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BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS WITH RESPECT TO GENERAL FILTRATIONS AND APPLICATIONS TO INSIDER FINANCE

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Abstract. In this paper, we study backward stochastic differential equations (BSDEs) with respect to general filtrations. We prove existence and uniqueness theorems for such BSDEs and we establish a comparison theorem. Reflected BSDEs with general filtration are also studied. The results are used to find the optimal consumption rate for an insider from a cash flow modeled as a generalized geometric Itô-Lévy process.

1. Introduction

The classical backward stochastic differential equation (BSDE) consists in finding a pair (Y_t, Z_t) of $\mathcal{F}_t$ -adapted processes such that

$$\begin{cases} dY_t &= -f(t, Y_t, Z_t)dt + Z_t dB_t; \quad t \in [0, T] \\ Y_T &= \xi. \end{cases} \tag{1.1}$$

where B_t is a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, ξ is a given $\mathcal{F}_T$ -measurable random variable and $f : [0, T] \times R \times R \to R$ is a given function.

If $f(t, y, z) = f(t, y)$ does not depend on z, then an equivalent way of writing (1.1) is

$$Y_t = E\big[\xi + \int_t^T f(s, Y_s)ds|\mathcal{F}_t]; \quad t \in [0, T]. \tag{1.2}$$

In this paper we extend (1.2) to a general filtration $\mathcal{H}_t$ and consider the problem to find an $\mathcal{H}_t$-adapted process Y_t such that

$$Y_t = E\big[\xi + \int_t^T f(s, Y_s)ds|\mathcal{H}_t]; \quad t \in [0, T], \tag{1.3}$$

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where ξ now is a given $\mathcal{H}_T$-measurable random variable. Thus we arrive at a BSDE based on a general filtration $\mathcal{H}_t$, not necessarily the filtration $\mathcal{F}_t$ of Brownian motion. This turns out to be a useful generalization for certain applications, for example in connection with insider trading in finance, which was the main motivation for this study. (See Section 5)

Here is an outline of the paper. In Section 2 we give a more detailed presentation of our BSDE based on a given filtration. In Section 3 we prove existence and uniqueness of solutions of such equations. In Section 4 we study reflected BSDEs based on a given filtration. We prove existence and uniqueness of solution and we show that it coincides with the solution of an optimal stopping problem (for $\mathcal{H}$-stopping times). In Section 5 we give an application to finance. We show that the optimal consumption problem for an insider can be transformed into a BSDE with respect to the information filtration $\mathcal{H}_t$ of the insider. Then we apply results from previous sections to find the optimal consumption rate explicitly.

2. Statement of the Problem

Let $(\Omega, \mathcal{H}, \mathcal{H}_t, P)$ be a complete filtrated probability space with a right continuous filtration $\{\mathcal{H}_t, t \geq 0\}$. Let $T > 0$ and let ξ be an $\mathcal{H}_T$ measurable random variable with $E[|\xi|] < \infty$, where E denotes expectation with respect to P. Let $f(\omega, t, y) : \Omega \times [0, T] \times R^d \to R^d$ be a given $\mathcal{P} \times \mathcal{B}(R^d)$-measurable function, where $\mathcal{P}$ is the predictable σ-field associated with the filtration $\{\mathcal{H}_t, t \geq 0\}$. Consider the following backward stochastic differential equation (BSDE):

BSDE(1): Find an $\mathcal{H}_t$- optional process Y_t such that

$$E\big[\int_0^T |f(s, Y_s)|ds\big] < \infty. \tag{2.1}$$

and

$$Y_t = E\big[\xi + \int_t^T f(s, Y_s)ds|\mathcal{H}_t]; \quad t \in [0, T]. \tag{2.2}$$

Next, consider the following BSDE:

BSDE(2): Find an $\mathcal{H}_t$- optional process Y_t and an $\mathcal{H}_t$-local martingale M_t such that $M_0 = 0$ and

$$\begin{cases} dY_t &= -f(t, Y_t)dt + dM_t \\ Y_T &= \xi. \end{cases} \tag{2.3}$$

An equivalent formulation to (2.3) is that

$$\int_0^T |f(s, Y_s)|ds < \infty \quad a.s. \tag{2.4}$$

and

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - (M_T - M_t); \quad t \in [0, T]. \tag{2.5}$$

There is a close relation between BSDE(1) and BSDE(2): First note that if Y_t satisfies BSDE(1), then we can define

$$M_t = E[\xi + \int_0^T f(s, Y_s)ds|\mathcal{H}_t]$$

and we see from (2.2) that

$$\begin{aligned} Y_t &= E[\xi + \int_0^T f(s, Y_s)ds - \int_0^t f(s, Y_s)ds|\mathcal{H}_t] \\ &= -\int_0^t f(s, Y_s)ds + M_t. \end{aligned}$$

Moreover, $Y_T = \xi$. Hence (Y_t, M_t) satisfies BSDE(2).

Conversely, if (Y_t, M_t) satisfies (2.5) and M_t is an $\mathcal{H}_t$-martingale, then (2.2) follows by taking conditional expectation of (2.5) with respect to $\mathcal{H}_t$. Hence Y_t satisfies BSDE(1).

We now proceed to study BSDE(2).

Definition 2.1. We say that a pair $(Y_t, M_t, t \geq 0)$ is a *solution* to BSDE(2) if

(i). Y_t is an $\mathcal{H}_t$-optional, R^d-valued process.

(ii). $M_t, t \geq 0$ is a càdlàg R^d-valued $\mathcal{H}_t$-martingale.

(iii). For every $t \geq 0$,

$$Y_t = \xi + \int_t^T f(s, Y_s)ds - (M_T - M_t) \tag{2.6}$$

P-almost surely.

Remark 2.2. Our BSDE solution concept may be regarded as a generalization, both with respect to filtration and to jumps, of the weak BSDE solution concept introduced and studied in [2][3][4][5]. Our methods of proofs are similar to methods found in e.g. [2][9][10][15], but adapted to our more general situation.

3. Backward Stochastic Differential Equations

3.1. Existence and uniqueness.

Theorem 3.1. *Suppose $\xi \in L^2(\Omega)$ and $E[\int_0^T |f(t, 0)|^2 dt] < \infty$. Assume that f is uniformly Lipschitz with respect to y, i.e., there exists a constant C such that*

$$|f(t, y_1) - f(t, y_2)| \leq C|y_1 - y_2| \tag{3.1}$$

Then there exists a unique pair (Y, M) such that (Y, M) is a solution to the BSDE(2) and

$$E[\sup_{0 \leq t \leq T} |Y_t|^2] < \infty. \tag{3.2}$$

Proof. Let B denote the Banach space of R^d-valued, $\mathcal{H}_t$-optional processes X such that

$$||X||_B := \sup_{0 \leq t \leq T} (E[X_t^2])^{\frac{1}{2}} < \infty.$$

Define recursively a sequence $Y_t^n, t \geq 0$ of processes in B by $Y^0 = 0$ and

$$Y_t^{n+1} = E[\xi + \int_t^T f(s, Y_s^n)ds \big| \mathcal{H}_t] \tag{3.3}$$

It is easy to see that $Y^n \in B$ for all $n \geq 1$. Moreover,

$$\begin{aligned} E\big[|Y_t^{n+1} - Y_t^n|^2\big] &\leq TE\big[\int_t^T |f(s, Y_s^n) - f(s, Y_s^{n-1})|^2 ds\big] \\ &\leq CT \int_t^T E[|Y_s^n - Y_s^{n-1}|^2]ds \end{aligned} \tag{3.4}$$

Set $\phi_n(t) = E[|Y_t^n - Y_t^{n-1}|^2]$. Then (3.4) becomes

$$\phi_{n+1}(t) \le CT \int_t^T \phi_n(s)ds \tag{3.5}$$

Repeating the above inequality, we get

$$\sup_{0\le t\le T} \phi_{n+1}(t) \le \big(\sup_{0\le s\le T} \phi_1(s)\big)\frac{(CT)^n T^n}{n!} \tag{3.6}$$

This implies that $Y^n, n \ge 1$ is a Cauchy sequence in B. Denote the limit of Y^n by $\hat{Y}$. Letting $n \to \infty$ in (3.3) we obtain

$$\hat{Y}_t = E[\xi + \int_t^T f(s, \hat{Y}_s)ds\big|\mathcal{H}_t] \tag{3.7}$$

Next we show that $\hat{Y}_t, t \ge 0$ admits a right continuous version which will be the solution to BSDE(2). Let $M_t, t \ge 0$ be the right continuous version of the square integrable martingale $E[\xi + \int_0^T f(s, \hat{Y}_s)ds\big|\mathcal{H}_t]$. Put

$$Y_t = M_t - \int_0^t f(s, \hat{Y}_s)ds, t \ge 0$$

Then Y_t is right continuous and for every $t \ge 0$,

$$Y_t = E[\xi + \int_t^T f(s, \hat{Y}_s)ds\big|\mathcal{H}_t] = \hat{Y}_t$$

P-almost surely. By the Fubini theorem, it follows that

$$\begin{aligned} Y_t &= M_t - M_T + \xi + \int_0^T f(s, \hat{Y}_s)ds - \int_0^t f(s, \hat{Y}_s)ds \\ &= \xi + \int_t^T f(s, \hat{Y}_s)ds - (M_T - M_t) \\ &= \xi + \int_t^T f(s, Y_s)ds - (M_T - M_t) \end{aligned} \tag{3.8}$$

P-almost surely. This shows that (Y, M) is a solution to the BSDE(2). Let us now prove (3.2). Using Doob's inequality, we have

$$\begin{aligned} E[\sup_{0\le t\le T} |Y_t|^2] &\le 2E[\sup_{0\le t\le T} |M_t|^2] + 2TE[\int_0^T |f(s, Y_s)|^2 ds] \\ &\le C_2 E[|M_T|^2] + 4TE[\int_0^T |f(s,0)|^2 ds] + 4T\int_0^T E[|Y_s|^2]ds \\ &= C_2 E[|\xi + \int_0^T f(s, Y_s)ds|^2] \\ &\quad +4TE[\int_0^T |f(s,0)|^2 ds] + C_3 4T\int_0^T E[|Y_s|^2]ds \\ &\le C(E[|\xi|^2] + \sup_{0\le t\le T} E[|Y_t|^2] + E[\int_0^T |f(s,0)|^2 ds] < \infty. \end{aligned} \tag{3.9}$$

It remains to prove the uniqueness. Let (X, Z) be another solution to equation BSDE(2). Then both Y and X satisfy

$$Y_t = E[\xi + \int_t^T f(s, Y_s)ds \big| \mathcal{H}_t] \tag{3.10}$$

$$X_t = E[\xi + \int_t^T f(s, X_s)ds \big| \mathcal{H}_t] \tag{3.11}$$

Using the Lipschitz continuity of f, as the proof of (3.4), we have

$$E\big[|Y_t - X_t|^2\big] \le CT \int_t^T E[|Y_s - X_s|^2]ds \tag{3.12}$$

By Gronwall's inequality, it follows that $Y_t = X_t$, which in turn gives $M_t = Z_t$. The proof is complete. □

Next theorem states a result on existence and uniqueness under some monotone conditions on the coefficients.

Theorem 3.2. *Suppose*

1. $\xi \in L^2(\Omega)$ *and* $E[\int_0^T |f(t,0)|^2 dt] < \infty$.
2. *There exists a constant* C *such that*

$$(y_1 - y_2)(f(t, y_1) - f(t, y_2)) \le C|y_1 - y_2|^2 \tag{3.13}$$

3. $f(t, y)$ *is continuous in* y *and*

$$|f(t, y)| \le C_1(t), \tag{3.14}$$

with $E[\int_0^T C_1(s)ds] < \infty$.

Then there exists a unique solution (Y, M) *to the BSDE(2) satisfying*

$$E[\sup_{0 \le t \le T} |Y_t|^2] < \infty. \tag{3.15}$$

Proof. Take an even, non-negative function $\phi \in C_0^\infty(R)$ with $\int_R \phi(x)dx = 1$. Define

$$f_n(t, y) = \int_R f(t, z)\phi_n(y - z)dz,$$

where $\phi_n(z) = n\phi(nz)$. Since f is continuous in y, it is easy to see that $f_n(t, y) \to f(t, y)$ as $n \to \infty$. Furthermore, for every $n \ge 1$,

$$|f_n(t, y_1) - f_n(t, y_2)| \le C_n|y_1 - y_2|, \tag{3.16}$$

for some constant C_n. Consider the BSDE:

$$Y_t^n = \xi + \int_t^T f_n(s, Y_s^n)ds + M_T^n - M_t^n; \quad t \in [0, T]. \tag{3.17}$$

Equation (3.17) has a unique solution (Y^n, M^n) according to Theorem 2.1. Next we show that Y_t^n is a Cauchy sequence. By Itô's formula, we have

$$\begin{aligned} & |Y_t^n - Y_t^m|^2 + [Y^n - Y^m, Y^n - Y^m]_T - [Y^n - Y^m, Y^n - Y^m]_t \\ = \;& 2\int_t^T (Y_s^n - Y_s^m)(f_n(s, Y_s^n) - f_m(s, Y_s^m))ds \\ & -2\int_t^T (Y_{s-}^n - Y_{s-}^m)d(M_s^n - M_s^m) \end{aligned} \tag{3.18}$$

In view of (3.13), (3.14),

$$
\begin{aligned}
&(Y_s^n - Y_s^m)(f_n(s, Y_s^n) - f_m(s, Y_s^m)) \\
&= \int_R (Y_s^n - Y_s^m)(f(s, Y_s^n - \frac{1}{n}z) - f(s, Y_s^m - \frac{1}{m}z))\phi(z)dz \\
&= \int_R [(Y_s^n - \frac{1}{n}z) - (Y_s^m - \frac{1}{m}z)](f(s, Y_s^n - \frac{1}{n}z) - f(s, Y_s^m - \frac{1}{m}z))\phi(z)dz \\
&\quad + \int_R (\frac{1}{n}z - \frac{1}{m}z))(f(s, Y_s^n - \frac{1}{n}z) - f(s, Y_s^m - \frac{1}{m}z))\phi(z)dz \\
&\leq C \int_R ((Y_s^n - \frac{1}{n}z) - (Y_s^m - \frac{1}{m}z))^2\phi(z)dz + C_1(s)\int_R (\frac{1}{n}|z| + \frac{1}{m}|z|)\phi(z)dz \\
&\leq C(Y_s^n - Y_s^m)^2 + C\int_R (\frac{1}{n^2} + \frac{1}{m^2})z^2\phi(z)dz + C_1(s)\int_R (\frac{1}{n}|z| + \frac{1}{m}|z|)\phi(z)dz
\end{aligned}
\tag{3.19}
$$

Substitute (3.19) into (3.18), take expectation to obtain

$$
\begin{aligned}
&E[|Y_t^n - Y_t^m|^2] + E\{[Y^n - Y^m, Y^n - Y^m]_T - [Y^n - Y^m, Y^n - Y^m]_t\} \\
&\leq \quad C\int_t^T E[(Y_s^n - Y_s^m)^2]ds + CT\int_R (\frac{1}{n^2} + \frac{1}{m^2})z^2\phi(z)dz \\
&\qquad + CE[\int_t^T C_1(s)ds]\int_R (\frac{1}{n}|z| + \frac{1}{m}|z|)\phi(z)dz
\end{aligned}
\tag{3.20}
$$

Applying Gronwall's inequality, it follows from (3.20) that

$$
\begin{aligned}
&E[|Y_t^n - Y_t^m|^2] \\
&\leq C_T\{\int_R (\frac{1}{n^2} + \frac{1}{m^2})z^2\phi(z)dz + E[\int_t^T C_1(s)ds]\int_R (\frac{1}{n}|z| + \frac{1}{m}|z|)\phi(z)dz\}
\end{aligned}
\tag{3.21}
$$

Hence,

$$
\lim_{n,m\to\infty} \sup_{0\leq t\leq T} E[|Y_t^n - Y_t^m|^2] = 0 \tag{3.22}
$$

By (3.20) and the Burkholder inequality, (3.22) further implies

$$
\begin{aligned}
&\lim_{n,m\to\infty} E[\sup_{0\leq t\leq T} |M_t^n - M_t^m|^2] \\
&\leq \quad \lim_{n,m\to\infty} E([M^n - M^m]_T) \\
&= \quad \lim_{n,m\to\infty} E([Y^n - Y^m]_T) = 0.
\end{aligned}
\tag{3.23}
$$

Consequently, there exist a square integrable, predictable process Y_t and a square integrable, right continuous martingale M_t such that

$$
\lim_{n\to\infty} \sup_{0\leq t\leq T} E[|Y_t^n - Y_t|^2] = 0 \tag{3.24}
$$

$$
\lim_{n\to\infty} E[\sup_{0\leq t\leq T} |M_t^n - M_t|^2] = 0 \tag{3.25}
$$

In view of (3.14), use the dominated convergence theorem and let $n \to \infty$ in (3.17) to get

$$
Y_t = \xi + \int_t^T f(s, Y_s)ds + M_T - M_t; \quad t \in [0, T]. \tag{3.26}
$$

Since the right hand side of (3.26) is right continuous, we can take Y to be right continuous. Thus $Y_t, t \geq 0$ is a solution to BSDE(2).

Now we prove the uniqueness. Suppose that (Y^1, M^1) and (Y^2, M^2) are two solutions to BSDE(2). Similar to the calculations for (3.18), we have

$$\begin{aligned}&|Y_t^1 - Y_t^2|^2 + [M^1 - M^2, M^1 - M^2]_T - [M^1 - M^2, M^1 - M^2]_t \\ &= 2\int_t^T (Y_s^1 - Y_s^2)(f(s, Y_s^1) - f(s, Y_s^2))ds - 2\int_t^T (Y_{s-}^1 - Y_{s-}^2)d(M_s^1 - M_s^2)\end{aligned} \tag{3.27}$$

Taking expectation and keeping (3.13) in mind, we get from (3.27) that

$$\begin{aligned}&E\{|Y_t^1 - Y_t^2|^2 + [M^1 - M^2, M^1 - M^2]_T - [M^1 - M^2, M^1 - M^2]_t\} \\ &\leq CE[\int_t^T (Y_s^1 - Y_s^2)^2 ds]\end{aligned}$$

By Gronwall's inequality, we deduce that $Y_t^1 = Y_t^2, M_t^1 = M_t^2$ for $t \geq 0$, thereby completing the proof. □

3.2. Comparison theorem. Let (Y, M) be the solution to the following linear BSDE:

$$Y_t = \xi + (\phi_T - \phi_t) + \int_t^T \beta_s Y_s ds - (M_T - M_t), \tag{3.28}$$

where $\phi_t, t \geq 0$ is a given, right continuous process of bounded variation with $\phi_0 = 0$ and β_t is a bounded predictable process. We have the following result.

Theorem 3.3. *Assume the total variation of ϕ is integrable. The following representation holds*

$$Y_t = E[L_t^T \xi + \int_t^T L_t^s d\phi_s | \mathcal{H}_t], \tag{3.29}$$

where

$$L_t^s = \exp(\int_t^s \beta_u du)$$

In particular, if $\xi \geq 0$, then $Y_t \geq 0$. Moreover $Y_0 = 0$ implies $\xi = 0$ and $\phi = 0$.

Proof. Put $L_t = \exp(\int_0^t \beta_u du)$. By Itô's formula, we find that

$$Y_t L_t + \int_0^t L_s d\phi_s = Y_0 - \int_0^t L_s dM_s$$

is a martingale. Consequently,

$$\begin{aligned}Y_t L_t + \int_0^t L_s d\phi_s &= E[Y_T L_T + \int_0^T L_t^s d\phi_s | \mathcal{H}_t] \\ &= E[\xi L_T + \int_0^T L_t^s d\phi_s | \mathcal{H}_t].\end{aligned}$$

(3.29) follows. □

Let both $(\xi^1, f^1(s, y))$ and $(\xi^2, f^2(s, y))$ satisfy the conditions in Theorem 2.1. Denote by (Y^1, M^1) and (Y^2, M^2) the solutions of the BSDEs associated with $(\xi^1, f^1(s, y))$ and $(\xi^2, f^2(s, y))$, respectively.

Theorem 3.4. *Suppose $f^1(s,Y_s^2) \geq f^2(s,Y_s^2)$ almost surely on $\Omega \times [0,T]$ and $\xi^1 \geq \xi^2$. Then, $Y_t^1 \geq Y_t^2$ P-almost surely for all $t \geq 0$. Furthermore, if $Y_t^1 = Y_t^2$ P-almost surely on an event $A \in \mathcal{H}_t$, then $\xi^1 = \xi^2$ on A and $Y_s^1 = Y_s^2$ on A for $s \geq t$.*

Proof. Define

$$\beta_s = \begin{cases} \frac{f^1(s,Y_s^1)-f^1(s,Y_s^2)}{Y_s^1-Y_s^2} & \text{if } Y_s^1 \neq Y_s^2, \\ 0 & \text{otherwise.} \end{cases} \tag{3.30}$$

Then β_s is bounded. Moreover, we have

$$Y_t^1 - Y_t^2 = \xi^1 - \xi^2 + \int_t^T (f^1(s,Y_s^2) - f^2(s,Y_s^2))ds + \int_t^T \beta_s(Y_s^1 - Y_s^2)ds - [(M_T^1 - M_T^2) - (M_t^1 - M_t^2)] \tag{3.31}$$

Using Theorem 2.2, we have

$$Y_t^1 - Y_t^2 = E[L_t^T(\xi^1 - \xi^2) + \int_t^T L_t^s(f^1(s,Y_s^2) - f^2(s,Y_s^2))ds|\mathcal{H}_t] \tag{3.32}$$

(3.32) implies the desired results. □

As a corollary to Theorem 3.4, we have the following

Theorem 3.5. *If $f(t,0) \geq 0$ $dP \times dt$, then the solution $Y_t(\xi)$ gives rise a price system, that is,*

1. *At any time t, the price $Y_t(\xi)$ for a positive contingent claim ξ is positive.*
2. *At any time t, the price $Y_t(\xi)$ is an increasing function with respect to ξ.*
3. *No-arbitrage holds, i.e., if the prices Y_t^1 and Y_t^2 coincide on an event $A \in \mathcal{F}_t$, then on A, $\xi^1 = \xi^2$, a.s.*

4. Reflected Backward Stochastic Differential Equations

Consider the reflected backward stochastic differential equation:

$$dY_t = -f(t,Y_t)dt + dM_t - dK_t \tag{4.1}$$

Definition 4.1. Let $L_t; t \geq 0$ be a given càdlàg $\mathcal{H}_t$-adapted process. We say that $(Y_t, M_t, K_t, t \geq 0)$ is a solution to RBSDE(3.1) with lower barrier $L_t, t \geq 0$ if

(i). Y_t is an $\mathcal{H}_t$-adapted, càdlàg real-valued process,
(ii). $Y_t \geq L_t$ P-a.s. for every $t \geq 0$.
(iii). $M_t, t \geq 0$ is a càdlàg real-valued $\mathcal{H}_t$-martingale.
(iv). $K_t, t \geq 0$ is an increasing, continuous $\mathcal{H}_t$-adapted process with $K_0 = 0$.
(v). For every $t \geq 0$,

$$Y_t = \xi + \int_t^T f(s,Y_s)ds - (M_T - M_t) + K_T - K_t \quad P - \text{almostly surely.} \tag{4.2}$$

(vi). $\int_0^T (Y_t - L_t)dK_t = 0$.

In the following we let $\mathcal{T}_{t,T}^{\mathcal{H}}$ denote the set of $\mathcal{H}$-stopping times τ such that $t \leq \tau \leq T$ a.s. By combining the arguments of [9][10] we get the following result.

Theorem 4.2. *Let $f(t,y)$ and ξ be as in Theorem 3.1. Assume $\xi \geq L_T$ and one of the following conditions hold:*

(i). L_t is a right continuous, increasing, square integrable predictable process with $E[L_T^2] < \infty$.

(ii).L_t is absolutely continuous and $E[\int_0^T (L_t')^2 dt] < \infty$.

Then :

a) The RBSDE(4.1) admits a unique solution.

b) The solution process Y_t can be given the optimal stopping representation

$$Y_t = esssup_{\tau \in \mathcal{T}_{t,T}^{\mathcal{H}}} E[\int_t^\tau f(s, Y_s)ds + L_\tau \chi_{\tau < T} + \xi \chi_{\tau = T} | \mathcal{H}_t]; t \in [0, T] \tag{4.3}$$

c) The solution process K_t is given by

$$K_T - K_{T-t} = \max_{s \leq t} (\xi + \int_{T-s}^T f(u, Y_u)du - (M_T - M_{T-s}) - L_{T-s})^-; t \in [0, T] \tag{4.4}$$

where $x^- = \max(-x, 0)$.

Proof. a) We first prove the uniqueness. Suppose (Y_t^1, M_t^1, K_t^1) and (Y_t^2, M_t^2, K_t^2) are two solutions to the RBSDE(2). By Itô's formula, we have

$$\begin{aligned} & |Y_t^1 - Y_t^2|^2 + [Y^1 - Y^2, Y^1 - Y^2]_T - [Y^1 - Y^2, Y^1 - Y^2]_t \\ = \;& 2\int_t^T (Y_s^1 - Y_s^2)(f(s, Y_s^1) - f(s, Y_s^2))ds - 2\int_t^T (Y_{s-}^1 - Y_{s-}^2) d(M_s^1 - M_s^2) \\ & +2\int_t^T (Y_s^1 - Y_s^2) d(K_s^1 - K_s^2) \end{aligned} \tag{4.5}$$

Take expectation in the above equation, use (ii), (vi) in Definition 4.1 to obtain

$$\begin{aligned} & E[|Y_t^1 - Y_t^2|^2] + E\{[Y^1 - Y^2, Y^1 - Y^2]_T - [Y^1 - Y^2, Y^1 - Y^2]_t\} \\ \leq \;& C\int_t^T E[(Y_s^1 - Y_s^2)^2]ds - 2E[\int_t^T (Y_s^2 - L_s)dK_s^1] \\ & -2E[\int_t^T (Y_s^1 - L_s)dK_s^2] \\ \leq \;& C\int_t^T E[(Y_s^1 - Y_s^2)^2]ds \end{aligned} \tag{4.6}$$

(4.6) and Gronwall's inequality implies that $E[|Y_t^1 - Y_t^2|^2] = 0$ for $t \geq 0$, proving the uniqueness.

To prove the existence, we will use the penalization method. For $n \geq 1$, consider the penalized backward stochastic differential equation:

$$Y_t^n = \xi + \int_t^T f(s, Y_s^n)ds - (M_T^n - M_t^n) + n\int_t^T (Y_s^n - L_s)^- ds \tag{4.7}$$

Equation (4.7) admits a unique solution according to Theorem 2.1. By the comparison Theorem 2.4, we know that the sequence $Y^n, n \geq 1$ is increasing, i.e., $Y_t^n \leq Y_t^{n+1}$ P-a.s. Set $Y_t := \lim_{n\to\infty} Y_t^n$. Similar to the proof of Theorem 4.2 of [10], we next give an a priori estimate for the L^2 bound of Y^n. Put

$K_t^n = n\int_0^t (Y_s^n - L_s)^- ds$. By Itô's formula, we have

$$\begin{aligned} & |Y_t^n|^2 + [M^n, M^n]_T - [M^n, M^n]_t \\ = \quad & \xi^2 + 2\int_t^T Y_s^n (f(s, Y_s^n) ds - 2\int_t^T Y_{s-}^n dM_s^n \\ & + 2n\int_t^T Y_s^n (Y_s^n - L_s)^- ds \end{aligned} \tag{4.8}$$

As f has a linear growth in the variable y, it follows that

$$\int_t^T |Y_s^n (f(s, Y_s^n)| ds \le C_T (1 + \int_t^T (Y_s^n)^2 ds) \tag{4.9}$$

For any $\delta > 0$,

$$\begin{aligned} & 2nE\left[\int_t^T Y_s^n (Y_s^n - L_s)^- ds\right] \\ = \quad & 2nE\left[\int_t^T (Y_s^n - L_s)(Y_s^n - L_s)^- ds\right] + 2nE\left[\int_t^T L_s (Y_s^n - L_s)^- ds\right] \\ \le \quad & \frac{1}{\delta} E\left[\sup_{0\le s\le T} (L_s)^2\right] + \delta E\left[(K_T^n - K_t^n)^2\right] \end{aligned} \tag{4.10}$$

On the other hand, in view of (4.7), we see that

$$\begin{aligned} & E\left[(K_T^n - K_t^n)^2\right] \\ \le \quad & CE[|\xi|^2] + CE[|Y_t^n|^2] + C(1 + \int_t^T E[(Y_s^n)^2] ds) \\ & + CE\left[(M_T^n - M_t^n)^2\right] \\ \le \quad & CE[|\xi|^2] + CE[|Y_t^n|^2] + C(1 + \int_t^T E[(Y_s^n)^2] ds) \\ & + CE\left([M^n, M^n]_T - [M^n, M^n]_t\right) \end{aligned} \tag{4.11}$$

Take expectation in (4.8) and substitute (4.9)–(4.11) into (4.8) to get

$$\begin{aligned} & E[|Y_t^n|^2] + E\left([M^n, M^n]_T - [M^n, M^n]_t\right) \\ \le \quad & C_\delta E[|\xi|^2] + C_\delta E\left[\sup_{0\le s\le T} (L_s)^2\right] + C_\delta (1 + \int_t^T E[(Y_s^n)^2] ds) \\ & + C\delta\left\{E[|Y_t^n|^2] + E\left([M^n, M^n]_T - [M^n, M^n]_t\right)\right\} \end{aligned} \tag{4.12}$$

Select δ so that $C\delta < 1$ and apply Gronwall's inequality to deduce that

$$\sup_n \sup_{0\le t\le T} (E[|Y_t^n|^2] + E([M^n, M^n]_T)) \le C_T E[|\xi|^2] + C_T E\left[\sup_{0\le s\le T} (L_s)^2\right] \tag{4.13}$$

This implies $\sup_n E[(M_T^n)^2] < \infty$. Thus, there exists a subsequence n_k such that $M_T^{n_k}$ converges weakly to some random variable M_T in $L^2(\Omega)$ as $k \to \infty$. Let $M_t, t \ge 0$ denote the martingale with terminal value M_T. Then it is easy to see that $M_t^{n_k}$ converges weakly to M_t in $L^2(\Omega)$ for every $t \le T$. Replacing n by n_k in (4.7) we get

$$K_T^{n_k} - K_t^{n_k} = Y_t^{n_k} - \xi - \int_t^T f(s, Y_s^{n_k}) ds + (M_T^{n_k} - M_t^{n_k}) \tag{4.14}$$

Since each term on the right hand side converges, we deduce that there exists an increasing process $K_t, t \ge 0$ such that $K_t^{n_k}$ converges weakly to K_t. Moreover, (Y, M, K) satisfies the following backward equation:

$$Y_t = \xi + \int_t^T f(s, Y_s) ds - (M_T - M_t) + K_T - K_t \tag{4.15}$$

By Lemma 2.2 in [16], it follows from the equation (4.15) that Y_t, K_t are right continuous with left limits. Furthermore, using Fatou Lemma it follows that

$$\begin{aligned} & E[\int_0^T (Y_t - L_t)^- dt] \\ \le \quad & \liminf_{n\to\infty} E[\int_0^T (Y_t^n - L_t)^- dt] \\ \le \quad & \liminf_{n\to\infty} \frac{1}{n} E[(K_T^n - K_t^n)] \le C \lim_{n\to\infty} \frac{1}{n} = 0 \end{aligned} \tag{4.16}$$

As both Y and L are right continuous, (4.16) implies that $Y_t \ge L_t$ P-a.s. for evert $t \ge 0$. To show that (Y, M, K) is a solution to the RBSDE(3.1), it remains to prove

$$\int_0^T (Y_t - L_t) dK_t = 0 \tag{4.17}$$

To this end, we need to strengthen the convergence of K^n to K. Define

$$\phi(u, x) = n[(x - L_u)^-]^2$$

Then $\phi(u, x)$ is convex in x for every $u \ge 0$. By smooth approximation, we may assume $\phi''(u, x)$ exists and $\phi''(u, x) \ge 0$, where ϕ' stands for the derivative of ϕ w.r.t. x. By Itô's formula, we have

$$\begin{aligned} d\phi(t, Y_t^n) \quad = \quad & \partial_t \phi(t, Y_t^n) dt + \phi'(t, Y_t^n) dY_t^n \\ & + \frac{1}{2} \phi''(t, Y_t^n) d[Y^n, Y^n]_t^c \\ & + d\Big(\sum_{0 < s \le t} \{\phi(s, Y_s^n) - \phi(s, Y_{s-}^n) - \phi'(s, Y_{s-}^n) \Delta Y_s^n\} \Big) \end{aligned} \tag{4.18}$$

Hence,

$$\begin{aligned} & \phi(t, Y_t^n) + \int_t^T [n(Y_u^n - L_u)^-]^2 du + \int_t^T \frac{1}{2} \phi''(u, Y_u^n) d[Y^n, Y^n]_u^c \\ & \quad + \sum_{0 < s \le t} \{\phi(s, Y_s^n) - \phi(s, Y_{s-}^n) - \phi'(s, Y_{s-}^n) \Delta Y_s^n\} \\ = \quad & -2n \int_t^T |_{\{L_u > Y_u^n\}} (L_u - Y_u^n) dL_u - 2n \int_t^T (Y_u^n - L_u)^- f(u, Y_u^n) du \\ & -2n \int_t^T (Y_u^n - L_u)^- dM_u^n \end{aligned} \tag{4.19}$$

Since $\phi(u,x)$ is convex in x, we have

$$\int_t^T \frac{1}{2}\phi''(u,Y_u^n)d[Y^n,Y^n]_u^c \geq 0, \qquad \sum_{0<s\leq t}\{\phi(s,Y_s^n)-\phi(s,Y_{s-}^n)-\phi'(s,Y_{s-}^n)\Delta Y_s^n\} \geq 0 \tag{4.20}$$

By virtue of the linear growth of f, it is easy to see that

$$-2n\int_t^T (Y_u^n-L_u)^- f(u,Y_u^n)du \leq \frac{1}{3}\int_t^T [n(Y_u^n-L_u)^-]^2du + C_T + C_T\int_t^T (Y_u^n)^2du \tag{4.21}$$

If condition (i) holds, $-2n\int_t^T \chi_{\{L_u>Y_u^n\}}(L_u-Y_u^n)dL_u \leq 0$. In this case, it follows from (4.19)–(4.21) that

$$\frac{2}{3}E\bigg[\int_t^T [n(Y_u^n-L_u)^-]^2du\bigg] \leq C + E\bigg[\int_t^T (Y_u^n)^2du\bigg] \tag{4.22}$$

On the other hand, if condition (ii) is true, then

$$-2n\int_t^T \chi_{\{L_u>Y_u^n\}}(L_u-Y_u^n)dL_u \leq \frac{1}{3}\int_t^T [n(Y_u^n-L_u)^-]^2du + C\int_t^T (L_u')^2du$$

In this case, we deduce from (4.19)–(4.21) that

$$\frac{1}{3}E\bigg[\int_t^T [n(Y_u^n-L_u)^-]^2du\bigg] \leq C + CE\bigg[\int_t^T (Y_u^n)^2du\bigg] + CE\bigg[\int_t^T (L_u')^2du\bigg] \tag{4.23}$$

In view of (4.13), we obtain both from (4.22) and (4.23) that

$$\sup_n E\bigg[\int_t^T [n(Y_u^n-L_u)^-]^2du\bigg] < \infty. \tag{4.24}$$

Choosing a further subsequence if necessary, (4.24) implies that $n_k(Y_u^{n_k}-L_u)^-$ converges weakly to some function g_u in $L^2(\Omega\times[0,T],P\times dt)$ and K_t defined above is given by $K_t=\int_0^t g_u du$. Now we are in a position to prove (4.17). Write

$$\begin{aligned}
&\int_0^T (Y_u-L_u)dK_u - \int_0^T (Y_u^{n_k}-L_u)dK_u^{n_k}\\
&= \int_0^T (Y_u-L_u)[n_k(Y_u^{n_k}-L_u)^- - g_u]du\\
&\quad + \int_0^T (Y_u-Y_u^{n_k})[n_k(Y_u^{n_k}-L_u)^-]du
\end{aligned} \tag{4.25}$$

Because of the weak convergence, we have

$$\lim_{k\to\infty}\int_0^T (Y_u-L_u)[n_k(Y_u^{n_k}-L_u)^- - g_u]du = 0 \tag{4.26}$$

By the monotone convergence theorem and (4.24), it follows that

$$\begin{aligned}
&\lim_{k\to\infty}\Big|\int_0^T (Y_u-Y_u^{n_k})[n_k(Y_u^{n_k}-L_u)^-]du\Big|\\
&\leq \lim_{k\to\infty}\bigg(\int_0^T (Y_u-Y_u^{n_k})^2du\bigg)^{\frac{1}{2}}\bigg(\int_0^T [n_k(Y_u^{n_k}-L_u)^-]^2du\bigg)^{\frac{1}{2}} = 0
\end{aligned} \tag{4.27}$$

Combining (4.26) and (4.27) we obtain

$$\int_0^T (Y_u - L_u)dK_u = \lim_{k\to\infty} \int_0^T (Y_u^n - L_u)dK_u^{n_k} \leq 0$$

As $Y_u \geq L_u$, (4.17) follows. The proof of a) is complete.

b) Next we prove that the unique solution process Y_t of (4.3) can be given the representation (4.4). We do this by adapting the argument used in [9] to our setting: First note that if $\tau \in \mathcal{T}_{t,T}^{\mathcal{H}}$, then by (4.2) we have

$$Y_\tau = \xi + \int_\tau^T f(s, Y_s)ds - (M_T - M_\tau) + K_T - K_\tau \tag{4.28}$$

Subtracting (4.28) from (4.2) and taking conditional expectation with respect to $\mathcal{H}_t$ we get

$$\begin{aligned} Y_t &= E[\int_t^\tau f(s,Y_s)ds + Y_\tau + K_\tau - K_t|\mathcal{H}_t] \\ &\geq E[\int_t^\tau f(s,Y_s)ds + L_\tau \chi_{\tau<T} + \xi\chi_{\tau=T}|\mathcal{H}_t]. \end{aligned}$$

Since $\tau \in \mathcal{T}_{t,T}^{\mathcal{H}}$ was arbitrary, this proves that

$$Y_t \geq esssup_{\tau\in\mathcal{T}_{t,T}^{\mathcal{H}}} E[\int_t^\tau f(s,Y_s)ds + L_\tau\chi_{\tau<T} + \xi\chi_{\tau=T}|\mathcal{H}_t]; t \in [0,T] \tag{4.29}$$

On the other hand, if we define

$$\hat{\tau}_t = inf\{s \in [t,T]; Y_s = L_s\}$$

then $\hat{\tau}_t \in \mathcal{T}_{t,T}^{\mathcal{H}}$ and

$$\begin{aligned} &E[\int_t^{\hat{\tau}_t} f(s,Y_s)ds + L_{\hat{\tau}_t}\chi_{\hat{\tau}_t<T} + \xi\chi_{\hat{\tau}_t=T}|\mathcal{H}_t] \\ &= E[\int_t^{\hat{\tau}_t} f(s,Y_s)ds + Y_{\hat{\tau}_t} + K_{\hat{\tau}_t} - K_t|\mathcal{H}_t] = Y_t \end{aligned}$$

Here we have used that

$$K_{\hat{\tau}_t} - K_t = 0,$$

which is a consequence of the requirement (vi) of Definition 4.1, i.e. of the equation

$$\int_0^T (Y_t - L_t)dK_t = 0.$$

This completes the proof of b).

To prove c) we use the following result:

Skorohod Lemma. Let $x(t)$ be a real càdlàg function on $[0,\infty)$ such that $x(0) \geq 0$. Then there exists a unique pair $(y(t), k(t))$ of càdlàg functions on $[0,\infty)$ such that

(i) $y(t) = x(t) + k(t)$
(ii) $y(t) \geq 0$
(iii) $k(t)$ is càdlàg and nondecreasing, $k(0) = 0$
(iv) $\int_0^\infty y(t)k(dt) = 0$.

The function $k(t)$ is actually given by

$$k(t) = sup_{s\leq t} x^-(s) \tag{4.30}$$

where $x^-(s) = \max(-x(s), 0)$.

We say that (y,k) is *the solution of the Skorohod problem.* Comparing with Definition 4.1 we see that if we put

$$\check{y}(t) = Y_{T-t} - L_{T-t} = \xi + \int_{T-t}^{T} f(s,Y_s)ds - (M_T - M_{T-t}) - L_{T-t} + K_T - K_{T-t}, \tag{4.31}$$

$$\check{x}(t) = \xi + \int_{T-t}^{T} f(s,Y_s)ds - (M_T - M_{T-t}) - L_{T-t}, \tag{4.32}$$

$$k(t) = K_T - K_{T-t}, \tag{4.33}$$

then $(y(t) = \check{y}(t+), k(t), t \geq 0)$ solves the Skorohod problem described in Definition 4.1 for $x(t) = \check{x}(t+)$. By (4.30) we conclude that K_t is given by

$$\begin{aligned} &K_T - K_{T-t} \\ &= \max_{s \leq t}(\xi + \int_{T-s}^{T} f(u,Y_u)du - (M_T - M_{T-s}) - L_{T-s})^-; t \in [0,T] \end{aligned} \tag{4.34}$$

Since the unique solution K_t of the RBSDE (4.1) is in particular a solution of the corresponding Skorohod problem and this solution is unique and given by (4.34), we can conclude that (4.34) defines K_t as an $\mathcal{H}$-adapted process. This completes the proof of c) and hence the proof of Theorem 4.2. □

Remark 4.3. A discussion of the Snell envelope and the associated local time in a semimartingale context can be found in [11].

5. Application to Finance

Let $B_t = B_t(\omega)$ and $N(dt,dz)$ be a Brownian motion and an independent Poisson random measure, respectively, on a probability space $(\Omega, \mathcal{F}, P)$. We let $\nu(dz) := E[N([0,1],dz)]$ be the Lévy measure of N and we put

$$\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt.$$

We assume that

$$\int_{\mathbb{R}} \zeta^2 \nu(d\zeta) < \infty.$$

We refer to [14], Chapter 1 for more information about stochastic calculus for Itô-Lévy processes.

Let $\mathcal{F}_t = \sigma(B_s, \tilde{N}(ds,dz); s \leq t)$ be the σ-algebra generated by B_s and $\tilde{N}(ds,dz)$ for $s \leq t$, and let $F = \{\mathcal{F}_t\}_{t \geq 0}$ be the corresponding filtration. We may regard $\mathcal{F}_t$ as the information obtained by a person observing B_s and $\tilde{N}(ds,dz)$ up to time t. If a person has more information than this, she is called an insider (at least in a financial market context). We represent such inside information by a filtration $H = \{\mathcal{H}_t\}_{t \geq 0}$ where $\mathcal{H}_t \supseteq \mathcal{F}_t$ for all $t \geq 0$. We assume that $\{\mathcal{H}_t\}_{t \geq 0}$ and $\{\mathcal{F}_t\}_{t \geq 0}$ are right-continuous and contain all sets of P -measure 0.

If the initial condition or some of the coefficients of a stochastic differential equation driven by dB_t and $\tilde{N}(dt,dz)$ are $\mathcal{H}_t$-adapted and not necessarily $\mathcal{F}_t$-adapted, the corresponding stochastic integrals are anticipating. Then it is necessary to specify what type of anticipating integral one is using. In this section we study a problem about optimal consumption of an insider in a financial market, and in this context we choose to represent the anticipating integrals as *forward integrals.* See e.g. [6][13] or the monograph [7] for a motivation for the use of the forward integral in the context of insider trading, and see [17] for the basic properties of the forward integral.

Now suppose we have a cash flow $X_t = X^{(\lambda)}(t)$ given by

$$\begin{aligned} dX_t &= X_{t-}\big[(\mu_t - \lambda_t)dt + \sigma_t d^- B_t \\ &\quad + \int_{\mathbb{R}} \theta(t,z)\tilde{N}(d^-t,dz)\big]; X_0 > 0; 0 \le t \le T \end{aligned} \tag{5.1}$$

where μ_t, σ_t and $\theta(t,z)$ are given bounded $\mathcal{H}_t$-predictable processes, $\theta > -1$, and d^-B_t, $\tilde{N}(d^-t,dz)$ indicates that we use a *forward integral interpretation.* Here $c(t) := \lambda_t X_t$ is the consumption rate, λ_t being our relative consumption rate. We assume that we are given a family $\mathcal{A}_{\mathcal{H}}$ of admissible controls $\lambda_t \ge 0$ included in the set of $\mathcal{H}_t$-predictable processes, where $\mathcal{H}_t \supseteq \mathcal{F}_t$ is a given filtration.

By the Itô formula for forward integrals the solution X_t of (5.1) is given by

$$\begin{aligned} X_t &= x \exp\Big[\int_0^t \{\mu_s - \lambda_s - \frac{1}{2}\sigma_s^2 \\ &\quad + \int_{\mathbb{R}} [\log(1+\theta(s,z)) - \theta(s,z)]\nu(dz)\}ds + \int_0^t \sigma_s d^- B_s \\ &\quad + \int_0^t \int_{\mathbb{R}} \log(1+\theta(s,z))\tilde{N}(d^-s,dz)\Big]; 0 \le t \le T. \end{aligned} \tag{5.2}$$

We assume that $\mathcal{A}_{\mathcal{H}}$ satisfies the following conditions:

(i) For all $\lambda, \beta \in \mathcal{A}_{\mathcal{H}}$ with β bounded there exists $\delta > 0$ such that $\lambda + y\beta \in \mathcal{A}_{\mathcal{H}}$ for all $y \in (-\delta, \delta)$.

(ii) For all $t_0 \in [0,T], h > 0$ with $t_0 + h \le T$ and all bounded measurable $\alpha(\omega)$ the control $\beta_s(\omega) := \alpha(\omega)\xi_{[t_0,t_0+h]}(s)$ is in $\mathcal{A}_{\mathcal{H}}$.

Let U_1, U_2 be given utility functions. We assume that U_1, U_2 are increasing and concave C^1 functions on $(0,\infty)$ and that $x \to \frac{\partial U_i}{\partial x}$ is strictly decreasing with $\lim_{x\to\infty} \frac{\partial U_i}{\partial x} = 0$; $i = 1,2$. Consider the problem to find Φ and $\lambda^* \in \mathcal{A}_{\mathcal{H}}$ such that

$$\Phi = \sup_{\lambda \in \mathcal{A}_{\mathcal{H}}} J(\lambda) = J(\lambda^*), \tag{5.3}$$

where $J(\lambda)$ is given by

$$J(\lambda) = E[\int_0^T e^{-\rho s} U_1(\lambda_s X_s)ds + e^{-\rho T} U_2(X_T)];$$

where $T > 0, \rho > 0$ are given constants.

To study this problem we use a perturbation argument: Suppose λ is optimal. Choose $\beta \in \mathcal{A}_{\mathcal{H}}$, $\delta > 0$, and consider

$$g(y) := J(\lambda + y\beta) \quad \text{for} \quad y \in (-\delta, \delta)$$

Since λ is optimal we have $g'(0) = 0$. Hence

$$\begin{aligned} 0 &= \frac{d}{dy}E\big[\int_0^T e^{-\rho s}U_1\big((\lambda_s + y\beta_s)X_s^{(\lambda+y\beta)}\big)ds \\ &\quad + e^{-\rho T}U_2(X_T^{(\lambda+y\beta)})\big]_{y=0} \\ &= E\big[\int_0^T U_1'\big((\lambda_s + y\beta_s)X_s^{(\lambda+y\beta)}\big)e^{-\rho s} \\ &\quad \{\beta_s X_s^{(\lambda+y\beta)} + (\lambda_s + y\beta_s)\frac{d}{dy}X_s^{(\lambda+y\beta)}\}ds \\ &\quad + e^{-\rho T}U_2'(X_T^{(\lambda+y\beta)})\frac{d}{dy}X_T^{(\lambda+y\beta)}\big]_{y=0} \end{aligned} \tag{5.4}$$

Now, by (5.2),

$$\frac{d}{dy}X_t^{(\lambda+y\beta)} = X_t^{(\lambda+y\beta)}\big[-\int_0^t \beta_r dr\big] \tag{5.5}$$

Hence, (5.4) gives

$$\begin{aligned} &E\big[\int_0^T e^{-\rho s}U_1'\big(\lambda_s X_s^{(\lambda)}\big)\{\beta_s X_s^{(\lambda)} - \lambda_s X_s^{(\lambda)}\big[\int_0^s \beta_r dr\big]\}ds \\ &- e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)}\int_0^T \beta_r dr\big] = 0 \end{aligned} \tag{5.6}$$

By the Fubini theorem,

$$\int_0^T h_s \int_0^s \beta_r dr ds = \int_0^T (\int_s^T h_r dr)\beta_s ds$$

Hence (5.6) can be written as

$$\begin{aligned} &E\big[\int_0^T \{e^{-\rho s}U_1'\big(\lambda_s X_s^{(\lambda)}\big)X_s^{(\lambda)} - \int_s^T U_1'(\lambda_r X_r^{(\lambda)})\lambda_r X_r^{(\lambda)}e^{-\rho r}dr \\ &- e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)}\}\beta_s ds\big] = 0 \end{aligned} \tag{5.7}$$

Now apply this to

$$\beta_s := \alpha(\omega)\chi_{[t,t+h]}(s) \quad (\alpha \quad \mathcal{H}_t - measurable)$$

for a fixed $t \in [0, T)$. Then (5.7) becomes

$$\begin{aligned} &E\big[\int_t^{t+h} \{e^{-\rho s}U_1'\big(\lambda_s X_s^{(\lambda)}\big)X_s^{(\lambda)} - \int_s^T U_1'(\lambda_r X_r^{(\lambda)})\lambda_r X_r^{(\lambda)}e^{-\rho r}dr \\ &- e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)}\}\alpha ds\big] = 0 \end{aligned} \tag{5.8}$$

Differentiating w.r.t. h at $h = 0$ and using that (5.8) holds for all $\mathcal{H}_t$ -measurable α, we get

$$\begin{aligned} &E\big[\{e^{-\rho t}U_1'\big(\lambda_t X_t^{(\lambda)}\big)X_t^{(\lambda)} - \int_t^T U_1'(\lambda_r X_r^{(\lambda)})\lambda_r X_r^{(\lambda)}e^{-\rho r}dr \\ &- e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)}\}|\mathcal{H}_t\big] = 0 \end{aligned} \tag{5.9}$$

Define

$$Y_t := e^{-\rho t}U_1'\big(\lambda_t X_t^{(\lambda)}\big)X_t^{(\lambda)} \tag{5.10}$$

$$\xi := e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)} \tag{5.11}$$

$$f(t,y,\omega)=\lambda_t y. \tag{5.12}$$

Then (5.9) can be written

$$Y_t=E[\xi+\int_t^T f(s,Y_s,\omega)ds|\mathcal{H}_t];\quad t\in[0,T]. \tag{5.13}$$

This is an equation of the type considered in Section 2. Hence we can apply the results of that section to study (5.13).

By Theorem 3.3 the solution of (5.13) is

$$\begin{aligned} Y_t &= E\big[\xi\exp(\int_t^T \lambda_s ds)|\mathcal{H}_t\big] \\ &= E\big[e^{-\rho T}U_2'(X_T^{(\lambda)})X_T^{(\lambda)}\exp(\int_t^T \lambda_s ds)|\mathcal{H}_t\big], \end{aligned}$$

which combined with (5.10) gives

$$\begin{aligned} &\exp(-\rho t+\int_0^t \lambda_s ds)U_1'(\lambda_t X_t^{(\lambda)})X_t^{(\lambda)} \\ &= E\big[\exp(-\rho T+\int_0^T \lambda_s ds)U_2'(X_T^{(\lambda)})X_T^{(\lambda)}|\mathcal{H}_t\big]; t\in[0,T]. \end{aligned}$$

Note that

$$\exp(\int_0^t \lambda_s ds)X_t^{(\lambda)}=X_t^{(0)},$$

where $X_t^{(0)}$ is the solution of (5.1) when there is no consumption ($\lambda=0$). Therefore, if we write $Z_t=X_t^{(0)}$ we have the following:

Theorem 5.1. *The relative consumption rate* λ *is optimal for problem* (5.3) *if and only if the following holds:*

$$\exp(-\rho t)U_1'(\lambda_t X_t^{(\lambda)})Z_t=E\big[\exp(-\rho T)U_2'(X_T^{(\lambda)})Z_T|\mathcal{H}_t\big]; t\in[0,T]. \tag{5.14}$$

Equation (5.14) gives a relation between the optimal consumption rate

$$c_t=\lambda_t X_t^{(\lambda)}$$

and the corresponding optimal terminal wealth $X_T^{(\lambda)}$. In some cases this can be used to find both. To see this, note that by (5.14) we get

$$U_1'(c_t)=\exp(\rho(t-T))E\big[U_2'(X_T^{(\lambda)})\frac{Z_T}{Z_t}|\mathcal{H}_t\big]$$

or

$$c_t=I_1(\exp(\rho(t-T))E\big[U_2'(X_T^{(\lambda)})\frac{Z_T}{Z_t}|\mathcal{H}_t\big]), \tag{5.15}$$

where $I_1=(U_1')^{-1}$, the inverse of U_1'. Substituting (5.15) into the equation (5.1) we get

$$dX_t^{(\lambda)}=X_{t-}^{(\lambda)}\big[\mu_t dt+\sigma_t d^-B_t+\int_{R_0}\theta(t,z)\tilde{N}(d^-t,dz)\big]-c_t dt. \tag{5.16}$$

The solution of this equation is

$$X_t^{(\lambda)}=X_0G_t-\int_0^t \frac{G_t}{G_s}c_s ds, \tag{5.17}$$

where

$$G_t = x\exp\Big[\int_0^t \{-\frac{1}{2}\sigma_s^2 + \int_{R_0}[\log(1+\theta(s,z)) - \theta(s,z)]\nu(dz)\}ds + \int_0^t \sigma_s d^- B_s + \int_0^t\int_{R_0}\log(1+\theta(s,z))\tilde{N}(d^- s, dz)\Big];\ t\geq 0. \quad (5.18)$$

Hence, putting $t = T$ in (5.17) we get

$$\begin{aligned} X_T^{(\lambda)} &= G_T(X_0 - \int_0^T \frac{c_s}{G_s}ds) \\ &= G_T(X_0 - \int_0^T \frac{1}{G_t} I_1(\frac{\exp(\rho(t-T))}{Z_t} E\big[U_2'(X_T^{(\lambda)})Z_T|\mathcal{H}_t\big])dt), \quad (5.19)\end{aligned}$$

which is an equation for the optimal terminal wealth $X_T^{(\lambda)}$. We do not know how to solve this equation in general. However, there are some solvable cases:

Corollary 5.2. *Suppose*

$$U_2(x,\omega) = K(\omega)x \quad (5.20)$$

where K is a bounded $\mathcal{F}_T$-measurable random variable. Then the optimal terminal wealth $X_T^{(\lambda)}$ is given by

$$X_T^{(\lambda)} = G_T(X_0 - \int_0^T \frac{1}{G_t} I_1(\frac{\exp(\rho(t-T))}{Z_t} E\big[Z_T K|\mathcal{H}_t\big])dt) \quad (5.21)$$

and the corresponding optimal consumption rate c_t is given by (5.15)

Corollary 5.3. *(Complete future information)*
Suppose that $\mathcal{H}_t = \mathcal{F}_T$ for all $t\in[0,T]$. Then the optimal terminal wealth $X_T^{(\lambda)}$ is a solution of the equation

$$X_T^{(\lambda)} = G_T(X_0 - \int_0^T \frac{1}{G_t} I_1(\exp(\rho(t-T))\frac{Z_T}{Z_t}U_2'(X_T^{(\lambda)}))dt) \quad (5.22)$$

and the corresponding optimal consumption rate c_t is given by (5.15)*.*

Example 5.4. Suppose $U_1(x) = K_1(\omega)\frac{1}{\gamma}x^\gamma$ and $U_2(x) = K_2(\omega)\frac{1}{\gamma}x^\gamma$, where $K_i(\omega)$ are bounded $\mathcal{F}_T$-measurable random variables and $\gamma\in(-\infty,1)\setminus\{0\}$. Suppose that $\mathcal{H}_t = \mathcal{F}_T$ for all $t\in[0,T]$. Then

$$I_1(y) = \big(\frac{y}{K_1}\big)^{\frac{1}{\gamma-1}}$$

So (5.22) becomes

$$X_T^{(\lambda)} = G_T\Big(X_0 - \int_0^T \frac{1}{G_t}\big(\frac{K_2}{K_1}\exp(\rho(t-T))\frac{Z_T}{Z_t}\big)^{\frac{1}{\gamma-1}}X_T^{(\lambda)})\big)dt\Big)$$

which gives

$$X_T^{(\lambda)} = \frac{G_T X_0}{1 + (\frac{K_2}{K_1})^{\frac{1}{\gamma-1}}\int_0^T \frac{G_T}{G_t}\big(\exp(\rho(t-T))\frac{Z_T}{Z_t}\big)^{\frac{1}{\gamma-1}}dt} \quad (5.23)$$

Thus we see that even with complete information about the future, the optimal consumption problem has a finite solution. This is in contrast with the optimal portfolio problem, which gives an infinite value even in the case of a slightly advanced information flow, i.e. with $\mathcal{H}_t = \mathcal{F}_{t+\delta(t)}$ for some $\delta(t) > 0$. See e.g. [1][6][12].

A special case:
If $U_1(x) = \ln x, U_2(x) = K \ln x$ (K constant) then (5.13) simplifies to

$$Y_t = E[Ke^{-\rho T} + \int_t^T \lambda_s Y_s ds | \mathcal{H}_t] \tag{5.24}$$

By (5.10)

$$Y_t = \frac{e^{-\rho t}}{\lambda_t}$$

Hence, by (5.24),

$$\frac{e^{-\rho t}}{\lambda_t} = Ke^{-\rho T} + \frac{1}{\rho}(e^{-\rho t} - e^{-\rho T})$$

This gives the optimal consumption rate

$$\lambda_t = \lambda_t^* = \frac{\rho}{1 + (\rho K - 1)e^{\rho(t-T)}} \tag{5.25}$$

This case was solved in [13]. However, the method in [13] does not apply to other cases than the logarithmic utility case.

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