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A REPRESENTATION FOR POSITIVE FUNCTIONALS OF A BROWNIAN MOTION AND AN APPLICATION

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Abstract. The aim of this work is to provide a simpler proof of the Boué-Dupuis variational representation for positive functionals of a Brownian motion. Earlier proofs have relied on results concerning measurable selections whereas our approach uses functional stochastic differential equations. An application of this variational formula to a Wentzell-Freidlin type large deviations result in the context of non-Newtonian fluid flow is briefly discussed.

1. Introduction

The Boué-Dupuis variational representation formula [1] states that if W is a d-dimensional Brownian motion, then for any bounded, Borel-measurable function $f : C([0,1] : \mathbb{R}^d) \to \mathbb{R}$,

$$-\log \mathbb{E}\left\{e^{-f(W)}\right\} = \inf_{v} \mathbb{E}\left[\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^{\cdot} v_s ds\right)\right], \tag{1.1}$$

where infimum is taken over all processes v that are progressively measurable with respect to the augmented filtration $\mathcal{F}_t$, defined in the next section. Here, the $|\cdot|$ refers to the d-dimensional Euclidean norm. Budhiraja and Dupuis [2] proved an infinite-dimensional extension of the representation when W is a Brownian motion that takes values in a separable Hilbert space H, and has a nuclear covariance form Q.

The impetus to study this representation arises from the large deviations theory. It is well-known that the large deviation principle (LDP) is equivalent to the Laplace principle (LP) when the underlying random variables take values in a Polish space. Indeed, the origins of the Laplace principle can be traced to a result of Laplace which states that given an $h \in C([0,1])$,

$$\lim_{n\to\infty} \frac{1}{n} \log \int_0^1 e^{-nh(x)}\, dx = -\min_{x\in[0,1]} h(x). \tag{1.2}$$

Motivated by Equation (1.2), Varadhan's Lemma and Bryc's converse [14] show that for a family $\{X^\epsilon : \epsilon > 0\}$ of random variables defined on a probability space

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$(\Omega, \mathcal{F}, P)$, and taking values in a Polish space (i.e. a separable, complete metric space) E, the following two statements are equivalent:
(i) The family $\{X^\epsilon : \epsilon > 0\}$ satisfies LDP with rate function I.

$$(ii)\ \ \forall \text{ real-valued } h \in C_b(E),\ \ \lim_{\epsilon \to 0} \epsilon \log \mathbb{E}\left\{ e^{-\frac{1}{\epsilon} h(X^\epsilon)} \right\} = -\inf_{x \in E} \{h(x) + I(x)\}. \tag{1.3}$$

If X^ϵ is a Borel measurable function of a Wiener process W, then the usefulness of the variational representation (1.1) in proving the Laplace principle (1.3) is quite clear. It is worthwhile to note that Borel measurable functions of W arise often as a representation for pathwise unique strong solutions of stochastic differential equations (driven by a Brownian motion W).

In the paper by Boué and Dupuis [1], the upper bound

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \leq \inf_v \mathbb{E}\left[\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^\cdot v_s ds\right)\right].$$

is proved using the Girsanov transformation and relative entropy in a relatively easy manner. However, the corresponding lower bound surprisingly requires results on measurable selections and more work. We bring about a slight simplification of the proof of the lower bound by using functional differential equations and standard techniques of stochastic analysis. Our proof holds in the infinite-dimensional context as well. The variational representation has led to fruitful studies on the LDP for equations of fluid dynamics (see [10, 13]). A new application of the variational formula to Wentzell-Freidlin type LDP for the two-dimensional non-Newtonian fluid flow in bounded domains is also briefly outlined in this paper.

In section 2, we recall the basic definitions and preliminary results. The proof of the upper bound for the representation formula is due to Boué and Dupuis, and is presented in section 3 to make the paper reasonably self-contained. The new proof of the lower bound is given in section 4. The large deviation principle is introduced in section 5. It also presents a set of sufficient conditions due to Budhiraja and Dupuis that yield the Laplace principle. In section 6, we establish the Wentzell-Freidlin type LDP for non-Newtonian fluid flow in the presence of a small multiplicative noise term.

2. Definitions and Basic Results

We will work on the Wiener space $(\Omega, \mathcal{F}, \mu)$ where $\Omega = C([0,1] : \mathbb{R}^d)$, $\mathcal{F}$ is the Borel σ-field for Ω under the topology of uniform convergence, and μ is the d-dimensional Wiener measure. Let W be the canonical d-dimensional Wiener process given by $W_t(\omega) = \omega(t)$ for all $t \in [0,1]$. Define $(\mathcal{F}_t)$ as the smallest filtration that contains the σ-field $\sigma(W_s : 0 \leq s \leq t)$ and subsets of μ-null sets of $\mathcal{F}$. Then W is a Wiener martingale with respect to $\mathcal{F}_t$.

Next, we build the necessary notation:

1. Let $\mathcal{A}$ denote the class of all d-dimensional $\mathcal{F}_t$-progressively measurable processes v such that

$$\int_0^1 \mathbb{E}(|v_s|^2) ds < \infty.$$

2. Let $\mathcal{A}_b$ denote the subset of bounded elements of $\mathcal{A}$ so that $v \in \mathcal{A}_b$ means that there exists a $K < \infty$ such that $|v_t| \leq K$ for all $t \in [0,1]$ almost surely.
3. A stochastic process $\{X_t\}$ on $(\Omega, \mathcal{F})$ is called a bounded simple process if it can be written in the form

$$X_t = H_0 1_{\{0\}}(t) + \sum_{i=0}^{n} H_i 1_{(t_i, t_{i+1}]}(t) \;\; \forall \;\; t \in [0,1]$$

where $H_i \in \mathcal{F}_{t_i}$ for all i, and $\{t_i\}$ satisfies $0 = t_0 < t_1 \cdots, t_n = 1$ for some $n \in \mathbb{N}$. Besides, there exists a finite constant C such that $|H_i| \leq C$ for all i. We denote the class of bounded simple processes by $\mathcal{A}_s$.

From the construction of stochastic integrals, we know that if $X := \{X_t\}$ belongs to $\mathcal{A}_b$, then there exists a sequence $X^n := \{X_t^n\}$ of processes from $\mathcal{A}_s$ that are bounded uniformly in n by the bound for X, and

$$\lim_{n\to\infty} \mathbb{E} \int_0^1 |X_s^n - X_s|^2 ds = 0. \tag{2.1}$$

Let E be a Polish space and $\mathcal{E}$, the Borel σ-field on E. It is also useful to recall the basic fact that a real-valued, Borel-measurable function f defined on a probability space $(E, \mathcal{E}, \eta)$ can be approximated by a sequence of continuous functions $\{f_n\}$ in the almost sure sense. If $|f| \leq K$, then, for all n, we can take $|f_n| \leq K$.

Next, we turn our attention to the concept known as relative entropy, and state a few useful results based on it.

Definition 2.1. Let $(E, \mathcal{E})$ be as above with $P(E)$ as the class of probability measures defined on it. For $\eta \in P(E)$, the *relative entropy function* $R(\cdot \,||\, \eta)$ is the mapping from $P(E)$ into the extended real numbers given by

$$R(\lambda \,||\, \eta) = \int_E \log \frac{d\lambda}{d\eta}(x) \lambda(dx) \tag{2.2}$$

if λ is absolutely continuous with respect to η and $\log \frac{d\lambda}{d\eta}(x)$ is λ-integrable. Otherwise, define $R(\lambda \,||\, \eta)$ to be infinity.

The following simple and elegant result gives an abstract variational representation using the relative entropy function. A proof of it can be found in [3].

Proposition 2.2. *Let $(E, \mathcal{E})$ be a measurable space, and f, a bounded, Borel-measurable function from E to $\mathbb{R}$. Suppose that η is a probability measure on E. Then*

(i) $-\log \int_E e^{-f(x)} \eta(dx) = \inf_{\lambda \in P(E)} \left\{ R(\lambda \,||\, \eta) + \int_E f(x) \lambda(dx) \right\}$.

(ii) *The infimum in the above equation is reached at a probability measure λ^* where*

$$\frac{d\lambda^*}{d\eta}(x) = C e^{-f(x)}$$

with C as the normalizing constant.

The next result is quite useful in the sequel, and its proof can be found in [1].

Proposition 2.3. *Consider the probability space $(E, \mathcal{E}, \eta)$, where E is a Polish space and $\mathcal{E}$ its Borel σ-field. Let f be a real-valued, bounded Borel-measurable function on E. Suppose that $\{\lambda_n\}$ is a sequence in $P(E)$ such that there exists a constant C satisfying*

$$\sup_n R(\lambda_n \,||\, \eta) \leq C < \infty$$

and $\lambda_n \to \lambda$ weakly as $n \to \infty$. Then the following hold:

(i) $\lim_{n\to\infty} \int_E f d\lambda_n = \int_E f d\lambda$, and

(ii) if $\{f_n\}$ is sequence of uniformly bounded, Borel-measurable functions that converges to f η-a.s., then

$$\lim_{n\to\infty} \int_E f_n d\lambda_n = \int_E f d\lambda.$$

Next, we recall the definition of a weak solution to a functional stochastic differential equation.

Definition 2.4. Let $b_i(t,x)$ and $\sigma_{ij}(t,x)$ for $1 \leq i \leq d$ and $1 \leq j \leq r$ be progressively measurable functionals from $[0,\infty) \times C([0,\infty) : \mathbb{R}^d)$ into $\mathbb{R}$. A *solution* to the functional stochastic differential equation

$$dX_t = b(t,X)dt + \sigma(t,X)dW_t \ \ \forall \ \ t \geq 0$$

with initial condition X_0, is a triple (X, W), $(\Omega, \mathcal{F}, P)$, $\{\mathcal{F}_t\}$ satisfying the following:

(i) $(\Omega, \mathcal{F}, P)$ is a probability space and $\{\mathcal{F}_t\}$ is a filtration satisfying the usual conditions,

(ii) X_t is an adapted $\mathbb{R}^d$-valued process with continuous paths, and W is an r-dimensional Wiener martingale,

(iii) $\int_0^t \left\{|b_i(s,X)| + \sigma_{ij}^2(s,X)\right\} ds < \infty \ \forall \ \ 1 \leq i \leq d$ and $1 \leq j \leq r \ \ t \geq 0$,

(iv) $X_t = X_0 + \int_0^t b(s,X)ds + \int_0^t \sigma(s,X)dW_s$ for all $t \geq 0$ a.s.

In the particular case where b_i is bounded for all i, and $\sigma_{ij}(t,x) = \delta_{ij}$, the Kronecker delta function, for all $t \geq 0$ and $x \in \mathbb{R}^d$, then by the Girsanov theorem one obtains the existence of a weak solution which is unique in law. Under more conditions on b and σ, one obtains stronger conclusions and moment estimates (see Karatzas and Shreve [4], pages 303-306). A strong solution of a functional stochastic differential equation is defined analogous to that of a stochastic differential equation. A pathwise unique strong solution, X_t, of a functional stochastic differential equation can be written as a Borel-measurable function of the initial variable and the path of the driving Wiener process upto time t, by a result of Kallenberg.

Next, we recall the definition of Wiener processes that take values in a separable Hilbert space H. Let $(\cdot,\cdot)$ denote the inner product for H. Let Q be a strictly positive, symmetric, nuclear operator on H.

Definition 2.5. Let $(\Omega, \mathcal{F}, \mu)$ be the Wiener space and $\{\mathcal{F}_t\}$, the augmented filtration. A stochastic process $\{W(t)\}_{0\leq t\leq T}$ on $(\Omega, \mathcal{F}, \mu)$ is said to be an *H-valued $\mathcal{F}_t$ adapted Wiener process with covariance operator Q* if

(1) for each non-zero $h \in H$, $|Q^{1/2}h|_H^{-1}(W(t), h)$ is a standard one-dimensional Wiener process, and
(2) for any $h \in H$, $(W(t), h)$ is a $\{\mathcal{F}_t\}$-adapted martingale.

The following notation is standard, and quite useful for us. Define the space $H_0 = Q^{1/2}H$ and equip it with the inner product $(f, g)_0 = (Q^{-1/2}f, Q^{-1/2}g)$. Then H_0 is a Hilbert space with its norm denoted by $|\cdot|_0$. Then H_0 is compactly embedded in H. Next, we state the Girsanov theorem in infinite dimensions.

Theorem 2.6. *Let h be an H_0-valued $\mathcal{F}_t$-predictable process satisfying the condition $\int_0^T |h(s)|_0^2 ds < \infty$ a.s. for some fixed T, and*

$$\mathbb{E}\left(\exp\left\{\int_0^T h(s)dW(s) - \frac{1}{2}\int_0^T |h(s)|_0^2 ds\right\}\right) = 1.$$

Then the process $\tilde{W}(t) := W(t) - \int_0^t h(s)ds$ for $t \in [0, T]$ is a Wiener process with covariance operator Q on $(\Omega, \mathcal{F}, \eta)$ where η is the probability measure given by

$$\frac{d\eta}{d\mu} = \exp\left\{\int_0^T h(s)dW(s) - \frac{1}{2}\int_0^T |h(s)|_0^2 ds\right\}$$

3. Proof of the Upper Bound

The main theorem on representation of positive functionals of a Brownian motion is stated below:

Theorem 3.1. *Let f be a bounded Borel-measurable function mapping $C([0,1] : \mathbb{R}^d)$ into $\mathbb{R}$. Then*

$$-\log \mathbb{E}\left[e^{-f(W)}\right] = \inf_{v \in \mathcal{A}} \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^{\cdot} v_s ds\right)\right\} \tag{3.1}$$

Remark 3.2. The upper bound refers to replacing the equality sign in (3.1) by $\leq$ sign. The result is due to Boué and Dupuis [1] We will give the main ideas of the proof since it contains a useful bound that we need later.

Proof. Take any v in $\mathcal{A}_b$. Define R_t by

$$R_t = \exp\left[\sum_{i=1}^d \int_0^t v_s^{(i)} dW_s^{(i)} - \frac{1}{2}\int_0^t |v_s|^2 ds\right], \tag{3.2}$$

then R_t is a martingale. Define a probability measure η_v on $\mathcal{F}_1$ by

$$\eta_v(A) = \int_A R_1 d\mu \quad \text{for } A \in \mathcal{F}_1. \tag{3.3}$$

By the Girsanov theorem, the process $\tilde{W} = \left\{\tilde{W}_t = (\tilde{W}_t^{(1)}, \cdots, \tilde{W}_t^{(d)}); 0 \leq t \leq 1\right\}$ given by $\tilde{W}_t = W_t - \int_0^t v_s ds$ is a d-dimensional Brownian motion under η_v. Let T_v be the operator defined on $C([0,1]; \mathbb{R}^d)$ by

$$T_v(\phi)_t = \phi_t - \int_0^t v_s(\phi) ds. \tag{3.4}$$

For any Borel set $A \subset C([0,1]:\mathbb{R}^d)$, it follows that

$$\mu(A) = \eta_v(T_v^{-1}(A)).$$

Using the definition of $R(\eta_v||\mu)$ and substituting (3.2) and (3.3), we obtain

$$\begin{aligned} R(\eta_v||\mu) &= \int \left(\log \frac{d\eta_v}{d\mu}\right) d\eta_v \\ &= \int \left\{ \sum_{i=1}^{d} \int_0^1 v_s^{(i)}(\phi) dW_s^{(i)} - \frac{1}{2}\int_0^1 |v_s(\phi)|^2 ds \right\} \eta_v(d\phi). \end{aligned} \tag{3.5}$$

Since $W_t = \tilde{W}_t + \int_0^t v_s ds$, we have

$$\begin{aligned} R(\eta_v||\mu) &= \mathbb{E}^v \left\{ \sum_{i=1}^{d} \int_0^1 v_s^{(i)} d\tilde{W}_s^{(i)} + \sum_{i=1}^{d} \int_0^1 (v_s^{(i)})^2 ds - \frac{1}{2}\int_0^1 |v_s|^2 ds \right\} \\ &= \mathbb{E}^v \left\{ \frac{1}{2} \int_0^1 |v_s|^2 ds \right\}, \end{aligned} \tag{3.6}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability measure η_v. Thus,

$$R(\eta_v||\mu) + \int f(\phi)\eta_v(d\phi) = \mathbb{E}^v \left\{ \frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s ds\right) \right\} \tag{3.7}$$

and we obtain

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \le \inf_{v\in\mathcal{A}_b} \mathbb{E}^v \left\{ \frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s ds\right) \right\}. \tag{3.8}$$

From (3.8), it follows that, for any $v \in \mathcal{A}$,

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \le \mathbb{E} \left\{ \frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^{\cdot} v_s ds\right) \right\} \tag{3.9}$$

where expectation is with respect to Wiener measure μ.

STEP 1: Suppose that v is in $\mathcal{A}_s$. Then $\tilde{v}$ can be recursively constructed such that $\tilde{v} \in \mathcal{A}_s$ and for $\phi \in C([0,1]:\mathbb{R}^d)$, $\tilde{v}(\phi) = v(T_{\tilde{v}}(\phi))$ with probability 1.

This implies that, for $\tilde{W}(\phi) = W(\phi) - \int_0^{\cdot} \tilde{v}_s(\phi) ds$ and $A \in \mathcal{B}(C([0,1];\mathbb{R}^d))$, $B \in \mathcal{B}(L^2([0,1];\mathbb{R}^d))$

$$\begin{aligned} \eta_{\tilde{v}}(\tilde{W} \in A, \tilde{v} \in B) &= \eta_{\tilde{v}}\left(\left\{\phi : \phi - \int_0^{\cdot} \tilde{v}_s(\phi) ds \in A, \tilde{v}(\phi) \in B\right\}\right) \\ &= \eta_{\tilde{v}}(\{\phi : T_{\tilde{v}}(\phi) \in A, v(T_{\tilde{v}}(\phi)) \in B\}) \\ &= \mu(\{\psi : \psi \in A, v(\psi) \in B\}) \\ &= \mu(W \in A, v \in B) \end{aligned} \tag{3.10}$$

which shows that the distribution of $(\tilde{W}, \tilde{v})$ under the measure $\eta_{\tilde{v}}$ is the same as the the distribution of (W, v) under μ. Using this equivalence and (3.8), we obtain

$$\begin{aligned} -\log \mathbb{E}\left[e^{-f(W)}\right] &\leq \mathbb{E}^{\tilde{v}}\left\{\frac{1}{2}\int_0^1 |\tilde{v}_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} \tilde{v}_s ds\right)\right\} \\ &= \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^{\cdot} v_s ds\right)\right\} \end{aligned} \tag{3.11}$$

which implies (3.9) for all $v \in \mathcal{A}_s$.

Let $L_\mu(W + \int_0^{\cdot} v_s ds)$ denote the measure on $C([0,1] : \mathbb{R}^d)$ that is induced by $W + \int_0^{\cdot} v_s ds$ under μ. For $A \in \mathcal{B}(C([0,1]; \mathbb{R}^d))$, we have

$$\begin{aligned} \mu(W(\phi) + \int_0^{\cdot} v_s(\phi) ds \in A) &= \eta_{\tilde{v}}(\tilde{W}(\phi) + \int_0^{\cdot} \tilde{v}_s(\phi) ds \in A) \\ &= \eta_{\tilde{v}}(W(\phi) \in A) \end{aligned} \tag{3.12}$$

which implies that $L_\mu(W + \int_0^{\cdot} v_s ds) = \eta_{\tilde{v}}$. Using (3.6) and taking $f = 0$ in the equality of (3.11), we have

$$\begin{aligned} R\left(L_\mu(W + \int_0^{\cdot} v_s ds) || \mu\right) &= R\left(\eta_{\tilde{v}} || \mu\right) \\ &= \mathbb{E}^{\tilde{v}}\left\{\frac{1}{2}\int_0^1 |\tilde{v}_s|^2 ds\right\} \\ &= \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds\right\} \end{aligned} \tag{3.13}$$

for all $v \in \mathcal{A}_s$.

STEP 2: Bounded v. Let $v \in \mathcal{A}_b$, so that $|v_s(\omega)| \leq M < \infty$ for $0 \leq s \leq 1$, $\omega \in \Omega$. According to [4] Lemma 3.2.4, there exists a sequence of simple processes $\{v^n, n \in \mathbb{N}\}$ such that $|v_s^n(\omega)| \leq M < \infty$ for all $0 \leq s \leq 1$, $\omega \in \Omega$, and

$$\lim_{n \to \infty} \mathbb{E} \int_0^1 |v_s^n - v_s|^2 ds = 0. \tag{3.14}$$

Thus $(W, \int_0^{\cdot} v_s^n ds)$ converges in distribution to $(W, \int_0^{\cdot} v_s ds)$ in $(C([0,1]; \mathbb{R}^d))^2$.

By Step 1, for each $n \in \mathbb{N}$,

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \leq \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s^n|^2 ds + f\left(W + \int_0^{\cdot} v_s^n ds\right)\right\}. \tag{3.15}$$

The inequality above continues to hold in the limit as $n \to \infty$ by an application of Proposition 2.3 (a). By the lower semicontinuity of $R(\cdot || \mu)$, we obtain

$$R\left(L_\mu(W + \int_0^{\cdot} v_s ds) || \mu\right) \leq \frac{1}{2}\mathbb{E}\left[\int_0^1 |v_s|^2 ds\right]. \tag{3.16}$$

STEP 3: General $v \in \mathcal{A}$. We define

$$v_s^n(\phi) = v_s(\phi) 1_{\{|v_s(\phi)| \leq n\}}, \quad 0 \leq s \leq 1, \;\; \phi \in C([0,1]; \mathbb{R}^d). \tag{3.17}$$

Let $\mu_n = L_\mu(W + \int_0^\cdot v_s^n ds)$, then (3.13) implies that

$$\sup_{n\in\mathbb{N}} R(\mu_n||\mu) = \sup_{n\in\mathbb{N}} \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s^n|^2 ds\right\} \le \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds\right\} < \infty. \tag{3.18}$$

As in Step 2, Proposition 2.3 and dominated convergence theorem yield (3.9) for any $v \in \mathcal{A}$, which finishes the proof. As before, we obtain the useful bound (3.16) when $v \in \mathcal{A}$. □

4. Proof of the Lower Bound

In this section we will give a proof of the lower bound in the variational representation formula. That is,

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \ge \inf_{v\in\mathcal{A}} \mathbb{E}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(W + \int_0^\cdot v_s ds\right)\right\}.$$

Proof. Step 1: Let $(\Omega, \mathcal{F})$ be the Wiener space and f be a bounded measurable function mapping Ω into $\mathbb{R}$. Let μ be the Wiener measure on Ω and $\Pi(\Omega)$ be the set of probabilities on Ω. Consider the measure η_0 where infimum is attained in the variational formula

$$-\log \int_\Omega e^{-f(x)} d\mu = \inf_{\eta\in\Pi(\Omega)} \left\{R(\eta||\mu) + \int_\Omega f(x) d\eta\right\}. \tag{4.1}$$

Then η_0 is not only absolutely continuous with respect to μ, but it is in fact equivalent to μ on $\mathcal{F}$. It follows that, for each $t \in [0,1]$, the restriction of η_0 to $\mathcal{F}_t$ is a probability measure which is equivalent to the restriction of μ to $\mathcal{F}_t$. Let R_t be the corresponding Radon-Nikodym derivative

$$R_t = \mathbb{E}\left[\frac{d\eta_0}{d\mu}|\mathcal{F}_t\right] = \mathbb{E}\left[\frac{e^{-f(x)}}{\int_\Omega e^{-f(x)}\mu(dx)}|\mathcal{F}_t\right]. \tag{4.2}$$

Then $\{R_t; 0 \le t \le 1\}$ forms a μ-martingale that is bounded from below and above μ-a.s. respectively by constants $\exp(-2||f||_\infty)$ and $\exp(2||f||_\infty)$. Moreover, since R_t is a martingale with respect to the augmentation under μ of the filtration generated by a Brownian motion, it can be represented as a stochastic integral $R_t = 1 + \int_0^t u_s dW_s$, where u_s is progressively measurable.

Since R_t is bounded from below, we can define $v_t = u_t/R_t$ and write

$$R_t = 1 + \int_0^t v_s R_s dW_s. \tag{4.3}$$

The random variable R_1 is bounded by a constant, and hence $\mathbb{E}(R_1^2) < \infty$. This observation and Equation (4.3) yield $\mathbb{E}\int_0^1 |v_s|^2 R_s^2 ds < \infty$. Since R_t is bounded below by a constant, we have $\mathbb{E}\int_0^1 |v_s|^2 ds < \infty$. Also, $d\eta_0/d\mu$ is bounded so that one obtains

$$\int_{C([0,1]:\mathbb{R}^d)} \int_0^1 |v_s|^2 ds d\eta_0 < \infty. \tag{4.4}$$

These bounds and Equation (4.3) allow us to write

$$R_t = \exp\left[\int_0^t v_s dW_s - \frac{1}{2}\int_0^t |v_s|^2 ds\right]. \tag{4.5}$$

Since R_t is a martingale, the Girsanov theorem identifies η_0 as the measure under which the process $\tilde{W} := W - \int_0^{\cdot} v_s ds$ is a Brownian motion. Analogous to the derivation of Equation (3.7) as in the proof of the upper bound to evaluate $R(\eta_0||\mu)$, we obtain

$$-\log \mathbb{E}\left[e^{-f(W)}\right] = \mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s\right)\right\}. \tag{4.6}$$

STEP 2: Let us first assume that f is continuous. Since progressively measurable processes can be approximated by bounded, simple processes in the L^2-sense, given $\varepsilon > 0$, there exists a process v^* be a bounded, simple process such that

$$\mathbb{E}^{\eta_0}\left\{\int_0^1 |v_s^* - v_s|^2 ds\right\} < \frac{\varepsilon}{2}. \tag{4.7}$$

Let us write the process v^* in the form

$$v_t^*(\omega) = \xi_0(\omega)1_{\{0\}}(t) + \sum_{i=0}^{l-1} \xi_i(\omega)1_{(t_i,t_{i+1}]}(t), \qquad 0 \le t \le 1, \ \ \omega \in \Omega, \tag{4.8}$$

where $0 = t_0 < t_1 < \cdots < t_l = 1$ and ξ_i is $\mathcal{F}_{t_i}$-measurable for each $i = 0, \cdots, l-1$. Each ξ_i can be approximated in $L^2(\mu)$ (and hence equivalently in $L^2(\eta_0)$ as well) by a smooth cylindrical functional with compact support, namely, $g_i(\omega_{s_1}, \cdots, \omega_{s_n})$, where $s_1 < s_2 < \cdots s_n \le t_i$ (see Nualart [12], Page 24). Replacing each ξ_i by g_i, and then using polygonalization in the time variable s, we can find a smooth progressively measurable functional z with continuous sample paths which approximates v^* in the sense that

$$\mathbb{E}^{\eta_0}\left\{\int_0^1 |z_s - v_s^*|^2 ds\right\} < \frac{\varepsilon}{2}.$$

It follows that given $\epsilon > 0$, we can choose a progressively measurable process z as constructed above such that

$$\begin{aligned} &\mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |v_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} v_s ds\right)\right\} \\ &\ge \mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |z_s|^2 ds + f\left(\tilde{W} + \int_0^{\cdot} z_s ds\right)\right\} - \varepsilon. \end{aligned} \tag{4.9}$$

Consider the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \eta_0)$. Under the measure η_0, we have $\tilde{W}(\omega) = \omega - \int_0^{\cdot} v_s(\omega) ds$ is a Brownian motion. Define

$$X(\omega) := \tilde{W}(\omega) + \int_0^{\cdot} v_s(\omega) ds,$$

and note that $X(\omega) = \omega$. The process X_t solves

$$X_t = \tilde{W}_t + \int_0^t v_s(X) ds. \tag{4.10}$$

In general, we can only assert that Equation (4.10) has a weak solution which is unique in law.

Define a probability measure η_1 on $(\Omega, \mathcal{F})$ by

$$\frac{d\eta_1}{d\eta_0} = \exp\left\{\int_0^1 (z_s - v_s) d\tilde{W}_s - \frac{1}{2}\int_0^1 |z_s - v_s|^2 ds\right\}.$$

Then $\eta_1 \equiv \eta_0$, and η_1 a.s., we can write, for all $0 \le t \le 1$,

$$\tilde{W}_t = \hat{W}_t + \int_0^t (z_s - v_s) ds,$$

where $\hat{W}$ is a Brownian motion on $(\Omega, \mathcal{F}, (\mathcal{F}_t), \eta_1)$. Thus η_1 a.s., we have

$$W_t = \hat{W}_t + \int_0^t z_s ds. \tag{4.11}$$

We can rewrite Equation (4.11) as

$$Y_t = \hat{W}_t + \int_0^t z_s(Y) ds. \tag{4.12}$$

Equation (4.12) has a strong pathwise unique solution by the choice of z. Therefore $Y = h(\hat{W})$ for some Borel measurable function h.

Note that η_1 depends on ϵ. Taking $\epsilon = 1/n$, let us denote the corresponding sequence of probability measures by $\eta^{(n)}$. Then $\eta^{(n)}$ is the law of the solution of the equation

$$Y_t^{(n)} = \hat{W}_t^{(n)} + \int_0^t z_s^{(n)}(Y^{(n)}) ds,$$

where $\hat{W}^{(n)}$ is a Wiener process with respect to $\eta^{(n)}$. Also,

$$\eta^{(n)} \to \eta_0$$

weakly as $n \to \infty$. Thus, for any fixed constant $K > 0$ and any given $\epsilon > 0$, there exists an n such that

$$\begin{aligned} &\mathbb{E}^{\eta_0}\left\{\frac{1}{2}[K \wedge \int_0^1 |v_s(X)|^2 ds] + f(X)\right\} \\ &\quad \ge \mathbb{E}^{\eta^{(n)}}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(Y_n)|^2 ds] + f(Y)\right\} - \varepsilon. \end{aligned} \tag{4.13}$$

Recalling that that $Y_n = h_n(\hat{W}^{(n)})$, let us introduce the following notation:

$$\mathcal{L}_n(\cdot) := \frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(h_n(\cdot))|^2 ds] + f(h_n(\cdot)).$$

Since $\hat{W}^{(n)}$ is a $\eta^{(n)}$-Brownian motion and W is a μ-Brownian motion, we have $\mathbb{E}^{\eta^{(n)}}(\mathcal{L}_n(\hat{W}^{(n)})) = E(\mathcal{L}_n(W))$. Then

$$\begin{aligned}
&\mathbb{E}^{\eta_0}\left\{\frac{1}{2}\int_0^1 |v_s(X)|^2 ds + f(X)\right\} \\
&\geq \mathbb{E}^{\eta^{(n)}}(\mathcal{L}(\hat{W}^{(n)})) - \varepsilon \\
&= \mathbb{E}(\mathcal{L}_n(W)) - \varepsilon \\
&= \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^{(n)}(h_n(W))|^2 ds] + f(h_n(W))\right\} - \varepsilon \\
&= \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^*(W)|^2 ds] + f(h_n(W))\right\} - \varepsilon \\
&= \mathbb{E}\left\{\frac{1}{2}[K \wedge \int_0^1 |z_s^*|^2 ds] + f\left(W + \int_0^\cdot z_s^* ds\right)\right\} - \varepsilon,
\end{aligned} \tag{4.14}$$

where $z_s^* := z_s^{(n)} \circ h_n$ is progressively measurable. Now allow $K \to \infty$ using monotone convergence. Recalling equation (4.6), the inequality (4.14) yields the lower bound for continuous f.

Step 3: If f is not continuous, let $\{f_j\}$ be a sequence of bounded continuous functions such that $||f_j||_\infty \leq ||f||_\infty < \infty$ and $\lim_{j\to\infty} f_j = f$, μ-a.s. The proceeding argument applied to each of the functions f_j implies that there exists a sequence of progressively measurable processes $\left\{z^{j*}, j \in \mathbb{N}\right\}$ satisfies (4.14) for each j but with f replaced by f_j, that is

$$-\log \mathbb{E}e^{-f_j(W)} \geq \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}(W))|^2 ds + f_j(W + \int_0^\cdot z_s^{j*} ds)\right\} - \varepsilon. \tag{4.15}$$

Thanks to (3.16), we have

$$\sup_j R\left(L_\mu(W + \int_0^\cdot z_s^{j*} ds)||\mu\right) \leq \sup_j \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}|^2 ds\right\} \leq ||f||_\infty. \tag{4.16}$$

It follows from this bound that the pair $(\int_0^\cdot z_s^{j*}, W)$ is tight,and hence there exists a subsequence such that $(\int_0^\cdot z_s^{j*}, W)$ converges in distribution to $(\int_0^\cdot z_s^*, W)$. It follows from (4.15), the dominated convergence theorem and Proposition 2.3 that, for all sufficiently large j,

$$-\log \mathbb{E}\left[e^{-f(W)}\right] \geq \mathbb{E}\left\{\frac{1}{2}\int_0^1 |z_s^{j*}(W))|^2 ds + f\left(W + \int_0^\cdot z_s^{j*} ds\right)\right\} - 2\varepsilon. \tag{4.17}$$

This completes the proof of the lower bound. □

In infinite-dimensions, the above proofs carry over with the necessary modifications, which allows us to state the following representation formula as in [2]. Let $\mathcal{P}$ denote the class of H_0-valued $\{\mathcal{F}_t\}$-predictable processes ϕ which satisfy $\int_0^T |\phi(s)|_0^2\, ds < \infty$ a.s.

Theorem 4.1. *Let f be a bounded, Borel measurable function mapping $C([0,T]: H)$ into $\mathbb{R}$. Then*

$$-\log \mathbb{E}\left[e^{-f(W)}\right] = \inf_{v\in\mathcal{P}} \mathbb{E}\left(\frac{1}{2}\int_0^T |v_s|_0^2 ds + g\left(W + \int_0^{\cdot} v_s ds\right)\right). \tag{4.18}$$

5. Large Deviation Principle

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{0\le t\le T}$ of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and P-completeness.

In what follows, the notation and terminology are built in order to state the large deviations result of Budhiraja and Dupuis [1] for Polish-space valued random elements:

Let $S_N = \left\{v \in L^2([0,T]:H_0) : \int_0^T |v(s)|_0^2\, ds \le N\right\}$. The set S_N endowed with the weak topology is a Polish space. Define $\mathcal{P}_N = \{\phi \in \mathcal{P} : \phi(\omega) \in S_N,\ P - a.s.\}$

Let E denote a Polish space, and let $g^\epsilon : C([0,T];H) \to E$ be a measurable map. Define $X^\epsilon = g^\epsilon(W(\cdot))$. We are interested in the large deviation principle for X^ϵ as $\epsilon \to 0$. Since $\{X^\epsilon\}$ are Polish space valued random elements, the Laplace principle and the large deviation principle are equivalent.

Definition 5.1. A function I mapping E to $[0,\infty]$ is called *a rate function* if I is lower semicontinuous. A rate function I is called *a good rate function* if for each $M < \infty$, the level set $\{x \in E : I(x) \le M\}$ is compact.

Definition 5.2. Let I be a rate function on E. A family $\{X^\epsilon : \epsilon > 0\}$ of E-valued random elements is said to *satisfy the Laplace principle* on E with rate function I if for each real-valued, bounded and continuous function h defined on E,

$$\lim_{\epsilon\to 0} \epsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\epsilon}h(X^\epsilon)\right]\right\} = -\inf_{x\in E}\{h(x) + I(x)\}. \tag{5.1}$$

Hypotheses H: There exists a measurable map $g^0 : C([0,T]:H) \to E$ such that the following hold:

1. Let $\{v^\epsilon : \epsilon > 0\} \subset \mathcal{P}_M$ for some $M < \infty$. Let v^ϵ converge in distribution as S_M-valued random elements to v. Then $g^\epsilon(W(\cdot) + \frac{1}{\sqrt{\epsilon}}\int_0^{\cdot} v^\epsilon(s)\, ds)$ converges in distribution to $g^0(\int_0^{\cdot} v(s)\, ds)$.
2. For every $M < \infty$, the set $K_M = \left\{g^0(\int_0^{\cdot} v(s)\, ds) : v \in S_M\right\}$ is a compact subset of E.

For each $f \in E$, define

$$I(f) = \inf_{\left\{v\in L^2([0,T]:H_0): f = g^0(\int_0^{\cdot} v(s)ds)\right\}} \frac{1}{2}\left\{\int_0^T |v(s)|_0^2\, ds\right\}, \tag{5.2}$$

where infimum over an empty set is taken as ∞.

The following theorem was proven by Budhiraja and Dupuis [1]. The variational representation theorem allows one to prove the sufficiency of Hypotheses H to establish the Laplace principle.

Theorem 5.3. *Let $X^\epsilon = g^\epsilon(W(\cdot))$. If $\{g^\epsilon\}$ satisfies the Hypotheses H, then the family $\{X^\epsilon : \epsilon > 0\}$ satisfies the Laplace principle in E with rate function I given by (5.2).*

6. Non-Newtonian Stochastic Navier-Stokes Equations

Let $G \subset \mathbb{R}^2$, be an arbitrary bounded open domain with a smooth boundary ∂G, and $(\Omega, \mathcal{F}, P)$be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0\le t\le T}$ of sub-σ-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and P-completeness. For $t \in [0, T]$, we consider the stochastic Navier-Stokes equation for a viscous impressible flow with a non-slip condition at the boundary

$$\begin{aligned}
&\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\eta(\mathbf{u}) = \mathbf{f}(t) + \sigma(t,\mathbf{u})\frac{dW}{dt}\\
&\nabla\cdot\mathbf{u} = 0, \qquad \forall (x,t,\omega) \in G\times(0,T)\times\Omega\\
&\mathbf{u}(x,t,\omega) = 0, \qquad \forall (x,t,\omega) \in \partial G\times(0,T)\times\Omega\\
&\mathbf{u}(x,0,\omega) = \mathbf{u}_0(x,\omega), \qquad \forall (x,\omega) \in G\times\Omega.
\end{aligned} \tag{6.1}$$

In the above, $\mathbf{u} = (u_1, u_2)$ is the two dimensional velocity and $\eta(\mathbf{u})$ denotes the (possibly nonlinear) stress tensor. Convective acceleration is represented by the nonlinear quantity: $(\mathbf{u}\cdot\nabla)\mathbf{u}$. The vector field $\mathbf{f}(t)$ represents the external body force, and typically these consist of only gravity forces, but may include other types(such as electromagnetic forces). The process $\{W_t\}$ is an infinite-dimensional Hilbert space-valued Wiener process (see Definition 2.5).

We now consider the following nonlinear and hyperviscosities cases for the stress tensor $\eta(\mathbf{u})$.

Case 1: Nonlinear Constitutive Relationship [5, 6, 8, 9]

$$\eta_1(\mathbf{u}) := -p\mathbf{I} + \nu_0\nabla\mathbf{u} + \nu_1|\nabla\mathbf{u}|^{q-2}\nabla\mathbf{u}, \tag{6.2}$$

where p denotes the pressure and is a scalar-valued function, $\nu_0, \nu_1 > 0$ and $q \ge 3$. In this case,

$$\nabla\cdot\eta_1(\mathbf{u}) = -\nabla p + \nu_0\Delta\mathbf{u} + \nu_1\nabla\cdot(|\nabla\mathbf{u}|^{q-2}\nabla\mathbf{u}).$$

Case 2: Nonlinear Nonlocal Viscosity [6]

$$\eta_2(\mathbf{u}) = -p\mathbf{I} + (\nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)})\nabla\mathbf{u}. \tag{6.3}$$

In this case, the nonlinear viscosity is given by

$$\nu(\|\nabla\mathbf{u}\|_{L^2(G)}) := \nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)},$$

where $\nu_0, \nu_1 > 0$ and $\|\nabla\mathbf{u}\|^2_{L^2(G)} := \int_G |\nabla\mathbf{u}|^2 dx$, so that

$$\nabla\cdot\eta_2(\mathbf{u}) = -\nabla p + (\nu_0 + \nu_1\|\nabla\mathbf{u}\|^2_{L^2(G)})\Delta\mathbf{u}.$$

Case 3: Hyperviscosity [9]

$$\eta_3(\mathbf{u}) = -p\mathbf{I} + \nu_0\nabla\mathbf{u} - \nu_1(-1)^m\nabla(\Delta^{m-1}\mathbf{u}) \tag{6.4}$$

with $m \ge 2$ and $\nu_0, \nu_1 > 0$. In this case, we prescribe additional boundary conditions $(\partial\mathbf{u}/\partial n)\mid_{\partial G} = \cdots = (\partial^{m-1}\mathbf{u}/\partial n^{m-1})\mid_{\partial G} = 0$ and we have

$$\nabla\cdot\eta_3(\mathbf{u}) = -\nabla p + \nu_0\Delta\mathbf{u} - \nu_1(-1)^m\Delta^m\mathbf{u}.$$

This type of regularization has been used in atomspheric dynamics models and also in the study of vortex reconnections [11, 7].

The stochastic Navier-Stokes equations can be written in the abstract evolution form for bounded domains by introducing the following functional spaces. Let

$$\begin{aligned} H &:= \left\{\mathbf{u} \in L^2(G); \nabla\cdot\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}\mid_{\partial\mathbf{G}} = \mathbf{0}\right\} \\ V_{r,q} &:= \{\mathbf{u} \in W_0^{r,q}(G); \nabla\cdot\mathbf{u} = 0\} \end{aligned}$$

where $\mathbf{n}$ is the outward normal. In *Case 1*: $r = 1, q \geq 3$, *in Case 2* : $r = 1, q = 2$ and in *Case 3*: $r = m, q = 2$.

We define the operators $\mathcal{A}_i (i = 1, 2, 3)$ as follows:

$$\begin{aligned} \langle \mathcal{A}_1(\mathbf{u}), \mathbf{v}\rangle &:= \int_G |\nabla\mathbf{u}|^{q-2}\nabla\mathbf{u}\cdot\nabla\mathbf{v}dx, && \forall \mathbf{u},\mathbf{v} \in V_{1,q} \\ \langle \mathcal{A}_2(\mathbf{u}), \mathbf{v}\rangle &:= \|\nabla\mathbf{u}\|^2_{L^2(G)}\int_G \nabla\mathbf{u}\cdot\nabla\mathbf{v}dx, && \forall \mathbf{u},\mathbf{v} \in V_{1,2} \\ \langle \mathcal{A}_3(\mathbf{u}), \mathbf{v}\rangle &:= \sum_{\alpha\in N, |\alpha|=m}\int_G D^\alpha\mathbf{u}\cdot D^\alpha\mathbf{v}dx, && \forall \mathbf{u},\mathbf{v} \in V_{m,2}. \end{aligned}$$

Let us denote V, for ease of notation, as the space $V_{r,q}$ in *Case 1, 2* and *3*, and V' be the dual of V, we have the dense, continuous embedding $V \subset H$, then for its dual space V' it follows that $H' \subset V'$ continuously and densely. Identifying H and its dual H' via the Riesz isomorphism we have that

$$V \subset\rightharpoonup H = H' \subset\rightharpoonup V'$$

continuously and densely and if $\langle , \rangle$ denotes the dualization between V' and V(i.e. $\langle z, v\rangle := z(v)$ for $z \in V', v \in V$), it follows that

$$\langle z, v\rangle = (z, v)_H, \qquad \forall z \in H, v \in V$$

and (V, H, V') is called a *Gelfand triple*.

Let us define the operator $\mathbf{A} : V_{1,2} \to V'_{1,2}$ by

$$\mathbf{Au} = -\Pi_H\Delta\mathbf{u}$$

for $\mathbf{u} \in D(\mathbf{A}) = W^{2,2}(G) \cap V_{1,2}$, where $\Pi_H : L^2(G) \to H$ is the Leray projector. The operator $\mathbf{A}$ is known as the *Stokes operator* and is positive, self-adjoint.

Define $b(\cdot,\cdot,\cdot) : V_{1,2} \times V_{1,2} \times V_{1,2} \to \mathbb{R}$ by

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = \sum_{i,j=1}^{2}\int_G u_i\frac{\partial v_j}{\partial x_i}w_jdx \tag{6.5}$$

and then we can define the continuous bilinear operator $\mathbf{B} : V_{1,2} \times V_{1,2} \to V'$ such that

$$\langle \mathbf{B}(\mathbf{u},\mathbf{v}), \mathbf{w}\rangle = b(\mathbf{u},\mathbf{v},\mathbf{w}) \tag{6.6}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in V_{1,2}$. We will use $\mathbf{B}(\mathbf{u})$ to denote $\mathbf{B}(\mathbf{u},\mathbf{u})$. Since $\mathbf{u} \in V_{1,2}$, it follows that $b(\mathbf{u},\mathbf{v},\mathbf{w}) = -b(\mathbf{u},\mathbf{w},\mathbf{v})$ and hence $b(\mathbf{u},\mathbf{v},\mathbf{v}) = 0$.

Let Q be a strictly positive definite, symmetric trace class operator on H and $H_0 = Q^{1/2}H$, then H_0 is a Hilbert space with the inner product

$$(\mathbf{u},\mathbf{v})_0 = (Q^{-1/2}\mathbf{u}, Q^{-1/2}\mathbf{v}) \quad \forall\ \mathbf{u},\mathbf{v} \in H_0, \tag{6.7}$$

where $Q^{-1/2}$ is the pseudo inverse of $Q^{1/2}$. Let $|\cdot|_0$ denote the norm in H_0. Clearly, the imbedding of H_0 in H is Hilbert-Schmidt since Q is a trace class operator.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H. Define the norm on the space L_Q by $|S|^2_{L_Q} = \mathbf{tr}\,(SQS^*)$. The noise coefficient $\sigma : [0,T]\times H \to L_Q(H_0; H)$ is such that it satisfies the following assumptions :

(A.1). The function $\sigma \in C([0,T] \times H; L_Q(H_0; H))$.

(A.2). For all $t \in (0,T)$, there exists a positive constant K such that

$$|\sigma(t,\mathbf{u})|^2_{L_Q} \le K(1 + |\mathbf{u}|^2_H), \qquad \forall\, \mathbf{u} \in H.$$

(A.3). For all $t \in (0,T)$, there exists a positive constant L such that

$$|\sigma(t,\mathbf{u}) - \sigma(t,\mathbf{v})|^2_{L_Q} \le L|\mathbf{u} - \mathbf{v}|^2_H, \qquad \forall\, \mathbf{u},\mathbf{v} \in H.$$

By applying the Leray projection Π_H to each term of the stochastic Navier-Stokes system, and employing the result of Helmholtz that $L^2(G)$ admits an orthogonal decomposition into divergence free and irrotational components,

$$L^2(G) = H + H^{\perp}, \tag{6.8}$$

where the irrotational component can be characterized by

$$H^{\perp} = \left\{ g \in L^2(G) : g = \nabla h, h \in W^{1,2}(G) \right\}. \tag{6.9}$$

The system (6.1) can be written as the abstract evolution form

$$d\mathbf{u} + [\nu_0 \mathbf{A}\mathbf{u} + \nu_1 \mathcal{A}_i(\mathbf{u}) + \mathbf{B}(\mathbf{u})]dt = \mathbf{f}(t)dt + \sigma(t,\mathbf{u})dW \tag{6.10}$$

If we replace the noise coefficient σ in the (6.10) by $\sqrt{\varepsilon}\sigma$ for $\varepsilon > 0$, then the resulting solution is denoted by $\mathbf{u}^{\varepsilon}$. Our main goal is to establish the Laplace principle for the family $\{\mathbf{u}^{\varepsilon}\}$.

Let us consider the Navier-Stokes equations with small noise diffusions

$$\begin{aligned} &d\mathbf{u}^{\varepsilon} + [\nu_0 \mathbf{A}\mathbf{u}^{\varepsilon} + \nu_1 \mathcal{A}_i(\mathbf{u}^{\varepsilon}) + \mathbf{B}(\mathbf{u}^{\varepsilon})]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t,\mathbf{u}^{\varepsilon})dW_t, \\ &\mathbf{u}^{\varepsilon}(0) = \xi \in H. \end{aligned} \tag{6.11}$$

Monotonicity method can be employed to show that there exists a strong solution of Eq.(6.11) with values in the Polish space $C([0,T];H) \cap L^q(0,T;V_{r,q})$, and it is pathwise unique. It follows that there exists a Borel-measurable function

$$g^{\varepsilon} : C([0,T];H) \to C([0,T];H) \cap L^q(0,T;V_{r,q})$$

such that $\mathbf{u}^{\varepsilon}(\cdot) = g^{\varepsilon}(W(\cdot))$ a.s.

Our aim is to verify that the family $\{g^{\varepsilon}\}$ satisfies the Hypothesis H in order to obtain the Laplace principle for $\{\mathbf{u}^{\varepsilon} : \varepsilon > 0\}$ in $C([0,T];H) \cap L^q(0,T;V_{r,q})$.

In the following Lemma and its corollary, we show certain results which help to prove the last two main Propositions on the compactness of the level sets and weak convergence as stated in the Hypothesis H.

Lemma 6.1. *Let $\{g^{\varepsilon}\}$ be defined as above. For any $\mathbf{v} \in \mathcal{P}_M$ where $0 < M < \infty$, let $g^{\varepsilon}(W(\cdot) + \frac{1}{\sqrt{\varepsilon}}\int_0^{\cdot} \mathbf{v}(s)ds)$ be denoted by $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$, then $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$ is the unique strong solution of the equation*

$$d\mathbf{u}^{\varepsilon}_{\mathbf{v}} + [\nu_0 \mathbf{A}\mathbf{u}^{\varepsilon}_{\mathbf{v}} + \nu_1 \mathcal{A}_i(\mathbf{u}^{\varepsilon}_{\mathbf{v}}) + \mathbf{B}(\mathbf{u}^{\varepsilon}_{\mathbf{v}})]dt = [\mathbf{f}(t) + \sigma(t,\mathbf{u}^{\varepsilon}_{\mathbf{v}})\mathbf{v}]dt + \sqrt{\varepsilon}\sigma(t,\mathbf{u}^{\varepsilon}_{\mathbf{v}})dW_t \tag{6.12}$$

with $\mathbf{u}^{\varepsilon}_{\mathbf{v}}(0) = \xi \in H$.

Proof. Since $\mathbf{v} \in \mathcal{P}_M$, $\int_0^T |\mathbf{v}(s)|_0^2 ds < M$ a.s., and $\tilde{W}(\cdot) := W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \mathbf{v}(s) ds$ is a Wiener process with covariance form Q under the probability measure

$$d\tilde{P}^{\varepsilon}_{\mathbf{v}} := \exp\left\{ -\frac{1}{\sqrt{\varepsilon}} \int_0^T \mathbf{v}(s) dW(s) - \frac{1}{2\varepsilon} \int_0^T |\mathbf{v}(s)|_0^2 ds \right\} dP. \tag{6.13}$$

Applying the Girsanov argument: let $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$ be the unique solution of Eq. (6.11) on $(\Omega, \mathcal{F}, \tilde{P}^{\varepsilon}_{\mathbf{v}})$ with $\tilde{W}$ in place of W. Then $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$ solves Eq. (6.12) P-a.s., and $\mathbf{u}^{\varepsilon}_{\mathbf{v}}(\cdot) = g^{\varepsilon}(\tilde{W}(\cdot))$.

If $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$ and $\mathbf{w}$ are solutions of Eq. (6.12) on $(\Omega, \mathcal{F}, P)$, then $\mathbf{u}^{\varepsilon}_{\mathbf{v}}$ and $\mathbf{w}$ would solve Eq. (6.11) on $(\Omega, \mathcal{F}, \tilde{P}^{\varepsilon}_{\mathbf{v}})$ with $\tilde{W}$ in place of W. Thus $\mathbf{u}^{\varepsilon}_{\mathbf{v}} = \mathbf{w}$ $\tilde{P}^{\varepsilon}_{\mathbf{v}}$-a.s. so that $\mathbf{u}^{\varepsilon}_{\mathbf{v}} = \mathbf{w}$ P-a.s., and hence uniqueness of solutions to Eq. (6.12) is obtained. □

Corollary 6.2. *Let* $\mathbf{v} \in L^2(0, T; H_0)$ *and* $\mathbf{f} \in L^4(0, T; H)$ *and* σ *satisfies (A.1)-(A.3). Then the equation*

$$\begin{aligned} &d\mathbf{u}_{\mathbf{v}} + [\nu_0 \mathbf{A}\mathbf{u}_{\mathbf{v}} + \nu_1 \mathcal{A}_i(\mathbf{u}_{\mathbf{v}}) + \mathbf{B}(\mathbf{u}_{\mathbf{v}})]dt = [\mathbf{f}(t) + \sigma(t, \mathbf{u}_{\mathbf{v}})\mathbf{v}]dt, \\ &\mathbf{u}_{\mathbf{v}}(0) = \xi \in H \end{aligned} \tag{6.14}$$

has a unique strong solution in $C([0, T]; H) \cap L^q(0, T; V_{r,q})$.

Proof. This result can be considered as a particular case of the previous Lemma, where the diffusion coefficient is absent. □

Now we are ready to verify the Hypothesis H.

Proposition 6.3. *(Compactness) Let* $M < \infty$ *be any fixed positive number. Let*

$$K_M := \{\mathbf{u}_{\mathbf{v}} \in C([0, T]; H) \cap L^q(0, T; V_{r,q}) : \mathbf{v} \in S_M\}, \tag{6.15}$$

where $\mathbf{u}_{\mathbf{v}}$ *is the unique solution of* (6.14) *in* $C([0, T]; H) \cap L^q(0, T; V_{r,q})$. *Then* K_M *is compact in* $C([0, T]; H) \cap L^q(0, T; V_{r,q})$.

Proof. We will outline the ideas of the proof since a full proof is routine and rather long. Let $\{\mathbf{u}_n\}$ be a sequence in K_M, where $\mathbf{u}_n$ corresponds to the solution of (6.14) with $\mathbf{v}_n$ in place of $\mathbf{v}$. By the weak compactness of S_M, there exists a subsequence of $\{\mathbf{v}_n\}$ which converges to a limit $\mathbf{v}$ weakly in $L^2(0, T; H_0)$. For ease of notation, the subsequence is still indexed by n, and the solution $\mathbf{u}_{\mathbf{v}}$ of (6.14) is denoted as $\mathbf{u}$ in the context of this proof.

We need to show $\mathbf{u}_n \to \mathbf{u}$ in $C([0, T]; H) \cap L^q(0, T; V_{r,q})$ as $n \to \infty$, i.e.,

$$\sup_{0 \le t \le T} |\mathbf{u}_n(t) - \mathbf{u}(t)| + \int_0^T ||\mathbf{u}_n(t) - \mathbf{u}(t)||^q_{V_{r,q}} dt \to 0. \tag{6.16}$$

Using the definitions of $\mathcal{A}_i$, the following energy estimate, and a Gronwall argument, the proof is completed. □

The following proposition verifies the first condition in Hypothesis H.

Proposition 6.4. *(Weak Convergence) Let* $\{\mathbf{v}_{\epsilon} : \epsilon > 0\} \subset \mathcal{P}_M$, *for some* $M < \infty$, *converge in distribution as* S_M*-valued random elements to* $\mathbf{v}$. *Then the process* $g^{\epsilon}(W(\cdot) + \frac{1}{\sqrt{\epsilon}} \int_0^{\cdot} \mathbf{v}_{\epsilon}(s)\, ds)$ *converges in distribution to* $g^0(\int_0^{\cdot} \mathbf{v}(s)\, ds)$.

Proof. Let $\mathbf{v}_\varepsilon$ converge to $\mathbf{v}$ in distribution as random elements taking values in S_M where S_M is equipped with the weak topology. Let $\mathbf{u}_\varepsilon$ solve

$$\begin{aligned} &d\mathbf{u}_\varepsilon + [\nu_0 \mathbf{A}\mathbf{u}_\varepsilon + \nu_1 \mathcal{A}(\mathbf{u}_\varepsilon) + \mathbf{B}(\mathbf{u}_\varepsilon)]dt \\ &= [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\varepsilon(t))\mathbf{v}_\varepsilon(t)]dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}_\varepsilon)dW_t \end{aligned} \tag{6.17}$$

with $\mathbf{u}_\varepsilon(0) = \xi \in H$.

Let $\mathbf{u}_\mathbf{v}$ be the solution of

$$d\mathbf{u}_\mathbf{v} + [\nu_0 \mathbf{A}\mathbf{u}_\mathbf{v} + \nu_1 \mathcal{A}(\mathbf{u}_\mathbf{v}) + \mathbf{B}(\mathbf{u}_\mathbf{v})]dt = [\mathbf{f}(t) + \sigma(t, \mathbf{u}_\mathbf{v}(t))\mathbf{v}(t)]dt \tag{6.18}$$

with $\mathbf{u}_\mathbf{v}(0) = \xi \in H$. Since pathwise unique strong solutions exist for the above equations, the Borel measurable function g^ε mentioned earlier satisfies the equality

$$g^\varepsilon(W(\cdot) + \frac{1}{\sqrt{\varepsilon}} \int_0^\cdot \mathbf{v}_\varepsilon(s)ds) = \mathbf{u}_\varepsilon.$$

For all $\mathbf{v} \in L^2([0,T]; H_0)$, note that $\int_0^\cdot \mathbf{v}(s)ds \in C([0,T]; H_0)$. Define $g^0 : C([0,T]; H_0) \to C([0,T]; H) \cap L^2(0,T; V)$ by

$$g^0(h) = \mathbf{u}_\mathbf{v}, \text{if } \ h = \int_0^\cdot \mathbf{v}(s)ds \quad \text{for some} \quad \mathbf{v} \in L^2([0,T]; H_0).$$

If h cannot be represented as above, then define $g^0(h) = 0$.

Since S_M endowed with the weak topology is Polish, the Skorokhod representation theorem can be invoked to construct processes $(\tilde{\mathbf{v}}_\varepsilon, \tilde{\mathbf{v}}, \tilde{W}_\varepsilon)$ such that the joint distribution of $(\tilde{\mathbf{v}}_\varepsilon, \tilde{W}_\varepsilon)$ is the same as that of $(\mathbf{v}_\varepsilon, W)$, and the distribution of $\tilde{\mathbf{v}}$ coincides with that of $\mathbf{v}$, and $\tilde{\mathbf{v}}_\varepsilon \to \tilde{\mathbf{v}}$ a.s. in the topology.

Define $\mathbf{w}(t) := \mathbf{u}_\varepsilon(t) - \mathbf{u}_\mathbf{v}(t)$. The notation $|\cdot|_{HS}$ will denote the Hilbert-Schmidt norm in what follows. Define the stopping time

$$\begin{aligned} \tau_{N,\varepsilon} \quad := \quad & T \wedge \inf\{t : \int_0^t \left\{ ||\mathbf{u}_\mathbf{v}(s)||^2 + |\mathbf{u}_\varepsilon(s)|^2 \right\} ds > N \\ & \text{or} \sup_{0 \le s \le t} |\mathbf{u}_\mathbf{v}(s)|^2 > N \text{or} \sup_{0 \le s \le t} |\mathbf{u}_\varepsilon(s)|^2 > N\}. \end{aligned} \tag{6.19}$$

Using a Gronwall argument, we obtain

$$\begin{aligned} &\frac{1}{2}(\sup_{0 \le t \le T \wedge \tau_{N,\varepsilon}} |\mathbf{w}(t)|^2) + \nu_1 \int_0^t ||\mathbf{w}(s)||^q_{V_{r,q}} ds \\ &\le \{\frac{1}{\nu_0} \int_0^{T \wedge \tau_{N,\varepsilon}} |\sigma(s, \mathbf{u}_\mathbf{v}(s))(\mathbf{v}_\varepsilon(s) - \mathbf{v}(s))|^2 ds \\ &\quad + \frac{\varepsilon}{2} K \int_0^{T \wedge \tau_{N,\varepsilon}} (1 + |\mathbf{u}_\varepsilon(s)|^2) ds \\ &\quad + \sqrt{\varepsilon}(\sup_{0 \le t \le T \wedge \tau_{N,\varepsilon}} | \int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s)) dW(s))|)\} e^{\frac{N}{\nu_0} + LM}. \end{aligned} \tag{6.20}$$

Let N be fixed, then for suitable constant C,

$$\liminf_{\varepsilon\to 0} P\left\{\tau_{N,\varepsilon} = T\right\} \geq 1 - \frac{C}{N}. \tag{6.21}$$

One can easily show that

$$\sqrt{\varepsilon} \sup_{0\leq t\leq T\wedge\tau N,\varepsilon} \left| \int_0^t (\mathbf{w}(s), \sigma(s, \mathbf{u}_\varepsilon(s) dW(s))) \right| \to 0 \quad \text{in probability}$$

as ε tends to zero. These two observations along with the weak convergence of $\mathbf{v}_\varepsilon \to \mathbf{v}$ in S_M, we obtain that

$$\frac{1}{2}(\sup_{0\leq t\leq T_0\wedge\tau_{N,\varepsilon}} |\mathbf{w}(t)|^2) + \nu_1 \int_0^t ||\mathbf{w}(s)||^q_{V_{r,q}} ds \to 0 \quad \text{in probability} \tag{6.22}$$

as $\varepsilon \to 0$, which completes the proof. □

The above two propositions 6.3 and 6.4 show that the family $\{g^\varepsilon\}$ satisfies the Hypothesis H, so that the Laplace principle is obtained for $\{\mathbf{u}^\varepsilon : \varepsilon > 0\}$ in $C([0,T];H) \cap L^q(0,T;V_{r,q})$.

Theorem 6.5. *Let* $\{\mathbf{u}^\varepsilon(\cdot)\}$ *be the solution of the equation*

$$\begin{aligned} &d\mathbf{u}^\varepsilon + [\nu_0 \mathbf{A}\mathbf{u}^\varepsilon + \nu_1 \mathcal{A}_i(\mathbf{u}^\varepsilon) + \mathbf{B}(\mathbf{u}^\varepsilon)]dt = \mathbf{f}(t)dt + \sqrt{\varepsilon}\sigma(t, \mathbf{u}^\varepsilon)dW_t \\ &\mathbf{u}^\varepsilon(0) = \xi \in H. \end{aligned} \tag{6.23}$$

Then $\{\mathbf{u}^\varepsilon\}$ *satisfies the Laplace principle in* $C([0,T];H) \cap L^q(0,T;V_{r,q})$ *with good rate function*

$$I(f) = \inf_{\left\{v\in L^2([0,T];H_0): f = g^0(\int_0^\cdot v(s)ds)\right\}} \left\{ \frac{1}{2} \int_0^T |v(s)|_0^2 ds \right\} \tag{6.24}$$

with the convention that the infimum of an empty set is infinity.

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