbb{H}_{-\beta}$. Let $Y = \delta^{(n+i)}\left[\widehat{\otimes}_{j=1}^{n+i} y_j\right]$ be an elementary object in $\mathcal{H}_{n+i,\beta}$. Then we have

$$\ll \delta^{(i)}\left[\delta^{(n)}[f]\otimes g\right], Y \gg_{\mp\beta} = \ll \delta^{(n)}[f]\otimes g, \mathcal{D}^i Y \gg_{\mp\beta,\otimes^i\mathbb{H}_{\mp\beta}}$$

$$
\begin{aligned}
&= (n+i)\cdots(n+1) \ll \delta^{(n)}[f], \langle g, \delta^{(n)}\left[\widehat{\otimes}_{j=1}^{n+i} y_j\right]\rangle_{\mathbb{H}_{\mp\beta}^{\otimes i}} \gg_{\mp\beta} \\
&= i! \sum_{\substack{A\subseteq\{1,\cdots,n+i\}\\ \#A=n}} \ll \delta^{(n)}[f], \delta^{(n)}[\widehat{\otimes}_{j\in A} y_j] \gg_{\mp\beta} \langle g, \widehat{\otimes}_{j\notin A} y_j\rangle_{\mathbb{H}_{\mp\beta}^{\otimes i}} \\
&= i!n! \sum_{\substack{A\subseteq\{1,\cdots,n+i\}\\ \#A=n}} \langle f, \widehat{\otimes}_{j\in A} y_j\rangle_{\mathbb{H}_{\mp\beta}^{\otimes n}} \langle g, \widehat{\otimes}_{j\notin A} y_j\rangle_{\mathbb{H}_{\mp\beta}^{\otimes i}} \\
&= i!n! \langle f\otimes g, \sum_{\substack{A\subseteq\{1,\cdots,n+i\}\\ \#A=n}} [\widehat{\otimes}_{j\in A} y_j]\otimes[\widehat{\otimes}_{j\notin A} y_j]\rangle_{\mathbb{H}_{\mp\beta}^{\otimes n+i}} \\
&= (n+i)! \langle f\otimes g, \widehat{\otimes}^{n+i} y_j\rangle_{\mathbb{H}_{\mp\beta}^{\otimes n+i}} \\
&= \ll \delta^{(n)}[f]\diamond\delta^{(i)}[g], Y \gg_{\mp\beta} .
\end{aligned}
$$

Note that we have used

$$
\widehat{\otimes}_{j=1}^{n+i} y_j = \frac{n!i!}{(n+i)!} \sum_{\substack{A\subseteq\{1,\cdots,n+i\}\\ \#A=n}} [\widehat{\otimes}_{j\in A} y_j]\otimes[\widehat{\otimes}_{j\notin A} y_j].
$$

The equality holds if Y is a simple element (finite sum of elementary objects). By a passage to limit, it is true for every $Y\in\mathcal{H}_{n+i,\beta}$. The proof is now complete. □

Corollary 2.22. *Let $\beta\geq 0$. If $f\in\mathbb{H}_{-\beta}^{\otimes n}$ and $g\in\mathbb{H}_{-\beta}^{\otimes i}$ are mutually orthogonal (Definition 2.6), then $\delta^{(i)}[\delta^{(n)}[f]\otimes g] = \delta^{(n)}[f]\delta^{(i)}[g]$.*

Corollary 2.23. *Let $\beta\geq 0$ and $f\in\mathbb{H}_{-\beta}^{\otimes n}$. If g is an antisymmetric element in $\mathbb{H}_{-\beta}^{\otimes i}$, then $\delta^{(i)}[\delta^{(n)}[f]\otimes g] = 0$.*

Definition 2.24. Le $\beta\geq 0$, and let $F=\sum_{n=0}^{\infty}\delta^{(n)}[f_n]$ be an element of the tensor product space $\mathcal{L}^{2,-\beta}\otimes(\mathbb{H}_{-\beta}^{\otimes i})$. We define $\delta^{(i)}[F]=\sum_{n=0}^{\infty}\delta^{(n+i)}[f_n]$, provided this sum converges in $\mathcal{L}^{2,-\beta}$.

Remark 2.25. The domain of $\delta^{(i)}$ in $\mathcal{L}^{2,-\beta}\otimes(\mathbb{H}_{-\beta}^{\otimes i})$ consists of all functions $F=\sum_{n=0}^{\infty}\delta^{(n)}[f_n]$ satisfying the condition $\sum_{n=0}^{\infty}(n+i)!|\mathcal{S}_{n+i}(f_n)|^2_{\mathbb{H}_{-\beta}^{\otimes n+i}}<\infty$.

3. Connection Between Wick and Ordinary Inner Products

In subsection 2.4 we defined the Wick inner product of elements in $\mathcal{L}^{2,\beta}\otimes\mathbb{K}$ (Definition 2.16). In this section we explore Wick inner products of vector-valued test functions and identify the spaces to which these products belong. Moreover, we generalize the results in [5] by proving identities describing the relation between Wick and ordinary inner products of Hilbert space-valued Wiener maps and by proving the closure property of $\mathcal{L}^{2,\infty}\otimes\mathbb{K}$ under both operations.

Proposition 3.1. *Let $f\in\mathbb{H}^{\widehat{\otimes} n}\otimes\mathbb{K}$ and $g\in\mathbb{H}^{\widehat{\otimes} m}\otimes\mathbb{K}$. Then*

$$
\langle\delta^{(n)}[f],\delta^{(m)}[g]\rangle_{\mathbb{K}} = \sum_{i=0}^{n\wedge m} i!\binom{n}{i}\binom{m}{i}\langle\delta^{(n-i)}[f]\diamond\delta^{(m-i)}[g]\rangle_{\mathbb{H}^{\otimes i}\otimes\mathbb{K}}
$$

$$= \sum_{i=0}^{n\wedge m} \frac{1}{i!} \langle \mathcal{D}^i \delta^{(n)}[f] \diamond \mathcal{D}^i \delta^{(m)}[g] \rangle_{\mathbb{H}^{\otimes i} \otimes \mathbb{K}}, \quad \textit{and} \tag{3.1}$$

$$\langle \delta^{(n)}[f] \diamond \delta^{(m)}[g] \rangle_{\mathbb{K}} = \sum_{i=0}^{n\wedge m} (-1)^i i! \binom{n}{i}\binom{m}{i} \langle \delta^{(n-i)}[f], \delta^{(m-i)}[g] \rangle_{\mathbb{H}^{\otimes i} \otimes \mathbb{K}}$$

$$= \sum_{i=0}^{n\wedge m} \frac{(-1)^i}{i!} \langle \mathcal{D}^i \delta^{(n)}[f], \mathcal{D}^i \delta^{(m)}[g] \rangle_{\mathbb{H}^{\otimes i} \otimes \mathbb{K}}. \tag{3.2}$$

Proof. We begin by proving identity 3.1 for the simple case in which $f = h_1^n$ and $g = h_2^m$, where h_1 and h_2 are unit vectors in $\mathbb{H}$. We use the decomposition $h_2 = \langle h_1, h_2 \rangle_{\mathbb{H}} h_1 + l$, where $l \perp h_1$. If $m \leq n$, then

$$\delta^{(n)}[h_1^n]\delta^{(m)}[h_2^m] = \sum_{i=0}^{m} \binom{m}{i} \langle h_1, h_2 \rangle_{\mathbb{H}}^i \delta^{(n)}[h_1^n] \delta^{(m)}[h_1^i \otimes l^{m-i}]$$

$$= \sum_{i=0}^{m} \binom{m}{i} \langle h_1, h_2 \rangle_{\mathbb{H}}^i \delta^{(n)}[h_1^n] \delta^{(i)}[h_1^i] \delta^{(m-i)}[l^{m-i}] \qquad \text{(Lemma 2.21)}$$

$$= \sum_{i=0}^{m} \sum_{k=0}^{i} \langle h_1, h_2 \rangle_{\mathbb{H}}^i k! \binom{m}{i}\binom{n}{k}\binom{i}{k} \delta^{(n+i-2k)}[h_1^{n+i-2k}] \delta^{m-i}[l^{m-i}] \quad \text{(Remark 2.10)}$$

$$= \sum_{k=0}^{m} k! \binom{n}{k} \sum_{j=0}^{m-k} \langle h_1, h_2 \rangle_{\mathbb{H}}^{j+k} \binom{m}{k+j}\binom{j+k}{k} \delta^{(n+j-k)}[h_1^{n+j-k}] \delta^{(m-j-k)}[l^{m-j-k}]$$

$$= \sum_{k=0}^{m} k! \binom{n}{k}\binom{m}{k} \sum_{j=0}^{m-k} \binom{m-k}{j} \langle h_1, h_2 \rangle_{\mathbb{H}}^{j+k} \delta^{(n+m-2k)}[h_1^{n+j-k} \otimes l^{m-j-k}]$$

$$= \sum_{k=0}^{m} k! \binom{n}{k}\binom{m}{k} \delta^{(n+m-2k)} \Big[\langle h_1, h_2 \rangle_{\mathbb{H}}^k h_1^{n-k} \otimes \sum_{j=0}^{m-k} \binom{m-k}{j} (\langle h_1, h_2 \rangle_{\mathbb{H}} h_1)^j \otimes l^{m-j-k} \Big]$$

$$= \sum_{k=0}^{m} k! \binom{n}{k}\binom{m}{k} \delta^{(n+m-2k)} \left[\langle h_1, h_2 \rangle_{\mathbb{H}}^k h_1^{n-k} \otimes h_2^{m-k} \right]$$

$$= \sum_{k=0}^{m} \frac{n!m!}{k!(n-k)!(m-k)!} \langle \delta^{(n-k)}[h_1^n] \diamond \delta^{(m-k)}[h_2^m] \rangle_{\mathbb{H}^{\otimes k}}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \langle \mathcal{D}^{(k)} \delta^{(n)}[h_1^n] \diamond \mathcal{D}^{(k)} \delta^{(m)}[h_2^m] \rangle_{\mathbb{H}^{\otimes k}}.$$

In the last equation we have used the fact $\mathcal{D}^k \delta^{(m)}[h_2^m] = 0$ for every $k > m$. Invoking the Polarization Identity and the continuity property of $\delta^{(n)}$ and $\delta^{(m)}$ we arrive at

$$\delta^{(n)}[f]\delta^{(m)}[g] = \sum_{k=0}^{\infty} \frac{1}{k!} \langle \mathcal{D}^{(k)} \delta^{(n)}[f] \diamond \mathcal{D}^{(k)} \delta^{(m)}[g] \rangle_{\mathbb{H}^{\otimes k}}$$

for every $f \in \mathbb{H}^{\widehat{\otimes} n}$ and $g \in \mathbb{H}^{\widehat{\otimes} m}$.

Now we use identity 3.1 to prove assertion 3.2. Let $f \in \mathbb{H}^{\widehat{\otimes} n}$ and $g \in \mathbb{H}^{\widehat{\otimes} m}$. Assume $m \leq n$. Then

$$\begin{aligned}
\sum_{i=0}^{n\wedge m} (-1)^i i! \binom{n}{i}\binom{m}{i} \langle \delta^{(n-i)}[f], \delta^{(m-i)}[g]\rangle_{\mathbb{H}^{\otimes i}} \\
&= \sum_{i=0}^{m} \frac{(-1)^i n! m!}{i!(n-i)!(m-i)!} \langle \delta^{(n-i)}[f], \delta^{(m-i)}[g]\rangle_{\mathbb{H}^{\otimes i}} \\
&= \sum_{i=0}^{m}\sum_{k=i}^{m} \frac{(-1)^i n! m!}{i!(k-i)!(n-k)!(m-k)!} \langle \delta^{(n-k)}[f] \diamond \delta^{(m-k)}[g]\rangle_{\mathbb{H}^{\otimes k}} \\
&= \sum_{k=0}^{m} \left(\sum_{i=0}^{k} \frac{(-1)^i k!}{i!(k-i)!}\right) \frac{n! m!}{k!(n-k)!(m-k)!} \langle \delta^{(n-k)}[f] \diamond \delta^{(m-k)}[g]\rangle_{\mathbb{H}^{\otimes k}}.
\end{aligned}$$

Identity 3.2 follows since $\sum_{i=0}^{k} \frac{(-1)^i k!}{i!(k-i)!} = 0$ if $k > 0$.

Now let $f \in \mathbb{H}^{\widehat{\otimes} n} \otimes \mathbb{K}$ and $g \in \mathbb{H}^{\widehat{\otimes} m} \otimes \mathbb{K}$. Then for an orthonormal basis $\{k_j\,;\, j \in \mathbb{N}\}$ of $\mathbb{K}$, there are sequences $\{f_j\,;\, j \in \mathbb{N}\}$ and $\{g_j\,;\, j \in \mathbb{N}\}$ in $\mathbb{H}^{\widehat{\otimes} n}$ and $\mathbb{H}^{\widehat{\otimes} m}$, respectively, such that $f = \sum_{j=1}^{\infty} f_j \otimes k_j$ and $g = \sum_{j=1}^{\infty} g_j \otimes k_j$. Then

$$\begin{aligned}
\langle \delta^{(n)}[f], \delta^{(m)}[g]\rangle_{\mathbb{K}} &= \sum_{j=1}^{\infty} \delta^{(n)}[f_j]\delta^{(m)}[g_j] \\
&= \sum_{j=1}^{\infty}\sum_{i=0}^{n\wedge m} i! \binom{n}{i}\binom{m}{i} \langle \delta^{(n-i)}[f_j] \diamond \delta^{(m-i)}[g_j]\rangle_{\mathbb{H}^{\otimes i}} \\
&= \sum_{i=0}^{n\wedge m} i! \binom{n}{i}\binom{m}{i} \sum_{j=1}^{\infty} \langle \delta^{(n-i)}[f_j] \diamond \delta^{(m-i)}[g_j]\rangle_{\mathbb{H}^{\otimes i}} \\
&= \sum_{i=0}^{n\wedge m} i! \binom{n}{i}\binom{m}{i} \langle \delta^{(n-i)}[f] \diamond \delta^{(m-i)}[g]\rangle_{\mathbb{H}^{\otimes i}\otimes\mathbb{K}}.
\end{aligned}$$

Identity 3.2 can be proved in the same way. □

Corollary 3.2. *Let n be a positive integer. For every $h \in \mathbb{H}$ and $f \in \mathbb{H}^{\widehat{\otimes} n}$ we have*

$$\delta[h]\delta^{(n)}[f] = \delta^{(n+1)}[h \widehat{\otimes} f] + n\langle \delta^{(n-1)}[f], h\rangle_{\mathbb{H}}.$$

Theorem 3.3. *Let $\beta > 0$, and let $\beta' < \log_{\frac{1}{\mathbb{M}}}\left(\frac{1}{2}\left(\frac{1}{\mathbb{M}}\right)^{\beta} - \frac{1}{2}\right)$. If $F, G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$, then the series $\sum_{k=0}^{\infty} \frac{1}{k!}\langle \mathcal{D}^k F \diamond \mathcal{D}^k G\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}}$ converges in $\mathcal{L}^{2,\beta'}$. Moreover*

$$\langle F, G\rangle_{\mathbb{K}} = \sum_{k=0}^{\infty} \frac{1}{k!}\langle \mathcal{D}^k F \diamond \mathcal{D}^k G\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}}. \tag{3.3}$$

Proof. Let $\beta' \in \mathbb{R}$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{k!} \parallel \langle \mathcal{D}^k F \diamond \mathcal{D}^k G\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}} \parallel_{\beta'} \\
&\quad\le \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i,j=k}^{\infty}\frac{i!j!}{(i-k)!(j-k)!} \parallel \langle \delta^{(i-k)}[f_i]\diamond\delta^{(j-k)}[g_j]\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}} \parallel_{\beta'} \\
&\quad\le \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i,j=k}^{\infty}\frac{i!j!}{(i-k)!(j-k)!}\sqrt{\binom{i+j-2k}{i-k}} \\
&\qquad\qquad \times \parallel \delta^{(i-k)}[f_i] \parallel_{\beta',\mathbb{H}^{\otimes k}\otimes\mathbb{K}} \parallel \delta^{(j-k)}[g_j] \parallel_{\beta',\mathbb{H}^{\otimes k}\otimes\mathbb{K}}, \quad \text{(Lemma 2.17)} \\
&\quad= \sum_{k=0}^{\infty}\frac{1}{k!}\sum_{i,j=k}^{\infty}\frac{i!j!}{(i-k)!(j-k)!}\sqrt{\binom{i+j-2k}{i-k}(i-k)!(j-k)!} \\
&\qquad\qquad \times |f_i|_{\mathbb{H}_{\beta'}^{\otimes(i-k)}\otimes\mathbb{H}^{\otimes k}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_{\beta'}^{\otimes(j-k)}\otimes\mathbb{H}^{\otimes k}\otimes\mathbb{K}}.
\end{aligned}$$

Suppose $0 \le \beta' < \beta$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{k!} \parallel \langle \mathcal{D}^k F \diamond \mathcal{D}^k G\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}} \parallel_{\beta'} \\
&\quad\le \sum_{k=0}^{\infty}\sum_{i,j=k}^{\infty}\sqrt{\binom{i}{k}\binom{j}{k}\binom{i+j-2k}{i-k}}\mathbb{M}^{\beta' k}\sqrt{i!}|f_i|_{\mathbb{H}_{\beta'}^{\otimes i}\otimes\mathbb{K}}\sqrt{j!}|g_j|_{\mathbb{H}_{\beta'}^{\otimes j}\otimes\mathbb{K}} \\
&\quad\le \sum_{k=0}^{\infty}\sum_{i,j=k}^{\infty}\sqrt{\binom{i}{k}\binom{j}{k}}2^{\frac{i+j}{2}-k}\mathbb{M}^{\beta' k} \parallel \delta^{(i)}[f_i] \parallel_{\beta',\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\beta',\mathbb{K}} \\
&\quad= \sum_{i,j=0}^{\infty}\Bigg(2^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\beta',\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\beta',\mathbb{K}} \\
&\qquad\qquad \times \sum_{k=0}^{i\wedge j}\sqrt{\binom{i}{k}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^k}\sqrt{\binom{j}{k}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^k}\Bigg) \\
&\quad\le \sum_{i,j=0}^{\infty}\Bigg(2^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\beta',\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\beta',\mathbb{K}} \\
&\qquad\qquad \times \sqrt{\sum_{k=0}^{i}\binom{i}{k}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^k}\sqrt{\sum_{k=0}^{j}\binom{j}{k}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^k}\Bigg) \\
&\quad= \left(\sum_{i=0}^{\infty}\left(2+\mathbb{M}^{\beta'}\right)^{i/2} \parallel \delta^{(i)}[f_i] \parallel_{\beta',\mathbb{K}}\right)\left(\sum_{j=0}^{\infty}\left(2+\mathbb{M}^{\beta'}\right)^{j/2} \parallel \delta^{(j)}[g_j] \parallel_{\beta',\mathbb{K}}\right)
\end{aligned}$$

Now using inequality 2.1 we get

$$\sum_{k=0}^{\infty} \frac{1}{k!} \parallel \langle \mathcal{D}^k F \diamond \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}} \parallel_{\beta'}$$

$$\leq \left(\sum_{i=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'} + \mathbb{M}^{\beta} \right)^{i/2} \parallel \delta^{(i)}[f_i] \parallel_{\beta,\mathbb{K}} \right) \times \left(\sum_{j=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'} + \mathbb{M}^{\beta} \right)^{j/2} \parallel \delta^{(j)}[g_j] \parallel_{\beta,\mathbb{K}} \right). \quad (3.4)$$

Now suppose $\beta' < 0$. Then

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{1}{k!} \parallel \langle \mathcal{D}^k F \diamond \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}} \parallel_{\beta'} \\
&\leq \sum_{k=0}^{\infty} \sum_{i,j=k}^{\infty} \sqrt{\binom{i}{k}\binom{j}{k}\binom{i+j-2k}{i-k}} \mathbb{M}^{-\beta'\left(\frac{i+j}{2}-k\right)} \sqrt{i!} |f_i|_{\mathbb{H}^{\otimes i} \otimes \mathbb{K}} \sqrt{j!} |g_j|_{\mathbb{H}^{\otimes j} \otimes \mathbb{K}} \\
&\leq \sum_{i,j=0}^{\infty} \left(2\mathbb{M}^{-\beta'} \right)^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\mathbb{K}} \sum_{k=0}^{i \wedge j} \sqrt{\binom{i}{k}\binom{j}{k}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^{2k}} \\
&\leq \sum_{i,j=0}^{\infty} \left(2\mathbb{M}^{-\beta'} \right)^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\mathbb{K}} \\
&\qquad \times \sqrt{\sum_{k=0}^{i} \binom{i}{k} \left(\frac{\mathbb{M}^{\beta'}}{2}\right)^{k}} \sqrt{\sum_{k=0}^{j} \binom{j}{k} \left(\frac{\mathbb{M}^{\beta'}}{2}\right)^{k}} \\
&= \sum_{i,j=0}^{\infty} \left(2\mathbb{M}^{-\beta'} \right)^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\mathbb{K}} \left(1 + \frac{\mathbb{M}^{\beta'}}{2} \right)^{(i+j)/2} \\
&= \left(\sum_{i=0}^{\infty} \left(2\mathbb{M}^{-\beta'} + 1 \right)^{i/2} \parallel \delta^{(i)}[f_i] \parallel_{\mathbb{K}} \right) \left(\sum_{j=0}^{\infty} \left(2\mathbb{M}^{-\beta'} + 1 \right)^{j/2} \parallel \delta^{(j)}[g_j] \parallel_{\mathbb{K}} \right) \\
&\leq \left(\sum_{i=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'} + \mathbb{M}^{\beta} \right)^{i/2} \parallel \delta^{(i)}[f_i] \parallel_{\beta,\mathbb{K}} \right) \\
&\qquad \times \left(\sum_{j=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'} + \mathbb{M}^{\beta} \right)^{j/2} \parallel \delta^{(j)}[g_j] \parallel_{\beta,\mathbb{K}} \right)
\end{aligned}$$

Therefore, inequality 3.4 holds for every $\beta' < \beta$. It follows from the assumption $\beta' < \log_{\frac{1}{\mathbb{M}}} \left(\frac{1}{2} \left(\frac{1}{\mathbb{M}}\right)^{\beta} - \frac{1}{2} \right)$ that $2\mathbb{M}^{\beta-\beta'} + \mathbb{M} \in (0,1)$. Therefore,

$$\sum_{k=0}^{\infty} \frac{1}{k!} \parallel \langle \mathcal{D}^k F \diamond \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}} \parallel_{\beta'}$$

$$\leq \left(\sum_{i=0}^{\infty} \left(2\mathbb{M}^{(\beta-\beta')} + \mathbb{M}^{\beta} \right)^{i/2} \parallel \delta^{(i)}[f_i] \parallel_{\beta,\mathbb{K}} \right)$$
$$\times \left(\sum_{j=0}^{\infty} \left(2\mathbb{M}^{(\beta-\beta')} + \mathbb{M}^{\beta} \right)^{j/2} \parallel \delta^{(j)}[g_j] \parallel_{\beta,\mathbb{K}} \right)$$
$$\leq \frac{1}{1 - 2\mathbb{M}^{\beta-\beta'} - \mathbb{M}^{\beta}} \parallel F \parallel_{\beta,\mathbb{K}} \parallel G \parallel_{\beta,\mathbb{K}} .$$

Now let $F = \sum_{n=0}^{\infty} \delta^{(n)}[f_n]$ and $G = \sum_{n=0}^{\infty} \delta^{(n)}[g_n]$ be elements in $\mathcal{L}^2$. Then

$$\begin{aligned} F \cdot G &= \sum_{n,m=0}^{\infty} \delta^{(n)}[f_n]\delta^{(m)}[g_m] \\ &= \sum_{n,m=0}^{\infty} \sum_{k=0}^{\infty} \frac{1}{k!} \langle \mathcal{D}^k \delta^{(n)}[f_n] \diamond \mathcal{D}^k \delta^{(m)}[g_m] \rangle_{\mathbb{H}^{\otimes k}} \quad \text{(Proposition 3.1)} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} \langle \mathcal{D}^k F \diamond \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k}}. \end{aligned}$$

The proof of the assertion for the case in which F and G are both real valued is complete.

If F and G are vector-valued elements in $\mathcal{L}^{2,\beta} \otimes \mathbb{K}$, then we can prove the assertion by applying identity 3.3 ($\mathbb{K} = \mathbb{R}$) to each term of the expansion of $\langle F, G \rangle_{\mathbb{K}}$ obtained using the Fourier coefficients of F and G with respect to an orthonormal basis of $\mathbb{K}$. □

Corollary 3.4. *Let $F, G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$, where $\beta > \log_{\frac{1}{\mathbb{M}}} 3$. Then $\langle F, G \rangle_{\mathbb{K}} \in \mathcal{L}^2$ and equation 3.3 holds in this space. Moreover, the map $(F, G) \mapsto \langle F, G \rangle_{\mathbb{K}}$ from the product space $(\mathcal{L}^{2,\beta} \otimes \mathbb{K}) \times (\mathcal{L}^{2,\beta} \otimes \mathbb{K})$ to $\mathcal{L}^2$ is continuous.*

Corollary 3.5. *If $F, G \in \mathcal{L}^{2,\infty} \otimes \mathbb{K}$, then $\langle F, G \rangle_{\mathbb{K}} \in \mathcal{L}^{2,\infty}$ and equation 3.3 holds in this space. Furthermore, the map $(F, G) \mapsto \langle F, G \rangle_{\mathbb{K}}$ from the product space $(\mathcal{L}^{2,\infty} \otimes \mathbb{K}) \times (\mathcal{L}^{2,\infty} \otimes \mathbb{K})$ to $\mathcal{L}^{2,\infty}$ is continuous.*

Theorem 3.6. *Let $\beta > 0$ and let $\beta' < \log_{\frac{1}{\mathbb{M}}} \left(\frac{1}{2} \left(\frac{1}{\mathbb{M}} \right)^{\beta} - 1 \right)$. If $F, G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$, then the series $\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \mathcal{D}^k F, \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}}$ converges in $\mathcal{L}^{2,\beta'}$, and in this space,*

$$\langle F \diamond G \rangle_{\mathbb{K}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \mathcal{D}^k F, \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}}. \tag{3.5}$$

Proof. Let $\beta > 0$ and let $\beta' < \beta$. Let $F = \sum_{i=0}^{\infty} \delta^{(i)}[f_i]$ and $G = \sum_{i=0}^{\infty} \delta^{(i)}[g_i]$ be elements of $\mathcal{L}^{2,\beta'} \otimes \mathbb{K}$. For every i and j and $k \leq i \wedge j$ we have

$$\begin{aligned} &\parallel \langle \delta^{(i-k)}[f_i], \delta^{(j-k)}[g_j] \rangle_{\mathbb{H}^{\otimes k} \otimes \mathbb{K}} \parallel_{\beta'} \\ &\leq \sum_{l=0}^{(i \wedge j)-k} l! \binom{i-k}{l} \binom{j-k}{l} \parallel \langle \delta^{(i-k-l)}[f_i] \diamond \delta^{(j-k-l)}[g_j] \rangle_{\mathbb{H}^{\otimes k+l} \otimes \mathbb{K}} \parallel_{\beta'} \end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{l=0}^{(i\wedge j)-k} l!\binom{i-k}{l}\binom{j-k}{l}\sqrt{\binom{i+j-2k-2l}{i-k-l}} \\
&\qquad \times \parallel \delta^{(i-k-l)}[f_i] \parallel_{\beta',\mathbb{H}^{\otimes k+l}\otimes\mathbb{K}} \parallel \delta^{(j-k-l)}[g_j] \parallel_{\beta',\mathbb{H}^{\otimes k+l}\otimes\mathbb{K}} \\
&\leq \sum_{l=0}^{(i\wedge j)-k} l!\binom{i-k}{l}\binom{j-k}{l}\sqrt{\binom{i+j-2k-2l}{i-k-l}}\mathbb{M}^{(\beta-\beta')(i+j-2k-2l)/2}\mathbb{M}^{\beta(k+l)} \\
&\qquad \times \sqrt{(i-k-l)!(j-k-l)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&\leq \left(2\mathbb{M}^{\beta-\beta'}\right)^{\frac{i+j}{2}-k}\mathbb{M}^{\beta k}\sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&\qquad \times \sum_{l=0}^{(i\wedge j)-k}\sqrt{\binom{i-k}{l}\binom{j-k}{l}\left(\frac{\mathbb{M}^{2\beta'}}{4}\right)^l} \\
&\leq \left(2\mathbb{M}^{\beta-\beta'}\right)^{\frac{i+j}{2}-k}\mathbb{M}^{\beta k}\sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&\qquad \times \sqrt{\sum_{l=0}^{i-k}\binom{i-k}{l}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^l}\sqrt{\sum_{l=0}^{j-k}\binom{j-k}{l}\left(\frac{\mathbb{M}^{\beta'}}{2}\right)^l} \\
&= \left(2\mathbb{M}^{\beta-\beta'}\right)^{\frac{i+j}{2}-k}\mathbb{M}^{\beta k}\sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}}\left(1+\frac{\mathbb{M}^{\beta'}}{2}\right)^{\frac{i+j}{2}-k} \\
&= \left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}-k}\mathbb{M}^{\beta k}\sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&= \left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}}\left(2\mathbb{M}^{-\beta'}+1\right)^{-k}\sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}}.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\sum_{k=0}^{\infty}\frac{1}{k!}\parallel \langle\mathcal{D}^k F,\mathcal{D}^k G\rangle_{\mathbb{H}^{\otimes k}\otimes\mathbb{K}} \parallel_{\beta'} \\
&\leq \sum_{k=0}^{\infty}\sum_{i,j=k}^{\infty}\frac{i!j!}{k!(i-k)!(j-k)!}\left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}}\left(2\mathbb{M}^{-\beta'}+1\right)^{-k} \\
&\qquad \times \sqrt{(i-k)!(j-k)!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&= \sum_{i,j=0}^{\infty}\Bigg(\left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}}\sqrt{i!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}\sqrt{j!}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}} \\
&\qquad \times \sum_{k=0}^{i\wedge j}\sqrt{\binom{i}{k}\binom{j}{k}(2\mathbb{M}^{-\beta'}+1)^{-2k}}\Bigg) \\
&\leq \sum_{i,j=0}^{\infty}\Bigg(\left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}}\sqrt{i!}|f_i|_{\mathbb{H}_\beta^{\otimes i}\otimes\mathbb{K}}\sqrt{j!}|g_j|_{\mathbb{H}_\beta^{\otimes j}\otimes\mathbb{K}}
\end{aligned}
$$

$$\times \sqrt{\sum_{k=0}^{i} \binom{i}{k} (2\mathbb{M}^{-\beta'}+1)^{-k}} \sqrt{\sum_{k=0}^{j} \binom{j}{k} (2\mathbb{M}^{-\beta'}+1)^{-k}} \Bigg)$$

$$\leq \sum_{i,j=0}^{\infty} \left(\left(2\mathbb{M}^{\beta-\beta'}+\mathbb{M}^{\beta}\right)^{\frac{i+j}{2}} \left(1+\frac{1}{2\mathbb{M}^{-\beta'}+1}\right)^{\frac{i+j}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\beta,\mathbb{K}} \parallel \delta^{(j)}[g_j] \parallel_{\beta,\mathbb{K}}\right.$$

$$= \sum_{i=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'}+2\mathbb{M}^{\beta}\right)^{\frac{i}{2}} \parallel \delta^{(i)}[f_i] \parallel_{\beta,\mathbb{K}} \sum_{j=0}^{\infty} \left(2\mathbb{M}^{\beta-\beta'}+2\mathbb{M}^{\beta}\right)^{\frac{j}{2}} \parallel \delta^{(j)}[g_j] \parallel_{\beta,\mathbb{K}}$$

$$\leq \frac{1}{1-2\mathbb{M}^{\beta-\beta'}-2\mathbb{M}^{\beta}} \parallel F \parallel_{\beta,\mathbb{K}} \parallel G \parallel_{\beta,\mathbb{K}} .$$

Note that condition $\beta' < \log_{\frac{1}{\mathbb{M}}} \left(\frac{1}{2}\left(\frac{1}{\mathbb{M}}\right)^{\beta} - 1\right)$ ensures that $2\mathbb{M}^{\beta-\beta'} + 2\mathbb{M}^{\beta} < 1$. If F and G are real-valued, then

$$\begin{aligned} F \diamond G &= \sum_{i,j=0}^{\infty} \delta^{(i)}[f_i] \diamond \delta^{(j)}[g_j] \\ &= \sum_{i,j=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \mathcal{D}^k \delta^{(i)}[f_i], \mathcal{D}^k \delta^{(j)}[g_j] \rangle_{\mathbb{H}^{\otimes k}} \qquad \text{(Proposition 3.1)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \langle \mathcal{D}^k F, \mathcal{D}^k G \rangle_{\mathbb{H}^{\otimes k}}. \end{aligned}$$

The proof for the case $\mathbb{K} = \mathbb{R}$ is complete. This case can be used to prove the assertion for vector-valued elements in $\mathcal{L}^{2,\beta} \otimes \mathbb{K}$ by considering expansions of F and G with respect to an orthonormal basis of $\mathbb{K}$. □

Corollary 3.7. *Let $F, G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$, where $\beta > \log_{\frac{1}{\mathbb{M}}} 4$. Then $\langle F \diamond G \rangle_{\mathbb{K}} \in \mathcal{L}^2$, and in this space, equation 3.5 holds. Moreover, the map $(F, G) \mapsto \langle F \diamond G \rangle_{\mathbb{K}}$ from the product space $(\mathcal{L}^{2,\beta} \otimes \mathbb{K}) \times (\mathcal{L}^{2,\beta} \otimes \mathbb{K})$ to $\mathcal{L}^2$ is continuous.*

Corollary 3.8. *If $F, G \in \mathcal{L}^{2,\infty} \otimes \mathbb{K}$, then $\langle F \diamond G \rangle_{\mathbb{K}} \in \mathcal{L}^{2,\infty}$, and in this space, equation 3.5 holds. Furthermore, the map $(F, G) \mapsto \langle F \diamond G \rangle_{\mathbb{K}}$ from the product space $(\mathcal{L}^{2,\infty} \otimes \mathbb{K}) \times (\mathcal{L}^{2,\infty} \otimes \mathbb{K})$ to $\mathcal{L}^{2,\infty}$ is continuous.*

4. Extension of Clark-Ocone Theorem in Abstract Wiener Space

Let H be a real separable Hilbert space, and let $L^2[0,T]$ be the space of square Lebesgue integrable real-valued functions defined on the interval $[0,T]$. Throughout the rest of this article we let $\mathbb{H} = L^2[0,T] \otimes H$, and in this space, we denote the tensor product of an element $\phi \in L^2[0,T]$ with an element $h \in H$ by ϕh. We use the symbol $\bigotimes$ for tensor product of vectors in $\mathbb{H}$ and the symbol $\otimes$ for tensor product of elements in H. Each elementary object $(\psi_1 h_1) \bigotimes \cdots \bigotimes (\psi_n h_n)$ in $\mathbb{H}^{\bigotimes n}$ can be identified with the element $\psi_1 \cdots \psi_n h_1 \otimes \cdots \otimes h_n$ in $L^2([0,T]^n, H^{\otimes n})$.

We let $\{\phi_i\,;\, i \in \mathbb{N}\}$ be an orthonormal basis of $L^2[0,T]$ and $\{e_i\,;\, i \in \mathbb{N}\}$ an orthonormal basis of H. Therefore, in this case, the index set $\mathcal{A}$ is $\mathbb{N} \times \mathbb{N}$. So each $\gamma \in \Gamma$ is a set of nonnegative integers indexed by $\mathbb{N} \times \mathbb{N}$ and containing a finite number of nonzero elements. For each $\gamma \in \Gamma$ and $i \in \mathbb{N}$ we denote the sequence

$\{\gamma_{(i,j)}\ ;\ j \in \mathbb{N}\}$ by γ_i. We denote the sum $\sum_{j\in\mathbb{N}} \gamma_{(i,j)}$ and the product $\prod_{j\in\mathbb{N}} \gamma_{(i,j)}!$ by $|\gamma_i|$ and $\gamma_i!$, respectively.

We let the operator $Q : \mathbb{H} \to \mathbb{H}$ be defined by

$$Q(\phi_i \otimes e_j) = \lambda_j \phi_i \otimes e_j$$

for every $(i, j) \in \mathbb{N} \times \mathbb{N}$, where $\lambda_j > 1$ for every j. For each $\beta \in \mathbb{R}$, the spaces $\mathbb{H}_\beta$ and H_β are defined as in subsection 2.1.

We note that each elementary object $[\phi_{i_1} e_{j_1}] \otimes \cdots \otimes [\phi_{i_n} e_{j_n}]$ in $\mathbb{H}^{\bigotimes n}$ is in equivalence relation (via $\mathcal{S}_n$) with a unique product $[\phi^{k_1}_{m_1} f_1] \otimes \cdots \otimes [\phi^{k_l}_{m_l} f_l]$, where $m_1 < m_2 < \cdots < m_l$, $k_1 + \cdots + k_l = n$, $f_1 \in H^{\widehat{\otimes} m_1}, \cdots, f_l \in H^{\widehat{\otimes} m_l}$. From this fact we get the following proposition:

Proposition 4.1. *For each nonnegative integer n and each $\beta \in \mathbb{R}$, the set*

$$\left\{ \sqrt{\frac{n!}{\gamma_{m_1}! \cdots \gamma_{m_k}!}}\ \widehat{\bigotimes}_{l=1}^{k} \left[\phi^{|\gamma_{m_l}|}_{m_l} \left(\widehat{\otimes}_{j\in\mathbb{N}} \left(\lambda_j^{-\beta/2} e_j\right)^{\gamma_{(m_l,j)}}\right)\right]\ ;\ \gamma \in \Gamma, \right.$$
$$\left. m_1, \cdots, m_k \in \mathbb{N}, m_1 < \cdots < m_k,\ |\gamma| = |\gamma_1| + \cdots + |\gamma_{m_k}| = n \right\}$$

is an orthonormal basis of $\mathbb{H}_\beta^{\widehat{\bigotimes} n}$.

Now we identify a convenient dense subspace of $\mathbb{H}_\beta^{\widehat{\bigotimes} n}$ that will be used in the proof of the extension of Clark-Ocone Theorem.

Proposition 4.2. *Let $\beta \geq 0$. The span of the collection consisting of elements of the type $\widehat{\bigotimes}_{m=1}^{k} \left[1_{(t_m,t'_m]^{j_m}} h_m^{j_m}\right]$, where $0 \leq t_1 < t'_1 \leq t_2 < t'_2 \leq \cdots \leq t_k < t'_k \leq T$, $j_1 + \cdots + j_k = n$, $h_1, \cdots, h_k \in H$, is dense in $\mathbb{H}_{-\beta}^{\widehat{\bigotimes} n}$.*

Proof. Since H is dense in $H_{-\beta}$, the span of the collection consisting of elements of the type $(1_{(t_1,t'_1]} h_1) \otimes \cdots \otimes (1_{(t_n,t'_n]} h_n)$, where $h_1, \cdots, h_n \in H$, and the intervals $(t_1, t'_1], \cdots, (t_n, t'_n]$ are (pairwise) either identical or disjoint, is dense in $\mathbb{H}_{-\beta}^{\bigotimes n}$. Each element of the type $(1_{(t_1,t'_1]} h_1) \otimes \cdots \otimes (1_{(t_n,t'_n]} h_n)$ is related (via $\mathcal{S}_n$) to a unique element of the type $(1_{(t_{i_1},t'_{i_1}]^{j_1}} f_1) \otimes \cdots \otimes (1_{(t_{i_k},t'_{i_k}]^{j_k}} f_k)$, where $0 \leq t_{i_1} < t'_{i_1} \leq t_{i_2} < t'_{i_2} \leq \cdots \leq t_{i_k} < t'_{i_k} \leq T$, $f_1 \in H^{\widehat{\otimes} j_1}, \cdots, f_k \in H^{\widehat{\otimes} j_k}$ are simple elements, and $j_1 + \cdots + j_k = n$. Now, applying the Polarization Identity to each of the elements $f_1, \cdots, f_k$ we arrive at the assertion of the proposition. □

4.1. Projection properties. For each $t \in [0, T]$, we let $\mathcal{F}_t$ be the completion of the σ-algebra generated by random variables of the type $\delta[1_{[s,s']} h]$, where $0 \leq s < s' \leq t$ and $h \in H$. We denote the set $\bigcup_{0\leq t\leq T} \mathcal{F}_t$ by $\mathcal{F}$.

Proposition 4.3. *Let n be a positive integer, and let $t \in [0, T]$. For each $f \in \mathbb{H}^{\otimes n}$ we have*

$$\mathbb{E}\Big[\delta^{(n)}[f]\ \Big|\ \mathcal{F}_t\Big] = \delta^{(n)}[f 1_{[0,t]\times\cdots\times[0,t]}]. \qquad (4.1)$$

Proof. We first prove the assertion for the case in which f is a simple element of the type $1_{(s,s']^n} h^n$, where $0 \le s < s' \le T$ and $h \in H$. In this case, if $t \le s$, the assertion is true since both sides of (4.1) are zero. If $t \ge s'$, the assertion is true since $\delta^{(n)}[f]$ is $\mathcal{F}_t$-measurable. So we assume $s \le t < s'$. If $n = 1$, the assertion is obviously true. Suppose

$$\mathbb{E}\left(\delta^{(i)}\left[1_{(s,s']^i} h^i\right] \middle| \mathcal{F}_t\right) = \delta^{(i)}\left[1_{(s,t]^i} h^i\right]$$

for $i = 1, \cdots, k$, where k is some positive integer. We know from Corollary 3.2 that

$$\begin{aligned}\delta^{(k+1)}\left[1_{(s,s']^{k+1}} h^{k+1}\right] = \delta^{(k)}\left[1_{(s,s']^k} h^k\right] \delta\left[1_{(s,s']} h\right] \\ - k(s'-s)|h|_H^2 \delta^{(k-1)}\left[1_{(s,s']^{k-1}} h^{k-1}\right].\end{aligned}$$

Using the decomposition $1_{(s,s']} h = 1_{(s,t]} h + 1_{(t,s']} h$ and the inductive assumption we get

$$\begin{aligned}
&\mathbb{E}\left(\delta^{(k+1)}\left[1_{(s,s']^{k+1}} h^{k+1}\right] \middle| \mathcal{F}_t\right) \\
&\quad = \mathbb{E}\left(\delta^{(k)}\left[1_{(s,s']^k} h^k\right] \delta\left[1_{(s,s']} h\right] \middle| \mathcal{F}_t\right) \\
&\qquad\quad - k(s'-s)|h|_H^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] \\
&\quad = \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}\left(\delta^{(i)}\left[1_{(t,s']^i} h^i\right] \delta\left[1_{(t,s']} h\right]\right) \delta^{(k-i)}\left[1_{(s,t]^{k-i}} h^{k-i}\right] \\
&\qquad\quad + \sum_{i=0}^{k} \binom{k}{i} \mathbb{E}\left(\delta^{(i)}\left[1_{(t,s']^i} h^i\right]\right) \delta^{(k-i)}\left[1_{(s,t]^{k-i}} h^{k-i}\right] \delta\left[1_{(s,t]} h\right] \\
&\qquad\quad - k(s'-s)|h|_H^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] \\
&\quad = k\mathbb{E}\left(\delta\left[1_{(t,s']} h\right] \delta\left[1_{(t,s']} h\right]\right) \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] \\
&\qquad\quad + \delta^{(k)}\left[1_{(s,t]^k} h^k\right] \delta\left[1_{(s,t]} h\right] - k(s'-s)|h|_H^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] \\
&\quad = k(s'-t)|h|_H^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] + \delta^{(k)}\left[1_{(s,t]^k} h^k\right] \delta\left[1_{(s,t]} h\right] \\
&\qquad\quad - k(s'-s)|h|_H^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] \\
&\quad = -k(t-s)|h|^2 \delta^{(k-1)}\left[1_{(s,t]^{k-1}} h^{k-1}\right] + \delta^{(k)}\left[1_{(s,t]^k} h^k\right] \delta\left[1_{(s,t]} h\right] \\
&\quad = \delta^{(k+1)}\left[1_{(s,t]^{k+1}} h^{k+1}\right].
\end{aligned}$$

Now we prove the theorem for an arbitrary $f \in \mathbb{H}^{\bigotimes n}$. In light of Proposition 3.2 and the fact that $\delta^{(n)}[f] = \delta^{(n)}[\mathcal{S}_n f]$ for every $f \in \mathbb{H}^{\bigotimes n}$, it suffices to prove the theorem for a simple element of the type $\left[1_{(t_1,t_1']^{i_1}} h_1^{i_1}\right] \otimes \cdots \otimes \left[1_{(t_j,t_j']^{i_j}} h_j^{i_j}\right]$, where $i_1 + \cdots + i_j = n$, $0 \le t_1 < t_1' \le t_2 < t_2' \le \cdots \le t_j < t_j' \le T$, and $h_1, \cdots, h_j \in H$. The assertion is clearly true if $t \le t_1$ or $t \ge t_j'$. If $t_1 < t < t_j'$, the assertion follows from the case already proved and the fact that $\delta^{(n)}\left[\left[1_{[t_1,t_1')^{i_1}} h_1^{i_1}\right] \otimes \cdots \otimes \left[1_{[t_j,t_j')^{i_j}} h_j^{i_j}\right]\right]$ is the product of independent elements $\delta^{(i_m)}\left[1_{[t_m,t_m')^{i_m}} h_m^{i_m}\right]$, $m = 1, \cdots, j$ (Lemma 2.21). □

Corollary 4.4. *If* $F = \sum_{n=0}^{\infty} \delta^{(n)}[f_n] \in \mathcal{L}^2 \otimes \mathbb{K}$ *and* $t \in [0,T]$*, then* $\mathbb{E}(F \,|\, \mathcal{F}_t) = \sum_n \delta^{(n)}[f_n 1_{[0,t]^n}]$.

For $\beta > 0$ and $G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$ the element $\mathbb{E}(G \,|\, \mathcal{F}_t)$ is a well defined object in $\mathcal{L}^{2,\beta} \otimes \mathbb{K}$. Proposition 4.3 provides a way to extend the concept of $\mathcal{F}_t$-measurability to a generalized function. Let $F = \sum_{n=0}^{\infty} \delta^{(n)}[f_n]$ be an element in $\mathcal{L}^{2,-\beta} \otimes \mathbb{K}$ for some $\beta \in [0, \infty)$. We say that F is $\mathcal{F}_t$-measurable for $t \in [0,T]$ if for every n, $f_n 1_{[0,t]^n} = f_n$ almost everywhere (with respect to Lebesgue measure in $[0,T]^n$). We note that in this case

$$\ll F, G \gg_{\mp\beta,\mathbb{K}} = \ll F, \mathbb{E}(G|\mathcal{F}_t) \gg_{\mp\beta,\mathbb{K}}$$

for every $G \in \mathcal{L}^{2,\beta} \otimes \mathbb{K}$.

It is easy to verify that the collection of $\mathcal{F}_t$-measurable elements is a closed subspace of $\mathcal{L}^{2,-\beta} \otimes \mathbb{K}$. We now define the conditional expectation of an arbitrary element $F \in \mathcal{L}^{2,-\beta} \otimes \mathbb{K}$ to be the orthogonal projection of F onto this subspace, and we denote this element by $\mathbb{E}(F \,|\, \mathcal{F}_t)$. Thus $\mathbb{E}(F \,|\, \mathcal{F}_t)$ is the unique $\mathcal{F}_t$-measurable element in $\mathcal{L}^{2,-\beta} \otimes \mathbb{K}$ such that

$$\ll F, G \gg_{\mp\beta,\mathbb{K}} = \ll \mathbb{E}(F|\mathcal{F}_t), G \gg_{\mp\beta,\mathbb{K}}$$

for every $\mathcal{F}_t$-measurable G in $\mathcal{L}^{2,\beta} \otimes \mathbb{K}$.

4.2. Extension of Clark-Ocone theorem.

Every $F \in \mathcal{L}^2$ has Malliavin derivative in $\mathcal{L}^{2,-\beta} \otimes \mathbb{H}$ if $\beta > \log_{\frac{1}{\mathbb{M}}} 4$. This result follows from Proposition 2.19 in subsection 2.5. Moreover, in section 2.6 we saw that for every $\beta > 0$, the domain of δ can be expanded to the entire space $\mathcal{L}^{2,-\beta} \otimes \mathbb{H}_{-\beta}$. These results make it possible for us to generalize the Clark-Ocone Theorem so that it can be applied to every Wiener function in $\mathcal{L}^2$, and not only to the ones that are Malliavin differentiable.

Lemma 4.5. *Let* n *be a nonnegative integer, and let* $f = 1_{(s,s']^{n+1}} h^{n+1}$*, where* $0 \le s < s' \le T$ *and* $h \in H$*. The map* $t \mapsto \mathbb{E}\left(\delta^{(n)}[f] \,\middle|\, \mathcal{F}_t\right)(t)$ *belongs to the domain of* δ*, and*

$$\delta\left[\mathbb{E}\left(\delta^{(n)}[f] \,\middle|\, \mathcal{F}_\cdot\right)(\cdot)\right] = \frac{1}{n+1}\delta^{(n+1)}[f].$$

Proof. Let $F = \delta^{(n)}[f]$ and $G = \delta^{(n+1)}[g]$, where $g \in \mathbb{H}^{\widehat{\otimes} n+1}$. Then

$$\begin{aligned}
\ll \mathbb{E}\left(F \middle| \mathcal{F}_\cdot\right)(\cdot), \mathcal{D}G \gg_{\mathbb{H}} &= (n+1)\int_s^{s'} \mathbb{E}\left(\delta^{(n)}[1_{(s,t]^n} h^n] \langle h, \delta^{(n)}[g](t)\rangle_{\mathbb{H}}\right) dt \\
&= (n+1)! \int_s^{s'} \int_s^t \cdots \int_s^t \langle h^{n+1}, g(s_1, \cdots, s_n, t)\rangle_{\mathbb{H}^{\otimes n+1}} ds_1 \cdots ds_n dt \\
&= n! \int_s^{s'} \int_s^{s'} \cdots \int_s^{s'} \langle h^{n+1}, g(s_1, \cdots, s_n, t)\rangle_{\mathbb{H}^{\otimes n+1}} ds_1 \cdots ds_n dt \\
&= \ll \frac{1}{n+1}\delta^{(n+1)}[f], G \gg,
\end{aligned}$$

where the fourth equation is a consequence of the symmetry of g. □

Theorem 4.6. *Let F be an element of $\mathcal{L}^2$, and let $\beta > \log_{\frac{1}{\mathbb{M}}} 4$. Then the map $(t,\omega) \mapsto \mathbb{E}(\mathcal{D}F \mid \mathcal{F}_t)(\omega, t)$ belongs to the domain of (the extension of) δ in the space $\mathcal{L}^{2,-\beta} \otimes \mathbb{H}_{-\beta}$. Furthermore, in $\mathcal{L}^{2,-\beta}$,*

$$F = \mathbb{J}_0 F + \delta\left[\mathbb{E}\left(\mathcal{D}F \mid \mathcal{F}_\cdot\right)(\cdot)\right].$$

Proof. Let F and β be as in the statement of this theorem. By Proposition 2.19, $\mathcal{D}F \in \mathcal{L}^{2,-\beta}\widehat{\widehat{\otimes}}\mathbb{H}$. Using estimate 2.1 we can see that if F has chaos decomposition $\sum_{n=0}^{\infty} \delta^{(n)}[f_n]$, then $\sum_{n=1}^{\infty} n^2 n! |f_n|^2_{\mathbb{H}_{-\beta}^{\otimes n}} < \infty$. This implies that $\mathcal{D}F$ belongs to the domain of (the extension of) δ in $\mathcal{L}^{2,-\beta} \otimes \mathbb{H}_{-\beta}$. (See Remark 2.26.) Moreover, the map $\delta \circ \mathcal{D} : \mathcal{L}^2 \to \mathcal{L}^{2,-\beta}$ is continuous. Hence it suffices to prove the assertion for elements in a dense subspace of $\mathcal{L}^2$, such as the one identified in Proposition 4.2. In view of linearity of $\delta^{(n)}$, we only need to prove the theorem for an element of the type $\delta^{(n)}[f]$, where $f = \otimes_{j=1}^{i} 1_{(t_1,t_1']^{m_j}} h_j^{m_j}$, $0 \le t_1 < t_1' \le t_2 < t_2' \le \cdots \le t_i < t_i' \le T$, $m_1 + \cdots + m_i = n$, $h_1, \cdots, h_i \in H$. In this case, by Lemma 2.21,

$$\delta^{(n)}[f] = \prod_{j=1}^{i} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right].$$

Then

$$\mathcal{D}\delta^{(n)}[f] = \sum_{k=1}^{i} m_k \Bigg(\prod_{j \in \{1,\cdots,i\}, j \neq k} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right] \times \delta^{(m_k-1)}\left[1_{(t_k,t_k']^{m_k-1}} h_k^{m_k-1}\right] 1_{(t_k,t_k']} h_k \Bigg).$$

In view of Proposition 4.3, $\mathbb{E}\left(\mathcal{D}\delta^{(n)}[f] \mid \mathcal{F}_t\right)(t) = 0$ if $t \le t_i$. Let $t_i < t \le t_i'$. In this case the first $i-1$ terms of $\mathbb{E}\left(\mathcal{D}\delta^{(n)}[f] \mid \mathcal{F}_t\right)$ vanish when evaluated at t and hence we only need to consider the last term. By Proposition 4.3,

$$\mathbb{E}\left(\delta^{(m_i-1)}\left[1_{(t_i,t_i']^{m_i-1}} h_k^{m_i-1}\right] 1_{(t_i,t_i']} h_i \prod_{j=1}^{i-1} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right] \mid \mathcal{F}_t\right)(t)$$
$$= \delta^{(m_i-1)}\left[1_{(t_i,t]^{m_i-1}} h_k^{m_i-1}\right] 1_{(t_i,t_i']}(t) h_i \prod_{j=1}^{i-1} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right].$$

The map $t \mapsto \mathbb{E}\left[\mathcal{D}\delta^{(n)}(f) \mid \mathcal{F}_t\right](t)$ belongs to the domain of δ. By Corollary 2.23 and Lemma 4.5 we have

$$\begin{aligned}
&\delta\Big[\mathbb{E}\big(\mathcal{D}\delta^{(n)}[f] \mid \mathcal{F}_\cdot\big)(\cdot)\Big] \\
&= \prod_{j=1}^{i-1} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right] m_i \delta\left[\delta^{(m_i-1)}\left[1_{(t_i,\cdot]^{m_i-1}} h_i^{m_i-1}\right] 1_{(t_i,t_i']}(\cdot) h_i\right] \\
&= \prod_{j=1}^{i} \delta^{(m_j)}\left[1_{(t_j,t_j']^{m_j}} h_j^{m_j}\right].
\end{aligned}$$

□

References

1. Holden, H., Øksendal, B., Ubøe, J., and Zhang, T.: *Stochastic Partial Differential Equations*, Second Edition, Springer, 2010.
2. Körezlioglu, H. and Üstünel, A. S.: New class of distributions on Wiener spaces, *Lecture Notes in Mathematics, Springer* **1444** (1990) 106–121.
3. Kuo, H.-H.: *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics, Springer **463** (1975).
4. Kuo, H.-H.: *White Noise Distribution Theory*, CRC Press, Boca Raton, FL, 1996.
5. Lanconelli, A: On the extension of a basic property of conditional expectations to second quantization operators, *Communications on Stochastic Analysis* **3** (2009) 369–381.
6. Potthof, J. and Timpel, M.: On a dual pair of spaces of smooth and generalized random variables, *Potential Analysis* **4** (1995) 637-654.

Fariborz Asadian: Department of Mathematics and Computer Science, Fort Valley State University, Fort Valley, GA 31030, USA

E-mail address: asadianf@fvsu.edu