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AN ESTIMATE FOR BOUNDED SOLUTIONS OF THE HERMITE HEAT EQUATION

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Abstract. An estimate result on the partial derivatives of the Mehler kernel $E(x, \xi, t)$ for $t > 0$ is first established. Particularly for $0 < t < 1$, it extends the estimate result given by S. Thangavelu in his monograph *A lecture notes on Hermite and Laguerre expansions* on the order of the partial derivative of the Mehler kernel with respect to the space variable. Furthermore, for each $m \in \mathbf{N}_0$, a growth estimate on the partial derivative $\frac{\partial^m U(x,t)}{\partial x^m}$ of all bounded solutions $U(x, t)$ of the Cauchy Dirichlet problem for the Hermite heat equation is established.

1. Introduction

As introduced in [1], we denote by $E(x, \xi, t)$ the Mehler kernel defined by

$$E(x, \xi, t) = \begin{cases} \sum_{k=0}^{\infty} e^{-(2k+1)t} h_k(x) h_k(\xi), & t > 0, \\ 0, & t \leq 0, \end{cases}$$

where h_k's are L^2 − normalized Hermite functions defined by

$$h_k(x) = \frac{(-1)^k \; e^{x^2/2}}{\sqrt{2^k k! \sqrt{\pi}}} \frac{d^k}{dx^k} e^{-x^2}, \quad x \in \mathbf{R}.$$

Moreover the explicit form of $E(x, \xi, t)$ for $t > 0$ is

$$E(x, \xi, t) = \frac{e^{-t} \; e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi)^2 - \frac{1-e^{-2t}}{1+e^{-2t}} x\xi}}{\sqrt{\pi}(1 - e^{-4t})^{\frac{1}{2}}}.$$

We note that for each $\xi \in \mathbf{R}$, $E(x, \xi, t)$ satisfies the Hermite heat equation. In (Theorem 3.1, [2]), we proved that

$$U(x, t) = \int_0^{\infty} \{E(x, \xi, t) - E(x, -\xi, t)\} \phi(\xi) d\xi \tag{1.1}$$

is a unique bounded solution of the following Cauchy Dirichlet problem for the Hermite heat equation

$$\begin{cases} (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2) U(x, t) = 0, & x > 0, \; t > 0, \\ U(x, 0) = \phi(x), & x > 0, \\ U(0, t) = 0, & t > 0, \end{cases} \tag{1.2}$$

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where ϕ is a continuous and bounded function on $[0,\infty)$ with $\phi(0)=0$.

It is not necessary that every bounded solution of the Hermite heat equation should satisfy a fixed growth behavior on its m^{th} partial derivative with respect to the space variable. However, since the solution $U(x,t)$ in (1.1) is a unique solution of (1.2), it is natural to make an effort for obtaining a fixed growth estimate on $\frac{\partial^m U(x,t)}{\partial x^m}$. But it is not as easy as we anticipate. To find a growth estimate on $\frac{\partial^m U(x,t)}{\partial x^m}$, we require first to obtain an estimate on $\frac{\partial^m E(x,\xi,t)}{\partial x^m}$. Note that an estimate on the partial derivatives of the heat kernel

$$E(x,t)=\begin{cases}(4\pi t)^{-\frac{1}{2}}\ e^{-\frac{x^2}{4t}}, & t>0,\\ 0, & t\le 0,\end{cases}$$

with respect to the space variable has been given in [3]:

$$\left|\frac{\partial^m E(x,t)}{\partial x^m}\right|\le C^m t^{-\frac{(1+m)}{2}} m!^{\frac{1}{2}}\ e^{-\frac{ax^2}{4t}},\qquad t>0,$$

where C is some constant and a can be taken as close as desired to 1 such that $0<a<1$.

Though the estimates of the following types on the Mehler kernel for $0<t<1$ and B independent of x,ξ and t

$$\left|\frac{\partial E(x,\xi,t)}{\partial x}\right|\le Ct^{-1}\ e^{-\frac{B}{t}|x-\xi|^2}, \tag{1.3}$$

$$\left|\frac{\partial^2 E(x,\xi,t)}{\partial x\ \partial\xi}\right|\le Ct^{-\frac{3}{2}}\ e^{-\frac{B}{t}|x-\xi|^2},$$

are provided in [4], the estimate on the partial derivatives of the Mehler kernel of all order with respect to the space variable is yet to be established.

Lemma 2.1 that gives an estimate on $\frac{\partial^m E(x,\xi,t)}{\partial x^m}$ for each nonnegative integer m, is therefore a novelty of this paper which as an application yields

$$t^{\frac{m}{2}}e^{-mt}\left|\frac{\partial^m U(x,t)}{\partial x^m}\right|\le M \text{ in } [0,\infty)\times[0,\infty)$$

for some constant M, the main objective and the final part of this paper.

2. Main Results

Lemma 2.1. *Let $E(x,\xi,t)$ be the Mehler kernel and $m\in\mathbf{N}_0$. Then for some constants a with $0<a<1$ and $A:=A(a)>0$*

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right|\le\frac{\sqrt{m!\ e^{(A+1)m}}}{\sqrt{\pi}\ 2^{1+\frac{m}{2}}}\frac{e^{mt}}{t^{\frac{m+1}{2}}}\ e^{-\frac{ae^{-2t}(x-\xi)^2}{1-e^{-4t}}}.$$

Proof. By the Cauchy integral formula, we have

$$\begin{aligned}
&\frac{\partial^m E(x,\xi,t)}{\partial x^m} \\
&= \frac{m!}{2\pi i}\int_{\Gamma_R} \frac{E(\zeta,\xi,t)}{(\zeta-x)^{m+1}} d\zeta \\
&= \frac{m!}{2\pi^{\frac{3}{2}} i}\int_{\Gamma_R} \frac{e^{-t} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(\zeta-\xi)^2 - \frac{1-e^{-2t}}{1+e^{-2t}}\zeta\xi}}{(\zeta-x)^{m+1}\,(1-e^{-4t})^{\frac{1}{2}}} d\zeta,
\end{aligned}$$

where Γ_R is a cirle of radius R in the complex plane $\mathbf{C}$ with center at x. With $\zeta = x + Re^{i\theta}$, we have

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \le \frac{m! e^{-t}}{2\pi^{\frac{3}{2}} R^m \sqrt{1-e^{-4t}}} \int_0^{2\pi} e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}(x-\xi+Re^{i\theta})^2 - \frac{1-e^{-2t}}{1+e^{-2t}}(x+Re^{i\theta})\xi} d\theta.$$

Then, writing S for $x + R\cos\theta$, we have

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \le \frac{m!\ e^{-t}}{2\pi^{\frac{3}{2}} R^m \sqrt{1-e^{-4t}}} \int_0^{2\pi} \frac{e^{-\left[\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}\{\xi-S\}^2 + \frac{1-e^{-2t}}{1+e^{-2t}}\xi S\right]}}{e^{-\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}R^2}} d\theta.$$

Let $P = \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}$ and $Q = \frac{1-e^{-2t}}{1+e^{-2t}}$. Then $P > 0$ and $Q > 0$ since t is positive. Now using the inequality

$$P\left\{\xi - (x + R\cos\theta)\right\}^2 + Q\xi(x + R\cos\theta) \ge \left(P - \frac{Q}{2}\right)\left\{\xi - (x + R\cos\theta)\right\}^2,$$

we have

$$\begin{aligned}
\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| &\le \frac{m!\ e^{-t}}{2\pi^{\frac{3}{2}} R^m \sqrt{1-e^{-4t}}} \int_0^{2\pi} e^{-\frac{e^{-2t}}{1-e^{-4t}}(x-\xi+R\cos\theta)^2 + \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}R^2} d\theta \\
&\le \frac{m!\ e^{-t}}{\sqrt{\pi} R^m \sqrt{1-e^{-4t}}} e^{-\frac{e^{-2t}}{1-e^{-4t}}\tilde{x}^2 + \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}R^2},
\end{aligned}$$

where $\tilde{x} = x - \xi - R$ or 0 or $x - \xi + R$. Since the ratio $\frac{\exp\left(\frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}R^2\right)}{R^m}$ attains its minimum at $R = \sqrt{\frac{m}{2b}}$ where $b = \frac{1}{2}\frac{1+e^{-4t}}{1-e^{-4t}}$, we have

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \le \frac{m!\ e^{-t}\ e^{\frac{m}{2}}}{\sqrt{\pi}\sqrt{1-e^{-4t}} m^{\frac{m}{2}}} \left(\frac{1+e^{-4t}}{1-e^{-4t}}\right)^{\frac{m}{2}} e^{-\frac{e^{-2t}}{1-e^{-4t}}\tilde{x}^2}. \tag{2.1}$$

But with $0 < a < 1$ and $|\beta| \le 1$

$$\begin{aligned}
e^{-\frac{e^{-2t}}{1-e^{-4t}}(x-\xi+\beta R)^2} &= e^{-\frac{ae^{-2t}}{1-e^{-4t}}(x-\xi)^2} e^{-\frac{e^{-2t}}{1-e^{-4t}}\left[(1-a)(x-\xi)^2 + 2(x-\xi)\beta R + \beta^2 R^2\right]} \\
&= e^{-\frac{ae^{-2t}}{1-e^{-4t}}(x-\xi)^2} e^{-\frac{(1-a)e^{-2t}}{1-e^{-4t}}\left[(x-\xi+\frac{\beta R}{1-a})^2 - \frac{a\beta^2 R^2}{(1-a)^2}\right]} \\
&\le e^{-\frac{ae^{-2t}}{1-e^{-4t}}(x-\xi)^2} e^{\frac{Ae^{-2t}}{1-e^{-4t}}R^2},
\end{aligned}$$

where $A = \frac{a}{1-a}$. Then clearly

$$e^{-\frac{e^{-2t}}{1-e^{-4t}}\tilde{x}^2} \le e^{-\frac{ae^{-2t}}{1-e^{-4t}}(x-\xi)^2}\ e^{\frac{Ae^{-2t}}{1-e^{-4t}}R^2}.$$

Using $R^2 = \frac{m(1-e^{-4t})}{1+e^{-4t}}$ and the inequalities $\frac{e^{-2t}}{1+e^{-4t}} \leq \frac{1}{2}$, $\frac{1+e^{-4t}}{1-e^{-4t}} \leq \frac{e^{2t}}{2t}$ for every $t > 0$, (2.1) reduces to

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \quad \leq \quad \frac{\sqrt{m!\ e^{(A+1)m}}}{\sqrt{\pi}} \frac{e^{-t}}{\sqrt{1-e^{-4t}}} \frac{e^{mt}}{2^{\frac{m}{2}} t^{\frac{m}{2}}}\ e^{-\frac{ae^{-2t}(x-\xi)^2}{1-e^{-4t}}}. \qquad (2.2)$$

Furthermore, since $\frac{e^{-t}}{\sqrt{1-e^{-4t}}} \leq \frac{1}{2\sqrt{t}}$ for every $t > 0$ we obtain

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \leq \frac{\sqrt{m!\ e^{(A+1)m}}}{\sqrt{\pi}\ 2^{1+\frac{m}{2}}} \frac{e^{mt}}{t^{\frac{m+1}{2}}}\ e^{-\frac{ae^{-2t}(x-\xi)^2}{1-e^{-4t}}}. \qquad (2.3)$$

This completes the proof. □

Remark 2.2. For $0 < t < 1$, in view of (2.3) and $-\frac{e^{-2t}}{1-e^{-4t}} \leq -\frac{1}{8t}$ it is easy to see that

$$\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| \leq \frac{\sqrt{m!\ e^{(A+3)m}}}{\sqrt{\pi}\ 2^{1+\frac{m}{2}} t^{\frac{m+1}{2}}}\ e^{-\frac{a(x-\xi)^2}{8t}}$$

which extends the estimate result (1.3) on the order $m > 1$ of the partial derivative of $E(x,\xi,t)$ with respect to the variable x.

Theorem 2.3. *Every bounded solution of the Cauchy Dirichlet problem for the Hermite heat equation*

$$\begin{cases} (\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2} + x^2)U(x,t) = 0 & x > 0,\ t > 0, \\ U(x,0) = \phi(x) & x > 0, \\ U(0,t) = 0 & t > 0, \end{cases} \qquad (2.4)$$

satisfies the following growth estimate

$$t^{\frac{m}{2}} e^{-mt} \left|\frac{\partial^m U(x,t)}{\partial x^m}\right| \leq M, \quad \text{in } [0,\infty) \times [0,\infty),$$

where $m \in \mathbf{N_0}$ and M is some constant.

Proof. From (Theorem 3.1, [2]), every bounded solution of the Cauchy Dirichlet problem (2.4) for the Hermite heat equation is of the form

$$U(x,t) = \int_0^\infty \{E(x,\xi,t) - E(x,-\xi,t)\}\, \phi(\xi) d\xi,$$

where ϕ is a continuous and bounded function on $[0,\infty)$ with $\phi(0) = 0$ and $E(x,\xi,t)$, the Mehler kernel. We write

$$\begin{aligned} U(x,t) &= \int_0^\infty \{E(x,\xi,t) - E(x,-\xi,t)\}\, \phi(\xi) d\xi \\ &= \int_{\mathbf{R}} E(x,\xi,t) h(\xi) d\xi, \end{aligned}$$

where

$$h(\xi) = \begin{cases} \phi(\xi), & \xi \geq 0, \\ -\phi(-\xi), & \xi < 0. \end{cases}$$

From (2.2), we have

$$\begin{aligned}\left|\frac{\partial^m U(x,t)}{\partial x^m}\right| &\leq \int_{\mathbf{R}}\left|\frac{\partial^m E(x,\xi,t)}{\partial x^m}\right| |h(\xi)|d\xi \\ &\leq \frac{\|h\|_\infty\sqrt{m!e^{(A+1)m}}\ e^{(m-1)t}}{\sqrt{\pi}\sqrt{1-e^{-4t}}(2t)^{\frac{m}{2}}}\int_{\mathbf{R}} e^{-\frac{ae^{-2t}(x-\xi)^2}{1-e^{-4t}}}d\xi.\end{aligned}$$

Under the change of variable $\frac{\sqrt{a}\ e^{-t}}{\sqrt{1-e^{-4t}}}(\xi - x) = s$ and integrating, we have

$$\left|\frac{\partial^m U(x,t)}{\partial x^m}\right| \leq \frac{\|h\|_\infty\ \sqrt{m!e^{(A+1)m}}}{2^{\frac{m}{2}}\sqrt{a}}\ \frac{e^{mt}}{t^{\frac{m}{2}}}.$$

Clearly

$$t^{\frac{m}{2}}e^{-mt}\left|\frac{\partial^m U(x,t)}{\partial x^m}\right| \leq M \text{ in } [0,\infty)\times[0,\infty)$$

if we take $M = \frac{\|h\|_\infty\ \sqrt{m!e^{(A+1)m}}}{2^{\frac{m}{2}}\sqrt{a}}$. □

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