

DIFFERENTIABILITY OF STOCHASTIC REFLECTING FLOW WITH RESPECT TO STARTING POINT

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ABSTRACT. Let $\varphi_t(x), t \geq 0, x \in \mathbb{R}_+^d$ be a flow generated by SDE in a half-space \mathbb{R}_+^d with normal reflection at the hyperplane $\mathbb{R}^{d-1} \times \{0\}$. We prove that the flow φ_t is Sobolev differentiable with respect to a starting point x with probability 1 if coefficients of the corresponding SDE are Lipschitz continuous and diffusion is not degenerate. Stochastic equations for derivative $\nabla_x \varphi_t(x)$ are obtained. We also discuss an existence of Fréchet derivative (not only Sobolev).

1. Introduction

Let us consider a flow $\varphi_t(x), t \geq 0, x \in \mathbb{R}_+^d = \mathbb{R}^{d-1} \times [0, \infty)$ generated by Skorokhod SDE in a half-space \mathbb{R}_+^d with normal reflection at the hyperplane $\partial\mathbb{R}_+^d = \mathbb{R}^{d-1} \times \{0\}$:

$$d\varphi_t(x) = a(\varphi_t(x))dt + \sum_{k=1}^m \sigma_k(\varphi_t(x))dw_k(t) + \bar{n}L(dt, x), \quad t \geq 0, \quad (1.1)$$

$$\varphi_0(x) = x, \quad \varphi_t(x) \in \mathbb{R}_+^d, \quad t \geq 0, \quad x \in \mathbb{R}_+^d, \quad (1.2)$$

where $\bar{n} = (0, \dots, 0, 1)$ is a normal vector to the hyperplane, $\{w_k(t), t \geq 0\}, k = 1, \dots, m$ are independent one-dimensional Wiener processes, $L(t, x)$ is continuous and non-decreasing in t process for any fixed x , and

$$L(0, x) = 0, \quad (1.3)$$

$$L(t, x) = \int_0^t \mathbb{I}_{\{\varphi_s(x) \in \partial\mathbb{R}_+^d\}} L(ds, x), \quad t \geq 0. \quad (1.4)$$

Condition (1.4) means that $L(\cdot, x)$ may increase only at instants when $\{\varphi_t(x), t \geq 0\}$ hits the hyperplane $\mathbb{R}^{d-1} \times \{0\}$.

Assume that functions $a, \sigma_k, k = 1, \dots, m$ are Lipschitz continuous. It is well-known that there exists a unique solution to the system of SDEs (1.1)-(1.4). It can be also proved [19] that continuous in (t, x) modification of $(\varphi_t(x), L(t, x))$ exists. Only this modification will be considered further. Moreover, it was proved in [20] that the mapping $\varphi_t(\cdot, \omega)$ belongs to the Sobolev class $\bigcap_{p \geq 1} W_{p, \text{loc}}^1(\mathbb{R}_+^d, \mathbb{R}_+^d)$ for a.a.

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ω . Stochastic differential equations for Sobolev derivative $\nabla\varphi_t(x)$ were derived in a case when $a, \sigma_k, k = 1, \dots, m$ are continuous differentiable and diffusion coefficient is non-degenerate at $\mathbb{R}^{d-1} \times \{0\}$, see [21, 22].

The main purpose of this paper is to derive SDEs for a derivative of the flow with respect to the initial data if coefficients are not C^1 but satisfy Lipschitz condition only. Further we give some conditions when Fréchet derivative exists also.

The problem of differentiability in initial data of a stochastic flow generated by an SDE with smooth coefficients in \mathbb{R}^d is a classical subject of stochastic analysis, see for ex. [12, 14]. An equation for the derivative can be obtained by formal differentiation of the initial SDE. If coefficients of an SDE in \mathbb{R}^d are only Lipschitzian, then the flow is only Sobolev differentiable w.r.t. the initial data and the equation for the derivative can also be obtained by formal differentiation. However this problem is non-trivial one [4] and the corresponding proof requires delicate results of geometric measure theory and theory of Sobolev spaces.

The problem of (Gateaux or Fréchet) differentiability in initial data of reflecting flow was studying comparatively recently, see papers [2, 3, 6, 7] for various types of domains (octants, polyhedrons, domains with smooth boundary). However only a constant diffusion term was considered in cited papers. This assumption as usual essentially simplifies the corresponding proofs. Note that the equations for $\nabla\varphi_t(x)$ introduced in mentioned papers have a specific structure. Assume for a moment that coefficients in (1.1) are smooth and let us try to guess a form of an equation for $\nabla\varphi_t(x)$.

The reasoning below is only formal! The rigorous statements are given in §2.

Assume that t_1, t_2, ω , and open set $U \in \mathbb{R}_+^d$ are such that

$$\varphi_t(x) \notin \partial\mathbb{R}_+^d, \quad t \in [t_1, t_2], x \in U.$$

Then $L(t_1, x) = L(t, x)$, $t \in [t_1, t_2]$, and therefore $\varphi_t(x)$ satisfies SDE (without reflection) for $t \in [t_1, t_2]$ with coefficients a, σ_k . So, if derivative exists, then it should satisfy SDE

$$d\nabla\varphi_t(x) = \nabla a(\varphi_t(x))\nabla\varphi_t(x)dt + \sum_{k=1}^m \nabla\sigma_k(\varphi_t(x))\nabla\varphi_t(x)dw_k(t), \quad t \in [t_1, t_2].$$

Assume now that $\varphi_{t_0}(x_0) \in \partial\mathbb{R}_+^d$. Then the d -th coordinate of $\varphi_{t_0}(x_0)$ equals zero. So a function

$$\mathbb{R}_+^d \ni x \rightarrow \varphi_{t_0}^d(x) \in [0, \infty)$$

attains a minimum at the point x_0 and a derivative $\nabla\varphi_{t_0}^d(x_0)$ (if it exists) has to be equal zero.

Summarizing two observations above, the derivative $\nabla\varphi_t(x)$ should satisfy equations:

$$\nabla\varphi_t(x) = E + \int_0^t \nabla a(\varphi_s(x))\nabla\varphi_s(x)ds + \int_0^t \nabla\sigma_k(\varphi_s(x))\nabla\varphi_s(x)dw_k(s), \quad t < \sigma, \quad (1.5)$$

$$\nabla\varphi_t(x) = P_{\varphi_{\tau(t)-}}(x) + \int_{\tau(t)}^t \nabla a(\varphi_s(x))\nabla\varphi_s(x)ds \quad (1.6)$$

$$+ \int_{\tau(t)}^t \nabla \sigma_k(\varphi_s(x)) \nabla \varphi_s(x) dw_k(s), t \geq \sigma$$

where

$$\begin{aligned} \sigma &:= \sigma(x) = \inf\{t \geq 0 : \varphi_t(x) \in \partial \mathbb{R}_+^d\}, \\ \tau(t) &:= \tau(t, x) = \sup\{s \in [0; t] \mid \varphi_s(x) \in \partial \mathbb{R}_+^d\}, \\ P &= (p_{ij})_{i,j=1}^d = \begin{cases} 1, & i = j \neq d \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

P is orthoprojection on the first $(d-1)$ coordinates of \mathbb{R}^d .

It appears that if $a, \sigma_k, k = 1, \dots, m$ are continuous differentiable and diffusion coefficient is non-degenerate at $\mathbb{R}^{d-1} \times \{0\}$, then the Sobolev derivative is a unique solution of the above equations (see [21, 22] or §2 for details). We will prove that the Sobolev derivative satisfies (1.5), (1.6) even if a, σ_k satisfies only the Lipschitz property.

The idea of the proof is the following. Let us approximate functions a, σ_k by a sequence of smooth functions $\{a^n, \sigma_k^n\}$. By $\varphi_t^n(x)$ denote the solution of the corresponding reflecting SDEs. We already know that the Sobolev derivative $\nabla \varphi_t^n(x)$ exists. Let $g_t(x)$ be a solution of (1.5), (1.6) (we don't know that $g_t(x) = \nabla \varphi_t(x)$ yet). To proof that $g_t(x) = \nabla \varphi_t(x)$ it suffices to show that

$$E \int_U \|\varphi_t^n(x) - \varphi_t(x)\|^p dx \rightarrow 0, n \rightarrow \infty, \quad (1.7)$$

$$E \int_U \|\nabla \varphi_t^n(x) - g_t(x)\|^p dx \rightarrow 0, n \rightarrow \infty, \quad (1.8)$$

for any bounded measurable set $U \subset \mathbb{R}_+^d$.

The proof of (1.7) is quite standard result. However in proving (1.8) we meet some difficulties. Namely, one of them is to prove approximation theorem for equations of the type (1.5), (1.6). Another one is the following. Even if $\varphi_t^n(x) \rightarrow \varphi_t(x)$, $n \rightarrow \infty$, then it is not obvious that $\nabla a^n(\varphi_t^n(x)) \rightarrow \nabla a(\varphi_t(x))$, $n \rightarrow \infty$, because ∇a may be discontinuous and ∇a^n does not converge uniformly to ∇a .

If the diffusion coefficient is non-degenerate in (1.1), then the distribution of $\varphi_t(x)$ is absolutely continuous. By Rademacher's theorem Lipschitz functions are differentiable almost everywhere w.r.t. the Lebesgue measure. So the compositions $\nabla a(\varphi_t(x))$, $\nabla \sigma_k(\varphi_t(x))$ are defined up to sets of probability zero.

Assume now that a drift coefficient a is smooth, the noise is additive and the diffusion term is a Wiener process, i.e. $\sigma = \mathbb{I}$. In this case, for various types of domains (octants, polyhedrons, domains with smooth boundary) it was proved a stronger result than in [21], see [2, 3, 6, 7]:

$$\forall x \forall t \geq 0 : P(\text{Fréchet derivative } \nabla \varphi_t(x) \text{ exists}) = 1. \quad (1.9)$$

It should be noted that (1.9) does not imply the existence of a set of probability 1 such that $\varphi_t(x)$ is differentiable for all x simultaneously. Indeed, let us consider the one-dimensional case, when $\varphi_t(x)$ is reflected Brownian motion with reflection

at zero, i.e., $d = m = 1$, $\sigma = 1$, $a = 0$. It is easy to verify that

$$\varphi_t(x) = \begin{cases} w(t) - \min_{0 \leq s \leq t} w(s), & x = 0, \\ w(t) + x, & x > 0 \text{ and } x + \min_{0 \leq s \leq t} w(s) > 0, \\ \varphi_t(0), & x > 0 \text{ and } x + \min_{0 \leq s \leq t} w(s) \leq 0. \end{cases}$$

Therefore, $d\varphi_t(x)/dx = 0$ if $x < -\min_{0 \leq s \leq t} w(s)$, $d\varphi_t(x)/dx = 1$ if $x > -\min_{0 \leq s \leq t} w(s)$, and derivative does not exist at the point $x = -\min_{0 \leq s \leq t} w(s)$. The condition (1.9) is obviously true, but $\varphi_t \notin C^1([0; \infty))$ with probability 1.

Moreover, it can be proved that the global continuous differentiability does not hold for any reflected flow in a half-space with C^∞ coefficients and non-degenerate diffusion at the hyperplane $\mathbb{R}^{d-1} \times \{0\}$. This statement is the consequence from the following reasoning. The Jacobian $\det \nabla \varphi_t(x)$ equals zero with probability 1 for λ^d -a.a. x such that the process $\varphi_t(x)$ hits the hyperplane $\partial \mathbb{R}_+^d$ before instant t (see [21]). On the other hand, for all other x the values of $\varphi_t(x)$ coincide with the values of the flow generated by the stochastic flow $\varphi_t^{ord}(x)$ (without reflection) in \mathbb{R}^d where coefficients a, σ_k are smoothly extended to the whole Euclidean space. It is well-known that $P(\forall x : \det \nabla \varphi_t^{ord}(x) \neq 0) = 1$. Derivative $\nabla \varphi_t^{ord}(x)$ is continuous in t, x . So $\inf_{\|x\| \leq R} \det \nabla \varphi_t^{ord}(x) > 0$ a.s. for any $R > 0$. Therefore $\det \nabla \varphi_t$ have a jump-type discontinuity if the sets $\{x : \exists s \in [0, t] \text{ such that } \varphi_s(x) \in \partial \mathbb{R}_+^d\}$ and $\{x : \forall s \in [0, t] \text{ such that } \varphi_s(x) \notin \partial \mathbb{R}_+^d\}$ have positive Lebesgue measure. If the diffusion is non-degenerate in a neighborhood of the hyperplane, then this requirement holds true with positive probability. For more counterexamples, see [23].

Let us discuss a relationship between the Sobolev derivative and the usual (Fréchet) derivative. Existence of Sobolev derivative $\nabla \varphi_t(x)$ implies existence of directional derivatives $\frac{\partial \varphi_t(x)}{\partial x_k}$ for almost all x with respect to the Lebesgue measure (see for ex. [18]). Assume that we succeed to prove that the flow satisfies Lipschitz property w.r.t. the initial point. Then Rademacher's theorem [9] implies the existence of the Fréchet derivative for almost all x w.r.t. the Lebesgue measure on \mathbb{R}_+^d (the Fréchet derivative will certainly coincide with the Sobolev one). Hence

$$P(\text{Fréchet derivative } \nabla \varphi_t(x) \text{ exists for } \lambda^d\text{-a.a. } x) = 1. \quad (1.10)$$

So by Fubini's theorem

$$P(\text{Fréchet derivative } \nabla \varphi_t(x) \text{ exists}) = 1 \text{ for } \lambda^d\text{-a.a. } x \quad (1.11)$$

Note, that flows in [2, 3, 6, 7] satisfy Lipschitz condition.

The paper is organized as follows. It was mentioned that the equations for derivatives w.r.t. the initial data have a specific form, see (1.5), (1.6). The corresponding definitions and statements are given in §2. We formulate and prove the main results in §3. The proof of continuous dependence on coefficients for the equations of type (1.5), (1.6) is given in §4. In §5 we discuss and compare results of the paper with results on Fréchet differentiability in the case when the diffusion coefficient is constant. In particular we discuss conditions that ensure (1.9) (not only (1.10) or (1.11)).

2. Stochastic Equations With Nulling

In this Section we give the definition and formulate some results on a stochastic equation which describe the derivative $\nabla\varphi_t(x)$ (see (1.5), (1.6)).

Let $(w(t) = (w_1(t), \dots, w_m(t)), \mathcal{F}_t)$ be m -dimensional Wiener process, $x(t)$ be continuous \mathcal{F}_t -adapted stochastic process, and functions

$$a_k : \Omega \times [0, T] \times \mathbb{R}^l \times \mathbb{R}^p \rightarrow \mathbb{R}^l, \quad k = \overline{0, m},$$

$$b_k : \Omega \times [0, T] \times \mathbb{R}^l \times \mathbb{R}^p \rightarrow \mathbb{R}^p, \quad k = \overline{0, m}$$

be such that for any $t \in [0, T]$ the restriction of functions a_k, b_k to the set $\Omega \times [0, t] \times \mathbb{R}^l \times \mathbb{R}^p$ are $\mathcal{F}_t \times \mathcal{B}([0, t]) \times \mathcal{B}(\mathbb{R}^l) \times \mathcal{B}(\mathbb{R}^p)$ -measurable mappings.

Consider a point random measure ν ,

$$\nu(A) = \sum_{t \in A} \mathbb{I}_{\{x(t)=0\}} \delta_{\{t\}},$$

where $\delta_{\{t\}}$ is an atomic measure concentrated at t .

Definition 2.1. A pair of \mathcal{F}_t -adapted stochastic processes (y_t, z_t) is said to be a *solution* to the system of stochastic differential equations with nulling

$$\begin{cases} dy_t = a_0(t, y_t, z_t)dt + \sum_{k=1}^m a_k(t, y_t, z_t)dw_k(t), \\ dz_t = b_0(t, y_t, z_t)dt + \sum_{k=1}^m b_k(t, y_t, z_t)dw_k(t) - z_t \nu(dt), \quad t \in [0, T], \end{cases} \quad (2.1)$$

if the following conditions are satisfied

- 1) $z_t, t \geq 0$ has a càdlàg trajectories;
- 2) $y_t, t \geq 0$ is a continuous process;
- 3) for a.a. ω the following equality holds:

$$y_t = y_0 + \int_0^t a_0(s, y_s, z_s)ds + \sum_{k=1}^m \int_0^t a_k(s, y_s, z_s)dw_k(s), \quad t \in [0, T];$$

$$4) \quad z_t = z_0 \mathbb{I}_{\{\tilde{\tau}(t) > 0\} \cup \{x_0 = 0\}} + \int_{\tilde{\tau}(t)}^t b_0(s, y_s, z_s)ds + \sum_{k=1}^m \int_{\tilde{\tau}(t)}^t b_k(s, y_s, z_s)dw_k(s), \quad (2.2)$$

where $\tilde{\tau}(t) = \sup\{s : s \in [0, t], x_s = 0\} \vee 0$ is the last hitting zero by a process x that does not exceed t .

Remark 2.2. An instant $\tilde{\tau}(t)$ is not a Markov moment, thus stochastic integrals in the last equality are not well-defined in general. Equation (2.2) is treated as follows. Define a random map $\pi : C([0, T]) \rightarrow D([0, T])$

$$(\pi f)(t) := f(t) - f(\tilde{\tau}(t)).$$

Observe that if $f(t), t \in [0, T]$ is \mathcal{F}_t -adapted continuous process, then $(\pi f)(t)$ is \mathcal{F}_t -adapted càdlàg process. If $\xi(t), t \in [0, T]$ is \mathcal{F}_t -adapted process with square integrable trajectories, then put $\int_{\tilde{\tau}(t)}^t \xi(s)dw(s) := (\pi(I\xi))(t)$, where $(I\xi)(s) = \int_0^s \xi(u)dw(u)$. Note that $\int_{\tilde{\tau}(t)}^t \xi(s)dw(s), t \in [0, T]$ is \mathcal{F}_t -adapted càdlàg process.

So, item 4) in Definition 2.1 can be replaced by the following equivalent condition:

$$z_t = \pi(z_0 + \int_0^t b_0(s, y_s, z_s) ds + \sum_{k=1}^m \int_0^t b_k(s, y_s, z_s) dw_k(s))(t).$$

This approach was considered by Andres [3]. In paper [21] item 4) in Definition 2.1 initially was replaced by the equivalent, but more complicated for investigation conditions

4') for a.a. ω a set $\{t \geq 0 : x_t = 0\}$ is contained in $\{t \geq 0 : z_t = 0\}$, i.e. z_t equals zero if x_t hits zero;

4'') for any stopping time τ the following equality holds with probability 1

$$z_t = z_\tau + \int_\tau^t b_0(s, y_s, z_s) ds + \sum_{k=1}^m \int_\tau^t b_k(s, y_s, z_s) dw_k(s), \quad t \in [\tau, \tau^0),$$

where $\tau^0 = \inf\{t \geq \tau : x_t = 0\}$.

Remark 2.3. If x_t has finite number of zeroes in $t \in [0, T]$, then (2.2) is equivalent to

$$z_t = z_0 + \int_0^t b_0(s, y_s, z_s) ds + \sum_{k=1}^m \int_0^t b_k(s, y_s, z_s) dw_k(s) - \int_0^t z_{s-} \nu(ds).$$

If the set of zeros for the process x_t has a complicated structure, for example, if x_t is a Wiener process, then the integral $\int_0^t z_{s-} \nu(ds)$ and the second equation in (2.1) are only formal.

Theorem 2.4. *Assume that functions a_k, b_k satisfy Lipschitz condition and linear growth property:*

1) $\exists L > 0 \quad \forall y_1, y_2, z_1, z_2 \quad \forall t \quad \forall \omega :$

$$\begin{aligned} & \|a_k(t, y_1, z_1) - a_k(t, y_2, z_2)\| + \|b_k(t, y_1, z_1) - b_k(t, y_2, z_2)\| \\ & \leq L(\|y_1 - y_2\| + \|z_1 - z_2\|), \quad k = \overline{0, m}; \end{aligned}$$

2) $\exists C > 0 \quad \forall \omega, t \quad \forall y \in \mathbb{R}^l, \quad \forall z \in \mathbb{R}^p :$

$$\|a_k(t, y, z)\| + \|b_k(t, y, z)\| \leq C(1 + \|y\| + \|z\|), \quad k = \overline{0, m}.$$

Then the system (2.1) has a unique solution. Moreover, if $E(\|y_0\|^p + \|z_0\|^p) < \infty$, $p \geq 2$ then for any $T > 0$ there exists a constant $K = K(p, T, C, L)$ such that

$$E \sup_{t \in [0; T]} (\|y(t)\|^p + \|z(t)\|^p) \leq KE(\|y(0)\|^p + \|z(0)\|^p + 1). \quad (2.3)$$

Observe that $\sup_{s \in [0, t]} |(\pi(f))(s)| \leq 2 \sup_{s \in [0, t]} |(f)(s)|$ for any f . So we can use standard methods for the moments estimation of $\sup_{s \in [0, t]} |(\pi(I\xi))(s)|$. Hence existence and uniqueness theorem and moments estimate (2.3) can be obtained analogously to the proof of similar result for Ito equation. This approach seems to be easier than the method used in [21], where the coefficients a_k, b_k were non-random.

Theorem 2.5. *Assume that conditions of Theorem 2.4 are satisfied and $E(\|y_0\|^p + \|z_0\|^p) < \infty$. Then for any $p \geq 2$ and $\alpha \in (0, p/2 - 1)$ there exists a constant $K = K(p, \alpha, T, C, L, E(\|y_0\|^p + \|z_0\|^p))$, such that*

$$\forall \delta > 0 : E \sup_{|s_1 - s_2| < \delta, s_1, s_2 \in [0; T]} \|y(s_2) - y(s_1)\|^p \leq K\delta^\alpha; \quad (2.4)$$

$$\forall \delta > 0 : E \sup_{\substack{\{(s_1, s_2): |s_1 - s_2| < \delta, \\ x(s) \neq 0, s \in [s_1, s_2] \subset [0; T]\}}} \|z(s_2) - z(s_1)\|^p \leq K\delta^\alpha \quad (2.5)$$

Proof. Using inequality (2.3), Definition 2.1 and Burkholder's inequality, it is easy to deduce that

$$\exists K_1 \forall \delta > 0 \sup_{|s_1 - s_2| < \delta, s_1, s_2 \in [0; T]} E \|y(s_2) - y(s_1)\|^p \leq K_1 \delta^{p/2}.$$

The inequality (2.4) follows from the last inequality and the well-known statement about an estimate of Hoelder constant for a stochastic process that satisfies moment inequality of Kolmogorov's continuous modification theorem (cf. [5]).

To prove (2.5) observe that Definition 2.1 implies

$$\begin{aligned} & \sup_{|s_1 - s_2| < \delta, x(s) \neq 0, s \in [s_1, s_2] \subset [0; T]} \|z(s_2) - z(s_1)\| \leq \\ & \sup_{|s_1 - s_2| < \delta, s_1, s_2 \in [0; T]} \left(\left\| \int_{s_1}^{s_2} b_0(s, y_s, z_s) ds \right\| + \sum_{k=1}^m \left\| \int_{s_1}^{s_2} b_k(s, y_s, z_s) d\omega_k(s) \right\| \right). \end{aligned} \quad (2.6)$$

The rest proof of inequality (2.5) is similar to the proof of (2.4). \square

We need the following statement about continuous dependence on coefficients for solutions of SDEs with nulling.

Theorem 2.6. *Let $(x^n(t), y^n(t), z^n(t))$ be a sequence of \mathcal{F}_t -adapted processes taking values in $\mathbb{R} \times \mathbb{R}^l \times \mathbb{R}^p$. Assume that $x^n(t)$ have continuous in t trajectories, and a process $(y^n(t), z^n(t))$ is a solution of the SDE with nulling:*

$$\begin{cases} dy^n(t) = a_{n,0}(x^n(t), y^n(t), z^n(t))dt + \sum_{k=1}^m a_{n,k}(x^n(t), y^n(t), z^n(t))d\omega_k(t), \\ dz^n(t) = b_{n,0}(x^n(t), y^n(t), z^n(t))dt \\ \quad + \sum_{k=1}^m b_{n,k}(x^n(t), y^n(t), z^n(t))d\omega_k(t) - z_{t-}^n \nu^n(dt), \\ y^n(0) = y_0^n, z^n(0) = z_0^n, \end{cases} \quad (2.7)$$

where $\nu^n(\{t\}) = \mathbb{1}_{\{x^n(t)=0\}} \delta_{\{t\}}$. Suppose that functions $a_{n,k}, b_{n,k}$ satisfy the following linear growth and Lipschitz conditions with constants independent of n .

1) *Linear growth condition:*

$$\begin{aligned} & \exists C \forall n \forall x, y, z \forall \omega : \sum_{k=0}^m \|a_{n,k}(x, y, z)\| \\ & + \sum_{k=0}^m \|b_{n,k}(x, y, z)\| \leq C(1 + \|y\| + \|z\|). \end{aligned}$$

2) *Lipschitz condition:*

$$\begin{aligned} & \exists C \forall n \forall x, y_1, z_1, y_2, z_2 \forall \omega : \\ & \sum_{k=0}^m (\|a_{n,k}(x, y_1, z_1) - a_{n,k}(x, y_2, z_2)\| + \|b_{n,k}(x, y_1, z_1) - b_{n,k}(x, y_2, z_2)\|) \\ & \leq C(\|y_1 - y_2\| + \|z_1 - z_2\|). \end{aligned}$$

Suppose also that:

- 3) for any y, z sequences $\{a_{n,k}(x^n(t), y, z)\}_{n \geq 1}$ and $\{b_{n,k}(x^n(t), y, z)\}_{n \geq 1}$ converge in probability as $n \rightarrow \infty$ to $a_{0,k}(x^0(t), y, z)$ and $b_{0,k}(x^0(t), y, z)$, respectively;
 4) x^n converges in probability to x^0 uniformly on compact sets:

$$\forall T > 0 \forall \varepsilon > 0 : P\left(\sup_{t \in [0, T]} |x^n(t) - x^0(t)| \geq \varepsilon\right) \rightarrow 0, n \rightarrow \infty; \quad (2.8)$$

5)

$$\forall t > 0 : \tilde{\tau}^n(t) \xrightarrow{P} \tilde{\tau}^0(t), n \rightarrow \infty, \quad (2.9)$$

where $\tilde{\tau}^n(t) = \sup\{s \leq t : x^n(s) = 0\} \vee 0$ is the last hitting zero by a process x^n that does not exceed t ;

6)

$$\forall t \geq 0 : P(x^0(t) = 0) = 0; \quad (2.10)$$

7) $\sup_n E(\|y^n(0)\|^4 + \|z^n(0)\|^4) < \infty$, and $\lim_{n \rightarrow \infty} y^n(0) = y^0(0)$, $\lim_{n \rightarrow \infty} z^n(0) = z^0(0)$ in mean-square.

Then for any $t \geq 0$:

$$E(\|y^n(t) - y^0(t)\|^2 + \|z^n(t) - z^0(t)\|^2) \rightarrow 0, n \rightarrow \infty.$$

The proof of this theorem is given in §4.

Remark 2.7. Generally speaking, in condition 7) we may assume the uniform boundedness of $(2 + \varepsilon)$ th moment instead of 4th. In this case we have to use Hoelder inequality in (4.4). We don't formulate and prove more general results, because we don't need them for an investigation of reflecting flow derivatives.

3. Main Results

The main result of this paper is the following theorem.

Theorem 3.1. *Assume that coefficients $a, \sigma_k, k = 1, \dots, m$ of equation (1.1) satisfy the Lipschitz condition and for any $x \in \mathbb{R}_+^d$ the matrix $\sum_{k=1}^m \sigma_k(x) \sigma_k^*(x)$ is positive defined.*

Then for a.a. ω the map $x \mapsto \varphi_t(x, \omega)$ belongs to Sobolev class $\bigcap_{p \geq 1} W_{p, \text{loc}}^1(\mathbb{R}_+^d, \mathbb{R}_+^d)$ for all $t \geq 0$ and there exists a modification $\psi_t(x)$ of $\nabla \varphi_t(x)$ such that

- 1) $\psi_t(x, \omega)$ is measurable in (t, x, ω) , $\psi_t(x)$ is \mathcal{F}_t -measurable;
- 2) $\psi_t = \nabla \varphi_t$, $t \geq 0$ for a.a. ω ;

3) a process $\psi_t(x)$ satisfies the equation:

$$\begin{cases} d\psi_t(x) = \nabla a(\varphi_t(x))\psi_t(x)dt + \sum_{k=1}^m \nabla \sigma_k(\varphi_t(x))\psi_t(x)dw_k(t) \\ -P\psi_{t-}\nu(ds, x), \quad t \geq 0, \\ \psi_0(x) = \mathbb{I}, \quad x \in \mathbb{R}_+^d, \end{cases} \quad (3.1)$$

where \mathbb{I} is unity $d \times d$ matrix, $P = \begin{pmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 0 & 1 \end{pmatrix}$, $\nu(dt, x)$ is a point measure such that $\nu(\{t\}, x) = \mathbb{I}_{\{\varphi_t(x) \in \mathbb{R}^{d-1} \times \{0\}\}} \delta_{\{t\}}$.

Remark 3.2. An equation (3.1) is understood in the sense of §2. Take $\varphi_t^d(x)$ (d th coordinate of the process $\varphi_t(x)$) in place of x_t , take $(\psi_t^{d,1}(x), \dots, \psi_t^{d,d}(x))$ in place of z_t , and a process $\{\psi_t^{i,j}(x), 1 \leq i \leq d-1, 1 \leq j \leq d\}$ in place of y_t .

Remark 3.3. An equality $\psi_t = \nabla \varphi_t$ in 2) is considered as equality of functions in $L_{p,\text{loc}}$, that is, for a.a. $x \in \mathbb{R}_+^d$.

Remark 3.4. Due to Rademacher's theorem (see [9]), Lipschitz continuous functions are differentiable almost surely with respect to the Lebesgue measure. We redefine ∇a and $\nabla \sigma_k$ by any measurable way at those points, where derivatives do not exist (for example, let them be equal to zero). Since $\sum_{k=1}^m \sigma_k(x)\sigma_k^*(x) > 0$, $x \in \mathbb{R}_+^d$, we have that the distribution of $\varphi_t(x)$ is absolute continuous for all $t > 0, x \in \mathbb{R}_+^d$. Therefore, random variable $\varphi_t(x)$ belongs to a set of non-differentiability of a or σ_k with probability 0. Hence, the solution of (3.1) is independent of a choice of modification for functions $\nabla a, \nabla \sigma_k$.

Proof. The proof of Sobolev differentiability of $\varphi_t(x)$ in x is given in [20] in the case when coefficients satisfy Lipschitz conditions only (no other additional assumptions).

Existence of equation (3.1) solution satisfying measurability condition 1) can be verified similarly to [21]. To do this, we have to apply the result on the existence of measurable modification for a limit of a random elements sequence that depends on a parameter (see for ex. [24, 25]), and observe that the process $\psi_t(x)$ can be obtained as a limit of measurable in (t, x, ω) iterations (see [21]).

Let us prove Theorem 3.1 at first, if coefficients of (1.1) satisfy the following condition:

$$\exists R_0 > 0 \forall x \in \mathbb{R}_+^d, \|x\| > R_0 : a(x) = 0, \sum_{k=1}^m \sigma_k(x)\sigma_k^*(x) = \mathbb{I}. \quad (3.2)$$

Let us select a sequence of continuously differentiable functions $\{a_n, \sigma_{n,k}, k = 1, \dots, m, n \geq 1\}$ such that

$$\sup_n \sup_x \left(\|a_n(x)\| + \|\nabla a_n(x)\| + \sum_{k=1}^m (\|\sigma_{n,k}(x)\| + \|\nabla \sigma_{n,k}(x)\|) \right) < \infty; \quad (3.3)$$

$$\sup_x \left(\|a_n(x) - a(x)\| + \sum_{k=1}^m \|\sigma_{n,k}(x) - \sigma_k(x)\| \right) \rightarrow 0, \quad n \rightarrow \infty; \quad (3.4)$$

$$\nabla a_n(x) \rightarrow \nabla a(x) \text{ and } \nabla \sigma_{n,k}(x) \rightarrow \nabla \sigma_k(x) \text{ as } n \rightarrow \infty \quad (3.5)$$

for a.a. x with respect to the Lebesgue measure.

For example, we may select $\{a_n, \sigma_{n,k}\}$ as some subsequence of convolutions $a_n = \tilde{a} * g_n$, $\sigma_{n,k} = \tilde{\sigma}_k * g_n$, where $\tilde{a}, \tilde{\sigma}_k$ are any bounded and Lipschitz continuous extensions to \mathbb{R}^d of functions a, σ_k , and

$$g_n(x) = g(nx)n^{-d}, \quad g \in C_0^\infty(\mathbb{R}^d), \quad g \geq 0, \quad \int_{\mathbb{R}^d} g(x)dx = 1.$$

Denote by $\varphi_t^n(x)$ a flow generated by (1.1)–(1.4) with coefficients $a_n, \sigma_{n,k}$, $k = 1, \dots, m$ instead of a, σ_k .

As it was mentioned in the Introduction, the Theorem holds true for the flow φ_t^n with continuously differentiable coefficients (see [21]).

Let us prove that

$$\forall t \geq 0 : P(\nabla \varphi_t = \psi_t) = 1. \quad (3.6)$$

It is sufficient to verify that

$$\forall p \geq 1 \quad \forall R > 0 \quad \forall T > 0 \quad \forall t \in [0, T] :$$

$$P\left(\int_{\substack{\|x\| \leq R \\ x \in \mathbb{R}_+^d}} (\|\varphi_t^n(x) - \varphi_t(x)\|^p + \|\nabla \varphi_t^n(x) - \psi_t(x)\|^p) dx \xrightarrow{n \rightarrow \infty} 0\right) = 1. \quad (3.7)$$

It follows from the standard moments estimates of SDEs with reflection (see, for example, [11]) that

$$\forall T > 0 \quad \forall R > 0 \quad \forall p > 1 :$$

$$\sup_n \sup_{\|x\| \leq R} E \sup_{t \in [0, T]} (\|\varphi_t^n(x)\|^p + \|\varphi_t(x)\|^p) < \infty. \quad (3.8)$$

It is also easy to verify that

$$\forall T > 0 \quad \forall p > 1 \quad \forall x \in \mathbb{R}^d : E\left(\sup_{t \in [0, T]} \|\varphi_t^n(x) - \varphi_t(x)\|^p\right) \rightarrow 0, \quad n \rightarrow \infty. \quad (3.9)$$

It follows from (3.9) and (1.1) that

$$\forall \zeta > 0 : P\left(\sup_{s \in [0, T]} |L_n(s) - L_0(s)| \geq \zeta\right) \rightarrow 0, \quad n \rightarrow \infty. \quad (3.10)$$

It follows from Theorem 2.4 that

$$\sup_x \sup_n (E \sup_{t \in [0, T]} \|\nabla \varphi_t^n(x)\|^p + E \sup_{t \in [0, T]} \|\psi_t(x)\|^p) < \infty. \quad (3.11)$$

So, to verify (3.6) it is sufficient to prove convergence in probability:

$$\forall t \geq 0 \quad \forall x \in \mathbb{R}_+^d : \nabla \varphi_t^n(x) \xrightarrow{P} \psi_t(x), \quad n \rightarrow \infty. \quad (3.12)$$

Further, we assume that x is fixed. Without loss of generality, it will be assumed that we have the uniform convergence with probability 1

$$P\left(\varphi_z^n(x) \xrightarrow{z \in [0, T]} \varphi_z(x), \quad n \rightarrow \infty\right) = 1. \quad (3.13)$$

Denote $\varphi_t^n := \varphi_t^n(x)$, $\varphi_t^0 := \varphi_t = \varphi_t(x)$, $\nabla\varphi_t^n := \nabla\varphi_t^n(x)$, and let $\nu_n(t)$ be a point measure constructed as in §2 using a process $\varphi_t^n(x)$.

Observe that we may solve equations for each column of ψ_t and $\nabla\varphi_t^n$ independently of other columns. Let $i \in \{1, \dots, d\}$ be fixed. Apply Theorem 2.6 for i th columns of ψ_t and $\nabla\varphi_t^n$. Take $\varphi_t^{n,d}$ (the d th coordinate of the process φ_t^n) in place of $x^n(t)$.

Put also

$$y^n(t) = \left(\frac{\partial\varphi_t^{n,j}}{\partial x_i}, j = \overline{1, d-1} \right),$$

$$z^n(t) = \frac{\partial\varphi_t^{n,d}}{\partial x_i}, y^0(t) = (\psi_t^{j,i}, j = \overline{1, d-1}), z^0(t) = \psi_t^{d,i},$$

$a_{n,0}(x, y, z) = \sum_{j=1}^{d-1} \frac{\partial a_n(\varphi_t^n)}{\partial x_j} y_j$, and so on.

Condition $P(\varphi_t^{0,d} = 0) = 0$ follows from the non-degeneracy of diffusion. Convergence (3.9) yields condition (2.8).

Let us verify convergence

$$\forall t \in [0, T]: \nabla a_n(\varphi_t^n) \xrightarrow{P} \nabla a(\varphi_t), n \rightarrow \infty, \quad (3.14)$$

$$\forall t \in [0, T]: \nabla\sigma_{n,k}(\varphi_t^n) \xrightarrow{P} \nabla\sigma_k(\varphi_t), n \rightarrow \infty. \quad (3.15)$$

Note that functions ∇a and $\nabla\sigma_k$ are discontinuous, in general. Also, ∇a_n and $\nabla\sigma_{n,k}$ do not uniformly converge to ∇a and $\nabla\sigma_k$. Therefore (3.14), (3.15) is not a consequence of (3.9) and (3.5), but more delicate result.

Lemma 3.5. *Let $(X, \rho_X), (Y, \rho_Y)$ be complete separable metric spaces, $\{\xi_n\}_{n \geq 0}$ be a sequence of random elements taking values in X , $\xi_n \xrightarrow{P} \xi_0, n \rightarrow \infty$, and $\{f_n\}_{n \geq 0}$ be a sequence of Borel functions, $f_n: X \rightarrow Y$.*

Assume that there exists a probability measure μ on X such that

- 1) for each $n \geq 1$ a distribution of ξ_n has a density ρ_n with respect to μ ;*
- 2) a sequence of densities $\{\rho_n\}_{n \geq 1}$ is uniformly integrable with respect to μ ;*
- 3) the following convergence in measure μ holds*

$$f_n \xrightarrow{\mu} f_0, n \rightarrow \infty.$$

Then $f_n(\xi_n) \xrightarrow{P} f_0(\xi_0), n \rightarrow \infty$.

This Theorem was proved in [13, Lemma 2] if $\Omega = X = Y$, $P = \mu$. Nothing changes in the proof if spaces are different, i.e., if Ω, X, Y do not coincide.

To apply Lemma 3.5 in a proof of (3.14), (3.15), we need the following estimates of the Green function for the solutions of the first boundary problem for parabolic equation in a half-space.

Lemma 3.6. [8] *Let $p(t, x, y)$ be Green's function for the solutions of the following boundary problem*

$$\begin{cases} \frac{\partial u(t, x)}{\partial t} = Lu(t, x), t > 0, x \in \mathbb{R}_+^d, \\ \frac{\partial u(t, x)}{\partial x_d} = 0, x_d = 0, \end{cases}$$

where

$$L = \sum_{i=1}^d a_i(x) \frac{\partial}{\partial x_i} + \sum_{i,j=1}^d b_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

a_i, b_{ij} are differentiable functions with bounded derivatives, the matrix $B(x) = \|b_{ij}(x)\|$ is symmetric, positive defined, and

$$\exists c > 0 \forall x \in \mathbb{R}_+^d : B(x) \geq c\mathbb{I}.$$

Then for any $t > 0$ there exist positive constants $K_1(t), K_2(t)$ such that

$$p(t, x, y) \leq K_1(t) \exp\{-K_2(t)\|x - y\|^2\}, \quad x, y \in \mathbb{R}_+^d.$$

This constants depends only on t, c and

$$\sup_x \max_{i,j} \{\|a_i(x)\|, \|b_{i,j}(x)\|, \|\nabla a_i(x)\|, \|\nabla b_{i,j}(x)\|\}.$$

Remark 3.7. Similar estimates for fundamental solutions of parabolic equations in \mathbb{R}^d are classical, see [10, 17]. Moreover, if

$$L = \sum_i a^i(x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j} \sum_k \sigma_k^j(x) \sigma_k^i(x) \frac{\partial^2}{\partial x_i \partial x_j},$$

then the Green's function $p(t, x, y)$ is a density of $\varphi_t(x)$ distribution.

Let us apply Lemma 3.5 to relations (3.14), (3.15). Put $\mu(dy) = f(y)dy$, where $f(y) = c(1 + \|y - x\|^{d+1})^{-1}$,

$$c = \left(\int_{\mathbb{R}_+^d} (1 + \|y\|^{d+1})^{-1} dy \right)^{-1}.$$

Let $p_n = p_n(y) = p_n(t, x, y)$ be a density (w.r.t. the Lebesgue measure) of $\varphi_t^n = \varphi_t^n(x)$ distribution.

By Lemma 3.6, we get $p_n(t, x, y) \leq K_1 e^{-K_2 \|y-x\|^2}$, where K_1, K_2 are independent of n .

The density of φ_t^n distribution with respect to μ equals

$$\frac{p_n(y)}{f(y)} \leq CK_1 e^{-K_2 \|y-x\|^2} (1 + \|y - x\|^{d+1}) \rightarrow 0, \quad \|y\| \rightarrow \infty.$$

This implies the uniform integrability of $\left\{ \frac{p_n}{f} \right\}_{n \geq 1}$ with respect to μ . The application of Lemma 3.5 yields (3.14) and (3.15).

To conclude the proof of (3.6), it remains to check (2.9), where $\tilde{\tau}^n(t) = \sup\{s \leq t : \varphi_s^n = 0\} \vee 0$ is the last hitting zero by a process φ^n that does not exceed t .

Let $\varepsilon > 0$ be given. It follows from (3.13) that

$$P(\exists s > \tilde{\tau}^0(t) + \varepsilon, \varphi_s^{n,d} = 0) \rightarrow 0, \quad n \rightarrow \infty.$$

Therefore

$$P(\tilde{\tau}^n(t) > \tilde{\tau}^0(t) + \varepsilon) \rightarrow 0, \quad n \rightarrow \infty. \quad (3.16)$$

The diffusion coefficient is not degenerate, so $\tilde{\tau}^0(t)$ is a point of growth for the process $L_0(\cdot) = L_0(\cdot, x)$ with probability 1. Thus

$$\forall \varepsilon > 0 : P(L_0(\tilde{\tau}^0(t)) - L_0(\tilde{\tau}^0(t) - \varepsilon) > 0) = 1. \quad (3.17)$$

It follows from (3.10) and (3.17) that

$$\forall \varepsilon > 0 : P(L_n \text{ has a point of growth in } [\tilde{\tau}^0(t) - \varepsilon, \tilde{\tau}^0(t) + \varepsilon]) \rightarrow 1, n \rightarrow \infty.$$

A process $\varphi^{n,d}$ hits zero in any point of growth of the process L_n . Hence

$$\forall \varepsilon > 0 : P(\tilde{\tau}^n(t) \geq \tilde{\tau}^0(t) - \varepsilon) \rightarrow 1, n \rightarrow \infty.$$

This and (3.16) imply (2.9).

Let us prove now that

$$P(\forall t \in [0, T] : \psi_t = \nabla \varphi_t) = 1. \quad (3.18)$$

First note that (3.6) implies

$$P(\forall t \in [0, T] \cap \mathbb{Q} : \psi_t = \nabla \varphi_t) = 1. \quad (3.19)$$

Let $p \geq 1$ and $R > 0$. Consider processes $\varphi_t(\cdot) = \varphi_t(x)$, $\psi_t(\cdot) = \psi_t(x)$, $t \in [0, T]$, $x \in \mathbb{R}_+^d$, $\|x\| \leq R$ as random elements with values in spaces $L_p(\{x \in \mathbb{R}_+^d, \|x\| \leq R\}; \mathbb{R}^d)$, $L_p(\{x \in \mathbb{R}_+^d, \|x\| \leq R\}; \mathbb{R}^{d \times d})$, respectively. For any x trajectories of the process $\varphi_t(x)$ are continuous in t , trajectories of $\psi_t(x)$ are continuous from the right. Therefore, by estimates (3.8), (3.11), it follows that trajectories of the processes $\varphi_t(\cdot) = \varphi_t(x)$ and $\psi_t(\cdot) = \psi_t(x)$, $t \in [0, T]$, $x \in \mathbb{R}_+^d$, $\|x\| \leq R$ are continuous in t in $L_p(\{x \in \mathbb{R}_+^d, \|x\| \leq R\}; \mathbb{R}^d)$, continuous from the right in t in $L_p(\{x \in \mathbb{R}_+^d, \|x\| \leq R\}; \mathbb{R}^{d \times d})$, respectively. Relation (3.18) follows from this, (3.19), and the fact that Sobolev derivative is closed operator.

Thus the Theorem is proved under the assumption (3.2).

To prove the general case let us apply localization technique. Put

$$\begin{aligned} a_n(x) &= a(x)h_n(x), \\ \sigma_{n,k}(x) &= \sigma_k(x)h_n(x) + e_k g_n(x), \end{aligned}$$

where functions $h_n, g_n \in C^\infty(\mathbb{R}^d)$ and vectors $e_k \in \mathbb{R}^d$ are such that $h_n(x) = 1$ for $\|x\| \leq n$, $h_n(x) = 0$ for $\|x\| \geq n+1$, $g_n(x) = 0$ for $\|x\| \leq n$, and $g_n(x) = 1$ for $\|x\| \geq n+1$, $\sum_k e_k e_k^* = \mathbb{I}$, and

$$\sum_k \sigma_{n,k}(x) \sigma_{n,k}^*(x) \geq c \mathbb{I}, x \in \mathbb{R}_+^d$$

where $c > 0$.

Remark 3.8. $\sum_k \sigma_{n,k}(x) \sigma_{n,k}^*(x) = \sum_k e_k e_k^* = \mathbb{I}$, $\|x\| \geq n+1$.

Denote the solution of (1.1) by $\varphi_t^n(x)$, where coefficients $a_n, \sigma_{n,k}$ are taken instead of a, σ_k . Similarly, let $\psi_t^n(x)$ be a solution of (3.1) with coefficients that are corrected by appropriate way. Remind that $\psi_t^n(x) = \nabla \varphi_t^n(x)$. Functions $a_n, \sigma_{n,k}$ satisfy the conditions of the Theorem and assumption (3.2). Moreover, by coincidence of coefficients inside the set $\{x \in \mathbb{R}_+^d : \|x\| \leq n\}$, we get

$$\forall x \in \mathbb{R}_+^d \forall n \geq 1 : P(\varphi_t(x) = \varphi_t^n(x), t \leq \tau_n(x)) = 1, \quad (3.20)$$

where $\tau_n(x) = \inf\{t \geq 0 : \|\varphi_t(x)\| \geq n\}$ is an exit moment from the hemisphere $\{y \in \mathbb{R}_+^d : \|y\| \leq n\}$ for the process $\varphi(x)$. It was mentioned above that flows $\varphi_t(x), \varphi_t^n(x)$ are continuous in (t, x) with probability 1 (see [19]). So, it follows

from (3.20) that for any bounded set $U \subset \mathbb{R}_+^d$ and a.a. ω there exists $n_0 = n_0(\omega)$ such that

$$\varphi_t^n(x, \omega) = \varphi_t(x, \omega), \quad t \in [0, T], n \geq n_0, x \in U. \quad (3.21)$$

The Sobolev derivative is a local operator. That is, if functions are equal at some domain, then their derivatives are equal for λ^d -almost all points of this domain. Let ω and n be such that (3.21) is satisfied. Then

$$\nabla \varphi_t^n(x, \omega) = \nabla \varphi_t(x, \omega), \quad t \in [0; T] \text{ for } \lambda^d\text{-a.a. } x \in U.$$

By the same argument as for usual SDE, it can be verified that if coefficients of equation with nulling of the type (2.7) coincide in some domain, then solutions coincide until the exit moment from this domain. Thus

$$\psi_t(x) = \psi_t^n(x) = \nabla \varphi_t^n(x) = \nabla \varphi_t(x), \quad t \in [0; T]$$

for λ^d -a.a. $x \in U$, a.a. ω and all n such that (3.21) is satisfied. So, the left hand side of the last equality is equal to the right hand side for λ^d -a.a. $x \in U$, a.a. ω . The Theorem is proved. \square

4. The Proof of Theorem 2.6

For the sake of simplicity, we will assume that processes $x_n(t), y_n(t), z_n(t)$ are one-dimensional and $m = 1$. The general case can be considered similarly but with more cumbersome formulas.

Further, we will use a numerous constants. If these constants are independent on n then we may omit an index and possibly understand different constants in the same formula denoted by the same symbol.

It follows from Theorem 2.4 that

$$\sup_n E \sup_{t \in [0, T]} (|y^n(t)|^2 + |z^n(t)|^2) < \infty. \quad (4.1)$$

Let us apply Ito's formula and then assumptions of Theorem 2.6 to the first equation in (2.7). Then

$$\begin{aligned} E(y^n(t) - y^0(t))^2 &\leq E(y^n(0) - y^0(0))^2 + K \int_0^t E(y^n(s) - y^0(s))^2 ds \\ &+ K \int_0^t E(z^n(s) - z^0(s))^2 ds + o_n(1), \quad t \in [0, T], \end{aligned} \quad (4.2)$$

where $K = \text{const}$,

$$o_n(1) = KE \int_0^T \sum_{i=0}^1 (a_{n,i}(x^n(t), y^0(t), z^0(t)) - a_{0,i}(x^0(t), y^0(t), z^0(t)))^2 dt.$$

Observe that $o_n(1)$ is independent of t and converges to 0 as $n \rightarrow \infty$. Indeed, it follows from the linear growth condition 1) of Theorem 2.6 and (4.1) that the integrand is bounded by integrable random variable $\text{const}(1 + y^0(t))^2 + (z^0(t))^2$. Assumption 3) of the Theorem and uniform Lipschitz condition 2) yields that the integrand converges to 0 in probability. So, $o_n(1)$ converges to zero by Lebesgue's dominated convergence theorem.

Let $c > 0$ be a constant. Introduce two sequences of stopping times $\sigma^{k,n} = \sigma_c^{k,n}$, $\tau^{k,n} = \tau_c^{k,n}$:

$$\begin{aligned}\sigma_c^{0,n} &= 0, \quad \tau_c^{k,n} = \inf\{t \geq \sigma_c^{k,n} : |x^n(t)| = c\} \wedge T, \\ \sigma_c^{k+1,n} &= \inf\{t \geq \tau_c^{k,n} : x^n(t) = 0\} \wedge T, \quad k \geq 0.\end{aligned}$$

Denote $\mathcal{I}_n(s, t) = \int_s^t b_{n,1}(x^n(u), y^n(u), z^n(u))dw(u)$. It will be supposed that continuous in (s, t) modification of $\mathcal{I}_n(s, t)$ is already selected.

It follows from Definition 2.1 that

$$\begin{aligned}\frac{1}{8}(z^n(t) - z^0(t))^2 &\leq (z^n(0) - z^0(0))^2 \tag{4.3} \\ &+ \sum_{k,m} \mathbb{1}_{t \in [\tau^{k,n}, \sigma^{k+1,n}] \cap [\tau^{m,0}, \sigma^{m+1,0})} \left\{ (\mathcal{I}_n(\tau^{k,n}, t) - \mathcal{I}_0(\tau^{k,n}, t))^2 \cdot \right. \\ &\quad \left. \cdot \mathbb{1}_{\tau^{k,n} \geq \tau^{m,0}} + (\mathcal{I}_n(\tau^{m,0}, t) - \mathcal{I}_0(\tau^{m,0}, t))^2 \mathbb{1}_{\tau^{m,0} > \tau^{k,n}} \right\} \\ &+ \left(\sum_k \mathbb{1}_{t \in [\tau^{k,n}, \sigma^{k+1,n})} (z^n(\tau^{k,n}))^2 + \sum_m \mathbb{1}_{t \in [\tau^{m,0}, \sigma^{m+1,0})} (z^0(\tau^{m,0}))^2 \right) \\ &\quad + \mathbb{1}_{t \in \cup_k [\sigma^{k,n}; \tau^{k,n}) \cup \cup_m [\sigma^{m,0}; \tau^{m,0})} \sup_{s \in [0, T]} ((z^n(s))^2 + (z^0(s))^2) \\ &\quad + \{ \dots ds \} = A_0 + A_1 + A_2 + A_3 + \{ \dots ds \},\end{aligned}$$

where $\{ \dots ds \}$ are expressions with integrals w.r.t. ds which are similar to sums of stochastic integrals in (4.3).

It follows from (2.3), assumption 7) of the Theorem, and Cauchy inequality that

$$EA_3 \leq K \sqrt{P(t \in \cup_k [\sigma_c^{k,n}, \tau_c^{k,n}) \cup \cup_m [\sigma_c^{m,0}, \tau_c^{m,0})} = K o_{c,n,t}(1), \tag{4.4}$$

where K is independent of n and c .

Using (2.8), (2.9), and (2.10) we get

$$\lim_{c \rightarrow 0+} \overline{\lim}_{n \rightarrow \infty} o_{c,n,t}(1) = 0.$$

Let us estimate EA_1 . Observe that for all $t_1, t_2, 0 \leq t_1 \leq t_2$ and arbitrary function f :

$$|f(t_2) - f(t_1)| \leq 2 \sup_{s \in [0, t_2]} |f(s) - f(0)|.$$

Combining this and Burkholder's inequality, we get the estimate:

$$\begin{aligned}EA_1 &\leq 8E \sup_{s \in [0, t]} (\mathcal{I}_n(0, s) - \mathcal{I}_0(0, s))^2 \\ &\leq 32E \int_0^t (b_{n,1}(x^n(u), y^n(u), z^n(u)) - b_{0,1}(x^0(u), y^0(u), z^0(u)))^2 du\end{aligned} \tag{4.5}$$

In the same way as (4.2), we obtain the inequality:

$$EA_1 \leq KE \int_0^t ((y^n(u) - y^0(u))^2 + (z^n(u) - z^0(u))^2) du + o_n(1),$$

where $o_n(1) = KE \int_0^T (b_{n,1}(x^n(t), y^0(t), z^0(t)) - b_{0,1}(x^0(t), y^0(t), z^0(t)))^2 dt$ is independent of t , $o_n(1)$ converges to 0 as $n \rightarrow \infty$, and a constant K is independent of n .

Let us estimate EA_2 . Let $\delta > 0$ be fixed. Then

$$\begin{aligned}
& E \sum_k \mathbb{I}_{t \in [\tau_c^{k,n}; \sigma_c^{k+1,n})} (z_n(\tau_c^{k,n}))^2 \\
& \leq E \sup_{s \in [0, T]} (z_n(s))^2 \sum_k \mathbb{I}_{\tau_c^{k,n} - \sigma_c^{k,n} > \delta} \mathbb{I}_{t \in [\tau_c^{k,n}; \sigma_c^{k+1,n})} \\
& + E \sup_{\substack{|s_1 - s_2| \leq \delta \\ x_n(s) \neq 0, s \in [s_1, s_2]}} |z_n(s_2) - z_n(s_1)|^2 \\
& \leq K \sqrt{P\left(\cup_k (\{\tau_c^{k,n} - \sigma_c^{k,n} > \delta\} \cap \{t \in [\tau_c^{k,n}, \sigma_c^{k+1,n})\})\right)} + o_\delta(1) \\
& = K \sqrt{\varepsilon_{n,c,t}(\delta)} + o_\delta(1),
\end{aligned} \tag{4.6}$$

where K is independent of c, n, δ (see Theorem 2.4), and $o_\delta(1) \rightarrow 0$ as $\delta \rightarrow 0+$ uniformly in n (see Theorem 2.5).

Verify that

$$\forall \delta > 0: \lim_{c \rightarrow 0+} \overline{\lim}_{n \rightarrow \infty} \varepsilon_{n,c,t}(\delta) = 0. \tag{4.7}$$

Indeed, it follows from (2.9) and (2.8) that

$$\begin{aligned}
& \overline{\lim}_{n \rightarrow \infty} \varepsilon_{n,c,t}(\delta) \\
& \leq \overline{\lim}_{n \rightarrow \infty} P\left(\cup_k (\{\tau_c^{k,n} - \tilde{\tau}^n(t) > \frac{\delta}{2}\} \cap \{t \in [\tau_c^{k,n}, \sigma_c^{k+1,n})\})\right) \\
& + \overline{\lim}_{n \rightarrow \infty} P\left(\cup_k (\{\tilde{\tau}^n(t) - \sigma_c^{k,n} > \frac{\delta}{2}\} \cap \{t \in [\tau_c^{k,n}, \sigma_c^{k+1,n})\})\right) \\
& \leq \overline{\lim}_{n \rightarrow \infty} P\left(\forall s \in [\tilde{\tau}^n(t), \tilde{\tau}^n(t) + \frac{\delta}{2}] : |x^n(s)| < c\right) \\
& + \overline{\lim}_{n \rightarrow \infty} P\left(\forall s \in [\tilde{\tau}^n(t) - \frac{\delta}{2}, \tilde{\tau}^n(t)] : |x^n(s)| < c\right) \\
& \leq P\left(\forall s \in [\tilde{\tau}^0(t), \tilde{\tau}^0(t) + \frac{\delta}{2}] : |x^0(s)| < \frac{c}{2}\right) \\
& + P\left(\forall s \in [\tilde{\tau}^0(t) - \frac{\delta}{2}, \tilde{\tau}^0(t)] : |x^0(s)| < \frac{c}{2}\right).
\end{aligned} \tag{4.8}$$

It follows from condition (2.10) that the right hand side of (4.8) converges to zero as $c \rightarrow 0+$.

Similarly we can estimate the expectation of the second term in A_2 and also a term $\{\dots ds\}$ in (4.3).

Combining the above estimates, we get

$\exists K > 0 \forall \delta > 0 \forall c > 0 :$

$$\begin{aligned}
& E(z^n(t) - z^0(t))^2 + E(y^n(t) - y^0(t))^2 \\
& \leq K \left[E((z^n(0) - z^0(0))^2 + E(y^n(0) - y^0(0))^2 \right. \\
& + E \int_0^t ((z^n(s) - z^0(s))^2 + (y^n(s) - y^0(s))^2) ds + o_{c,n,t}(1) \\
& \left. + o_n(1) + o_\delta(1) + \sqrt{\varepsilon_{n,c,t}(\delta)} \right], \tag{4.9}
\end{aligned}$$

where

$$\lim_{n \rightarrow \infty} o_n(1) = 0 \text{ and } o_n(1) \text{ is independent of } c, t, \delta;$$

$$\lim_{\delta \rightarrow 0+} o_\delta(1) = 0 \text{ and } o_\delta(1) \text{ is independent of } n, t, c;$$

for any $t \in [0, T]$ and any $\delta > 0 :$ $\lim_{c \rightarrow 0+} \overline{\lim}_{n \rightarrow \infty} \varepsilon_{n,c,t}(\delta) = 0, 0 \leq \varepsilon_{n,c,t}(\delta) \leq 1;$

for any $t \in [0, T] :$ $\lim_{c \rightarrow 0+} \overline{\lim}_{n \rightarrow \infty} o_{c,n,t}(1) = 0, 0 \leq o_{c,n,t}(1) \leq 1.$

Lemma 4.1. *Let $f, \varepsilon : [0, \infty) \rightarrow [0, \infty)$ be non-negative measurable locally bounded functions, $K > 0,$*

$$f(t) \leq K \int_0^t f(s) ds + \varepsilon(t), t \geq 0.$$

Then

$$f(t) \leq \varepsilon(t) + K e^{Kt} \int_0^t \varepsilon(s) ds, t \geq 0.$$

The proof of the Lemma is similar to the proof of the Gronwall-Bellman lemma. It follows from (4.9) and Lemma 4.1 that

$$\begin{aligned}
& E((y^n(t) - y^0(t))^2 + (z^n(t) - z^0(t))^2) \\
& \leq K_1 e^{K_2 t} \left(o_{c,n,t}(1) + o_n(1) + o_\delta(1) + \sqrt{\varepsilon_{n,c,t}(\delta)} + E(y^n(0) - y^0(0))^2 \right. \\
& \left. + E(z^n(0) - z^0(0))^2 + \int_0^t \left(\sqrt{\varepsilon_{n,c,s}(\delta)} + o_{c,n,s}(1) \right) ds \right).
\end{aligned}$$

Since $0 \leq o_{c,n,t}(1) \leq 1, 0 \leq \varepsilon_{n,c,t}(\delta) \leq 1,$ then it follows from Lebesgue's dominated convergence theorem that for any $\delta > 0 :$

$$\lim_{c \rightarrow 0+} \overline{\lim}_{n \rightarrow \infty} \int_0^t \left(\sqrt{\varepsilon_{n,c,s}(\delta)} + o_{c,n,s}(1) \right) ds = 0.$$

Therefore

$$\overline{\lim}_{n \rightarrow \infty} E((y^n(t) - y^0(t))^2 + (z^n(t) - z^0(t))^2) \leq K_1 e^{K_2 t} o_\delta(1).$$

Note that $o_\delta(1) \rightarrow 0, \delta \rightarrow 0+.$ This completes the proof of Theorem 2.6.

5. Comparison of Different Types of Differentiability for Stochastic Reflecting Flow

As it was mentioned in the Introduction, Sobolev's differentiability implies existence of usual partial derivatives for a.a. x with respect to the Lebesgue measure. Therefore, if coefficients of the SDE with reflection satisfy the Lipschitz condition, then $\varphi_t \in \cap_{p>1} W_{p,\text{loc}}^1$ a.s. (see [20]), and

$$P(\text{partial derivatives } \frac{\partial \varphi_t(x)}{\partial x_k}, k = 1, \dots, m \text{ exist for } \lambda^d\text{-a.a. } x) = 1. \quad (5.1)$$

Remark. We do not assume here that diffusion coefficient is non-degenerative.

If in addition conditions of Theorem 3.1 are satisfied, i.e., for any $x \in \mathbb{R}_+^d$ the matrix $\sum_{k=1}^m \sigma_k(x) \sigma_k^*(x)$ is positive defined, then

$$P(\frac{\partial \varphi_t(x)}{\partial x_k} = \psi_t^k(x), k = 1, \dots, m \text{ for } \lambda^d\text{-a.a. } x) = 1,$$

where $\psi_t^k(x)$ is k th column of a process $\psi_t(x)$ constructed in Theorem 3.1.

Fubini's theorem and (5.1) imply that for λ^d -a.a. x

$$P(\text{partial derivatives } \frac{\partial \varphi_t(x)}{\partial x_k}, k = 1, \dots, m \text{ exist}) = 1.$$

It is natural to investigate the following problem. Does the last equality hold for any $x \in \mathbb{R}^{d-1} \times (0; \infty)$?

Let us extend coefficients a and σ_k to \mathbb{R}^d taking into account smoothness of these functions on \mathbb{R}_+^d . I.e., if coefficients are Lipschitz continuous then choose Lipschitz continuous extensions, if they are continuously differentiable then select continuously differentiable extension, etc. Denote by φ_t^{ord} the flow on \mathbb{R}^d generated by SDE (without reflection) with extended coefficients.

Assume that ω is such that $\varphi_z(x)$, $z \in [0, s]$ have not visited the hyperplane $\partial \mathbb{R}_+^d$. Then φ_s coincides a.s. with the flow φ_s^{ord} in some (random) neighborhood $U(x)$ of x . Denote by $\varphi_{s,t}$ a solution of SDE with reflection started from x at an instant s . Observe that $\varphi_t = \varphi_{s,t} \circ \varphi_s$. Hence, $\varphi_t(y) = \varphi_{s,t}(\varphi_s^{\text{ord}}(y))$, $y \in U(x)$.

Suppose that φ_s^{ord} is differentiable at the point x . If $\varphi_s^{\text{ord}}(x)$ gets into a set where $\varphi_{s,t}$ is differentiable, then this will imply differentiability of φ_t at x . Note that the distributions of random maps $\varphi_{s,t}$ and φ_{t-s} are equal. So for λ^d -a.a. y partial derivatives $\frac{\partial \varphi_{s,t}(y)}{\partial x_k}$, $k = 1, \dots, m$ exist.

Since random maps φ_{st} and φ_s are independent, the reasoning above implies the following statement.

Lemma 5.1. *Assume that functions a and σ_k satisfy the Lipschitz conditions. Suppose also that for some $s > 0$ the distribution of $\varphi_s(x)$ or $\varphi_s^{\text{ord}}(x)$ are absolute continuous, and a random map $\varphi_s^{\text{ord}}(\cdot)$ is differentiable at x with probability one. Then partial derivatives $\frac{\partial \varphi_t(x)}{\partial x_k}$, $k = 1, \dots, m$ exist and $\frac{\partial \varphi_t(x)}{\partial x_k} = \psi_t^k(x)$ for a.a. ω such that $\varphi_z(x)$, $z \in [0, s]$ has not visited hyperplane $\partial \mathbb{R}_+^d$.*

The following conditions that ensure continuous differentiability of φ_s^{ord} w.r.t. the initial condition are well-known:

(A) there exists $\varepsilon > 0$ such that coefficients of the equation belong to a class $C^{1+\varepsilon}$,

or

(B) diffusion coefficient is constant and a drift coefficient belongs to a class C^1 .

A sufficient condition that guarantees the absolute continuity of $\varphi_s^{ord}(x)$ distribution is the non-degeneracy of the diffusion.

Note that, if $x \in \mathbb{R}^{d-1} \times (0; \infty)$ and s is sufficiently small, then the probability that $\varphi_z(x) \notin \partial\mathbb{R}_+^d, z \in [0; s]$ is close to 1.

Combining the reasoning above, we get the following statement on differentiability.

Lemma 5.2. *Assume that functions a, σ_k satisfy the Lipschitz condition and at least one of the conditions (A) or (B) holds. Let $t \geq 0$ be fixed and $x \in \mathbb{R}^{d-1} \times (0; \infty)$ be such that the matrix $\sum_{k=1}^m \sigma_k(x)\sigma_k^*(x)$ is positive defined. Then partial derivatives $\frac{\partial\varphi_t(x)}{\partial x_k}, k = 1, \dots, m$ exist and are equal to $\psi_t^k(x)$ with probability 1.*

Recall that Lipschitz continuous functions are differentiable for a.a. x (see Rademacher's theorem, [9]). Therefore, if we find conditions such that the reflecting flow satisfies the local Lipschitz condition w.r.t. the initial data and if assumptions of Lemma 5.2 are satisfied, then this will imply existence of Fréchet derivative $\nabla\varphi_t(x)$ with probability 1. This derivative certainly coincides with the Sobolev derivative. The reasoning above fits for any reflecting flow in any domain G (not only for $G = \mathbb{R}_+^d$). Let us formulate the general result.

Lemma 5.3. *Assume that there exists a unique solution φ_t to an SDE with reflection in a closed domain G . Suppose that for any $t > 0$ and a.a. ω a map $\varphi_t(\cdot, \omega)$ is locally Lipschitz continuous. Assume that functions a and σ_k satisfy one of the conditions (A) or (B), and $x \in \text{int}G$ is such that a matrix $\sum_{k=1}^m \sigma_k(x)\sigma_k^*(x)$ is positively defined. Then*

$$\forall t > 0 : P(\text{Fréchet derivative } \nabla\varphi_t(x) \text{ exists}) = 1.$$

Moreover, if $G = \mathbb{R}_+^d$ then

$$\forall t > 0 : P(\nabla\varphi_t(x) = \psi_t(x)) = 1,$$

where $\psi_t(x)$ is constructed in Theorem 3.1.

Remark 5.4. We don't formulate the precise assumptions about the domain, because to prove the Lemma it is sufficient to verify that the following two facts are true:

a) $\varphi_t(y) = \varphi_{s,t}(\varphi_s^{ord}(y)), y \in U(x)$ if $\varphi_z(x), z \in [0, s]$ have not visited the boundary of the domain,

b) the distribution of $\varphi_s(x)$ or $\varphi_s^{ord}(x)$ is absolute continuity for any $s > 0$.

For example, the assumptions of the Lemma are satisfied if

1. $G = \mathbb{R}_+^d, a \in C^1, a$ is of linear growth, $\sigma_k = \text{const}, \det \sum_{k=1}^m \sigma_k\sigma_k^* > 0$.

2. $G = [0; \infty)^d, a \in C^1, a$ is of linear growth, $\sigma_k = \text{const}, \sum_{k=1}^m \sigma_k\sigma_k^* = \mathbb{I}$, and initial point x belongs to $(0; \infty)^d$. In this case, the solution $\varphi_t(x), t \geq 0$ with probability 1 does not get into "bad" points of the boundary which have a form $x_j = x_k = 0$. This case was considered in [7].

3. A domain G has a sufficiently smooth boundary, and $\varphi_t(x), t \geq 0$ is a Brownian reflecting flow, i.e., $a = 0$, $\sigma_k = \text{const}$, $\det \sum_{k=1}^m \sigma_k \sigma_k^* > 0$. This case was considered in [6].

Remark 5.5. Assume that φ_t is a reflecting flow in a “nice” domain G with possible oblique reflection at the boundary, where the direction of the reflection and the boundary of G are smooth enough. Equations for derivatives may be obtained using localization technique.

Suppose at first that there exists C^2 diffeomorphism $f : G \rightarrow \mathbb{R}_+^d$ such that $\bar{\varphi}_t(x) := f(\varphi_t(x))$ satisfies conditions of Theorem 3.1. Then we know an equation for $\nabla \bar{\varphi}_t(x)$, so $\nabla \varphi_t(x) = \nabla(f^{-1}(\bar{\varphi}_t(x))) = (\nabla f^{-1})(\bar{\varphi}_t(x)) \nabla \bar{\varphi}_t(x)$. It would be interesting to obtain a direct equation for a derivative $\nabla \varphi_t(x)$ or some representation in a form of some multiplicative functional of $\varphi_t(x)$ as it was done by Burdzy [6] for Brownian reflecting flow. It seems that similar representation can be obtained, but the proof will be non-trivial.

Now suppose that the domain can be represented as a finite union of relatively open sets $G = \cup_{k=1}^n O_k$ which are C^2 -diffeomorphic to a ball $\{x \in \mathbb{R}^d : \|x\| < 1\}$ or a half-ball $\{x \in \mathbb{R}^d : \|x\| < 1, x_d \geq 0\}$. There is no problem to define a derivative when a solution belongs to a set diffeomorphic to a ball (because it does not visit ∂G). In the second case, it is possible to make a local change of variables such that it transforms initial SDE in O_k into SDE in half-ball $\{x \in \mathbb{R}^d : \|x\| < 1, x_d \geq 0\}$ with normal reflection at $\{x \in \mathbb{R}^d : \|x\| < 1, x_d = 0\}$ (cf. [1] for details).

To obtain a representation of $\nabla \varphi_t$, we have to trace a walk of φ_t between different sets O_k , obtain a representation for $\nabla \varphi_t(x)$ in each O_k and glue derivatives. Unfortunately, this method does not give an SDE for the derivative expressed through intrinsic properties of G . The obtained expression will depend on the representation $G = \cup_{k=1}^n O_k$.

Remark 5.6. Assume now that domain G is a polyhedron or a cone. Consider reflecting SDE with possible oblique reflection at the boundary. Suppose that for any x a process $\varphi_t(x)$ does not hit “bad” points of the boundary (edges of a polyhedron or vertex of a cone) with probability 1. Then such flows can be treated similarly to the case of reflecting flow in smooth domain (see previous remark). However, it should be noted that the verification of non-hitting of bad points may be a serious problem. Some sufficient conditions are given in [15, 16, 26].

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