FUNCTIONAL ITÔ'S CALCULUS AND DYNAMIC CONVEX RISK MEASURES FOR DERIVATIVE SECURITIES

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ABSTRACT. Using the functional Itô's calculus and forward-backward stochastic differential equations (FBSDEs), a new approach for evaluating dynamic convex risk measures for European-style derivative securities is proposed in a general, non-Markovian, continuous-time financial market. Firstly a dynamic convex risk measure for an unhedged position of derivative securities is represented as the conditional g-expectation which is given by the solution of the backward system in a FBSDE. Then we use the functional Itô's calculus, a martingale representation and the unique decomposition of special semimartingales to identify the solution of the backward system in the FBSDE. In particular, the control component in the backward system is identified using functional derivatives. Whereas the first component of the backward system satisfies a functional partial differential equation.

1. Introduction

Derivative securities contribute significantly to the post Global Financial Crisis (GFC) starting in the United States in 2008. Major banks, financial institutions and insurance companies, such as Lehman Brothers, American International Group (AIG) and Merrill Lynch, collapsed or were bailouts due to large amounts of losses resulting from massive trading in derivative securities. In the aftermath of the GFC, there have been ongoing international discussions on the challenges in risk management and financial regulation for trading activities involving derivative securities in the ever changing and sophisticated global financial market. These discussions have also stimulated regulators, academics and market practitioners to rethink the appropriateness of the current practice of risk measurement and management for derivative securities in the finance and insurance industries.

Value at Risk (VaR) has emerged as a popular measure for risk in the finance and insurance industries. It has been widely used to measure risks of trading positions involving derivative securities. The use of VaR may be traced back to the famous "4:15 report" which was called for by J.P. Morgan CEO Dennis Weatherstone and provided a one-page summary for all daily risk exposure of a firm, available within 15 minutes of the market close. The rapid development

of the VaR methodology for practical use was mainly attributed to the Risk-Metrics Group, which was an integral part of J.P. Morgan and now becomes an independent business organization. VaR is an estimate of the maximum amount of loss one is prepared to incur at a given probability level and in a fixed time horizon. For example, if the 95% daily VaR of a trading position is three million British pounds sterling, this means that one is prepared to incur an actual loss of three million British pounds sterling, or less, with a probability of 95% in a single trading day. Despite its popularity, as observed by Artzner et al. (1997), VaR has some shortcomings. In particular, VaR is, in general, not subadditive, which means that allocating assets over risky positions may increase risk. This is counter-intuitive and is not consistent with a key notion in modern finance, namely, diversification. Another shortcoming of VaR is that it is not time-consistent which can then lead to dynamically inconsistent risk behavior, (see, for example, Cheridito and Stadje, (2008)). Due to its shortcomings, it is not unreasonable to question, at least from a theoretical perspective, whether the widespread use of VaR for measuring risks of derivative positions is appropriate. If not, what else one may use?

Artzner et al. (1997) proposed an axiomatic approach to construct theoretically consistent risk measures and a set of properties, (including the subadditive property), a theoretically consistent risk measure should satisfy. They introduced a class of coherent risk measures satisfying the set of properties and provided a representation for any coherent risk measure. Two typical examples of coherent risk measures are the generalized scenario expectation (GSE) and the expected shortfall (ES). The generalized scenario expectation is defined as the supremum of the expected loss over a set of probability measures, or generalized scenarios. It comes from the representation of coherent risk measures and generalizes scenario-based risk measures used for stress testing in the Chicago Mercantile Exchange (CME). The ES, which is also called the conditional tail expectation (CTE), is defined as the conditional expectation of a loss given that the loss exceeds a certain threshold level specified by the VaR level. It describes the tail risk of a risky position which cannot be described by VaR. The use of coherent risk measures for describing risks for unhedged derivative positions has been studied by the author and his collaborators, (see Siu and Yang (2000), Yang and Siu (2001), Siu et al. (2001), Boyle et al. (2002), Elliott et al. (2008)).

Föllmer and Schied (2002) and Frittelli and Rosazza-Gianin (2002) argued that in practice, the risk of a trading position may increase, in a nonlinear fashion, with the size of the position, where the nonlinearity is attributed to the lack of liquidity of a large trading position. They introduced, independently, a class of convex risk measures which is a generalization of coherent risk measures. To incorporate the nonlinearity due to the lack of liquidity, they relaxed the subadditive and positively homogeneous properties of coherent risk measures and replaced them by a convex property. Indeed, the idea of convex risk measures is related to that of the convex premium principles first introduced to the actuarial science literature by Deprez and Gerber (1985), where the intrinsic notion is the convexity of a, (premium or risk), functional. Liquidity risk has been highlighted as an important source of risk in the post GFC. Its importance to risk measurement and management cannot be undermined, especially for large unwinding trading positions which are usually

the case for derivative positions. Convex risk measures may provide some insights, at least from a theoretical perspective, how one can incorporate liquidity risk in evaluating the riskiness of trading positions involving derivative securities.

Both the coherent risk measures of Artzner et al. (1997) and the convex risk measures of Föllmer and Schied (2002) and Frittelli and Rosazza-Gianin (2002) are static one-period risk measures. However, in practical situations, one may adjust the evaluation of risk of a trading position dynamically over time as new market information emerges. Riedel (2004) and Detlefsen and Scandolo (2005) introduced dynamic coherent risk measures and dynamic convex risk measures, respectively. They provided representations for dynamic coherent risk measures and dynamic convex risk measures. One of the key concepts of dynamic risk measures is time-consistency which is also a fundamental notion in inter-temporal economic and financial theories. Informally speaking, time consistency means that two risky positions having the same value at a future time must have the same risk any time before that time. Riedel (2004) and Detlefsen and Scandolo (2005) discussed under what conditions dynamic coherent risk measures and dynamic convex risk measures, respectively, are time-consistent. Peng (2004) and Rosazza-Gianin (2006) discussed the relationship between conditional g-expectations and dynamic risk measures, (including both coherent and convex risk measures). This provides a link between the solutions of backward stochastic differential equations, (BSDEs), and dynamic risk measures and a mathematical framework to discuss the time-consistency of dynamic risk measures. Elliott and Siu (2010) used a BSDE approach to evaluate a convex risk measure for an unhedged position of derivative securities which extended the early work of Siu and Yang (2000) to an non-Markovian modeling framework. Elliott and Siu represented the convex risk measure as the solution of a BSDE and used the Clark-Ocone representation result along with Malliavin calculus to identify the control component of the solution of the BSDE. In the Markovian case, they related the BSDE solution to the partial differential equation solution for evaluating the convex risk measure.

In this paper, we propose a new approach to evaluate dynamic convex risk measures for unhedged positions of European-style derivative securities in a general, non-Markovian, continuous-time financial market. In such a market, the price process of an underlying risky asset is governed by a, (functional), geometric Brownian motion, where the drift and the volatility depend not merely on the current price of the asset, but also its past values. The proposed approach for nonlinear risk evaluation is established using the functional Itô's calculus and forward-backward stochastic differential equations (FBSDEs). The functional Itô's calculus was introduced by Dupire (2009) and further developed by Cont and Fournié (2011). It extends the classical Itô's calculus to "differentiate" functionals of stochastic processes. The proposed approach consists of two stages. Initially, a dynamic convex risk measure for an unhedged position of derivative securities is represented as the conditional g-expectation which is given by the solution of the backward system in a FBSDE. Note that the forward system in the FBSDE is given by the price process of the underlying risky asset. Then, at the second stage, we identify the solution of the backward system in the FBSDE which, in turns, gives rise to the dynamic convex risk measure. More specifically we first relate the martingale

component of the backward system in the FBSDE to a "smooth" functional of the solution of the forward system in the FBSDE. Applying the functional Itô's formula, a martingale representation and the unique decomposition of a special semimartingale, we identify the solution of the backward system in the FBSDE. In particular, the control component of the backward system is identified using functional derivatives. Whereas the first component of the backward system satisfies a functional partial differential equation. We also consider a parametric case where the space of probability measures generating the dynamic convex risk measure is specified by a family of stochastic exponentials generated by a Brownian motion. In this case, the, (nonlinear), evaluation of the dynamic convex risk measure is formulated as a stochastic optimal control problem. We then apply the results obtained from the functional Itô's calculus to discuss this stochastic optimal control problem. Besides contributing to the literature about nonlinear evaluation of dynamic convex risk measures, these results developed using the functional Itô's calculus provide a novel approach to discuss the solutions of FBSDEs. By exploring the relationship between convex risk measures and convex premium principles, the results obtained here based on the functional Itô's calculus may also be applied to establish a dynamic convex premium principle, (i.e., a dynamic extension of the convex premium principle of Deprez and Gerber (1985)), and its nonlinear evaluation via a conditional g-expectation.

The rest of the paper is structured as follows. The next section presents the model dynamics and a dynamic convex risk measure for an unhedged position of European-style derivative securities. Section 3 presents a FBSDE and the conditional g-expectation corresponding to the dynamic convex risk measure. In Section 4, we discuss some key results of the functional Itô's calculus. Then we identify the solution of the backward system in the FBSDE in Section 5. We discuss the parametric case and its associated control problem in Section 6. The final section summarizes the results.

2. The Risk Measurement Model

We consider a non-Markovian extension to the continuous-time Black-Scholes-Merton economy with two primitive investment assets, namely, a bond and a share. These assets are tradeable continuously over time in a finite time horizon $\mathcal{T} := [0, T]$, for $T < \infty$. The results discussed here hold for a multi-dimensional case involving several correlated shares. However, to simplify our discussion and the notation, we consider here one share. We suppose that the only source of uncertainty in our model is described by a standard Brownian motion and that the price process of the share is described by a stochastic, (functional), differential equation driven by the standard Brownian motion. Consequently, as in Elliott (1982), it is convenient to consider the canonical probability space defined by the path space of continuous functions. More precisely, let $\Omega := \mathcal{C}(\mathcal{T}; \Re)$, the space of real-valued continuous functions $\omega : \mathcal{T} \to \Re$ defined on the time domain $\mathcal{T}$. The coordinate mapping process $W : \mathcal{T} \times \Omega \to \Re$ is defined by:

$$W(t, \omega) := \omega(t) , \quad (t, \omega) \in \mathcal{T} \times \Omega .$$

Write $\mathbb{F}^W := \{\mathcal{F}^W(t)|t \in \mathcal{T}\}$ for the natural filtration generated by the process W. Then we endow the measurable space $(\Omega, \mathcal{F}^W(T))$ with the Wiener measure $\mathbb{P}$ under which the process W is a standard Brownian motion. Let $\mathbb{F} := \{\mathcal{F}(t)|t \in \mathcal{T}\}$ be the $\mathbb{P}$-augmentation of the natural filtration $\mathbb{F}^W$ and $\mathcal{F} := \mathcal{F}(T)$. Then the canonical probability space is defined as the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

To simplify the notation, we can take the interest rate of the bond to be zero, so all of the price processes we introduce later can be interpreted as discounted price processes. We also suppress the notation "ω" in $W(t, \omega)$ and write $W(t) := W(t, \omega)$, for each $t \in \mathcal{T}$.

For each $t \in \mathcal{T}$, let $\mu(t)$ and $\sigma(t)$ be the appreciation rate and the volatility of the share at time t under the measure $\mathbb{P}$. We assume that $\{\mu(t)|t \in \mathcal{T}\}$ and $\{\sigma(t)|t \in \mathcal{T}\}$ are $\mathbb{F}$-adapted, bounded, real-valued processes on $(\Omega, \mathcal{F}, \mathbb{P})$ such that for each $t \in \mathcal{T}$, $\mu(t) \in \Re$ and $\sigma(t) > 0$, $\mathbb{P}$-a.s. Furthermore, we suppose that under $\mathbb{P}$, the price process of the share $X := \{X(t)|t \in \mathcal{T}\}$ is governed by the following, (functional), geometric Brownian motion (FGBM):

$$
\begin{aligned}
dX(t) &= \mu(t)X(t)dt + \sigma(t)X(t)dW(t) , \\
X(0) &= x_0 > 0 .
\end{aligned}
\tag{2.1}
$$

Note that the paths of X lie in the space $\mathcal{C}(\mathcal{T}; \Re)$ For each $t \in \mathcal{T}$, write $X_{0,t} := \{X(u)|u \in [0,t]\}$ for the restriction of $X \in \mathcal{C}(\mathcal{T}; \Re)$ to $\mathcal{C}([0,t]; \Re)$, where $\mathcal{C}([0,t]; \Re)$ is the space of real-valued, continuous functions on $[0,t]$. Let $\mathbb{F}^X := \{\mathcal{F}^X(t)|t \in \mathcal{T}\}$ be the $\mathbb{P}$-augmentation of the natural filtration generated by the share price process X. Then for each $t \in \mathcal{T}$, $\mathcal{F}^X(t) = \mathcal{F}(t)$. Consequently, for each $t \in \mathcal{T}$, $\mu(t)$ and $\sigma(t)$ are $\mathcal{F}^X(t)$-measurable. Then for each $t \in \mathcal{T}$, $\mu(t)$ and $\sigma(t)$ are functionals of $X_{0,t}$; that is,

$$
\mu(t) := \mu^\dagger(X_{0,t}) , \quad \sigma(t) := \sigma^\dagger(X_{0,t}),
$$

for some functionals $\mu^\dagger : \mathcal{C}([0,t]; \Re) \to \Re$ and $\sigma^\dagger : \mathcal{C}([0,t]; \Re) \to (0, \infty)$. To simplify our notation, we do not distinguish notationally μ and σ from $\mu^\dagger$ and $\sigma^\dagger$, respectively.

We consider a European-style contingent claim with payoff $G(X(T))$ and maturity at time T, where $G(X(T)) \in L^2(\Omega, \mathcal{F}^X(T), \mathbb{P})$, the space of square-integrable, $\mathcal{F}^X(T)$-measurable random variables with respect to the measure $\mathbb{P}$. Our object is to provide a "dynamic" nonlinear evaluation for the risk of an unhedged position of such a claim $G(X(T))$ using a version of "dynamic" convex risk measures discussed in Rosazza-Gianin (2006). This "dynamic" convex risk measure is a "dynamic" extension of the static convex risk measure introduced by Föllmer and Schied (2002) and Frittelli and Rosazza-Gianin (2002) independently.

Firstly, we describe some essential concepts and results for "dynamic" convex risk measures by adapting results in Rosazza-Gianin (2006) to our current setting. For each $p \in [1, +\infty]$, let $L^p(\Omega, \mathcal{F}^X(T), \mathbb{P})$ be the space of p-integrable, $\mathcal{F}^X(T)$-measurable, random variables. Write $\mathcal{H}^p := L^p(\Omega, \mathcal{F}^X(T), \mathbb{P})$, the space of random variables representing the future values of risky financial positions which will be realized at time T. For each $t \in \mathcal{T}$, let $L^0(\mathcal{F}^X(t)) := L^0(\Omega, \mathcal{F}^X(t), \mathbb{P})$, the space of bounded, $\mathcal{F}^X(t)$-measurable random variables. We now wish to evaluate the riskiness of a financial position $H \in \mathcal{H}^p$ at any intermediate time t between the

initial time 0 and the terminal time T. Consequently, given the information $\mathcal{F}^X(t)$ about the price process of the share up to time t, we can consider an $\mathcal{F}^X(t)$-measurable random variable $\rho_t(H)$ which is the value of a dynamic risk measure of the risky position H at time t. The definition of a dynamic risk measure is given as follows:

Definition 2.1. A dynamic risk measure is a family of maps $\{\rho_t | t \in \mathcal{T}\}$ such that
  (1) $\rho_t : \mathcal{H}^p \to L^0(\mathcal{F}^X(t))$;
  (2) ρ_0 is a static risk measure;
  (3) $\rho_T(H) = -H$, for each $H \in \mathcal{H}^p$.

In general, we may define the dynamic risk measure as a double-indexed family of random variables, say $\{\rho_{t,s} | t, s \in \mathcal{T}, t \leq s\}$. However, in our case, the terminal time T is fixed, so it suffices to consider a one-parameter family of random variables.

We now give the definition of a dynamic convex risk measure.

Definition 2.2. A dynamic risk measure $\{\rho_t | t \in \mathcal{T}\}$ is said to be convex if it satisfies the following three properties:
  (1) (Translation Invariance): for each $t \in \mathcal{T}$, $H \in \mathcal{H}^p$ and $K \in L^0(\mathcal{F}^X(t))$,

$$\rho_t(H + K) = \rho_t(H) - K \ ;$$

  (2) (Monotonicity): for each $t \in \mathcal{T}$ and $H_1, H_2 \in \mathcal{H}^p$, if $H_1(\omega) \geq H_2(\omega)$ for each $\omega \in \Omega$, $\rho_t(H_1) \leq \rho_t(H_2)$;
  (3) (Convexity): for each $t \in \mathcal{T}$, $\lambda \in L^0(\mathcal{F}^X(t))$ with $\lambda \in (0,1)$ and $H_1, H_2 \in \mathcal{H}^p$,

$$\rho_t(\lambda H_1 + (1 - \lambda)H_2) \leq \lambda \rho_t(H_1) + (1 - \lambda)\rho_t(H_2) \ ;$$

  (4) (Normalization): for each $t \in \mathcal{T}$, $\rho_t(0) = 0$.

An important property for dynamic risk measures is time-consistency. It is formally defined as follows:

Definition 2.3. A dynamic risk measure $\{\rho_t | t \in \mathcal{T}\}$ is said to be time-consistent if for each $t \in \mathcal{T}$, $H \in \mathcal{H}^p$ and $A \in \mathcal{F}^X(t)$,

$$\rho_0(HI_A) = \rho_0(-\rho_t(H)I_A) \ ,$$

where I_A is the indicator function of the event A.

In general, a dynamic convex risk measure is not necessarily time-consistent. Some conditions are required to ensure that a dynamic convex risk measure is time-consistent, (see, for example, Detlefsen and Scandolo (2005)).

As in the static case, a representation for a dynamic convex risk measure is available. The following theorem gives the representation and is due to Detlefsen and Scandolo (2005).

Theorem 2.4. *Let $\mathcal{P}$ be a convex set of probability measures on $(\Omega, \mathcal{F}^X(T))$ which are absolutely continuous with respect to $\mathbb{P}$. For each $t \in \mathcal{T}$, let $\eta_t : \mathcal{P} \to \Re \cup \{+\infty\}$ be a convex functional such that*

$$\inf_{\mathbb{Q} \in \mathcal{P}} \eta_t(\mathbb{Q}) = 0 \ .$$

Then a dynamic risk measure $\{\rho_t | t \in \mathcal{T}\}$ defined by:

$$\rho_t(H) := ess - \sup_{\mathbb{Q} \in \mathcal{P}} \{E_{\mathbb{Q}}[-H|\mathcal{F}(t)] - \eta_t(\mathbb{Q})\} \ , \quad t \in \mathcal{T} \ , \quad H \in \mathcal{H}^p \ , \qquad (2.2)$$

is a dynamic convex risk measure, where $E_{\mathbb{Q}}$ is expectation under $\mathbb{Q}$.
Suppose that for each $t \in \mathcal{T}$,

$$\eta_t(\mathbb{Q}) := ess - \sup_{H \in \mathcal{H}^p} \{E[-H|\mathcal{F}^X(t)] - \rho_t(H)\} \ .$$

Here E is expectation under $\mathbb{P}$.
Then any dynamic convex risk measure can be represented in the form (2.2).

We are interested in evaluating a dynamic convex risk measure of the claim $G(X(T))$. We shall relate the dynamic convex risk measure to the conditional g-expectation which is the solution of the backward system in a FBSDE in the next section. Then in Section 5 we determine the solution of the backward system, and hence, the nonlinear evaluation of the dynamic convex risk measure using an approach based on the functional Itô's calculus.

3. FBSDEs and Conditional g-Expectations for Dynamic Convex Risk Measures

The forward-backward stochastic differential equation, (FBSDE), we consider here has the following form:
Forward system in the share price process $X(t)$:

$$\begin{aligned} dX(t) &= \mu(X_{0,t})X(t)dt + \sigma(X_{0,t})X(t)dW(t) \ , \\ X(0) &= x_0 \ . \end{aligned} \qquad (3.1)$$

Backward system in the unknown processes $Y(t)$ and $Z(t)$:

$$\begin{aligned} -dY(t) &= g(t, X_{0,t}, Y(t), Z(t))dt - Z(t)dW(t) \ , \\ Y(T) &= -G(X(T)) \ . \end{aligned} \qquad (3.2)$$

Here $g : \mathcal{T} \times \mathcal{C}([0,t]; \Re) \times \Re \times \Re \to \Re$ is the driver function and $-G(X(T))$ is the terminal condition of the backward stochastic differential equation, (BSDE), in the backward system of the FBSDE, where $G(X(T))$ is the terminal payoff of the claim at time T described in the last section.

Since the functionals $\mu^\dagger(X_{0,t})$ and $\sigma^\dagger(X_{0,t})$ are bounded, it is obvious that the coefficients in the forward system satisfy the Lipschitz and growth conditions to ensure the existence and uniqueness of its solution. We suppose that the driver function g and the terminal condition $-G(X(T))$ satisfies the Lipschitz and growth conditions so that the BSDE has a unique solution $\{(Y(t), Z(t)) | t \in \mathcal{T}\}$. For details, please refer to the monograph by Ma and Yong (1999). We also assume that the driver function g is square-integrable so that the BSDE has a unique, square-integrable solution $\{(Y(t), Z(t)) | t \in \mathcal{T}\}$.

The triplet $\{(X(t), Y(t), Z(t)) | t \in \mathcal{T}\}$ is the solution of the FBSDE. In our case, $\{Y(t) | t \in \mathcal{T}\}$ and $\{Z(t) | t \in \mathcal{T}\}$ are unknown processes and shall be determined by the approach based on the functional Itô's calculus in Section 5. In the sequel, we relate the solution of the BSDE (3.2) to the dynamic convex risk measure

$\{\rho_t(G(X(T)))|t \in \mathcal{T}\}$ of the contingent claim $G(X(T))$ described in the last section via a conditional g-expectation.

Note that the terminal condition of the BSDE (3.2) is in $L^2(\Omega, \mathcal{F}^X(T), \mathbb{P})$. Then following the definition of the conditional g-expectation in Peng (1997), the conditional g-expectation of $-G(X(T))$ given $\mathcal{F}^X(t)$ is defined by:

$$\mathcal{E}_g(-G(X(T))|\mathcal{F}^X(t)) := Y(t) ,$$

where $\{Y(t)|t \in \mathcal{T}\}$ is the first component of the solution of the BSDE (3.2).

In particular, if $t = 0$,

$$\mathcal{E}_g(-G(X(T))) := Y(0) ,$$

which is called the g-expectation of $-G(X(T))$.

Let $\{\rho_t^g(G(X(T)))|t \in \mathcal{T}\}$ be a dynamic risk measure of the claim $G(X(T))$ associated with the driver function g defined by:

$$\rho_t^g(G(X(T))) := \mathcal{E}_g(-G(X(T))|\mathcal{F}^X(t)) = Y(t) , \quad t \in \mathcal{T} . \qquad (3.3)$$

Then the following theorem presented in Frittelli and Rosazza-Gianin (2004), (see Proposition 23 therein), gives the sufficient condition to ensure that

$$\{\rho_t^g(G(X(T)))|t \in \mathcal{T}\}$$

is a time-consistent dynamic convex risk measure.

Theorem 3.1. *Suppose the driver function $g(t, x_{0,t}, y, z)$ in the BSDE (3.2) is convex in $(y, z) \in \Re^2$ for each $t \in \mathcal{T}$ and $x_{0,t} \in \mathcal{C}([0, t]; \Re)$. Then the dynamic risk measure $\{\rho_t^g(G(X(T)))|t \in \mathcal{T}\}$ of the claim $G(X(T))$ is a time-consistent dynamic convex risk measure.*

Indeed, the converse of Theorem 3.1 is also true, which gives the sufficient condition for a dynamic convex risk measure to come from a conditional g-expectation, (see, for example, Frittelli and Rosazza-Gianin (2004), Proposition 24 therein).

Note that the convex driver function g entails some interesting interpretations. It has been shown in Coquet et al. (2002) that if $g_1 \le g_2$, then $\rho_0^{g_1}(H) \le \rho_0^{g_1}(H)$, for each $H \in \mathcal{H}^p$. Consequently, g can be interpreted as a measure of risk aversion. If g does not depend on y, ρ_0^g satisfies the translation invariance property. Furthermore, if g has the following form:

$$g(t, x_{0,t}, y, z) = \frac{1}{2}y^2 , \quad t \in \mathcal{T} , \quad x_{0,t} \in \mathcal{C}([0, t]; \Re) ,$$

then $\rho_0^g(H)$, $H \in \mathcal{H}^p$, becomes the following entropic risk measure:

$$\rho_0^g(H) := \mathcal{E}_g(-H) = \sup_{\mathbb{Q} \in \mathcal{P}}\{E_\mathbb{Q}[-H] - R(\mathbb{Q}, \mathbb{P})\} .$$

Here $R(\mathbb{Q}, \mathbb{P})$ is the relative entropy between $\mathbb{Q}$ and $\mathbb{P}$ defined by:

$$R(\mathbb{Q}, \mathbb{P}) := E\left[\frac{d\mathbb{Q}}{d\mathbb{P}} \ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right] .$$

From Equation (3.3) and Theorem 3.1, to determine the dynamic convex risk measure $\{\rho_t^g(G(X(T)))|t \in \mathcal{T}\}$ of the claim $G(X(T))$, it suffices to determine the first component $\{Y(t)|t \in \mathcal{T}\}$ of the solution of the BSDE (3.2) with the convex driver function g.

4. Functional Ito's Calculus

In this section we present some key concepts and results of the functional Itô's calculus which will be used in this paper. For our purpose here, it suffices to consider the original version of the functional Itô's calculus introduced by Dupire (2009) though a more general version has been developed in Cont and Fournié (2011). Indeed, this version of the functional Itô's calculus is sufficient to deal with functionals of continuous semimartingales. Here we consider functionals of stochastic, (functional), differential equations driven by a standard Brownian motion which are particular cases of functionals of continuous semimartingales.

The key idea of the functional Itô's calculus developed by Dupire (2009) is to define the time and spatial derivatives of a functional by a perturbation, or variation, in the endpoint of a given current path of a process, (i.e., the restriction of a process up to the current time). This is different from the Malliavin derivative which is defined by a perturbation of the whole path of a process including its future values. Consequently, the functional Itô's calculus is non-anticipative while the Mallivian calculus is anticipative. More precisely, as in Dupire (2009), the functional derivatives in the functional Itô's calculus are defined in the sequel.

For each $t \in \mathcal{T}$, we define the time and spatial functional derivatives of a given current path $X_{0,t} \in \mathcal{C}([0,t]; \Re)$ by the following vertical perturbation and horizontal extension of the path.

Vertical Perturbation: Perturb the endpoint of the current path $X_{0,t}$ by a "small" quantity ϵ

$$X_{0,t}^{\epsilon}(s) := X_{0,t}(s) + \epsilon I_t(s) , \quad s \in [0,t] ,$$

where I_t is the indicator function of the singleton $\{t\}$ and $X_{0,t}(s)$, (resp. $X_{0,t}^{\epsilon}(s)$), is the evaluation of the path $X_{0,t} \in \mathcal{C}([0,t]; \Re)$, (resp. $X_{0,t}^{\epsilon} \in \mathcal{C}([0,t]; \Re)$), at time $s \in [0,t]$.

Horizontal Extension: Extend the path $X_{0,t}$ by freezing the endpoint over $[t, t + \delta t]$:

$$X_{t,\delta t}(s) := X_{0,t}(s)I_{[0,t]}(s) + X_{0,t}(t)I_{(t,t+\delta t]}(s) .$$

Here δt is a "small" positive quantity; I_E is the indicator function of the set E; $X_{t,\delta t} \in \mathcal{C}([0, t + \delta t]; \Re)$.

Then for each $X_{0,t} \in \mathcal{C}([0,t]; \Re)$, the time and spatial functional derivatives of a "smooth" functional f of $X_{0,t}$ are defined as follows:

$$\Delta_t f(X_{0,t}) := \lim_{\delta t \to 0^+} \frac{f(X_{t,\delta t}) - f(X_{0,t})}{\delta t} ,$$

$$\Delta_x f(X_{0,t}) := \lim_{\epsilon \to 0} \frac{f(X_{0,t}^{\epsilon}) - f(X_{0,t})}{\epsilon} ,$$

$$\Delta_{xx} f(X_{0,t}) := \lim_{\epsilon \to 0} \frac{\Delta_x f(X_{0,t}^{\epsilon}) - \Delta_x f(X_{0,t})}{\epsilon} ,$$

provided that the above limits exist.

Note that the above time and spatial functional derivatives are defined in a pathwise and non-anticipative sense. The time functional derivative is a right-derivative while the spatial functional derivatives are both sides. The time and

spatial functional derivatives have some desirable properties such as linearity, product rule, chain rule and continuity under an appropriate topology on the functional space $\mathcal{C}(\mathcal{T}; \Re)$.

When $f(X_{0,t}) := F(t, X(t))$, (i.e., a "smooth" function of the current time t and the current value $X(t)$ of the state process), the time and spatial functional derivatives reduce to their corresponding classical partial derivatives:

$$\Delta_t f(X_{0,t}) = \frac{\partial F}{\partial t}(t, X(t)),$$

$$\Delta_x f(X_{0,t}) = \frac{\partial F}{\partial x}(t, X(t)),$$

$$\Delta_{xx} f(X_{0,t}) = \frac{\partial^2 F}{\partial x^2}(t, X(t)).$$

The main result of the functional Itô's calculus of Dupire (2009) is the functional Itô's formula which is an extension of the classical Itô's formula to the case of functionals. Before presenting the functional Itô's formula, we must define the concept of continuity of a functional under a suitable topology.

For any two paths $X_{0,t}, Y_{0,s} \in \mathcal{C}(\mathcal{T}; \Re)$ of different lengths, (i.e., $t, s \in \mathcal{T}$ with $t \leq s$), the distance between $X_{0,t}$ and $Y_{0,s}$ is defined by:

$$d_{\mathcal{C}(\mathcal{T};\Re)}(X_{0,t}, Y_{0,s}) := ||X_{t,s-t} - Y_{0,t}||_\infty + s - t .$$

Here $|| \cdot ||_\infty$ is the usual sup-norm; $X_{t,s-t}$ is the horizontal extension of $X_{0,t}$ over the future time horizon $[t, s]$, (i.e., $X_{t,s-t} = X_{t,\delta t}$ when $\delta t = s - t$). Note that $d_{\mathcal{C}(\mathcal{T};\Re)}$ allows a distance between two paths of different lengths to be defined. The presence of the difference $s - t$ makes $d_{\mathcal{C}(\mathcal{T};\Re)}(X_{0,t}, Y_{0,s})$ a norm instead of a semi-norm.

Then the continuity of a functional on $\mathcal{C}(\mathcal{T}; \Re)$ can be defined by the topology associated with the norm $d_{\mathcal{C}(\mathcal{T};\Re)}$. This definition is due to Dupire (2009).

Definition 4.1. A functional $f : \mathcal{C}(\mathcal{T}; \Re) \to \Re$ is said to be $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuous at the "point" $X_{0,t} \in \mathcal{C}(\mathcal{T}; \Re)$ if for any $\epsilon > 0$, there exists a $\eta > 0$ such that for all $Y_{0,t} \in \mathcal{C}(\mathcal{T}; \Re)$,

$$d_{\mathcal{C}(\mathcal{T};\Re)}(X_{0,t}, Y_{0,s}) < \eta \Longrightarrow |f(Y_{0,s}) - f(X_{0,t})| < \epsilon .$$

Here $| \cdot |$ is the usual norm in $\Re$.
The functional $f : \mathcal{C}(\mathcal{T}; \Re) \to \Re$ is said to be $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuous if it is $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuous at all $X_{0,t} \in \mathcal{C}(\mathcal{T}; \Re)$.

Note that the $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuity is weaker than the L^∞-uniform continuity. Then the following theorem gives the functional Itô's formula for functionals of continuous semimartingales.

Theorem 4.2. *Suppose $X \in \mathcal{C}(\mathcal{T}; \Re)$ is a continuous semimartingale and $f : \mathcal{C}(\mathcal{T}; \Re) \to \Re$ is a smooth functional, (i.e., f is $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuous, $\mathcal{C}^2$ in x in the sense of spatial functional derivatives and $\mathcal{C}^1$ in t in the sense of time functional derivatives, with these functional derivatives being $d_{\mathcal{C}(\mathcal{T};\Re)}$-continuous). Then for any $t \in \mathcal{T}$,*

$$f(X_{0,t}) \quad = \quad f(X_{0,0}) + \int_0^t \Delta_x f(X_{0,s}) dX(s) + \int_0^t \Delta_s f(X_{0,s}) ds$$

$$+\frac{1}{2}\int_0^t \Delta_{xx}f(X_{0,s})d\langle X,X\rangle(s) \ .$$

Here $\{\langle X,X\rangle(t)|t \in \mathcal{T}\}$ is the predictable quadratic variation of the continuous semimartingale X.

Of course, when $f(X_{0,t}) = F(t, X(t))$ for a smooth function $F \in \mathcal{C}^{1,2}(\mathcal{T} \times \Re)$, the above functional Itô's formula reduces to the classical Itô's formula.

In the original contribution of Dupire (2009), he applied the functional Itô's calculus to derive functional partial differential equations for valuing path-dependent derivatives, such as Asian options. However, the application of the functional Itô's calculus to evaluate risk measures has not yet been explored.

5. Solution to the BSDE: A Functional Itô's Calculus Approach

In this section we present our main results which are obtained using the functional Itô's formula presented in the last section to discuss the solution of the backward system in the FBSDE described in Section 3. We show that the first component of the solution of the BSDE (3.2) is given by the solution of a functional partial differential equation, (FPDE), while the control component of the BSDE (3.2) is identified with the functional derivative of the first component.

Note that the backward system (3.2) can be written in the following integral form:

$$Y(t) - \int_t^T g(s, X_{0,s}, Y(s), Z(s))ds + \int_t^T Z(s)dW(s) = -G(X(T)) \ . \qquad (5.1)$$

Conditioning on $\mathcal{F}^X(t)$ under $\mathbb{P}$ and using the martingale property of the stochastic integral term give:

$$Y(t) \quad = \quad \mathrm{E}\left[-G(X(T)) + \int_t^T g(s, X_{0,s}, Y(s), Z(s))ds|\mathcal{F}^X(t)\right] \ . \qquad (5.2)$$

Note that for each $t \in \mathcal{T}$, $Y(t)$ is a functional, say $f(X_{0,t})$, of $X_{0,t} \in \mathcal{C}([0,t]; \Re)$.

For each $t \in \mathcal{T}$, let

$$U(t) := \int_0^t g(s, X_{0,s}, Y(s), Z(s))ds \ .$$

Since $\{(Y(t), Z(t))|t \in \mathcal{T}\}$ is the solution of the BSDE (3.2), it is adapted to the, (forward), filtration $\mathbb{F}^X$. Consequently, $g(s, X_{0,s}, Y(s), Z(s))$ is a functional of $X_{0,s} \in \mathcal{C}([0,s]; \Re)$. So we can write $g(s, X_{0,s}, Y(s), Z(s)) := \bar{g}(X_{0,s})$, for some functional $\bar{g} : \mathcal{C}([0,s]; \Re) \rightarrow \Re$. Then

$$U(t) = \int_0^t \bar{g}(X_{0,s})ds \ .$$

It is not difficult to see that

$$\Delta_t U(t) = \bar{g}(X_{0,t}) \ ,$$

and that

$$\Delta_x U(t) = \Delta_{xx}U(t) = 0 \ .$$

We now define another process $\bar{Y} := \{\bar{Y}(t)|t \in \mathcal{T}\}$ by putting:

$$\bar{Y}(t) \quad := \quad Y(t) + U(t)$$

$$= \quad \mathrm{E}\left[-G(X(T)) + \int_0^T g(s, X_{0,s}, Y(s), Z(s))ds | \mathcal{F}^X(t) \right] .$$

Clearly, due to the square-integrability of the terminal condition $-G$ and the driver function g, $\bar{Y}$ is a square-integrable, $(\mathbb{F}^X, \mathbb{P})$-martingale. Note also that $\bar{Y}(t)$ is a functional, say $\bar{f}(X_{0,t})$, of $X_{0,t} \in \mathcal{C}([0,t]; \Re)$. We suppose that $\bar{f}$ is a smooth functional in the sense defined in Theorem 4.2.

Now by the martingale representation theorem, $\bar{Y}$ has the following integral representation:

$$\bar{Y}(t) = \int_0^t \alpha(s)dW(s) ,$$

where $\alpha := \{\alpha(t)|t \in \mathcal{T}\}$ is an $\mathbb{F}^X$-predictable process such that

$$\mathrm{E}\left[\int_0^T |\alpha(t)|^2 dt \right] < \infty .$$

The following theorem presents the main result which gives a representation for the integrand process $\alpha := \{\alpha(t)|t \in \mathcal{T}\}$ and a FPDE for $Y(t) = f(X_{0,t})$, $t \in \mathcal{T}$.

Theorem 5.1. *For each $t \in \mathcal{T}$,*

$$\alpha(t) \quad = \quad \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t}) .$$

Furthermore, $\{f(X_{0,t})|t \in \mathcal{T}\}$ satisfies the following functional partial differential equation, (FPDE):

$$\Delta_t f(X_{0,t}) + \mu(X_{0,t})X(t)\Delta_x f(X_{0,t}) + \frac{1}{2}\sigma^2(X_{0,t})X^2(t)\Delta_{xx} f(X_{0,t})$$

$$+ g(t, X_{0,t}, f(X_{0,t}), Z(t)) = 0 ,$$

with terminal condition:

$$f(X_{0,T}) = -G(X(T)) .$$

Proof. Applying the functional Itô's formula in Theorem 4.2 to $\bar{f}(X_{0,t})$ gives:

$$\bar{f}(X_{0,t})$$

$$= \quad \bar{f}(X_{0,0}) + \int_0^t \Delta_s \bar{f}(X_{0,s})ds + \int_0^t \Delta_x \bar{f}(X_{0,s})dX(s)$$

$$+ \frac{1}{2}\int_0^t \Delta_{xx}\bar{f}(X_{0,s})d\langle X, X \rangle(s)$$

$$= \quad f(X_{0,0}) + \int_0^t \left(\Delta_s f(X_{0,s}) + g(s, X_{0,s}, Y(s), Z(s)) \right)ds$$

$$+ \int_0^t \Delta_x f(X_{0,s})dX(s) + \frac{1}{2}\int_0^t \Delta_{xx}f(X_{0,s})d\langle X, X \rangle(s)$$

$$= \quad f(X_{0,0}) + \int_0^t \left(\Delta_s f(X_{0,s}) + \mu(X_{0,s})X(s)\Delta_x f(X_{0,s}) \right.$$

$$+\frac{1}{2}\sigma^2(X_{0,s})X^2(s)\Delta_{xx}f(X_{0,s}) + g(s, X_{0,s}, Y(s), Z(s))\bigg)ds$$

$$+\int_0^t \sigma(X_{0,s})X(s)\Delta_x f(X_{0,s})dW(s) .$$

Note that $\bar{Y}(t) = \bar{f}(X_{0,t})$, $t \in \mathcal{T}$, is an $(\mathbb{F}^X, \mathbb{P})$-martingale, so it must be a special semimartingale. Consequently, the above decomposition for $\bar{f}(X_{0,t})$ must be unique. This implies that the bounded variation term in the above decomposition must be indistinguishable from the zero process which gives the FPDE for $f(X_{0,t})$ and that

$$\alpha(t) = \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t}) , \quad t \in \mathcal{T} .$$

$\square$

The following theorem provides a representation for the solution

$$\{(Y(t), Z(t))|t \in \mathcal{T}\}$$

of the backward system in the FBSDE.

Theorem 5.2. *The first component $Y(t) = f(X_{0,t})$, $t \in \mathcal{T}$, of the BSDE (3.2) satisfies the following semi-linear FPDE:*

$$\Delta_t f(X_{0,t}) + \mu(X_{0,t})X(t)\Delta_x f(X_{0,t}) + \frac{1}{2}\sigma^2(X_{0,t})X^2(t)\Delta_{xx}f(X_{0,t})$$

$$+g(t, X_{0,t}, f(X_{0,t}), \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t})) = 0 ,$$

with terminal condition:

$$f(X_{0,T}) = -G(X(T)) .$$

The control component $\{Z(t)|t \in \mathcal{T}\}$ of the BSDE (3.2) is given by:

$$Z(t) = \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t}) .$$

Proof. We first prove the second statement. By the martingale representation,

$$Y(t) + \int_0^t g(s, X_{0,s}, Y(s), Z(s))ds = \bar{Y}(t) = \int_0^t \alpha(s)dW(s) . \qquad (5.3)$$

Putting $t = T$ gives:

$$-G(X(T)) + \int_0^T g(s, X_{0,s}, Y(s), Z(s))ds = \int_0^T \alpha(s)dW(s) . \qquad (5.4)$$

Subtracting (5.4) from (5.3) gives:

$$Y(t) - \int_t^T g(s, X_{0,s}, Y(s), Z(s))ds + \int_t^T \alpha(s)dW(s) = -G(X(T)) .$$

By the uniqueness of the solution of the backward system in the FBSDE and Theorem 5.1, we must have:

$$Z(t) = \alpha(t) = \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t}) , \quad t \in \mathcal{T} .$$

The first statement follows from the second statement in Theorem 5.1 and by noting that $Z(t) = \sigma(X_{0,t})X(t)\Delta_x f(X_{0,t})$. $\square$

Theorem 5.2 gives a new approach to discuss the solution of a FBSDE. When the functional $f(X_{0,t}) = F(t, X(t))$ for a smooth function $F \in \mathcal{C}^{1,2}(\mathcal{T} \times \Re)$. The FPDE for the first component $\{Y(t)|t \in \mathcal{T}\}$ of the BSDE (3.2) becomes a classical semi-linear, second-order, parabolic partial differential equation and the control component becomes:

$$Z(t) = \sigma(X_{0,t})X(t)\frac{\partial F}{\partial x}(t, X(t)) \ .$$

A practical issue is the development of some efficient and practically useful numerical schemes to approximate the solution of the FPDE in Theorem 5.2, and hence, the two components in the solution of the backward system in the FBSDE. Can some standard numerical methods such as finite-difference methods and finite-element methods be modified to solve numerically the FPDE? This may represent a potential research topic for enthusiasts of numerical analysis.

6. A Parametric Case Based on Stochastic Optimal Control

In this section we consider a parametric case where the family of probability measures for risk measurement is specified by stochastic exponentials generated by the Brownian notion and the penalty function has the form given in Mataramvura and Oksendal (2007).

Firstly, we specify the family of probability measures for risk measurement. Let $\{\theta(t)|t \in \mathcal{T}\}$ be an $\mathbb{F}^X$-progressively measurable, bounded, real-valued process such that

   (1) for each $t \in \mathcal{T}$, $\theta(t) \in [\theta_1, \theta_2]$, where $\theta_1, \theta_2 \in \Re$ with $\theta_1 < \theta_2$;
   (2) the following Novikov's condition is satisfied:

$$\mathrm{E}\left[\exp\left(\frac{1}{2}\int_0^T \theta^2(t)dt \right) \right] < \infty \ .$$

Write Θ for the space of such processes $\{\theta(t)|t \in \mathcal{T}\}$. A process $\theta \in \Theta$ is called an admissible scenario.

For each $\theta \in \Theta$, let $\{\Lambda^\theta(t)|t \in \mathcal{T}\}$ be an $\mathbb{F}^X$-adapted process defined by:

$$\Lambda^\theta(t) = 1 + \int_0^t \Lambda^\theta(u)\theta(u)dW(u) \ .$$

Then $\{\Lambda^\theta(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}^X, \mathbb{P})$-(local)-martingale, and

$$\begin{aligned}
\Lambda^\theta(t) &= \mathcal{E}\left(\int_0^\cdot \theta(u)dW(u) \right)(t) \\
&= \exp\left(\int_0^t \theta(u)dW(u) - \frac{1}{2}\int_0^t \theta^2(u)du \right) \ .
\end{aligned}$$

Here $\{\mathcal{E}(\int_0^\cdot \theta(u)dW(u))(t)|t \in \mathcal{T}\}$ is the stochastic exponential of

$$\left\{ \int_0^t \theta(u)dW(u)|t \in \mathcal{T} \right\}.$$

Since $\theta \in \Theta$ satisfies the Novikov condition, $\{\Lambda^\theta(t)|t \in \mathcal{T}\}$ is an $(\mathbb{F}^X, \mathbb{P})$-martingale.

For each $\theta \in \Theta$, we define a new probability measure $\mathbb{P}^\theta$ absolutely continuous to $\mathbb{P}$ on $\mathcal{F}^X(T)$ associated with θ by putting:

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\bigg|_{\mathcal{F}^X(T)} := \Lambda^\theta(T) \ .$$

By Girsanov's theorem, the process $\{W^\theta(t)|t \in \mathcal{T}\}$ defined by:

$$W^\theta(t) := W(t) - \int_0^t \theta(u)du \ ,$$

is an $(\mathbb{F}^X, \mathbb{P}^\theta)$-martingale.

Consequently, under $\mathbb{P}^\theta$,

$$dX(t) = (\mu(X_{0,t}) + \sigma(X_{0,t})\theta(t))X(t)dt + \sigma(X_{0,t})X(t)dW^\theta(t) \ .$$

We then specify the family of probability measures $\mathcal{P} := \mathcal{P}(\Theta)$ as follows:

$$\mathcal{P}(\Theta) := \{\mathbb{P}^\theta|\theta \in \Theta\} \ .$$

We now specify the penalty function η_t. Suppose $\lambda : \mathcal{T} \times \mathcal{C}(\mathcal{T}; \Re) \times [\theta_1, \theta_2] \to \Re$ and $h : \Re \to \Re$ are two bounded, real-valued, measurable convex functions on $[\theta_1, \theta_2]$ and $\Re$, respectively. Note that for each $t \in \mathcal{T}$ and $\theta(t) \in [\theta_1, \theta_2]$, $\lambda(t, X_{0,t}, \theta(t))$ is the "running" penalty at time t which depends on the price path $X_{0,t}$ of the share up to and including time t; $h(X(T))$ is the "terminal" penalty which is a function of the terminal share price $X(T)$. We suppose that for each $\theta \in \Theta$,

$$\mathrm{E}^\theta\bigg[\int_0^T |\lambda(t, X_{0,t}, \theta(t))|dt + |h(X(T))|\bigg] < \infty \ .$$

Here E^θ is expectation under $\mathbb{P}^\theta$.

Then we suppose that the penalty function has the following form:

$$\eta_t(\mathbb{P}^\theta) := \mathrm{E}^\theta\bigg[\int_t^T \lambda(s, X_{0,s}, \theta(s))ds + h(X(T))|\mathcal{F}^X(t)\bigg] \ , \quad t \in \mathcal{T} \ , \quad \theta \in \Theta \ .$$

We consider the following parametric form of the dynamic convex risk measure $\{\rho_t(G(X(T)))|t \in \mathcal{T}\}$ of the claim $G(X(T))$:

$$\rho_t(G(X(T))) = ess - \sup_{\theta \in \Theta}\{\mathrm{E}^\theta[-G(X(T))|\mathcal{F}^X(t)] - \eta_t(\mathbb{P}^\theta)\} \ .$$

For each $\theta \in \Theta$ and each $t \in \mathcal{T}$, let the performance functional of θ at time t be:

$$J^\theta(t) := \mathrm{E}^\theta\bigg[- G(X(T)) - \int_t^T \lambda(s, X_{0,s}, \theta(s))ds - h(X(T))|\mathcal{F}^X(t)\bigg] \ .$$

Then the, (nonlinear), evaluation of the dynamic convex risk measure can be formulated as the following stochastic optimal control problem:

$$\rho_t(G(X(T))) = ess - \sup_{\theta \in \Theta} J^\theta(t) \ .$$

Here $\rho_t(G(X(T)))$ is the value function of the control problem evaluated at time t.

In what follows, we first relate the value function $\rho_t(G(X(T)))$ of the control problem to the solution of the backward system in a FBSDE. Then we use the approach based on the functional Itô's calculus presented in Section 5 to discuss the solution of the backward system, and hence, the control problem.

Firstly we need the following theorem which was presented in El Karoui et al. (1997).

Theorem 6.1. *Let $\{\beta_2(t)|t \in \mathcal{T}\}$ and $\{\gamma(t)|t \in \mathcal{T}\}$ be two bounded, real-valued, $\mathbb{F}^X$-progressively measurable processes. Write $\mathcal{K}^2$ for the space of real-valued, $\mathbb{F}^X$-progressively measurable processes $\{l(t)|t \in \mathcal{T}\}$ such that $E[\int_0^T |l(t)|^2 dt] < \infty$. Suppose $\{\beta_1(t)|t \in \mathcal{T}\} \in \mathcal{K}^2$ and $\xi \in L^2(\Omega, \mathcal{F}^X(T), \mathbb{P})$. Consider the following linear BSDE:*

$$dY(t) = -(\beta_1(t) + \beta_2(t)Y(t) + \gamma(t)Z(t))dt + Z(t)dW(t) , \quad Y(T) = \xi .$$

Then the BSDE has a unique, square-integrable solution $\{(Y(t), Z(t))|t \in \mathcal{T}\}$, and the first component of the soluton Y has the following expectation representation:

$$Y(t) = E\left[\xi\Gamma(t,T) + \int_t^T \Gamma(t,s)\beta_1(s)ds|\mathcal{F}^X(t)\right] .$$

Here the double-indexed process $\{\Gamma(t,s)|t,s \in \mathcal{T}, t \leq s\}$ is the adjoint process satisfying the following forward linear stochastic differential equation:

$$d\Gamma(t,s) = \Gamma(t,s)(\beta_2(s)ds + \gamma(s)dW(s)) , \quad \Gamma(t,t) = 1 .$$

Let $H : \mathcal{T} \times \mathcal{C}(\mathcal{T}; \Re) \times \Re \times [\theta_1, \theta_2] \to \Re$ be the Hamiltonian defined by:

$$H(t, X_{0,t}, Z(t), \theta(t)) := -\lambda(t, X_{0,t}, \theta(t)) + \theta(t)Z(t) .$$

Recall that λ is convex in $\theta(t)$, so H is concave in $\theta(t)$. Consequently,

$$H(t, X_{0,t}, Z(t), \theta^\dagger(t)) := \sup_{\theta(t) \in [\theta_1, \theta_2]} H(t, X_{0,t}, Z(t), \theta(t)) ,$$

where $\theta^\dagger(t)$ is a value of $\theta(t)$ maximizing the Hamiltonian H.

Due to the boundedness of λ and θ, $H(t, X_{0,t}, z, \theta^\dagger(t))$ is Lipschitz in $z \in \Re$, uniformly in $(t, X_{0,t}) \in \mathcal{T} \times \mathcal{C}(\mathcal{T}; \Re)$.

The following theorem represents the value function $\rho_t(G(X(T)))$ of the control problem as the first component of the solution of a BSDE.

Theorem 6.2. *Suppose $\{(Y(t), Z(t))|t \in \mathcal{T}\}$ is the unique, square-integrable solution of the following BSDE:*

$$\begin{aligned} -dY(t) &= H(t, X_{0,t}, Z(t), \theta^\dagger(t))dt - Z(t)dW(t) , \\ Y(T) &= -G(X(T)) - h(X(T)) . \end{aligned} \quad (6.1)$$

Here we assume that the conditions to ensure the existence and uniqueness of the square-integrable solution $\{(Y(t), Z(t))|t \in \mathcal{T}\}$ hold. Then

$$Y(t) = J^{\theta^\dagger}(t) = ess - \sup_{\theta \in \Theta} J^\theta(t) = \rho_t(G(X(T))) ,$$

and $\theta^\dagger(t)$ is an optimal control.

Proof. For each $t, s \in \mathcal{T}$ with $t \leq s$, we set

$$\Gamma^{\theta^\dagger}(t, s) := \frac{\Lambda^{\theta^\dagger}(s)}{\Lambda^{\theta^\dagger}(t)} \ ,$$

so that

$$d\Gamma^{\theta^\dagger}(t, s) = \Gamma^{\theta^\dagger}(t, s)\theta^\dagger(s)dW(s) \ , \quad \Gamma^{\theta^\dagger}(t, t) = 1 \ .$$

Then by Theorem 6.1, the first component $\{Y(t)|t \in \mathcal{T}\}$ of the solution of the BSDE (6.1) is explicitly characterized as the expectation:

$$
\begin{aligned}
Y(t) \ &= \ \mathrm{E}\bigg[-\Gamma^{\theta^\dagger}(t, T)\Big(G(X(T)) + h(X(T))\Big) \\
&\quad - \int_t^T \Gamma^{\theta^\dagger}(t, s)\lambda(s, X_{0,s}, \theta^\dagger(s))ds|\mathcal{F}^X(t)\bigg] \\
&= \ J^{\theta^\dagger}(t) \ , \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Let θ be another control in Θ and $\{(Y^\theta(t), Z^\theta(t))|t \in \mathcal{T}\}$ be the unique solution of the following BSDE:

$$
\begin{aligned}
-dY^\theta(t) \ &= \ H(t, X_{0,t}, Z^\theta(t), \theta(t))dt - Z^\theta(t)dW(t) \ , \\
Y^\theta(T) \ &= \ -G(X(T)) - h(X(T)) \ .
\end{aligned}
$$

Note that for any $t \in \mathcal{T}$ and any $\theta \in \Theta$,

$$H(t, X_{0,t}, Z^\theta(t), \theta(t)) \leq H(t, X_{0,t}, Z(t), \theta^\dagger(t)) \ .$$

Then by the standard comparsion theorem of the solutions of BSDEs, (see, for example, El Karoui et al. (1997)),

$$J^\theta(t) = Y^\theta(t) \leq J^{\theta^\dagger}(t) = Y(t) \ , \quad \forall \theta \in \Theta \ .$$

This completes the proof. □

For each $t \in \mathcal{T}$, the value of the dynamic convex risk measure $\rho_t(G(X(T)))$ of the claim $G(X(T))$ at time t is a functional of $X_{0,t}$. Consequently, we write:

$$\rho_t(G(X(T))) = \rho(X_{0,t}) \ , \quad t \in \mathcal{T} \ ,$$

for some functional $\rho : \mathcal{C}([0, t]; \Re) \to \Re$.

Then the following theorem gives the FPDE governing the evolution of the dynamic convex risk measure $\{\rho(X_{0,t})|t \in \mathcal{T}\}$ over time.

Theorem 6.3. *The dynamic convex risk measure $\{\rho(X_{0,t})|t \in \mathcal{T}\}$ of the claim $G(X(T))$ satisfies the following FPDE:*

$$
\begin{aligned}
&\Delta_t\rho(X_{0,t}) + (\mu(X_{0,t}) + \theta^\dagger(t)\sigma(X_{0,t}))X(t)\Delta_x\rho(X_{0,t}) \\
&+\frac{1}{2}\sigma^2(X_{0,t})X^2(t)\Delta_{xx}\rho(X_{0,t}) - \lambda(t, X_{0,t}, \theta^\dagger(t)) = 0 \ ,
\end{aligned}
$$

with terminal condition:

$$\rho(X_{0,T}) = -G(X(T)) - h(X(T)) \ .$$

Proof. By Theorem 5.2 and Theorem 6.2, $\{\rho(X_{0,t})|t \in \mathcal{T}\}$ satisfies the following FPDE:

$$\Delta_t \rho(X_{0,t}) + \mu(X_{0,t})X(t)\Delta_x \rho(X_{0,t})$$
$$+\frac{1}{2}\sigma^2(X_{0,t})X^2(t)\Delta_{xx}\rho(X_{0,t})$$
$$+H(t, X_{0,t}, Z(t), \theta^\dagger(t)) = 0 \ ,$$

with terminal condition:

$$\rho(X_{0,T}) = -G(X(T)) - h(X(T)) \ .$$

The result then follows by noting that

$$H(t, X_{0,t}, Z(t), \theta^\dagger(t)) = -\lambda(t, X_{0,t}, \theta^\dagger(t)) + \theta^\dagger(t)Z(t) \ ,$$

and that

$$Z(t) = \sigma(X_{0,t})X(t)\Delta_x \rho(X_{0,t}) \ .$$

$\square$

Indeed, Theorem 6.3 also provides an example to illustrate how the functional Itô's calculus can be applied to give a FPDE solution to a stochastic optimal control problem. The FPDE in Theorem 6.3 is different from the PDE for the dynamic convex risk measure in Elliott and Siu (2010) which is a classical PDE without involving functional derivatives and only holds true in the Markovian case.

In the case where $\lambda(t, X_{0,t}, \theta^\dagger(t)) = h(X(T)) = 0$, the dynamic convex risk measure $\{\rho(X_{0,t})|t \in \mathcal{T}\}$ becomes a dynamic coherent risk measure, say $\{\bar{\rho}(X_{0,t})|t \in \mathcal{T}\}$. Then by Theorem 6.3, the dynamic coherent risk measure $\{\bar{\rho}(X_{0,t})|t \in \mathcal{T}\}$ satisfies the following FPDE:

$$\Delta_t \bar{\rho}(X_{0,t}) + \left[\mu(X_{0,t}) + \left(\theta_1 I_{\{\Delta_x \bar{\rho}(X_{0,t})<0\}} + \theta_2 I_{\{\Delta_x \bar{\rho}(X_{0,t})\geq 0\}}\right)\sigma(X_{0,t})\right]$$
$$\times X(t)\Delta_x \bar{\rho}(X_{0,t}) + \frac{1}{2}\sigma^2(X_{0,t})X^2(t)\Delta_{xx}\bar{\rho}(X_{0,t}) = 0 \ ,$$

with terminal condition:

$$\bar{\rho}(X_{0,T}) = -G(X(T)) \ .$$

This gives a FPDE approach for a functional "bang-bang" type control problem and is a generalization of the PDE for a coherent risk measure in Siu and Yang (2000).

7. Conclusion

We developed an approach based on the functional Itô's calculus to discuss the, (nonlinear), evaluation of a dynamic convex risk measure for a standard European contingent claim. By relating the dynamic convex risk measure to the solution of the backward system in a FBSDE, we derive a FPDE governing the evolution of the dynamic convex risk measure over time using the functional Itô's formula together with the unique decomposition of a special semimartingale. This also identified the control component in the solution of the backward system in the FBSDE by, (pathwise), functional derivatives. A parametric case was discussed

in some details, where the, (nonlinear), evaluation of the dynamic convex risk measure is formulated as a stochastic optimal control problem. We then used the functional Itô's calculus to discuss the stochastic optimal control problem. In the special case of a dynamic coherent risk measure, a FPDE for a functional "bang-bang" control problem associated with the evaluation of the dynamic coherent risk measure was derived which generalized some existing results.

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