

SDE SOLUTIONS IN THE SPACE OF SMOOTH RANDOM VARIABLES

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ABSTRACT. In this paper we analyze properties of a dual pair $(\mathcal{G}, \mathcal{G}^*)$ of spaces of smooth and generalized random variables on a Lévy white noise space. We show that $\mathcal{G} \subset L^2(\mu)$ which shares properties with a Fréchet algebra contains a larger class of solutions of Itô equations driven by pure jump Lévy processes. Further a characterization of $(\mathcal{G}, \mathcal{G}^*)$ in terms of the S -transform is given. We propose $(\mathcal{G}, \mathcal{G}^*)$ as an attractive alternative to the Meyer-Watanabe test function and distribution space $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ [30] to study strong solutions of SDE's.

1. Introduction

Gel'fand triples or dual pairs of spaces of random variables have proved to be very useful in the study of various problems of stochastic analysis. Important applications pertain e.g. to the analysis of the regularity of the solutions of the Zakai equation in non-linear filtering theory, positive distributions in potential theory, the construction of local time of Lévy processes or the Clark-Ocone formula for the hedging of contingent claims in mathematical finance. See e.g. [4, 6, 8, 28] and the references therein.

The most prominent examples of dual pairs in stochastic and infinite dimensional analysis are $((S), (S)^*)$ of Hida and $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ of Meyer and Watanabe. See [6], [8] and [30]. The Hida test function and distribution space $((S), (S)^*)$ has been e.g. successfully applied to quantum field theory, the theory of stochastic partial differential equations or the construction of Feynman integrals ([6, 7]). One of the most interesting properties of the distribution space $(S)^*$ is that it accommodates the singular white noise which can be viewed as the time-derivative of the Brownian motion. The latter provides a favorable setting for the study of stochastic differential equations (see [24]). See also [15], where the authors derived an explicit representation for strong solutions of Itô equations. From an analytic point of view the pair $((S), (S)^*)$ also exhibits the nice feature that it can be characterized by the powerful tool of S -transform [6]. It is also worth mentioning that test functions in (S) admit continuous versions on the white noise probability space. However the Brownian motion is not contained in (S) since elements in (S) have chaos expansions with kernels in the Schwartz test function space. Therefore

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(S) does not seem to be suitable for the study of SDE's. It turns out that the test function space \mathbb{D}_∞ is more appropriate for the investigation of solutions of SDE's than (S) , since it carries a larger class of solutions of Itô equations. However a severe deficiency of the pair $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ compared to $((S), (S)^*)$ is that it lacks the availability of characterization-type theorems.

In this paper we propose a dual pair $(\mathcal{G}, \mathcal{G}^*)$ of smooth and generalized random variables on a Lévy white noise space which meets the following two important requirements: A richer class of solutions of (pure jump) Lévy noise driven Itô equations belongs to the test function space \mathcal{G} . On the other hand $(\mathcal{G}, \mathcal{G}^*)$ allows for a characterization-type theorem.

The pair $(\mathcal{G}, \mathcal{G}^*)$ has been studied in the Gaussian case by [2, 13, 22, 29]. See also [4] and the references therein for the case of Lévy processes. Similarly to the Gaussian case, \mathcal{G} is defined by means of exponential weights of the number operator on a Lévy white noise space. The space \mathcal{G} comprises the test functions in (S) and is included in the space $\mathbb{D}_{\infty,2} \supset \mathbb{D}_\infty$. The important question whether \mathcal{G} contains a bigger class of Itô jump diffusions has not been addressed so far in the literature. We will give an affirmative answer to this problem. For example, one can more or less directly show (Section 4) that solutions of compound Poisson driven SDE's are contained in \mathcal{G} . Furthermore we will discuss a characterization of Lévy noise functionals in terms of the S -transform by using the concept Bargmann-Segal spaces (see [5]). We believe that the pair $(\mathcal{G}, \mathcal{G}^*)$ could serve as an alternative tool to $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ for the study of Lévy noise functionals. By approximating general Lévy measures by finite ones and by using the "nice" topologies of \mathcal{G} and \mathcal{G}^* it is conceivable that one can construct solutions to SDE's (with discontinuous coefficients) driven by more general Lévy processes (as e.g. the variance gamma process or even Lévy processes of unbounded variation). See e.g. [17] in the Brownian motion case. See also [24].

The paper is organized as follows: In Section 2 we introduce the framework of our paper, that is we briefly elaborate some basic concepts of a white noise analysis for Lévy processes and give the definitions of the pairs $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$, $(\mathcal{G}, \mathcal{G}^*)$. In Section 3 we discuss some properties of $(\mathcal{G}, \mathcal{G}^*)$ and provide a characterization theorem. In Section 4 we verify that a bigger class of SDE solutions actually lives in \mathcal{G} .

2. Framework

In this section, we concisely recall some concepts of white noise analysis of pure jump Lévy processes which was developed in [14, 15]. This theory presents a framework which is suitable for all pure jump Lévy processes. For general information about white noise theory, see [6, 10, 11] and [19]. We conclude this section with a discussion of the dual pairs $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$, $(\mathcal{G}, \mathcal{G}^*)$ and $((S), (S)^*)$.

2.1. White noise analysis of Lévy processes. A Lévy process $L(t)$ is defined as a stochastic process on \mathbb{R}_+ which starts in zero and has stationary and independent increments. It is a canonical example of a semimartingale, which is uniquely determined by the characteristic triplet

$$(B_t, C_t, \hat{\mu}) = (a \cdot t, \sigma \cdot t, dt\nu(dz)), \quad (2.1)$$

where a, σ are constants and ν is the Lévy measure on $\mathbb{R}_0 = \mathbb{R} \setminus \{0\}$. We denote by π the product measure $\pi(dt, dz) := dt\nu(dz)$. For more information about such processes, see e.g. [1, 3, 9, 25, 26]. In this paper, we are only dealing with the case of pure jump Lévy processes without drift, i.e. (2.1) with $a = \sigma = 0$.

We want to work with a white noise measure, which is constructed on the nuclear algebra $\tilde{\mathcal{S}}(X)$ as introduced in [15]. Here $X := \mathbb{R} \times \mathbb{R}_0$. For that purpose, recall that $\mathcal{S}(\mathbb{R})$ is the Schwartz space of test functions on \mathbb{R} and the space $\mathcal{S}'(\mathbb{R})$ is its dual space, which is the space of tempered distributions. The space $\tilde{\mathcal{S}}(X)$ which is a variation of the Schwartz space on the space X is then defined as the quotient algebra

$$\tilde{\mathcal{S}}(X) = \mathcal{S}(X) / \mathcal{N}_\pi, \tag{2.2}$$

where $\mathcal{S}(X)$ is a closed subspace of $\mathcal{S}(\mathbb{R}^2)$, given by

$$\mathcal{S}(X) := \left\{ \varphi(t, z) \in \mathcal{S}(\mathbb{R}^2) : \varphi(t, 0) = \left(\frac{\partial}{\partial z}\varphi\right)(t, 0) = 0 \right\} \tag{2.3}$$

and the closed ideal \mathcal{N}_π in $\mathcal{S}(X)$ is defined as

$$\mathcal{N}_\pi := \{ \phi \in \mathcal{S}(X) : \|\phi\|_{L^2(\pi)} = 0 \}. \tag{2.4}$$

The space $\tilde{\mathcal{S}}(X)$ is a nuclear algebra with a compatible system of norms given by

$$\|\hat{\phi}\|_{p,\pi} := \inf_{\psi \in \mathcal{N}_\pi} \|\phi + \psi\|_p, \quad p \geq 0, \tag{2.5}$$

where $\|\cdot\|_p, p \geq 0$ are the norms of $\mathcal{S}(\mathbb{R}^2)$. Moreover the Cauchy-Bunjakowski inequality holds, that is for all $p \in \mathbb{N}$ there exists an M_p such that for all $\hat{\phi}, \hat{\psi} \in \tilde{\mathcal{S}}(X)$ we have

$$\|\hat{\phi}\hat{\psi}\|_{p,\pi} \leq M_p \|\hat{\phi}\|_{p,\pi} \|\hat{\psi}\|_{p,\pi}.$$

We indicate $\tilde{\mathcal{S}}'(X)$ as its dual. For further information, see [15].

Next, we define the (pure jump) Lévy white noise probability measure μ on the Borel sets of $\Omega = \tilde{\mathcal{S}}'(X)$, by means of Bochner-Minlos-Sazonov theorem

$$\int_{\tilde{\mathcal{S}}'(X)} e^{i\langle \omega, \phi \rangle} d\mu(\omega) = \exp \left(\int_X (e^{i\phi} - 1) d\pi \right) \tag{2.6}$$

for all $\phi \in \tilde{\mathcal{S}}(X)$, where $\langle \omega, \phi \rangle := \omega(\phi)$ denotes the action of $\omega \in \tilde{\mathcal{S}}'(X)$ on $\phi \in \tilde{\mathcal{S}}(X)$. For $\omega \in \tilde{\mathcal{S}}'(X)$ and $\phi \in \tilde{\mathcal{S}}(X)$, define the exponential functional

$$\tilde{e}(\phi, \omega) := (e(\cdot, \omega) \circ l)(\phi) = \exp \left(\langle \omega, \ln(1 + \phi) \rangle - \int_X \phi(x) \lambda \otimes \nu(dx) \right)$$

as a function of $\phi \in \tilde{\mathcal{S}}_{q_0}(X)$ for functions $\phi \in \tilde{\mathcal{S}}_{q_0}(X)$ satisfying $\phi(x) > -1$ and $l(x) = \ln(1 + x)$ is analytic in a neighborhood of zero with $l(0) = 0$ for all $x \in X$. See [15].

Denote by $\tilde{\mathcal{S}}(X)^{\hat{\otimes} n}$ the n -th completed symmetric tensor product of $\tilde{\mathcal{S}}(X)$ with itself. Since $\tilde{e}(\phi, \omega)$ is holomorphic in ϕ around zero for $\phi(x) > -1$, it can be

expanded into a power series. Furthermore, there exist generalized Charlier polynomials $C_n(\omega) \in (\tilde{\mathcal{S}}(X)^{\hat{\otimes} n})'$ such that

$$\tilde{e}(\phi, \omega) = \sum_{n \geq 0} \frac{1}{n!} \langle C_n(\omega), \phi^{\otimes n} \rangle \quad (2.7)$$

for ϕ in a certain neighborhood of zero. One shows that

$$\{\langle C_n(\cdot), \phi^{(n)} \rangle : \phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} n}, n \in \mathbb{N}_0\} \quad (2.8)$$

is a total set of $L^2(\mu)$. Further, one observes that for all $n, m, \phi^{(n)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} n}, \psi^{(m)} \in \tilde{\mathcal{S}}(X)^{\hat{\otimes} m}$ the orthogonality relation

$$\int_{\tilde{\mathcal{S}}'(X)} \langle C_n(\omega), \phi^{(n)} \rangle \langle C_m(\omega), \psi^{(m)} \rangle \mu(d\omega) = \delta_{n,m} n! \langle \phi^{(n)}, \psi^{(m)} \rangle_{L^2(X^n, \pi^n)} \quad (2.9)$$

holds, where

$$\delta_{n,m} = \begin{cases} 0, & n \neq m \\ 1, & \text{else} \end{cases}$$

is the Kronecker symbol. Using (2.9) and a density argument we can extend $\langle C_n(\omega), \phi^{(n)} \rangle$ to act on $\phi^{(n)} \in L^2(X^n, \pi^n)$ for ω a.e. The functionals $\langle C_n(\omega), \phi^{(n)} \rangle$ can be regarded as an n -fold iterated stochastic integral of functions $\phi^{(n)} \in L^2(X^n, \pi^n)$ with respect to the compensated Poisson random measure

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt,$$

where $N(\Lambda_1, \Lambda_2) := \langle \omega, 1_{\Lambda_1 \times \Lambda_2} \rangle$ for $\Lambda_1 \in \mathbb{R}$ and $\Lambda_2 \in \mathbb{R}$ such that zero is not in the closure of Λ_2 , defined on our white noise probability space

$$(\Omega, \mathcal{F}, P) = (\tilde{\mathcal{S}}'(X), \mathcal{B}(\tilde{\mathcal{S}}'(X)), \mu).$$

In this setting, a square integrable pure jump Lévy process $L(t)$ can be represented as

$$L(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(dt, dz).$$

Denote $\hat{L}^2(X^n, \pi^n)$ the space of square integrable functions $\phi^{(n)}(t_1, z_1, \dots, t_n, z_n)$ being symmetric in the n -pairs $(t_1, z_1), \dots, (t_n, z_n)$. Then one infers from (2.7) to (2.9) the Lévy-Itô chaos representation property of square integrable Lévy functionals: For all $F \in L^2(\mu)$, there exists a unique sequence of $\phi^{(n)} \in L^2(X^n, \pi^n)$ such that

$$F(\omega) = \sum_{n \geq 0} \langle C_n(\omega), \phi^{(n)} \rangle$$

for ω a.e. Moreover, we have the Itô-isometry

$$\|F\|_{L^2(\mu)}^2 = \sum_{n \geq 0} n! \|\phi^{(n)}\|_{L^2(X^n, \pi^n)}^2. \quad (2.10)$$

2.2. The spaces $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$, $(\mathcal{G}, \mathcal{G}^*)$ and $((S), (S)^*)$. In our search for appropriate candidates of subspaces of $L^2(\mu)$ in which strong solutions of SDE's live, we shall focus on the Meyer-Watanabe test function and distribution spaces $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ and the dual pair $(\mathcal{G}, \mathcal{G}^*)$ of smooth and generalized random variables on the Lévy white noise space.

The *Meyer-Watanabe test function* \mathbb{D}_∞ for pure jump Lévy process (see e.g. [4, 31, 32]) is defined as a dense subspace of $L^2(\mu)$ endowed with the topology given by the seminorms

$$\| F \|_{k,p} = \left(\mathbb{E}[| F |^p] + \sum_{j=1}^k \mathbb{E}[\| D_{\cdot, \cdot}^j F \|_{L^2(\pi^n)}^p] \right)^{1/p}, \tag{2.11}$$

$k \in \mathbb{N}$, $p \geq 1$, with

$$D_{t_1, z_1, \dots, t_j, z_j}^j F(\omega) := D_{t_1, z_1} D_{t_2, z_2} \dots D_{t_j, z_j} F(\omega)$$

for $F \in \mathbb{D}_\infty$, where $D_{t,z}$ stands for the *Malliavin derivative* in the direction of the (square integrable) pure jump Lévy process $L(t)$, $t \geq 0$. $D_{\cdot, \cdot}$ is defined as a mapping

$$D : \mathbb{D}_{1,2} \rightarrow L^2(\mu \times \pi)$$

given by

$$D_{t,z} F = \sum_{n \geq 1} n \cdot \langle C_{n-1}(\cdot), \phi^{(n)}(\cdot, t, z) \rangle, \tag{2.12}$$

if $F \in L^2(\mu)$ with chaos expansion

$$F = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle$$

satisfies

$$\sum_{n \geq 1} n \cdot n! \| \phi^{(n)} \|_{L^2(\pi^n)}^2 < \infty. \tag{2.13}$$

The domain $\mathbb{D}_{1,2}$ of $D_{\cdot, \cdot}$ is the space of all $F \in L^2(\mu)$ such that inequality (2.13) holds. See [4, 23, 31, 32] for further information.

The Meyer-Watanabe distribution space $\mathbb{D}_{-\infty}$ is defined as the (topological) dual of \mathbb{D}_∞ . If one combines the transfer principle from the Wiener space (or Gaussian white noise space) to the Poisson space as devised in [23] with the results of [30], one finds that solutions of non-degenerate jump SDE's exist in \mathbb{D}_∞ . This is a striking feature which pays off dividends in the analysis of Lévy functionals. However it seems not that easy to set up a characterization-type theorem for $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ in the sense of [21]. Consequently, other Gel'fand triples have been studied to overcome this deficiency. In [22] the authors study the pair $(\mathcal{G}, \mathcal{G}^*)$ and provide sufficient conditions in terms of the S -transform to characterize $(\mathcal{G}, \mathcal{G}^*)$. Using Bargmann-Segal spaces, a complete characterization of this pair (and for a scale of closely related pairs) is obtained by [5] in the Gaussian case.

We will show in Section 3 and 4 that $(\mathcal{G}, \mathcal{G}^*)$ can be characterized by means of the S -transform on the Lévy noise space and that \mathcal{G} contains a richer class of solutions of jump SDE's. These two properties make $(\mathcal{G}, \mathcal{G}^*)$ an interesting alternative to $(\mathbb{D}_\infty, \mathbb{D}_{-\infty})$ to analyze functionals of Lévy processes.

The test function space \mathcal{G} is a subspace of $L^2(\mu)$ which is constructed by means of exponential weights of the Ornstein-Uhlenbeck or number operator. Denoted by N , this operator acts on the elements of $L^2(\mu)$ by multiplying the n -th homogeneous chaos with $n \in \mathbb{N}_0$. The *space of smooth random variables* \mathcal{G} is defined as the collection of all

$$f = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle \in L^2(\mu) \tag{2.14}$$

such that

$$\|f\|_q^2 := \|e^{qN} f\|_{L^2(\mu)}^2 < \infty$$

for all $q \geq 0$. The latter condition is equivalent to

$$\|f\|_q^2 = \sum_{n \geq 0} n! e^{2qn} \|\phi^{(n)}\|_{L^2(X^n, \pi^n)}^2 \tag{2.15}$$

for all $q \geq 0$. The space \mathcal{G} is endowed with the topology given by the family of norms $\|\cdot\|_q, q \geq 0$. Its topological dual is the *space of generalized random variables* \mathcal{G}^* .

Let us turn our attention to the S -transform which is a fundamental concept of white noise distribution theory and serves as a tool to characterize elements of the *Hida test function space* (\mathcal{S}) and the *Hida distribution space* $(\mathcal{S})^*$. See [6] or [15] for a precise definition of the pair $((\mathcal{S}), (\mathcal{S})^*)$. The S -transform of $\Phi \in (\mathcal{S})^*$, denoted by $S(\Phi)$, is defined as the dual pairing

$$S(\Phi)(\phi) := \langle \Phi, \tilde{e}(\phi, \cdot) \rangle, \quad \phi \in \tilde{\mathcal{S}}_{\mathbb{C}}(X), \tag{2.16}$$

where $\|\tilde{e}(\phi, \cdot)\|^2 = \sum_{n=0}^{\infty} \|\phi\|_{p, \pi}^{2n}$ and $\tilde{\mathcal{S}}_{\mathbb{C}}(X)$ is the complexification of $\tilde{\mathcal{S}}(X)$. The S -transform is a monomorphism, that is, if

$$S(\Phi) = S(\Psi) \quad \text{for } \Phi, \Psi \in (\mathcal{S})^*$$

then

$$\Phi = \Psi.$$

One verifies, e.g. that

$$S(\dot{N}(t, z))(\phi) = \phi(t, z), \tag{2.17}$$

$\dot{N}(t, z)$ the *white noise* of the compensated Poisson random measure $\tilde{N}(dt, dz)$ in $(\mathcal{S})^*$ and $\phi \in \tilde{\mathcal{S}}_{\mathbb{C}}(X)$. We refer the reader to [6] or [15] for more information on the Hida test function space (\mathcal{S}) and Hida distribution space $(\mathcal{S})^*$.

Finally, we give the important definition of the *Wick* or *Wick-Grassmann product*, which can be considered a tensor algebra multiplication on the Fock space. The Wick product of two distributions $\Phi, \Psi \in (\mathcal{S})^*$, denoted by $\Phi \diamond \Psi$, is the unique element in $(\mathcal{S})^*$ such that

$$S(\Phi \diamond \Psi)(\phi) = S(\Phi)(\phi)S(\Psi)(\phi) \tag{2.18}$$

for all $\phi \in \tilde{\mathcal{S}}_{\mathbb{C}}(X)$. As an example one finds that

$$\langle C_n(\omega), \phi^{(n)} \rangle \diamond \langle C_m(\omega), \psi^{(m)} \rangle = \langle C_{n+m}(\omega), \phi^{(n)} \hat{\otimes} \psi^{(m)} \rangle \tag{2.19}$$

for $\phi^{(n)} \in (\tilde{\mathcal{S}}(X))^{\hat{\otimes} n}$ and $\psi^{(m)} \in (\tilde{\mathcal{S}}(X))^{\hat{\otimes} m}$. The latter and (2.7) imply that

$$\tilde{e}(\phi, \omega) = \exp^{\diamond}(\langle \omega, \phi \rangle) \tag{2.20}$$

for $\phi \in \tilde{\mathcal{S}}(X)$. The Wick exponential $\exp^\diamond(X)$ of an $X \in (\mathcal{S})^*$ is defined as

$$\exp^\diamond(X) = \sum_{n \geq 0} \frac{1}{n!} X^{\diamond n} \quad (2.21)$$

provided the sum converges in $(\mathcal{S})^*$, where $X^{\diamond n} = X \diamond \dots \diamond X$. We mention that the following chain of continuous inclusions is valid:

$$(\mathcal{S}) \hookrightarrow \mathcal{G} \hookrightarrow L^2(\mu) \hookrightarrow \mathcal{G}^* \hookrightarrow (\mathcal{S})^*.$$

3. Properties of the Spaces \mathcal{G} and \mathcal{G}^*

In the Gaussian case the space \mathcal{G} has the nice feature to be stable in the sense of pointwise multiplication of random variables. More precisely, \mathcal{G} is a Fréchet algebra. See [13, 22]. In the Lévy setting, we can show the following:

Theorem 3.1. *Suppose that our Lévy measure ν satisfies the moment condition*

$$\int_{\mathbb{R}_0} |z|^n \nu(dz) < \infty$$

for all $n \in \mathbb{N}$. Let F, G be in \mathcal{G} with chaos expansions $F = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle$ and $G = \sum_{n \geq 0} \langle C_n(\cdot), \varphi^{(n)} \rangle$. Define $K_R = \{(t, z) \in \mathbb{R} \times \mathbb{R}_0 : \|(t, z)\| < R\}$, $R > 0$. Assume that

$$\sup_{n \geq 0} \sqrt{n!} \|\phi^{(n)}\|_{L^\infty(X^n, \pi^n)} < \infty \quad (3.1)$$

and

$$\sup_{n \geq 0} \sqrt{n!} \|\varphi^{(n)}\|_{L^\infty(X^n, \pi^n)} < \infty. \quad (3.2)$$

In addition require that there exists a $R > 0$ such that the compact support of $\phi^{(n)}$ and $\varphi^{(n)}$ are in $(K_R)^n$, i.e.,

$$\text{supp } \phi^{(n)}, \text{supp } \varphi^{(n)} \subseteq (K_R)^n$$

for all $n \geq 0$. Then

$$F \cdot G \in \mathcal{G}.$$

In particular, let $\lambda_0 = \frac{\ln(\pi(K_R))}{4} + \ln(4R) + \ln(\sqrt{2 + \sqrt{2}})$ and assume that for $\lambda > 2\lambda_0$, $F, G \in \mathcal{G}_\lambda$. Then for all $\nu > \lambda_0 + \frac{\lambda}{2}$, $F \cdot G \in \mathcal{G}_{\lambda - \nu}$.

Proof. Let $F, G \in \mathcal{G}_\lambda \subset \mathcal{G}$ for some $\lambda \in \mathbb{R}$ with $F = \sum_{n \geq 0} \langle C_n(\cdot), \phi^{(n)} \rangle$ and $G = \sum_{m \geq 0} \langle C_m(\cdot), \varphi^{(m)} \rangle$. Then,

$$\|F\|_\lambda^2 = \sum_{n \geq 0} n! e^{2\lambda n} \|\phi^{(n)}\|_{L^2(\pi^n)}^2 < \infty,$$

$$\|G\|_\lambda^2 = \sum_{m \geq 0} m! e^{2\lambda m} \|\varphi^{(m)}\|_{L^2(\pi^m)}^2 < \infty.$$

By the product formula in [12], we obtain the following:

$$\begin{aligned} & \langle C_n(\cdot), \phi^{(n)} \rangle \cdot \langle C_m(\cdot), \varphi^{(m)} \rangle \\ &= \sum_{k=0}^{m \wedge n} \sum_{r=0}^{(m \wedge n) - k} k! r! \binom{m}{k} \binom{n}{k} \binom{m-k}{r} \binom{n-k}{r} \langle C_{m+n-2k-r}(\cdot), \phi^{(n)} \hat{\otimes}_k^r \varphi^{(m)} \rangle, \end{aligned}$$

where $\phi \hat{\otimes}_k^r \varphi$ for $\phi \in \hat{L}^2(X^n)$, $\varphi \in \hat{L}^2(X^m)$, $0 \leq k \leq m \wedge n$, $0 \leq r \leq m \wedge n - k$, is the symmetrization of the function $\phi \otimes_k^r \varphi$ on $X^{n-k-r} \times X^{m-k-r} \times X^r$ given by

$$\phi \otimes_k^r \varphi(A, B, Z) := \left(\prod_{z \in Z} p_2(z) \right) \int_{X^k} \phi(A, Z, Y) \varphi(Y, Z, B) d\pi^{\otimes k}(Y)$$

for $(A, B, Z) \in X^{n-k-r} \times X^{m-k-r} \times X^r$. Here $\left(\prod_{z \in Z} p_2(z) \right) := z_1 \cdot z_2 \cdots z_r$ when $Z = ((t_1, z_1), (t_2, z_2), \dots, (t_r, z_r))$. Because of Lemma 3.4 in [12], we know that

$$\begin{aligned} \|\phi^{(n)} \hat{\otimes}_k^r \varphi^{(m)}\|_{L^2(\mathbb{R}^{m+n-2k-r})} &\leq R^r \sqrt[4]{(\pi(K_R))^{(m+n-2r)}} \cdot \sqrt{\|\phi^{(n)}\|_{L^\infty}} \\ &\quad \cdot \sqrt{\|\varphi^{(m)}\|_{L^\infty}} \cdot \sqrt{\|\phi^{(n)}\|_{L^2(\mathbb{R}^n)}} \cdot \sqrt{\|\varphi^{(m)}\|_{L^2(\mathbb{R}^m)}}. \end{aligned}$$

Moreover, using the conditions (3.1) and (3.2), we obtain the following inequality:

$$\begin{aligned} \|\langle C_n(\cdot), \phi^{(n)} \rangle \cdot \langle C_m(\cdot), \varphi^{(m)} \rangle\|_{\lambda-\nu} &\leq \sum_{k=0}^{m \wedge n} \sum_{r=0}^{(m \wedge n)-k} k! r! \binom{m}{k} \binom{n}{k} \binom{m-k}{r} \\ &\quad \binom{n-k}{r} e^{(\lambda-\nu)(m+n-2k-r)} \|\langle C_{m+n-2k-r}(w), \phi^{(n)} \hat{\otimes}_k^r \varphi^{(m)} \rangle\|_{L^2(\mu)} \\ &\leq \sum_{k=0}^{m \wedge n} \sum_{r=0}^{(m \wedge n)-k} k! r! \binom{m}{k} \binom{n}{k} \binom{m-k}{r} \binom{n-k}{r} e^{\lambda(m+n)} e^{-\nu(m+n)} e^{-(\lambda-\nu)(2k+r)} \\ &\quad \sqrt{(m+n-2k-r)!} \|\phi^{(n)} \hat{\otimes}_k^r \varphi^{(m)}\|_{L^2(\mathbb{R}^{m+n-2k-r})} \\ &\leq \text{constant} \|\langle C_n(\cdot), \phi^{(n)} \rangle\|_{\lambda}^{1/2} \|\langle C_m(\cdot), \varphi^{(m)} \rangle\|_{\lambda}^{1/2} e^{-\nu(m+n)} e^{\frac{\lambda}{2}(m+n)} 2^{m+n} \\ &\quad \left(\sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} \sqrt{(m+n-2k)!} e^{-(\lambda-\nu)2k} \right) \\ &\quad \left(\sum_{k=0}^{m \wedge n} \sum_{r=0}^{(m \wedge n)-k} \frac{R^r \sqrt[4]{(\pi(K_R))^{(m+n-2r)}} r! \sqrt{(m+n-2k-r)!}}{\sqrt{n!} \sqrt{m!} \sqrt{(m+n-2k)!}} \right), \end{aligned}$$

for $\nu < \lambda$. From now, without loss of generality we assume that $\pi(K_R) > e$ and $R \geq 1$. It is clear that

$$\frac{r! \sqrt{(m+n-2k-r)!}}{\sqrt{n!} \sqrt{m!} \sqrt{(m+n-2k)!}} \leq \frac{r!}{(n \wedge m)!} \frac{\sqrt{(m+n-2k-r)!}}{\sqrt{(m+n-2k)!}} \leq 1,$$

$$\begin{aligned} \sum_{r=0}^{m \wedge n} R^r \sqrt[4]{(\pi(K_R))^{(m+n-2r)}} &\leq e^{\frac{m+n}{4} \ln(\pi(K_R))} \sum_{r=0}^{m \wedge n} R^r \\ &\leq (m \wedge n) R^{m+n} e^{\frac{m+n}{4} \ln(\pi(K_R))} \\ &< (2R)^{m+n} e^{\frac{m+n}{4} \ln(\pi(K_R))}, \end{aligned}$$

and

$$\left(\sum_{k=0}^{m \wedge n} k! \binom{m}{k} \binom{n}{k} \sqrt{(m+n-2k)!} e^{-(\lambda-\nu)2k} \right) \leq \left(\frac{\sqrt{2}}{\sqrt{2}-1} \right)^{1/2} (\sqrt{2+\sqrt{2}})^{m+n}.$$

Therefore,

$$\begin{aligned} & \| \langle C_n(\cdot), \phi^{(n)} \rangle \cdot \langle C_m(\cdot), \varphi^{(m)} \rangle \|_{\lambda-\nu} \\ & \leq \| \langle C_n(\cdot), \phi^{(n)} \rangle \|_{\lambda}^{1/2} \| \langle C_m(\cdot), \varphi^{(m)} \rangle \|_{\lambda}^{1/2} H \sigma_n \sigma_m, \end{aligned}$$

where H is a constant and $\sigma = 4R e^{-\nu+\lambda/2+\ln(\pi(K_R))/4} (\sqrt{2+\sqrt{2}})$. Then,

$$\begin{aligned} & \| F \cdot G \|_{\lambda-\nu} \\ & = \| \sum_{m,n=0}^{\infty} \langle C_n(\cdot), \phi^{(n)} \rangle \langle C_m(\cdot), \varphi^{(m)} \rangle \|_{\lambda-\nu} \\ & \leq \sum_{m,n=0}^{\infty} \| \langle C_n(\cdot), \phi^{(n)} \rangle \langle C_m(\cdot), \varphi^{(m)} \rangle \|_{\lambda-\nu} \\ & \leq H \left(\sum_{n=0}^{\infty} \sigma^n \| \langle C_n(\cdot), \phi^{(n)} \rangle \|_{\lambda}^{1/2} \right) \left(\sum_{n=0}^{\infty} \sigma^n \| \langle C_n(\cdot), \varphi^{(n)} \rangle \|_{\lambda}^{1/2} \right) \\ & \leq H \left(\sum_{n=0}^{\infty} \sigma^{4/3n} \right)^{3/4} \left(\sum_{n=0}^{\infty} \| \langle C_n(\cdot), \phi^{(n)} \rangle \|_{\lambda}^2 \right)^{1/4} \left(\sum_{n=0}^{\infty} \sigma^{4/3n} \right)^{3/4} \\ & \quad \left(\sum_{n=0}^{\infty} \| \langle C_n(\cdot), \varphi^{(n)} \rangle \|_{\lambda}^2 \right)^{1/4} \\ & \leq H \left(\frac{1}{1-\sigma^{4/3}} \right)^{3/2} \| F \|_{\lambda}^{1/2} \| G \|_{\lambda}^{1/2}, \end{aligned}$$

if $\sigma < 1$, i.e. $\nu > \frac{\lambda}{2} + \lambda_0$, where $\lambda_0 = \frac{\ln(\pi(K_R))}{4} + \ln(4R) + \ln(\sqrt{2+\sqrt{2}})$. □

Remark 3.2. Note that, in the conditions (3.1) and (3.2), $\sqrt{n!}$ can be replaced by $\sqrt[3]{n!}$.

Define the space $\mathbb{D}_{\infty,2} \supset \mathbb{D}_{\infty}$ as

$$\mathbb{D}_{\infty,2} = \text{proj} \lim_{k \rightarrow 0} \mathbb{D}_{k,2} \tag{3.3}$$

and denote by $\mathbb{D}_{-\infty,2}$ its topological dual. Then it is apparent from the definition of \mathcal{G} that

$$\mathcal{G} \subset \mathbb{D}_{\infty,2} \subset L^2(\mu) \subset \mathbb{D}_{-\infty,2} \subset \mathcal{G}^*. \tag{3.4}$$

If $L(t)$ is a Poisson process, then a transfer principle to Poisson spaces based on exponential distributions (see [23]) gives

$$\mathcal{G} \subset \mathbb{D}_{\infty} \subset L^2(\mu) \subset \mathbb{D}_{-\infty} \subset \mathcal{G}^*.$$

Finally, we want to discuss the characterization of the spaces \mathcal{G} and \mathcal{G}^* in terms of the S -transform. For this purpose assume a densely defined operator A on $L^2(X, \pi)$ such that

$$A \xi_j = \lambda_j \xi_j, j \geq 1,$$

where $1 < \lambda_1 \leq \lambda_2 \leq \dots$ and $\{\xi_j\}_{j \geq 1} \subset \widetilde{\mathcal{S}}(X)$ is an orthonormal basis of $L^2(X, \pi)$. Further we require that there exists a $\alpha > 0$ such that $A^{-\alpha/2}$ is Hilbert-Schmidt. Then let us denote by \mathcal{S} the standard countably Hilbert space constructed from A (see [19]). An application of the Bochner-Minlos theorem leads to a Gaussian measure μ_G on \mathcal{S}' (dual of \mathcal{S}) such that

$$\int_{\mathcal{S}'} e^{i\langle \omega, \phi \rangle} \mu_G(d\omega) = e^{-\frac{1}{2}\|\phi\|_{L^2(X, \pi)}^2}$$

for all $\xi \in \mathcal{S}$. It is well-known that each element f in $L^2(\mu_G)$ has the chaos representation

$$f = \sum_{n \geq 0} \langle H_n(\cdot), \phi^{(n)} \rangle, \tag{3.5}$$

for unique $\phi^{(n)} \in L^2(X^n, \pi^n), n \geq 0$, where $H_n(\omega) \in (\mathcal{S}^{\otimes n})'$ are generalized Hermite polynomials. Comparing (2.14) with (3.5) we observe that the mapping

$$\mathcal{U} : L^2(\mu) \longrightarrow L^2(\mu_G) \tag{3.6}$$

given by

$$\sum_{n \geq 0} \langle C_n(\omega), \phi^{(n)} \rangle \longmapsto \sum_{n \geq 0} \langle H_n(\omega), \phi^{(n)} \rangle$$

is a unitary isomorphism between the spaces $L^2(\mu)$ and $L^2(\mu_G)$. In the following let us denote by S_G the S -transform on the Gaussian Hida distribution space $(\mathcal{S})_{\mu_G}^*$ which is defined as

$$S_G(\phi) = \langle \Phi, \tilde{e}(\phi, \omega) \rangle, \phi \in (\mathcal{S})_{\mu_G}^*, \tag{3.7}$$

where

$$\tilde{e}(\phi, \omega) = e^{\langle \omega, \phi \rangle - 1/2\|\phi\|_{L^2(X, \pi)}^2}.$$

See [6]. Our characterization of $(\mathcal{G}, \mathcal{G}^*)$ requires the concept of Bargmann-Segal space (see [27], [5] and the references therein):

Definition 3.3. Let $\mu_{G, \frac{1}{2}}$ be the Gaussian measure on \mathcal{S}' associated with the characteristic function $C(\phi) := e^{-\frac{1}{4}\|\phi\|_{L^2(X, \pi)}^2}$. Introduce the measure ν on $\mathcal{S}'_{\mathbb{C}}$ given by

$$\nu(dz) = \mu_{G, \frac{1}{2}}(dx) \times \mu_{G, \frac{1}{2}}(dy),$$

where $z = x + iy$. Further denote by \mathbb{P} the collection of all projections P of the form

$$Pz = \sum_{j=1}^m \langle z, \xi_j \rangle \xi_j, z \in \mathcal{S}'_{\mathbb{C}}.$$

The *Bargmann-Segal space* $E^2(\nu)$ is the space consisting of all entire functions $f : L^2_{\mathbb{C}}(X, \mu_G) \longrightarrow \mathbb{C}$ such that

$$\sup_{P \in \mathbb{P}} \int_{\mathcal{S}'_{\mathbb{C}}} |f(Pz)| \nu(dz) < \infty.$$

So we obtain from Theorem 7.1 and 7.3 in [5] the following result:

Theorem 3.4. (i) The smooth random variable φ belongs to \mathcal{G} if and only if

$$S_G(\mathcal{U}(\varphi))(\lambda \cdot) \in E^2(\nu)$$

for all $\lambda > 0$.

(ii) The generalized random variable Φ is an element of \mathcal{G}^* if and only if there is a $\lambda > 0$ such that

$$S_G(\mathcal{U}(\varphi))(\lambda \cdot) \in E^2(\nu).$$

Remark 3.5. The connection between $S_G \circ \mathcal{U}$ and S in (2.16) is given by the following relation: Since

$$\mathcal{U} \left(\left\langle C_n(\cdot), \phi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \phi_k^{\widehat{\otimes} n_k} \right\rangle \right) = \left\langle H_n(\cdot), \phi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \phi_k^{\widehat{\otimes} n_k} \right\rangle$$

for $\phi_1, \dots, \phi_k \in L^2(X, \pi)$, $n_i \geq 1$ with $n_1 + \dots + n_k = n$ we find (see (2.19)) that

$$S \left(\left\langle C_n(\cdot), \phi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \phi_k^{\widehat{\otimes} n_k} \right\rangle \right) = (S(\langle C_1(\cdot), \phi_1 \rangle))^{n_1} \cdot \dots \cdot (S(\langle C_1(\cdot), \phi_k \rangle))^{n_k}$$

as well as

$$\begin{aligned} S_G \circ \mathcal{U} \left(\left\langle C_n(\cdot), \phi_1^{\widehat{\otimes} n_1} \widehat{\otimes} \dots \widehat{\otimes} \phi_k^{\widehat{\otimes} n_k} \right\rangle \right) \\ = (S_G \circ \mathcal{U}(\langle C_1(\cdot), \phi_1 \rangle))^{n_1} \cdot \dots \cdot (S_G \circ \mathcal{U}(\langle C_1(\cdot), \phi_k \rangle))^{n_k}. \end{aligned}$$

We conclude this section with a sufficient condition for a Hida distribution to be an element of \mathcal{G} :

Theorem 3.6. Let Q be a positive quadratic form on $L^2(X, \pi)$ with finite trace. Further let Φ be in $(S)^*$. Assume that for every $\epsilon > 0$ there exists a $K(\epsilon) > 0$ such that

$$|S_G(\mathcal{U}(\Phi))(z\phi)| \leq K(\epsilon)e^{\epsilon|z|^2Q(\phi, \phi)}$$

holds for all $\phi \in \mathcal{S}$ and $z \in \mathbb{C}$ for some constant $K > 0$. Then $\Phi \in \mathcal{G}$.

Proof. The proof is a direct consequence from the proof of Theorem 4.1 in [22]. \square

Example 3.7. Let $\gamma \in L^2(X, \pi)$ with $\gamma > -1$ and $\epsilon > 0$. Then

$$Y(t) := \exp^\circ(\langle C_1(\omega), \chi_{[0,t]}\gamma \rangle)$$

is the solution of

$$dY(t) = Y(t^-) \int_0^t \int_{\mathbb{R}_0} \gamma(t, u) \tilde{N}(dt, du).$$

So we get

$$\begin{aligned} |S_G(\mathcal{U}(Y(t)))(z\phi)| &\leq \exp \left(\left| \int_X \chi_{[0,t]} z\phi(x) \gamma(x) \pi(dx) \right| \right) \\ &\leq K(\epsilon) \exp(\epsilon|z|^2Q(\phi, \phi)), \end{aligned}$$

where $K(\epsilon) = e^{1/(4\epsilon)}$ and

$$Q(\phi, \phi) = \left(\int_X \chi_{[0,t]} \gamma(x) \phi(x) \pi(dx) \right)^2.$$

Thus $Y(t) \in \mathcal{G}$.

4. Solutions of SDE's in \mathcal{G}

In this section, we deal with strong solutions of pure jump Lévy stochastic differential equations of the type

$$X(t) = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X(s^-), z) \tilde{N}(ds, dz) \quad (4.1)$$

for $X(0) = x \in \mathbb{R}$, where $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R}_0 \rightarrow \mathbb{R}$ satisfies the linear growth and Lipschitz condition, i.e.,

$$\int_{\mathbb{R}_0} |\gamma(t, x, z)|^2 \nu(dz) \leq C(1 + |x|^2), \quad (4.2)$$

$$\int_{\mathbb{R}_0} |\gamma(t, x, z) - \gamma(t, y, z)|^2 \nu(dz) \leq K|x - y|^2, \quad (4.3)$$

where C, K and M are some constants such that $|\gamma(t, x, z)| < M$ for all $x, y \in \mathbb{R}$, $0 \leq t \leq T$. Note that since γ satisfies the conditions (4.2) and (4.3), there exists a unique solution $X = \{X(t), t \in [0, T]\}$ with the initial condition $X(0) = x$. Moreover, the process is adapted and càdlàg [1].

If $\nu(\mathbb{R}_0) < \infty$ (i.e. $X(t), t \geq 0$ is compound Poissonian), we will prove that $X(t) \in \mathcal{G}, t \geq 0$. To this end we need some auxiliary results:

Lemma 4.1. *Let $\{X_n\}_{n=0}^\infty$ be a sequence of random variables converging to X in $L^2(\mu)$. Suppose that*

$$\sup_n \| |X_n| \|_{k,2} < \infty$$

for some $k \geq 1$. Then $X \in \mathbb{D}_{k,2}$ and $D^k X_n, n \geq 0$ converges to $D^k X$ in the sense of the weak topology of $L^2((\lambda \times \nu \times \mu)^k)$.

Proof. First note that $\sup_n \| |X_n| \|_{k,2} < \infty$ is equivalent to

$$\sup_n \| (1 + N)^{\frac{k}{2}} X_n \|_{k,2} < \infty.$$

By weak compactness, there exists a subsequence $\{X_{n_i}\}_{i,n=1}^\infty$ such that $(1 + N)^{\frac{k}{2}} X_{n_i}$ converges weakly to some element $\alpha \in L^2(\mu \times (\lambda \times \nu)^k)$. Then for any Y in the domain of $(1 + N)^{\frac{k}{2}}$, it follows from the self-adjointness of N that

$$\begin{aligned} \mathbb{E}[X(1 + N)^{\frac{k}{2}} Y] &= \lim_{n \rightarrow \infty} \mathbb{E}[X_{n_i} (1 + N)^{\frac{k}{2}} Y] \\ &= \lim_{n \rightarrow \infty} \mathbb{E}[(1 + N)^{\frac{k}{2}} X_{n_i} Y] \\ &= \mathbb{E}[\lim_{n \rightarrow \infty} (1 + N)^{\frac{k}{2}} X_{n_i} Y] \\ &= \mathbb{E}[\alpha Y]. \end{aligned}$$

Therefore $\alpha = ((1 + N)^{\frac{k}{2}})^* X = (1 + N)^{\frac{k}{2}} X$. For the proof in Brownian motion case, see e.g. [18]. \square

For notational convenience, we shall identify from now on Malliavin derivatives of the same order, that is we set $D_{r,z}^N X(t) = D_{r_1, z_1, r_2, z_2, \dots, r_N, z_N}^N X(t)$.

Lemma 4.2. *Let $X(t)$, $0 \leq t \leq T$ be defined as in the Equation (4.1). Then $X(t) \in \mathbb{D}_{\infty,2}$, i.e. $D_{r,z}^N X(t)$ exists for all $N \geq 1$.*

We need the following results to prove this lemma:

Proposition 4.3. *Let $X \in \mathbb{D}_{1,2}$ and f be a real continuous function on \mathbb{R} . Then $f(X) \in \mathbb{D}_{1,2}$ and*

$$D_{t,z} f(X) = f(X + D_{t,z} X) - f(X). \quad (4.4)$$

Proof. See e.g. [4]. □

Lemma 4.4. *Let $X(t)$, $0 \leq t \leq T$ be defined as in the Equation (4.1). Then, the N -th Malliavin derivative of $X(t)$ can be written as*

$$\begin{aligned} D_{r,z}^N X(t) = & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-), \xi) \tilde{N}(ds, d\xi) \\ & + N \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X(r^-), z), \end{aligned} \quad (4.5)$$

for $N \geq 1$ and $D_{r,z}^0 X(t) := X(t)$.

Proof. We will prove the equality (4.5) by induction methodology. Since the proof is basically based on calculations, we include it in the Appendix. □

Now, we are ready to prove Lemma 4.2.

Proof. Let us consider the Picard approximations $X_n(t)$ to $X(t)$ given by

$$X_{n+1}(t) = x + \int_0^t \int_{\mathbb{R}_0} \gamma(s, X_n(s^-), z) \tilde{N}(ds, dz), \quad (4.6)$$

for $n \geq 0$ and $X_0(t) = x$. We want to show by induction on n that $X_n(t)$ belongs to $\mathbb{D}_{N,2}$ and

$$\varphi_{n+1,N}(t) \leq k_1 + k_2 \sum_{j=1}^N \int_0^t \varphi_{n,j}(u) du,$$

for all $n \geq 0$, $N \geq 1$ and $t \in [0, T]$ where

$$\varphi_{n+1,N}(t) := \sup_{0 \leq r \leq t} \mathbb{E} \left[\int_{\mathbb{R}_0^N} \sup_{r \leq s \leq t} |D_{r,z}^N X_{n+1}(s)|^2 \nu(dz) \dots \nu(dz) \right] < \infty.$$

Note that

$$\begin{aligned} D_{r,z}^N X_{n+1}(t) = & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X_n(s^-), \xi) \tilde{N}(ds, d\xi) \\ & + N \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X_n(r^-), z), \end{aligned} \quad (4.7)$$

with $D_{r,z}^0 X_n(s^-) := X_n(s^-)$. See Lemma 4.4 for a proof. Then, by Doob's maximal inequality, Fubini's theorem, Itô isometry, Equation (4.2) and (4.3), we get

$$\begin{aligned}
& \sum_{j=1}^N \mathbb{E} \left[\int_{\mathbb{R}_0^j} \sup_{r \leq s \leq t} \left(D_{r,z}^j X_{n+1}(s) \right)^2 (\nu(dz))^j \right] \\
&= \sum_{j=1}^N \mathbb{E} \left[\int_{\mathbb{R}_0^j} \sup_{r \leq s \leq t} \left(\int_r^s \int_{\mathbb{R}_0} \sum_{k=0}^j \binom{j}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(u, \sum_{i=0}^{j-k} \binom{j-k}{i} D_{r,z}^i X_n(u^-), \xi \right) \tilde{N}(du, d\xi) \right. \right. \\
&\quad \left. \left. + j \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{j-k-1} \binom{j-k-1}{i} D_{r,z}^i X_n(r^-), z \right) \right)^2 (\nu(dz))^j \right] \\
&\leq 2 \sum_{j=1}^N \int_{\mathbb{R}_0^j} \mathbb{E} \left[\sup_{r \leq s \leq t} \left(\int_r^s \int_{\mathbb{R}_0} \sum_{k=0}^j \binom{j}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(u, \sum_{i=0}^{j-k} \binom{j-k}{i} D_{r,z}^i X_n(u^-), \xi \right) \tilde{N}(du, d\xi) \right)^2 \right] (\nu(dz))^j \\
&\quad + 2 \sum_{j=1}^N \mathbb{E} \left[\int_{\mathbb{R}_0^j} \sup_{r \leq s \leq t} \left(j \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(r, \sum_{i=0}^{j-k-1} \binom{j-k-1}{i} D_{r,z}^i X_n(r^-), z \right) \right)^2 (\nu(dz))^j \right] \\
&\leq 8 \sum_{j=1}^N \int_{\mathbb{R}_0^j} \mathbb{E} \left[\left(\int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^j \binom{j}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(u, \sum_{i=0}^{j-k} \binom{j-k}{i} D_{r,z}^i X_n(u^-), \xi \right) \tilde{N}(du, d\xi) \right)^2 \right] (\nu(dz))^j \\
&\quad + 2 \sum_{j=1}^N \mathbb{E} \left[\int_{\mathbb{R}_0^j} \left(j \sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(r, \sum_{i=0}^{j-k-1} \binom{j-k-1}{i} D_{r,z}^i X_n(r^-), z \right) \right)^2 (\nu(dz))^j \right] \\
&= 8 \sum_{j=1}^N \int_{\mathbb{R}_0^j} \mathbb{E} \left[\int_r^t \int_{\mathbb{R}_0} \left(\sum_{k=0}^j \binom{j}{k} (-1)^k \right. \right. \\
&\quad \left. \left. \gamma \left(u, \sum_{i=0}^{j-k} \binom{j-k}{i} D_{r,z}^i X_n(u^-), \xi \right) \right)^2 \nu(d\xi) du \right] (\nu(dz))^j
\end{aligned}$$

$$\begin{aligned}
 &+ 2 \sum_{j=1}^N j^2 \mathbb{E} \left[\int_{\mathbb{R}_0^j} \left(\sum_{k=0}^{j-1} \binom{j-1}{k} (-1)^k \right. \right. \\
 &\quad \left. \left. \gamma \left(r, \sum_{i=0}^{j-k-1} \binom{j-k-1}{i} D_{r,z}^i X_n(r^-, z) \right) \right)^2 (\nu(dz))^j \right] \\
 &\leq k_1 + k_2 \sum_{j=1}^N \int_r^t \mathbb{E} \left[\int_{\mathbb{R}_0^j} \sum_{k=0}^j |D_{r,z}^k X_n(u^-)|^2 (\nu(dz))^j \right] du, \tag{4.8}
 \end{aligned}$$

for some constants k_1 and k_2 . Applying a discrete version of Gronwall’s inequality to Equation (4.8) we get

$$\sup_n \| | X_n | \|_{N,2} < \infty,$$

for all $N \geq 1$. Moreover, note that

$$\mathbb{E} \left[\sup_{0 \leq s \leq T} | X_n(s) - X(s) |^2 \right] \rightarrow 0$$

as n goes to infinity by the Picard approximation. Hence, by Lemma 4.1 we conclude that $X(t) \in D_{\infty,2}$. \square

Theorem 4.5. *Let $X(t)$ be the strong solution of the SDE,*

$$dX(t) = \int_{\mathbb{R}_0} \gamma(t, X(t^-), z) \tilde{N}(dt, dz), \tag{4.9}$$

with $X(0) = x \in \mathbb{R}$. Assume that $\gamma : [0, T] \times \mathbb{R} \times \mathbb{R}_0 \rightarrow \mathbb{R}$ satisfies the conditions (4.2) and (4.3). Then,

$$X(t) \in \mathcal{G}_q$$

for all $q \in \mathbb{R}$ and for all $0 \leq t \leq T$.

Proof. Using the isometry $\mathcal{U} : L^2(\mu) \rightarrow L^2(\mu_G)$ in (3.6) and Meyer’s inequality (see e.g. [20]), we obtain that

$$\| N^n X(t) \|^2 \leq C_n \left(\| D_{\cdot, \cdot}^{2n} X(t) \|_{L^2((\lambda \times \nu)^{2n} \times \mu)}^2 + \| X(t) \|_{L^2(\mu)}^2 \right),$$

where $C_n \geq 0$ is a constant depending on n . The proof of Meyer’s inequality in [20] or Theorem 1.5.1 in [18] shows that C_n is given by

$$C_n = M^{n-1} \prod_{j=1}^{n-1} \left(1 + \frac{1}{j} \right)^{\frac{j}{2}}, \quad n \geq 1$$

for a universal constant M . We see that

$$C_n \leq M^{n-1} e^{\frac{n-1}{2}}, \quad n \geq 1.$$

Thus we get

$$\begin{aligned} \|X(t)\|_q &= \|e^{qN}X(t)\|_{L^2(\mu)} \\ &\leq \sum_{n \geq 0} \frac{q^n}{n!} \|N^n X(t)\|_{L^2(\mu)} \\ &\leq \sum_{n \geq 0} \frac{q^n}{n!} M^{\frac{n-2}{2}} e^{\frac{n-1}{4}} (\|D_{\cdot, \cdot}^{2n} X(t)\|_{L^2((\lambda \times \nu)^{2n} \times \mu)} + \|X(t)\|_{L^2(\mu)}) \end{aligned}$$

On the other hand, it follows from Equation (4.5) that

$$\begin{aligned} \|D_{\cdot, \cdot}^{2n} X(t)\|_{L^2((\lambda \times \nu)^{2n} \times \mu)} &\leq L \cdot \left((n+1) \sum_{k=0}^n \binom{n}{k} + n^2 \sum_{k=0}^{n-1} \binom{n-k}{k} \right) \\ &\leq L \cdot ((n+1)2^n + n^3 2^{n-k}) \\ &\leq L \cdot 2^{3n+1} \end{aligned}$$

for a constant $L \geq 0$. Hence we get

$$\|X(t)\|_q \leq (L+1) \cdot e^{16\sqrt[4]{e}M \cdot q} \|X(t)\|_{L^2(\mu)} < \infty. \quad \square$$

Remark 4.6. We shall mention that the proof of Theorem 4.5 also carries over to backward stochastic differential equations (BSDE's) of the type

$$Y(t) = x + \int_t^T f(s, Y(s), Z(s, \cdot)) ds - \int_t^T \int_{\mathbb{R}_0} Z(s^-, z) \tilde{N}(ds, dz), \quad (4.10)$$

provided e.g. that the driver f is bounded and fulfills a linear growth and Lipschitz condition and $\nu(\mathbb{R}_0) < \infty$.

Appendix: Proof of Lemma 4.4

For $N = 1$, we have

$$\begin{aligned} D_{r,z} X(t) &= \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^1 \binom{1}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{1-k} \binom{1-k}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) + \gamma(r, X(r^-), z) \\ &= \int_r^t \int_{\mathbb{R}_0} \{ \gamma(s, X(s^-) + D_{r,z}^i X(s^-), \xi) - \gamma(s, X(s^-), \xi) \} \tilde{N}(ds, d\xi) + \gamma(r, X(r^-), z). \end{aligned}$$

Let us assume that it holds for $N \geq 1$. Hence

$$\begin{aligned} D_{r,z}^{N+1} X(t) &= D_{r,z}(D_{r,z}^N X(t)) \\ &= D_{r,z} \left[\int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \right] \end{aligned}$$

$$\begin{aligned}
& + N \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X(r^-), z \right) \Big] \\
= & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \\
& \gamma \left(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-) + \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^{i+1} X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& - \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \\
& + N \left[\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \right. \\
& \gamma \left(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X(r^-) + \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^{i+1} X(r^-), z \right) \\
& \left. - \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X(r^-), z \right) \right].
\end{aligned}$$

Note that

$$\sum_{i=0}^{N-k} \binom{N-k}{i} (D_{r,z}^i + D_{r,z}^{i+1}) X(s^-) = \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-)$$

and

$$\sum_{i=0}^{N-k-1} \binom{N-k-1}{i} (D_{r,z}^i + D_{r,z}^{i+1}) X(s^-) = \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-).$$

Hence,

$$\begin{aligned}
& D_{r,z}^{N+1} X(t) \\
= & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^{k+1} \gamma \left(s, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right)
\end{aligned}$$

$$\begin{aligned}
& + N \left[\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \right. \\
& \left. + \sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k-1} \binom{N-k-1}{i} D_{r,z}^i X(r^-), z \right) \right] \\
= & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \int_r^t \int_{\mathbb{R}_0} \sum_{k=1}^{N+1} \binom{N}{k-1} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \\
& + N \left[\sum_{k=0}^{N-1} \binom{N-1}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \right. \\
& \left. + \sum_{k=1}^N \binom{N-1}{k-1} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \right] \\
= & \int_r^t \int_{\mathbb{R}_0} \gamma \left(s, \sum_{i=0}^{N+1} \binom{N+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \int_r^t \int_{\mathbb{R}_0} \sum_{k=1}^N \left\{ \binom{N}{k} + \binom{N}{k-1} \right\} (-1)^k \\
& \quad \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \int_r^t \int_{\mathbb{R}_0} (-1)^{N+1} \gamma(s, X(s^-), \xi) \tilde{N}(ds, d\xi) \\
& + \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \\
& + N \left[\gamma \left(r, \sum_{i=0}^N \binom{N}{i} D_{r,z}^i X(r^-), z \right) + (-1)^N \gamma(r, X(r^-), z) \right. \\
& \left. + \sum_{k=1}^{N-1} \left\{ \binom{N-1}{k-1} + \binom{N-1}{k} \right\} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \right] \\
= & \int_r^t \int_{\mathbb{R}_0} \gamma \left(s, \sum_{i=0}^{N+1} \binom{N+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \int_r^t \int_{\mathbb{R}_0} (-1)^k \gamma(s, X(s^-), \xi) \tilde{N}(ds, d\xi)
\end{aligned}$$

$$\begin{aligned}
& + \int_r^t \int_{\mathbb{R}_0} \sum_{k=1}^N \binom{N+1}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \\
& + N \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right) \\
= & \int_r^t \int_{\mathbb{R}_0} \sum_{k=0}^{N+1} \binom{N+1}{k} (-1)^k \gamma \left(s, \sum_{i=0}^{N-k+1} \binom{N-k+1}{i} D_{r,z}^i X(s^-), \xi \right) \tilde{N}(ds, d\xi) \\
& + (N+1) \sum_{k=0}^N \binom{N}{k} (-1)^k \gamma \left(r, \sum_{i=0}^{N-k} \binom{N-k}{i} D_{r,z}^i X(r^-), z \right).
\end{aligned}$$

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