Communications on Stochastic Analysis
Vol. 7, No. 1 (2013) 39-63

Serials Publications
www.serialspublications.com

LARGE DEVIATIONS FOR THE SHELL MODEL OF TURBULENCE PERTURBED BY LÉVY NOISE

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ABSTRACT. The Laplace principle for the strong solution of the stochastic shell model of turbulence perturbed by Lévy noise is established in a suitable Polish space using weak convergence approach. The large deviation principle is proved using the well known results of Varadhan and Bryc.

1. Introduction

The large deviations theory is among the most classical areas in probability theory with many deep developments and applications. Several authors have established the Wentzell-Freidlin type large deviation estimates for a class of infinite dimensional stochastic differential equations (see for eg., Budhiraja and Dupuis [9], Da Prato and Zabczyk [12], Kallianpur and Xiong [20]). In these works the proofs of large deviation principle (LDP) usually rely on first approximating the original problem by time-discretization so that LDP can be shown for the resulting simpler problems via contraction principle, and then showing that LDP holds in the limit. The discretization method to establish LDP was introduced by Wentzell and Freidlin[17]. Dupuis and Ellis [14] have combined weak convergence methods to the stochastic control approach developed earlier by Fleming [16] to the large deviations theory.

The literature associated to the LDP of stochastic partial differential equations with Lévy noises is very few. De Acosta [2, 3] first studied the large deviations for Lévy processes on Banach spaces and large deviations for solutions of stochastic differential equations driven by Poisson measures. Recently Budhiraja, Dupuis, and Maroulas [7] and Maroulas [25], using the theorems of Varadhan and Bryc [13], have extended the result of Budhiraja and Dupuis [9] to prove the LDP for stochastic differential equations with Poisson noises by first establishing the Laplace principles in Polish spaces using the weak convergence approach. The other notable recent work is due to Swiech and Zabczyk [27], where the large deviation principle for solutions of abstract stochastic evolution equations perturbed by small Lévy noise is proved using the theorems of Varadhan and Bryc coupled with the techniques of Feng and Kurtz [15], viscosity solutions of Hamilton-Jacobi-Bellman equations in Hilbert spaces and control theory.

Received 2012-2-20; Communicated by P. Sundar.
2000 *Mathematics Subject Classification.* Primary 60F10; Secondary 60G51, 60H15, 76D06.
Key words and phrases. Large deviations, Lévy processes, GOY model.

To the best of our knowledge, the only work available in the area of LDP for the fluid dynamics models with jump processes is due to Xu and Zhang [30], where they applied the theory of De Acosta to establish the LDP for 2-D Navier-Stokes equations with additive Lévy noises.

This work deals with an infinite dimensional shell model, a mathematical turbulence model, that received increasing attention in recent years. Apparently there are only a few rigorous works on infinite dimensional shell model, namely Constantin, Levant and Titi [11], and Barbato, Barsanti, Bessaih and Flandoli [4] one in the deterministic case and the other in the stochastic case with additive noise respectively. In both of these works a variational semigroup formulation has been introduced. The work by Manna, Sritharan and Sundar [23] deals with the existence and uniqueness of the strong solutions of the stochastic shell model of turbulence perturbed by multiplicative noise. They have also established a LDP for the solution of the shell model by using weak convergence approach developed on the theory by Budhiraja and Dupuis [9]. The LDP for the inviscid shell models has been proved by Bessaih and Millet [5]. Recently Manna and Mohan [24] has proved the existence and uniqueness of the strong solutions of the shell model of turbulence perturbed by Lévy noise.

In this work, the authors established the LDP for the shell model of turbulence with Lévy noise by proving the Laplace principle for the strong solution in certain Polish space using the weak convergence approach developed by Budhiraja, Dupuis and Maroulas [7], and Maroulas [25] and finally applying the well known results by Varadhan and Bryc.

The main result of this paper is as follows:

Theorem 1.1 (Main Theorem)**.** *Let the stochastic shell model of turbulence perturbed by Lévy Noise described by*

$$\begin{aligned} \mathrm{d}u^\varepsilon + \big[\nu A u^\varepsilon + B(u^\varepsilon)\big]\mathrm{d}t &= f(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(t,u^\varepsilon)\mathrm{d}W(t) + \varepsilon\int_Z g(u^\varepsilon,z)\tilde{N}(dt,dz) \\ u^\varepsilon(0) &= \xi, \end{aligned} \tag{1.1}$$

has a unique strong solution in the Polish space $X = \mathcal{D}([0,T];H)\cap \mathrm{L}^2(0,T;V)$. Let the solution be denoted by $u^\varepsilon = \mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot), \varepsilon N^{\varepsilon^{-1}}\right)$. Then the family $\{u^\varepsilon : \varepsilon > 0\}$ satisfies Large Deviation Principle in X with the rate function I given by

$$I(\zeta) = \inf_{(\psi,\phi)\in\mathbb{S}_\zeta}\left\{\int_0^T\int_Z \ell(\phi(t,z))\lambda(dz)\mathrm{d}t + \frac{1}{2}\int_0^T \|\psi(s)\|_0^2 ds\right\}.$$

The construction of the paper is as follows. The next section is devoted to the formulation the abstract stochastic GOY model, the energy estimates, the existence and uniqueness of strong solutions. Proofs have been omitted as these results are already been proved by the authors [24]. In the last Section proof of the main theorem has been given in a systematic way.

2. The Stochastic GOY Model of Turbulence

2.1. Preliminaries. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space equipped with an increasing family $\{\mathcal{F}_t\}_{0\le t\le T}$ of sub-sigma-fields of $\mathcal{F}$ satisfying the usual conditions of right continuity and $\mathbb{P}$-completeness. Let H be a real separable Hilbert space and Q be a strictly positive, symmetric, trace class operator on H.

Definition 2.1. A stochastic process $\{W(t)\}_{0\leq t\leq T}$ is said to be an H-valued $\mathcal{F}_t$-adapted Wiener process with covariance operator Q if

- For each non-zero $h \in H$, $|Q^{1/2}h|^{-1}(W(t), h)$ is a standard one dimensional Wiener process,
- For any $h \in H, (W(t), h)$ is a martingale adapted to $\mathcal{F}_t$.

If W is a an H-valued Wiener process with covariance operator Q with $\mathrm{Tr}\, Q < \infty$, then W is a Gaussian process on H and $\mathbb{E}(W(t)) = 0, \quad \mathrm{Cov}\ (W(t)) = tQ, \quad t \geq 0$. Let $H_0 = Q^{1/2}H$. Then H_0 is a Hilbert space equipped with the inner product $(\cdot,\cdot)_0$, $(u, v)_0 = (Q^{-1/2}u, Q^{-1/2}v), \ \forall u, v \in H_0$, where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$. Since Q is a trace class operator, the imbedding of H_0 in H is Hilbert-Schmidt.

Let L_Q denote the space of linear operators S such that $SQ^{1/2}$ is a Hilbert-Schmidt operator from H to H. Define the norm on the space L_Q by $|S|^2_{\mathrm{L}_Q} = \mathrm{Tr}(SQS^*)$.

Definition 2.2. A càdlàg adapted process is called a Lévy process if it has stationary independent increments and is stochastically continuous.

The jump of X_t at $t \geq 0$ is given by $\triangle X_t = X_t - X_{t-}$. Let $Z \in \mathcal{B}(H)$, define

$$N(t, Z) = N(t, Z, \omega) = \sum_{s:0<s\leq t} \chi_Z(\triangle X_s).$$

In other words, $N(t, Z)$ is the number of jumps of size $\triangle X_s \in Z$ which occur before or at time t. $N(t, Z)$ is called the *Poisson random measure* (or *jump measure*) of $(X_t)_{t\geq 0}$. The differential form of this measure is written as $N(dt, dz)(\omega)$.

We call $\tilde{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt$ a *compensated Poisson random measure (cPrm)*, where $\lambda(dz)dt$ is known as *compensator* of the Lévy process $(X_t)_{t\geq 0}$. Here dt denotes the Lesbesgue measure on $\mathcal{B}(\mathbb{R}^+)$, and $\lambda(dz)$ is a σ-finite measure on $(Z, \mathcal{B}(Z))$.

Definition 2.3. Let H and F be separable Hilbert spaces. Let $F_t := \mathcal{B}(H) \otimes \mathcal{F}_t$ be the product σ-algebra generated by the semi-ring $\mathcal{B}(H) \times \mathcal{F}_t$ of the product sets $Z \times F, \quad Z \in \mathcal{B}(H), \quad F \in \mathcal{F}_t$ (where $\mathcal{F}_t$ is the filtration of the additive process $(X_t)_{t\geq 0}$). Let $T > 0$, define

$$\begin{aligned}\mathbb{H}(Z) = &\big\{g : \mathbb{R}^+ \times Z \times \Omega \to F, \text{ such that g is } F_T/\mathcal{B}(F) \ \textit{measurable and} \\ &g(t, z, \omega) \ \textit{is} \ \mathcal{F}_t - \textit{adapted} \ \forall z \in Z, \forall t \in (0, T]\big\}.\end{aligned}$$

For $p \geq 1$, let us define,

$$\mathbb{H}^p_\lambda([0, T] \times Z; F) = \left\{ g \in \mathbb{H}(Z) : \int_0^T \int_Z \mathbb{E}[\|g(t, z, \omega)\|^p_F]\lambda(dz)dt < \infty \right\}.$$

For more details see Mandrekar and Rüdiger [22].

Lemma 2.4. *Let $1 < p \leq 2$ and let H be a separable Hilbert space of martingale type p, i.e., there is a constant $K_p(H) > 0$ such that for all H-valued discrete martingale $\{M_n\}_{n=0}^N$ the following inequality holds*

$$\sup_n \mathbb{E}|M_n|^p \leq K_p(H) \sum_{n=0}^N \mathbb{E}|M_n - M_{n-1}|^p,$$

where set $M_{-1} = 0$. Assume that $g \in \mathbb{H}^p_\lambda((0,\infty) \times Z; H)$. Then there exists a constant $C = C_p(H)2^{2-p}$ only depending on H and p such that for $0 < q \leq p$

$$\mathbb{E} \sup_{t \in 0 < t \leq T} \left| \int_0^t \int_Z g(s,z,\omega)\tilde{N}(ds,dz) \right|^q \leq C\mathbb{E} \left(\int_0^T \int_Z |g(t,z,\omega)|^p \lambda(dz)dt \right)^{q/p}.$$

For proof see Corollary C.2 of [6].

Note: Let $C([0,T];H)$ and $\mathcal{D}([0,T];H)$ be the space of all continuous functions and the space of all cádlág paths (right continuous functions with left limits) from $[0,T]$ into H, where H is a Hilbert space, endowed with the uniform topology and Skorohod topology respectively.

Lemma 2.5. *(Kunita's Inequality) Let us consider the stochastic differential equations driven by Lévy noise of the form*

$$\mathrm{d}u(t) = b(u(t))\mathrm{d}t + \sigma(t,u(t))\mathrm{d}W(t) + \int_Z g(u(t-),z)\tilde{N}(dt,dz).$$

Then for all $p \geq 2$, there exists $C(p,t) > 0$ such that for each $t > t_0 \geq 0$,

$$\begin{aligned} &\mathbb{E}\left[\sup_{t_0 \leq s \leq t} |u(s)|^p\right] \\ &\leq C(p,t)\left\{\mathbb{E}|u(0)|^p + \mathbb{E}\left(\int_{t_0}^t |b(u(r))|^p \mathrm{d}r\right) + \mathbb{E}\left[\int_{t_0}^t |\sigma(r,u(r))|^p \mathrm{d}r\right]\right. \\ &\left. + \mathbb{E}\left[\int_{t_0}^t (\int_Z |g(u(r-),z)|^2 \lambda(dz))^{p/2} \mathrm{d}r\right] + \mathbb{E}\left[\int_{t_0}^t \int_Z |g(u(r-),z)|^p \lambda(dz)dr\right]\right\}. \end{aligned}$$

For proof see Corollary 4.4.24 of [1].

Remark 2.6. In this paper we will be frequently using the following form of Young's inequality with exponents p and q

$$ab \leq \varepsilon a^p + C(\varepsilon)b^q, \ \ (a,b > 0, \varepsilon > 0) \quad \text{for} \quad C(\varepsilon) = (\varepsilon p)^{-q/p}q^{-1}.$$

2.2. The GOY Model of Turbulence. The GOY model (Gledger-Ohkitani-Yamada) [26] is a particular case of so called "Shell model" (see, Frisch [18]). This model is the Navier-Stokes equation written in the Fourier space where the interaction between different modes is preserved between nearest modes. To be precise, the GOY model describes a one-dimensional cascade of energies among an infinite sequence of complex velocities, $\{u_n(t)\}$, on a one dimensional sequence of wave numbers $k_n = k_0 2^n, \quad k_0 > 0, \ n = 1, 2, \ldots$, where the discrete index n is referred to as the "shell index". The equations of motion of the GOY model of turbulence have the form

$$\frac{\mathrm{d}u_n}{\mathrm{d}t} + \nu k_n^2 u_n + i\big(a k_n u_{n+1}^\star u_{n+2}^\star + b k_{n-1} u_{n-1}^\star u_{n+1}^\star + \\ + c k_{n-2} u_{n-1}^\star u_{n-2}^\star\big) = f_n, \quad \text{for } n = 1, 2, \ldots, \tag{2.1}$$

along with the boundary conditions $u_{-1} = u_0 = 0$. Here $u_n^\star$ denotes the complex conjugate of u_n, $\nu > 0$ is the kinematic viscosity and f_n is the Fourier component of the forcing. a, b and c are real parameters such that energy conservation condition $a + b + c = 0$ holds (see Ohkitani and Yamada[26]).

2.3. Functional Setting. Let H be a real Hilbert space such that

$$H := \left\{ u = (u_1, u_2, \ldots) \in \mathbb{C}^\infty : \sum_{n=1}^{\infty} |u_n|^2 < \infty \right\}.$$

For every $u, v \in H$, the scalar product $(\cdot,\cdot)$ and norm $|\cdot|$ are defined on H as $(u, v)_H = Re\ \sum_{n=1}^\infty u_n v_n^\star, \quad |u| = \left(\sum_{n=1}^\infty |u_n|^2\right)^{1/2}$. Let us now define the space

$$V := \left\{ u \in H : \sum_{n=1}^{\infty} k_n^2 |u_n|^2 < \infty \right\},$$

which is a Hilbert space equipped with the norm $\|u\| = \left(\sum_{n=1}^\infty k_n^2 |u_n|^2\right)^{1/2}$. The linear operator $A : D(A) \to H$ is a positive definite, self adjoint linear operator defined by

$$Au = ((Au)_1, (Au)_2, \ldots), \text{ where } (Au)_n = k_n^2 u_n, \quad \forall u \in D(A). \tag{2.2}$$

The domain of A, $D(A) \subset H$, is a Hilbert space equipped with the norm

$$\|u\|_{D(A)} = |Au| = \left(\sum_{n=1}^{\infty} k_n^4 |u_n|^2\right)^{1/2}, \quad \forall u \in D(A).$$

Since the operator A is positive definite, we can define the power $A^{1/2}$,

$$A^{1/2} u = (k_1 u_1, k_2 u_2, \ldots), \quad \forall u = (u_1, u_2, \ldots).$$

Furthermore, we define the space

$$D(A^{1/2}) = \left\{ u = (u_1, u_2, \ldots) : \sum_{n=1}^{\infty} k_n^2 |u_n|^2 < \infty \right\},$$

which is a Hilbert space equipped with the scalar product

$$(u, v)_{D(A^{1/2})} = (A^{1/2} u, A^{1/2} v), \quad \forall u, v \in D(A^{1/2}),$$

and the norm

$$\|u\|_{D(A^{1/2})} = \left(\sum_{n=1}^{\infty} k_n^2 |u_n|^2\right)^{1/2}.$$

Note that $V = D(A^{1/2})$. We consider $V' = D(A^{-1/2})$ as the dual space of V. Then the following inclusion holds

$$V \subset H = H' \subset V'.$$

We will now introduce the sequence spaces analogue to Sobolev functional spaces. For $1 \leq p < \infty$ and $s \in \mathbb{R}$,

$$\mathrm{W}^{s,p} := \left\{ u = (u_1, u_2, \ldots) : \|A^{s/2}u\|_p = \left(\sum_{n=1}^{\infty} (k_n^s |u_n|)^p \right)^{1/p} < \infty \right\},$$

and for $p = \infty$,

$$\mathrm{W}^{s,\infty} := \left\{ u = (u_1, u_2, \ldots) : \|A^{s/2}u\|_\infty = \sup_{1 \leq n < \infty} (k_n^s |u_n|) < \infty \right\},$$

where for $u \in \mathrm{W}^{s,p}$ the norm is defined as $\|u\|_{\mathrm{W}^{s,p}} = \|A^{s/2}u\|_p$. Here $\|\cdot\|$ denotes the usual norm in the l^p sequence space. It is clear from the above definitions that $W^{1,2} = V = D(A^{1/2})$.

Remark 2.7. For the shell model we can reasonably assume that the complex velocities u_n are such that $|u_n| < 1$ for almost all n. Then

$$\|u\|_{l^4}^4 = \sum_{n=1}^{\infty} |u_n|^4 \leq \left(\sum_{n=1}^{\infty} |u_n|^2 \right)^2 = |u|^4,$$

which leads to $H \subset l^4$.

We now state a Lemma which is useful in this work. We omit the proof since it is quite simple.

Lemma 2.8. *For any smooth function $u \in H$, the following holds:*

$$\|u\|_{l^4}^4 \leq C|u|^2 \ \|u\|^2. \tag{2.3}$$

2.4. Properties of the Linear and Nonlinear Operators. We define the bilinear operator $B(\cdot,\cdot) : V \times H \to H$ as

$$B(u,v) = (B_1(u,v), B_2(u,v), \ldots),$$

where

$$B_n(u,v) = ik_n \left(\frac{1}{4} u_{n+1}^\star v_{n-1}^\star - \frac{1}{2}(u_{n+1}^\star v_{n+2}^\star + u_{n+2}^\star v_{n+1}^\star) + \frac{1}{8} u_{n-1}^\star v_{n-2}^\star \right).$$

In other words, if $\{e_n\}_{n=1}^{\infty}$ be a orthonormal basis of H, i.e. all the entries of e_n are zero except at the place n it is equal to 1, then

$$B(u,v) = i \sum_{n=1}^{\infty} k_n \left(\frac{1}{4} u_{n+1}^\star v_{n-1}^\star - \frac{1}{2}(u_{n+1}^\star v_{n+2}^\star + u_{n+2}^\star v_{n+1}^\star) + \frac{1}{8} u_{n-1}^\star v_{n-2}^\star \right) e_n. \tag{2.4}$$

The following lemma says that $B(u,v)$ makes sense as an element of H, whenever $u \in V$ and $v \in H$ or $u \in H$ and $v \in V$. It also says that $B(u,v)$ makes sense as an element of V'. Here we state the following lemma which has been proved in Constantin, Levant and Titi [11] for the Sabra shell model, but one can also prove the similar estimates for the GOY model (see Barbato, Barsanti, Bessaih, and Flandoli[4]).

Lemma 2.9. *(i) There exist constants $C_1 > 0, C_2 > 0$,*

$$|B(u,v)| \leq C_1 \|u\| |v|, \quad \forall u \in V, v \in H, \tag{2.5}$$

and

$$|B(u,v)| \leq C_2 |u| \|v\|, \quad \forall u \in H, v \in V. \tag{2.6}$$

(ii) $B : H \times H \to V'$ is a bounded bilinear operator and for a constant $C_3 > 0$

$$\|B(u,v)\|_{V'} \leq C_3 |u||v|, \quad \forall u, v \in H. \tag{2.7}$$

(iii) $B : H \times D(A) \to V$ is a bounded bilinear operator and for a constant $C_4 > 0$

$$\|B(u,v)\|_{V} \leq C_4 |u||Av|, \quad \forall u \in H, v \in D(A). \tag{2.8}$$

(iv) For every $u \in V$ and $v \in H$

$$(B(u,v), v) = 0. \tag{2.9}$$

We now present one more important property of the nonlinear operator B in the following lemma which will play an important role in the later part of this section and also in the next section. The proof is straightforward and uses the bilinearity property of B.

Lemma 2.10. *If $w = u - v$, then*

$$B(u,u) - B(v,v) = B(v,w) + B(w,v) + B(w,w).$$

With above functional setting and following the classical treatment of the Navier-Stokes equation, one can write the stochastic GOY model of turbulence (2.1) with the Lévy forcing as the following,

$$\mathrm{d}u + \big[\nu A u + B(u,u)\big]\mathrm{d}t = f(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(t,u)\mathrm{d}W(t) + \varepsilon \int_Z g(u,z)\tilde{N}(dt,dz) \tag{2.10}$$

$u(0) = u_0$, where $u \in H$, the operators A and B are defined through (2.2) and (2.4) respectively, $f = (f_1, f_2, \ldots), \sigma(t,u) = (\sigma_1(t,u_1), \sigma_2(t,u_2), \ldots)$. Here $(W(t)_{t\geq 0})$ is a H-valued Wiener process with trace class covariance Q, and the space L_Q has been defined in the beginning of this section. Here $g(u,z)$ is a measurable mapping from $H \times Z$ into H, where Z is a measurable space and $Z \in \mathcal{B}(H)$, and let $\mathcal{D}([0,T]; H)$ be the space of all càdlàg paths from $[0,T]$ into H.

Assume that σ and g satisfy the following hypotheses of joint continuity, linear growth and Lipschitz condition:

Hypothesis 2.11. The main hypothesis is the following,

H.1. The function $\sigma \in C([0,T] \times V; L_Q(H_0; H))$, and $g \in \mathbb{H}^2_\lambda([0,T] \times Z; H)$.

H.2. For all $t \in (0,T)$, there exists a positive constant K such that for all $u \in H$,

$$|\sigma(t,u)|^2_{L_Q} + \int_Z |g(u,z)|^2_H \lambda(dz) \leq K(1 + |u|^2).$$

H.3. For all $t \in (0,T)$, there exists a positive constant L such that for all $u, v \in H$,

$$|\sigma(t,u) - \sigma(t,v)|^2_{L_Q} + \int_Z |g(u,z) - g(v,z)|^2_H \lambda(dz) \leq L|u - v|^2.$$

The following lemma shows that sum of the linear and nonlinear operator is locally monotone in the l^4-ball.

Lemma 2.12. *For a given $r > 0$, let us denote by $\mathbb{B}_r$ the closed l^4-ball in V: $\mathbb{B}_r = \{v \in V; \|v\|_{l^4} \leq r\}$. Define the nonlinear operator F on V by $F(u) := -\nu Au - B(u,u)$. Then for $0 < \varepsilon < \frac{\nu}{2L}$, where L is the positive constant that appears in the condition (H.3), the pair $(F, \sqrt{\varepsilon}\sigma + \varepsilon \int_Z g(.,z)\lambda(dz))$ is monotone in $\mathbb{B}_r$, i.e. for any $u \in V$ and $v \in \mathbb{B}_r$, and $w = u - v$,*

$$(F(u) - F(v), w) - \frac{r^4}{\nu^3}|w|^2 + \varepsilon \Big[|\sigma(t,u) - \sigma(t,v)|^2_{L_Q} + \int_Z |g(u,z) - g(v,z)|^2 \lambda(dz)\Big] \leq 0. \tag{2.11}$$

For proof see Lemma 3.6 of [24].

2.5. Energy Estimates and Existence Result. Let H_n be defined as the span $\{e_1, e_2, \cdots, e_n\}$, where $\{e_j\}$ is any fixed orthonormal basis in H with each $e_j \in D(A)$. Let P_n denote the orthogonal projection of H to H_n. Define $u^n = P_n u$, to avoid confusion with earlier notation u_n. Let $W_n = P_n W$. Let $\sigma_n = P_n \sigma$ and $\int_Z g^n(u^{n,\varepsilon}(t-), z)\tilde{N}(dt,dz) = P_n \int_Z g(u(t-),z)\tilde{N}(dt,dz)$, where $g^n = P_n g$. Define $u^{n,\varepsilon}$ as the solution of the following stochastic differential equation in the variational form such that for each $v \in H_n$,

$$\mathrm{d}(u^{n,\varepsilon}(t), v) = (F(u^{n,\varepsilon}(t)), v)\mathrm{d}t + (f(t), v)\mathrm{d}t + \sqrt{\varepsilon}(\sigma_n(t, u^{n,\varepsilon}(t))\mathrm{d}W_n(t), v) + \varepsilon \int_Z \big(g^n(u^{n,\varepsilon}(t-), z), v\big)\tilde{N}(dt,dz), \tag{2.12}$$

with $u(0) = P_n u(0)$.

Theorem 2.13. *With the above mathematical setting let f be in $\mathrm{L}^2([0,T], V')$, $u(0)$ be $\mathcal{F}_0$ measurable, $\sigma \in C([0,T] \times V; L_Q(H_0; H))$, $g \in \mathbb{H}^2_\lambda([0,T] \times Z; H)$ and $\mathbb{E}|u(0)|^2 < \infty$. Let $u^{n,\varepsilon}$ denote the unique strong solution of the stochastic differential equation* (2.12) *in $\mathcal{D}([0,T]; H_n)$. Then with K as in condition (H.2), the following estimates hold:*
For all ε, and $0 \leq t \leq T$,

$$\mathbb{E}|u^{n,\varepsilon}(t)|^2 + \nu \int_0^t \mathbb{E}\|u^{n,\varepsilon}(s)\|^2 \mathrm{d}s \leq \left(1 + \varepsilon KT e^{\varepsilon KT}\right)\left(\mathbb{E}|u(0)|^2 + \frac{1}{\nu}\int_0^t \|f(s)\|^2_{V'} ds + \varepsilon KT\right), \tag{2.13}$$

and for all $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |u^{n,\varepsilon}(t)|^2 + \nu \int_0^T \|u^{n,\varepsilon}(t)\|^2 \mathrm{d}t\right] \leq C\left(\mathbb{E}|u(0)|^2, \int_0^T \|f(t)\|^2_{V'}\mathrm{d}t, \nu, T\right). \tag{2.14}$$

Assumption 2.14. Let $p \geq 2$, for all $t \in (0,T)$, and for all $u \in H$, there exists a positive constant K_1 such that

$$|\sigma(s, u(t))|^p + \int_Z |g(u(t),z)|^p \lambda(dz) \leq K_1 \left(1 + |u(t)|^p\right). \tag{2.15}$$

Theorem 2.15. *Let $p \geq 2$, $u(0)$ be $\mathcal{F}_0$ measurable, $f \in \mathrm{L}^p([0,T],V')$, $\sigma \in C([0,T]\times V; L_Q(H_0;H))$, $g \in \mathbb{H}^p_\lambda([0,T]\times Z;H)$ and let $\mathbb{E}|u(0)|^p < \infty$. Let $u^{n,\varepsilon}(t)$ denote the unique strong solution to finite system of equations (2.12) in $\mathcal{D}([0,T],H_n)$. Then with K as in condition (H.2) and K_1 as in Assumption, the following estimates hold:*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|u^{n,\varepsilon}(t)|^p\right]+\frac{p\nu}{2}\mathbb{E}\left(\int_0^T\|u^{n,\varepsilon}(t)\|^2|u^{n,\varepsilon}(t)|^{p-2}dt\right)$$
$$\leq C\left(\mathbb{E}|u(0)|^2,K,K_1,p,T,\nu,\int_0^T\|f(t)\|^p_{V'}\mathrm{d}t\right). \tag{2.16}$$

Proof. Define $\tau_N=\inf\left\{t:|u^{n,\varepsilon}(t)|^p+\int_0^t\|u^{n,\varepsilon}(s)\|^2|u^{n,\varepsilon}(s)|^{p-2}\mathrm{d}s>N\right\}$. Let us take the function $f(x)=|x|^p$ and apply the Itô's lemma to the process $u^{n,\varepsilon}(t)$. Use the property of the operators A and B, apply Cauchy-Schwartz inequality and Young's inequality (Remark 2.6) to the term $|u^{n,\varepsilon}(s)|^{p-2}\left(f(s),u^{n,\varepsilon}(s)\right)$ to obtain,

$$\begin{aligned}
&|u^{n,\varepsilon}(t\wedge\tau_N)|^p+\frac{p\nu}{2}\int_0^{t\wedge\tau_N}\|u^{n,\varepsilon}(s)\|^2|u^{n,\varepsilon}(s)|^{p-2}\mathrm{d}s\\
&\leq|u(0)|^p+\frac{p}{2\nu}\int_0^{t\wedge\tau_N}\|f(s)\|^p_{V'}\mathrm{d}s+C_1(p,\nu)\int_0^{t\wedge\tau_N}|u^{n,\varepsilon}(s)|^p\mathrm{d}s\\
&+p\int_0^{t\wedge\tau_N}|u^{n,\varepsilon}(s)|^{p-2}\left(\sigma(s,u^{n,\varepsilon}(s)),u^{n,\varepsilon}(s)\right)\mathrm{d}W_n(s)\\
&+\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N}|u^{n,\varepsilon}(s)|^{p-2}\mathrm{Tr}(\sigma(s,u^{n,\varepsilon}(s))Q\sigma(s,u^{n,\varepsilon}(s)))\mathrm{d}s\\
&+\int_0^{t\wedge\tau_N}\int_Z\left[|u^{n,\varepsilon}(s-)+g(u^{n,\varepsilon}(s-),z)|^p-|u^{n,\varepsilon}(s-)|^p\right]\tilde{N}(ds,dz)\\
&\quad+\int_0^{t\wedge\tau_N}\int_Z\Big[|u^{n,\varepsilon}(s-)+g(u^{n,\varepsilon}(s-),z)|^p-|u^{n,\varepsilon}(s-)|^p\\
&\qquad-p|u^{n,\varepsilon}(s-)|^{p-2}\left(g(u^{n,\varepsilon}(s-),z),u^{n,\varepsilon}(s-)\right)\Big]\lambda(dz)ds,
\end{aligned}\tag{2.17}$$

where $C_1(p,\nu)=\left(\frac{p-2}{2\nu}\right)\left(\frac{2}{p}\right)^{2/(p-2)}$. Let us denote the last two terms on RHS by I. Consider $\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N}|u^{n,\varepsilon}(s)|^{p-2}\mathrm{Tr}(\sigma(s,u^{n,\varepsilon}(s))Q\sigma(s,u^{n,\varepsilon}(s)))\mathrm{d}s$ in (2.17), apply Young's inequality and $|u^{n,\varepsilon}|\leq\|u^{n,\varepsilon}\|$ to obtain,

$$\begin{aligned}
&\frac{p(p-1)}{2}\int_0^{t\wedge\tau_N}|u^{n,\varepsilon}(s)|^{p-2}\mathrm{Tr}(\sigma(s,u^{n,\varepsilon}(s))Q\sigma(s,u^{n,\varepsilon}(s)))\mathrm{d}s\\
&\leq\frac{p\nu}{4}\int_0^{t\wedge\tau_N}\|u^{n,\varepsilon}(s)\|^2|u^{n,\varepsilon}(s)|^{p-2}\mathrm{d}s+C_2(p,\nu)\int_0^{t\wedge\tau_N}|\sigma(s,u^{n,\varepsilon}(s))|^p\mathrm{d}s,
\end{aligned}\tag{2.18}$$

where $C_2(p,\nu) = (p-1)^{p/2}\left(\frac{2(p-2)}{p\nu}\right)^{\frac{p-2}{2}}$. Now applying (2.18) in (2.17), taking supremum up to time $T\wedge\tau_N$, and then taking the expectation, one can get,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|u^{n,\varepsilon}(t)|^p + \frac{p\nu}{4}\int_0^{T\wedge\tau_N}\|u^{n,\varepsilon}(t)\|^2|u^{n,\varepsilon}(t)|^{p-2}\mathrm{d}t\right] \\
&\leq \mathbb{E}\left[|u(0)|^p\right] + \frac{p}{2\nu}\int_0^{t\wedge\tau_N}\|f(s)\|_{V'}^2\mathrm{d}s + C_1(p,\nu)\mathbb{E}\left[\int_0^{t\wedge\tau_N}\sup_{0\leq s\leq t}|u^{n,\varepsilon}(s)|^p\mathrm{d}s\right] \\
&\quad + p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |u^{n,\varepsilon}(s)|^{p-2}\left(\sigma(s,u^{n,\varepsilon}(s)),u^{n,\varepsilon}(s)\right)\mathrm{d}W_n(s)\right|\right] \\
&\quad + C_2(p,\nu)\mathbb{E}\left(\int_0^{T\wedge\tau_N}|\sigma(s,u^{n,\varepsilon}(s))|^p\mathrm{d}s\right) + \mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N} I\right]. \qquad (2.19)
\end{aligned}
$$

Let us consider $p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |u^{n,\varepsilon}(s)|^{p-2}\left(\sigma(s,u^{n,\varepsilon}(s)),u^{n,\varepsilon}(s)\right)\mathrm{d}W_n(s)\right|\right]$ from (2.19) and apply Burkholder-Davis-Gundy inequality, Young's inequality and Hölder's inequality to get,

$$
\begin{aligned}
&p\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}\left|\int_0^t |u^{n,\varepsilon}(s)|^{p-2}\left(\sigma(s,u^{n,\varepsilon}(s),u^{n,\varepsilon}(s))\right)\mathrm{d}W_n(s)\right|\right] \qquad (2.20) \\
&\leq \frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|u^{n,\varepsilon}(t)|^p\right] + (2(p-1))^{p-1}T^{\frac{p-2}{2}}\mathbb{E}\int_0^{T\wedge\tau_N}|\sigma(s,u^{n,\varepsilon}(s))|^p\mathrm{d}s.
\end{aligned}
$$

Apply Kunita's inequality (see Lemma 2.5) by taking $b=0$ and $\sigma=0$, we have,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|I(t)|\right] \leq\ & C_3(p,T)\left\{\mathbb{E}\int_0^{T\wedge\tau_N}\left(\int_Z |g(u^{n,\varepsilon}(s-),z)|^2\lambda(dz)\right)^{p/2}\mathrm{d}s\right. \\
&\left. + \mathbb{E}\left[\int_0^{T\wedge\tau_N}\int_Z |g(u^{n,\varepsilon}(s-),z)|^p\lambda(dz)\mathrm{d}s\right]\right\}. \qquad (2.21)
\end{aligned}
$$

Thus we can write (2.19) as,

$$
\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|u^{n,\varepsilon}(t)|^p\right] + \frac{p\nu}{4}\mathbb{E}\left(\int_0^{T\wedge\tau_N}\|u^{n,\varepsilon}(s)\|^2|u^{n,\varepsilon}(s)|^{p-2}\mathrm{d}s\right) \\
&\quad \leq \mathbb{E}\left[|u(0)|^p\right] + \frac{p}{2\nu}\int_0^{t\wedge\tau_N}\|f(s)\|_{V'}^2\mathrm{d}s + C_1(p,\nu)\mathbb{E}\int_0^{t\wedge\tau_N}\sup_{0\leq s\leq t}|u^{n,\varepsilon}(s)|^p\mathrm{d}s \\
&\quad + C_4(p,T,\nu)\mathbb{E}\left(\int_0^{T\wedge\tau_N}|\sigma(s,u^{n,\varepsilon}(s))|^p\mathrm{d}s\right) \\
&\quad + C_3(p,T)\left\{\mathbb{E}\left[\int_0^{T\wedge\tau_N}\left(\int_Z |g(u^{n,\varepsilon}(s-),z)|^2\lambda(dz)\right)^{p/2}\mathrm{d}s\right]\right. \\
&\quad \left. + \mathbb{E}\left[\int_0^{T\wedge\tau_N}\int_Z |g(u^{n,\varepsilon}(s-),z)|^p\lambda(dz)\mathrm{d}s\right]\right\}, \qquad (2.22)
\end{aligned}
$$

where $C_4(p,T,\nu) = C_2(p,\nu) + (2(p-1))^{p-1}T^{\frac{p-2}{2}}$. Let us take the last three terms of the above inequality and apply Hypothesis 2.11 and Assumption 2.14 to get,

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T\wedge\tau_N}|u^{n,\varepsilon}(t)|^p\right] + \frac{p\nu}{4}\mathbb{E}\left(\int_0^{T\wedge\tau_N}\|u^{n,\varepsilon}(s)\|^2|u^{n,\varepsilon}(s)|^{p-2}\mathrm{d}s\right)\\
&\leq \mathbb{E}\left[|u(0)|^p\right] + \frac{p}{2\nu}\int_0^{t\wedge\tau_N}\|f(s)\|_{V'}^2\mathrm{d}s + \left[C(K,K_1,p,\nu,T)\right]T\\
&\quad + \left[C(K,K_1,p,\nu,T)\right]\mathbb{E}\left(\int_0^{T\wedge\tau_N}\sup_{0\leq s\leq t}|u^{n,\varepsilon}(s)|^p\mathrm{d}t\right). \qquad (2.23)
\end{aligned}$$

Note that $T\wedge\tau_N \to T$ a.s. as $N\to\infty$. Finally taking the limit in the above estimate (2.23) and apply Gronwall's inequality to get the result. □

Definition 2.16. A strong solution u^ε of the stochastic GOY model is defined on a given probability space $(\Omega,\mathcal{F},\mathcal{F}_t,\mathbb{P})$ as a $\mathrm{L}^p\left(\Omega;\mathrm{L}^\infty(0,T;H)\cap\mathcal{D}(0,T;H)\right)\cap \mathrm{L}^2\left(\Omega;\mathrm{L}^2(0,T;V)\right)$ valued adapted process which satisfies

$$\mathrm{d}u^\varepsilon + \left[\nu A u^\varepsilon + B(u^\varepsilon,u^\varepsilon)\right]\mathrm{d}t = f(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(t,u^\varepsilon)\mathrm{d}W(t) + \varepsilon\int_Z g(u^\varepsilon,z)\tilde{N}(dt,dz) \qquad (2.24)$$

$u^\varepsilon(0) = u_0$, in the weak sense and also the energy inequality in Theorem 2.15.

Theorem 2.17. *Let $u(0)$ be $\mathcal{F}_0$ measurable and $\mathbb{E}|u_0|^2<\infty$. Let $f\in\mathrm{L}^p(0,T;V')$. We also assume that $0<\varepsilon<\frac{\nu}{L}$ and the diffusion coefficient satisfies the conditions (H.1)-(H.3). Then there exists unique adapted process $u^\varepsilon(t,x,w)$ with the regularity*

$$u^\varepsilon \in \mathrm{L}^p\left(\Omega;\mathcal{D}(0,T;H)\right)\cap\mathrm{L}^2\left(\Omega;\mathrm{L}^2(0,T;V)\right)$$

satisfying the stochastic GOY model (2.24) *and the a priori bounds in Theorem 2.15.*

Proof. The theorem can be proved using the local monotonicity property and the energy estimates. A version of the theorem with

$$u^\varepsilon\in\mathrm{L}^2(\Omega;\mathcal{D}(0,T;H)\cap\mathrm{L}^2(0,T;V))$$

has been proved in Theorem 4.4 of [24]. □

3. Large Deviation Principle

In this section we first give an abstract formulation and basic properties for a class of large deviation problems and then prove the Main Theorem 1.1.

Let us denote by X a complete separable metric space and $\{\mathbb{P}_\varepsilon : \varepsilon>0\}$ a family of probability measures on the Borel subsets of X.

Definition 3.1. A function $I : X\to[0,\infty]$ is called a *rate function* if I is lower semicontinuous. A rate function I is called a *good rate function* if for arbitrary $M\in[0,\infty)$, the level set $K_M=\{x : I(x)\leq M\}$ is compact in X.

Definition 3.2. (Large Deviation Principle) We say that a family of probability measures $\{\mathbb{P}_\varepsilon : \varepsilon>0\}$ satisfies the *large deviation principle* (LDP) with a good rate function I satisfying,

(i) for each closed set $F \subset X$

$$\limsup_{\varepsilon\to 0} \ \varepsilon \ \log \mathbb{P}_\varepsilon(F) \leq -\inf_{x\in F} I(x),$$

(ii) for each open set $G \subset X$

$$\liminf_{\varepsilon\to 0} \ \varepsilon \ \log \mathbb{P}_\varepsilon(G) \geq -\inf_{x\in G} I(x).$$

Lemma 3.3. *(Varadhan's Lemma [28]) Let E be a Polish space and $\{X^\varepsilon : \varepsilon > 0\}$ be a family of E-valued random elements satisfying LDP with rate function I. Then $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the Laplace principle on E with the same rate function I if for all $h \in C_b(E)$,*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\varepsilon}h(X^\varepsilon)\right]\right\} = -\inf_{x\in E}\{h(x) + I(x)\}. \tag{3.1}$$

Lemma 3.4. *(Bryc's Lemma [13]) The Laplace principle implies the LDP with the same rate function. More precisely, if $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the Laplace principle on the Polish space E with the rate function I and if the limit*

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}\left\{\exp\left[-\frac{1}{\varepsilon}h(X^\varepsilon)\right]\right\} = -\inf_{x\in E}\{h(x) + I(x)\}$$

is valid for all $h \in C_b(E)$, then $\{X^\varepsilon : \varepsilon > 0\}$ satisfies the LDP on E with rate function I.

Note that, Varadhan's Lemma together with Bryc's converse of Varadhan's Lemma state that for Polish space valued random elements, the Laplace principle and the large deviation principle are equivalent.

We will now define the function spaces required for the formulation of the large deviation problem. These spaces are defined based on the theory developed in Budhiraja, Dupuis and Maroulas [7].

Let X be a locally compact Polish space and let $X_T = [0, T] \times X$ for any $T \in (0, \infty)$. Let $\mathcal{M}(X)$ be the space of all measures μ on $(X, \mathcal{B}(X))$, satisfying $\mu(K) < \infty$ for every compact subset K of X. We endow $\mathcal{M}(X)$ with the weakest topology such that for every $f \in C_c(X)$ the function $\mu \to \langle f, \mu\rangle = \int_X f(x)\mu(dx)$, $\mu \in \mathcal{M}(X)$ is a continuous function. This topology can be metrized such that $\mathcal{M}(X)$ is a Polish space. Let $\mathbb{M} = \mathcal{M}(X_T)$ and let $\mathbb{P}$ be the unique probability measure on $(\mathbb{M}, \mathcal{B}(\mathbb{M}))$. Then $\mathcal{B}(\mathbb{M})$ will denote a Borel σ-field on the space $\mathcal{M}(X)$. For more details see Budhiraja, Dupuis and Maroulas [7]. Let us denote the product space $C([0, T]; H) \times \mathbb{M}$ by $\mathbb{V}(H)$. Define $\mathcal{G}_t = \sigma\big\{N(s, Z), W(s) : 0 \leq s \leq t, Z \in \mathcal{B}(X_T)\big\}$.

For $\theta > 0$, define $\mathbb{P}_\theta$ the unique probability measure on $(\mathbb{V}(H), \mathcal{B}(\mathbb{V}(H)))$ such that under $\mathbb{P}_\theta$

(i) $W(t)$ is an H-valued Q-Wiener process.
(ii) N is a Poisson Random Measure with intensity measure λ_T.
(iii) $\{W(t), t \in [0, T]\}$, $\{N(t, Z), t \in [0, T]\}$ are $\mathcal{G}_t$ martingales for every $Z \in \mathcal{B}(X_T)$.

Now let us consider the $\mathbb{P}$-completion of the filtration $\{\mathcal{G}_t\}$ and denote it by $\{\mathcal{F}_t\}$. We denote by $\mathcal{P}$ the predictable σ-field on $[0, T] \times \mathbb{V}(H)$ with the filtration

$\{\mathcal{F}_t : 0 \le t \le T\}$ on $(\mathbb{V}(H), \mathcal{B}(\mathbb{V}(H)))$. Let $\mathcal{A}$ be the class of all $(\mathcal{P}\otimes\mathcal{B}(X))/\mathcal{B}[0,\infty)$ measurable maps $\phi : X_T \times \mathbb{V} \to [0,\infty)$. For $\phi \in \mathcal{A}$, define the counting process N^ϕ on X_T as follows,

$$N^{\phi}(t,Z) = \int_{(0,t]\times Z}\int_0^\infty 1_{[0,\phi(s,z)]}(r)\tilde{N}(ds,dz)\mathrm{d}r, \quad t\in[0,T], \quad Z\in\mathcal{B}(X).$$

Here N^ϕ is to be thought of as a controlled random measure, with ϕ selecting the intensity for the points of location z and time s, in a possibly random but non-anticipating way. Let us define $\ell : [0,\infty) \to [0,\infty)$ by $\ell(r) = r\log r - r + 1, \quad r \in [0,\infty)$. For any $\phi \in \mathcal{A}$, let us define $L_T(\phi)$ by

$$L_T(\phi) = \int_0^T \int_Z \ell(\phi(t,z,\omega))\lambda(dz)dt.$$

Define $\mathcal{P}_2 \equiv \left\{\psi : \psi \text{ is } \mathcal{P}/\mathcal{B}(\mathbb{R}) \text{ measurable and } \int_0^T \|\psi(s)\|_0^2 ds < \infty \ \mathbb{P}-\text{a.s}\right\}$ and let us set $\mathcal{U}(H) = \mathcal{P}_2 \times \mathcal{A}$. For $\psi \in \mathcal{P}_2$ let us define $\tilde{L}(\psi)$ by

$$\tilde{L}_T(\psi) = \frac{1}{2}\int_0^T \|\psi(s)\|_0^2 ds,$$

and for $u = (\psi,\phi) \in \mathcal{U}$, set $\bar{L}_T(u) = L_T(\phi) + \tilde{L}_T(\psi)$. For $\psi \in \mathcal{P}_2$, let W^ψ be

$$W^\psi(t) = W(t) + \int_0^t \psi(s)ds, \ t\in[0,T].$$

Let us define for $N \in \mathbb{N}$, $\tilde{S}^N(H_0) = \left\{\psi \in \mathrm{L}^2([0,T] : H_0) : \tilde{L}_T(\psi) \le N\right\}$. Also let us define $S^N = \{\phi : X_T \to [0,\infty) : L_T(\phi) \le N\}$. The convergence in $\mathbb{M}$ is essentially equivalent to weak convergence on compact subsets. The super linear growth of ℓ implies that $\left\{\lambda_T^g : g \in S^N\right\}$ is a compact subset of $\mathbb{M}$ where

$$\lambda_T^g = \int_0^T \int_Z g(s,z)\lambda(dz)\mathrm{d}s, \ Z\in\mathcal{B}(X_T).$$

Throughout we consider the topology on S^N obtained through this identification which makes S^N a compact space. We set $\bar{S}^N = \tilde{S}^N(H_0) \times S^N$ with the usual product topology. Also let $\mathcal{U} = \mathcal{P}_2(H)\times\mathcal{A}$ and set $\mathbb{S} = \bigcup_{N\ge1}\bar{S}^N$ and let $\mathcal{U}^N$ be the space of all $\bar{S}^N$- valued controls, $\mathcal{U}^N = \left\{u = (\psi,\phi) \in \mathcal{U} : u(\omega) \in \bar{S}^N, \ \mathbb{P} \text{ a.e } \omega\right\}$.

Let X and X_0 denote Polish spaces and for $\varepsilon > 0$ let $\mathcal{G}^\varepsilon : X_0 \times \mathbb{V}(H) \to X$ be a measurable map. Define

$$u^\varepsilon = \mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot), \varepsilon N^{\varepsilon^{-1}}\right).$$

We are interested in the large deviation principle for u^ε as $\varepsilon \to 0$.

Assumption 3.5. There exists a measurable map $\mathcal{G}^0 : X_0 \times \mathbb{V}(H) \to X$ such that the following hold:

(i) Let $\left\{\theta^\varepsilon = (\psi^\varepsilon,\phi^\varepsilon) \in \mathcal{U}, \theta^\varepsilon(\omega) \in \bar{S}^M, \mathbb{P}- \text{ a.e. } \omega\right\} \subset \mathcal{U}^M$ for $M < \infty$, θ^ε converges in distribution on $\bar{S}^M$-valued random elements $\theta = (\psi,\phi)$. Then

$$\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot) + \int_0^\cdot \psi^\varepsilon(s)ds, \varepsilon N^{\varepsilon^{-1}\phi^\varepsilon}\right) \longrightarrow \mathcal{G}^0\left(\int_0^\cdot \psi(s)ds, \lambda_T^\phi\right).$$

(ii) For every $M < \infty$, the set

$$K_M = \left\{ \mathcal{G}^0 \left(\int_0^{\cdot} \psi(s)ds, \lambda_T^{\phi} \right) : (\phi, \psi) \in \mathcal{U}^M \right\}$$

is a compact subset of X.

For each $\zeta \in X$, define $\mathbb{S}_\zeta = \left\{ (\psi, \phi) \in \mathbb{S} : \zeta = \mathcal{G}^0 \left(\int_0^{\cdot} \psi(s)ds, \lambda_T^{\phi} \right) \right\}$. Let $I : X \to [0, \infty]$ be defined as

$$I(\zeta) = \inf_{q=(\psi,\phi)\in\mathbb{S}_\zeta} \left\{ \int_0^T \int_Z \ell(\phi(t,z))\lambda(dz)\mathrm{d}t + \frac{1}{2} \int_0^T \|\psi(s)\|_0^2 ds \right\}, \tag{3.2}$$

where infimum over an empty set is taken as ∞. Also here $Z \subset \mathcal{B}(X)$.

We now state an important result by Budhiraja, Dupuis and Maroulas [7] (see Theorem 4.2 of [7]).

Theorem 3.6. *Let* $u^\varepsilon = \mathcal{G}^\varepsilon \left(\sqrt{\varepsilon} W(\cdot), \varepsilon N^{\varepsilon^{-1}} \right)$. *If* $\{\mathcal{G}^\varepsilon\}$ *satisfies the Assumption 3.5, then the family* $\{u^\varepsilon : \varepsilon > 0\}$ *satisfies the Laplace principle in* X *with rate function* I *given by* (3.2).

Remark 3.7.
1. Notice that, since the underlying space X is Polish, the family $\{u^\varepsilon : \varepsilon > 0\}$ satisfies the LDP in X with the same rate function I.
2. Assumption 3.5(i) is a statement on the weak convergence of a certain family of random variables and is at the core of weak convergence approach to the study of large deviations. Assumption 3.5(ii) essentially says that the level sets of the rate function are compact.

Remark 3.8. The stochastic GOY model in consideration,

$$\mathrm{d}u^\varepsilon + \left[\nu A u^\varepsilon + B(u^\varepsilon) \right] \mathrm{d}t = f(t)\mathrm{d}t + \sqrt{\varepsilon}\sigma(t, u^\varepsilon)\mathrm{d}W(t) + \varepsilon \int_Z g(u^\varepsilon, z)\tilde{N}(dt, dz)$$

$u^\varepsilon(0) = \xi$ has a unique strong solution in the Polish space $X = \mathcal{D}([0,T]; H) \cap \mathrm{L}^2(0,T;V)$. The solution to the stochastic GOY model, denoted by u^ε, can be written as $\mathcal{G}^\varepsilon \left(\sqrt{\varepsilon} W(\cdot), \varepsilon N^{\varepsilon^{-1}} \right)$ for a Borel measurable function $\mathcal{G}^\varepsilon : \mathcal{D}([0,T]; H) \to X$ (see Karatzas and Shreve [21], page 310; Vishik and Fursikov [29], Chapter X, Corollary 4.2). For more details about this formulation see Chapter IV (classical Yamada-Watababe argument) of Ikeda and Watanabe [19] and Section 3.2 of Budhiraja, Chen and Dupuis [10].

The aim of this section is to verify that such a $\mathcal{G}^\varepsilon$ satisfies Assumption 3.5. Then applying the Theorem 3.6 the LDP for $\{u^\varepsilon : \varepsilon > 0\}$ in X can be established.

The LDP for $\{u^\varepsilon : \varepsilon > 0\}$ in X have been proved here systematically in four steps. In the first and second Theorems we show the well posedness of certain controlled stochastic and controlled deterministic equations in X. These results help to prove the last two main Theorems on the compactness of the level sets and weak convergence of the stochastic control equation stated in Assumption 3.5.

Theorem 3.9. *For any $\theta \in \mathcal{U}^M$, $0 < M < \infty$, the stochastic control equation*

$$\begin{aligned} \mathrm{d}u^\varepsilon_\theta(t) + [\nu A u^\varepsilon_\theta(t) + B(u^\varepsilon_\theta(t), u^\varepsilon_\theta(t))]\,\mathrm{d}t \\ = \left[f(t) + \sigma(t, u^\varepsilon_\theta(t))\psi(t) + \int_Z g(u^\varepsilon_\theta(t), z)\ell(\phi(t,z))\lambda(dz) \right] \mathrm{d}t \\ + \sqrt{\varepsilon}\sigma(t, u^\varepsilon_\theta(t))\mathrm{d}W(t) + \varepsilon \int_Z g(u^\varepsilon_\theta(t), z)\tilde{N}(dt, dz), \end{aligned} \tag{3.3}$$

$u^\varepsilon_\theta(0) = \xi \in H$ has a unique strong solution in $\mathrm{L}^2(\Omega; X)$, where $X = \mathcal{D}(0,T;H) \cap \mathrm{L}^2(0,T;V)$, $f \in \mathrm{L}^2(0,T;V')$ and $\sigma, \int_Z g(\cdot, z)\lambda(dz)$ will satisfy the hypotheses H.1.–H.3. in Section 2.

Proof. Following the proof of Theorem 2.15, we can prove that if $u^\varepsilon_\theta(t)$ is a strong solution of the stochastic controlled equation (3.3), the following energy estimate holds:

$$\mathbb{E}\left(\sup_{0\le t\le T} |u^\varepsilon_\theta(t)|^2 + \nu \int_0^T \|u^\varepsilon_\theta(t)\|^2 \mathrm{d}t \right) \le C, \tag{3.4}$$

where $C = C\left(|\xi|^2, \int_0^T \|f\|^2_{V'}\mathrm{d}t, \nu, K, T, M\right)$ is a positive constant.

The proofs of the existence and uniqueness of the strong solution of the stochastic controlled equation (3.3) follow from Theorem 2.17. The proofs can be obtained from Manna and Mohan [24] after a few modifications as needed due to the presence of the control term. The energy estimate obtained in (3.4) plays a crucial role in the proof of the existence and uniqueness theorems. □

Corollary 3.10. *Since $V \subset H$, $|u| \le \|u\|$, from (3.4), we have,*

$$\mathbb{E}\left(\sup_{0\le t\le T} |u^\varepsilon_\theta(t)|^2 + \nu \int_0^T |u^\varepsilon_\theta(t)|^2 \mathrm{d}t \right) \le C.$$

Theorem 3.11. *Let $\theta = (\psi, \phi) \in \mathcal{U}$; $f \in \mathrm{L}^2(0,T;V')$ and σ, g satisfy the hypotheses H.1.–H.3. in Section 2. Then the equation*

$$\begin{aligned} \mathrm{d}u_\theta(t) + [\nu A u_\theta(t) + B(u_\theta(t), u_\theta(t))]\mathrm{d}t = f(t)\mathrm{d}t + \sigma(t, u_\theta(t))\psi(t)\mathrm{d}t \\ + \int_Z g(u_\theta(t), z)\ell(\phi(t,z))\lambda(dz)\mathrm{d}t, \end{aligned} \tag{3.5}$$

where $u_\theta(0) = \xi \in H$, has a unique strong solution in $X = \mathcal{D}(0,T;H) \cap \mathrm{L}^2(0,T;V)$.

Proof. This result can be considered as a particular case of the previous Theorem 3.9, where the noise term or the diffusion co-efficient is absent. □

Next we state an important lemma from Budhiraja and Dupuis [9].

Lemma 3.12. *Let $\{\psi_n\}$ be a sequence of elements from $\tilde{S}^M$ for some finite $M > 0$. Let ψ_n converges in distribution to ψ with respect to the weak topology on $\mathrm{L}^2(0,T;H_0)$. Then $\int_0^\cdot \psi_n(s)\mathrm{d}s$ converges in distribution as $C(0,T;H)$- valued processes to $\int_0^\cdot \psi(s)\mathrm{d}s$ as $n \to \infty$.*

Now we are ready to check the Assumptions 3.5.

Theorem 3.13 (Compactness)**.** *Let $M<\infty$ be a fixed positive number and let $\xi\in H$ be deterministic. Let*

$$K_M := \left\{u_\theta\in \mathcal{D}(0,T;H)\cap \mathrm{L}^2(0,T;V);\theta\in\mathcal{U}^M\right\},$$

where u_θ is the unique solution in $X=\mathcal{D}(0,T;H)\cap \mathrm{L}^2(0,T;V)$ of the deterministic controlled equation (3.5), with $u_\theta(0)=\xi\in H$. Then K_M is compact in X.

Proof. Let us consider a sequence $\{u_{\theta_n}\}$ in K_M, where u_{θ_n} corresponds to the solution of (3.5) with control $\theta_n\in\mathcal{U}^M$ in place of θ, i.e.

$$\begin{aligned}\mathrm{d}u_{\theta_n}(t)+[\nu Au_{\theta_n}(t)+B(u_{\theta_n}(t),u_{\theta_n}(t))]\mathrm{d}t&=f(t)\mathrm{d}t+\sigma(t,u_{\theta_n}(t))\psi_n(t)\mathrm{d}t\\&\quad+\int_Z g(u_{\theta_n}(t),z)\ell(\phi_n(t,z))\lambda(dz)\mathrm{d}t,\end{aligned}\tag{3.6}$$

with $u_{\theta_n}(0)=\xi\in H$. Then by weak compactness of $\mathcal{U}^M$, there exists a subsequence of $\{\theta_n\}$, still denoted by $\{\theta_n\}$, which converges weakly to $\theta\in\mathcal{U}^M$ in $\mathcal{U}$. We need to prove $u_{\theta_n}\to u_\theta$ in X as $n\to\infty$, or in other words,

$$\sup_{0\le t\le T}|u_{\theta_n}(t)-u_\theta(t)|^2+\int_0^T\|u_{\theta_n}(t)-u_\theta(t)\|^2\mathrm{d}t\longrightarrow 0,\quad\text{as}\quad n\to\infty.\tag{3.7}$$

According to the Theorem 3.11, u_θ is unique strong solution in X of the deterministic controlled equation (3.5). Hence it is obvious to note that, u_θ satisfies the following a-priori estimate

$$\sup_{0\le t\le T}|u_\theta(t)|^2+\int_0^T\|u_\theta(t)\|^2\mathrm{d}t\le C,\tag{3.8}$$

where $C=C\left(|\xi|^2,\int_0^T\|f\|_{V'}^2\mathrm{d}t,\nu,K,T,M\right)$ is a positive constant.

For the proof, we refer the Theorem 3.9, where the stochastic version of the above a priori estimate has been worked out.

Let $w_{\theta_n}=u_{\theta_n}-u_\theta$. Then w_{θ_n} satisfies the following differential equation

$$\begin{aligned}&\mathrm{d}w_{\theta_n}(t)+[\nu Aw_{\theta_n}(t)+B(u_{\theta_n}(t),u_{\theta_n}(t))-B(u_\theta(t),u_\theta(t))]\mathrm{d}t\\&\quad=[\sigma(t,u_{\theta_n}(t))\psi_n(t)-\sigma(t,u_\theta(t))\psi(t)]\mathrm{d}t\\&\qquad+\int_Z\left[g(u_{\theta_n}(t),z)\ell(\phi_n(t,z))-g(u_\theta(t),z)\ell(\phi(t,z))\right]\lambda(dz)\mathrm{d}t.\end{aligned}\tag{3.9}$$

Let us multiply (3.9) by $w_{\theta_n}(t)$ and then integrating from $0\le s\le t$ to get,

$$\begin{aligned}&|w_{\theta_n}(t)|^2+2\nu\int_0^t\|w_{\theta_n}(s)\|^2\mathrm{d}s\\&\quad+2\int_0^t\big(B(u_{\theta_n}(s),u_{\theta_n}(s))-B(u_\theta(s),u_\theta(s)),w_{\theta_n}(s)\big)\mathrm{d}s=I_1,\end{aligned}\tag{3.10}$$

where

$$I_1 = 2\int_0^t \big(\sigma(s,u_{\theta_n}(s))\psi_n(s) - \sigma(s,u_\theta(s))\psi(s), w_{\theta_n}(s)\big)\mathrm{d}s$$
$$+ 2\int_0^t\int_Z \big(g(u_{\theta_n}(s),z)\ell(\phi_n(s,z)) - g(u_\theta(s),z)\ell(\phi(s,z)), w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s. \tag{3.11}$$

But $B(u_{\theta_n},u_{\theta_n}) - B(u_\theta,u_\theta) = B(u_\theta,w_{\theta_n}) + B(w_{\theta_n},u_\theta) + B(w_{\theta_n},w_{\theta_n})$, from Lemma 2.10. Using this, the properties (ii) and (iv) of the bilinear operator B given in Lemma 2.9 and using the inequality $2ab \leq \nu a^2 + \frac{1}{\nu}b^2$, one can obtain,

$$2\big|\big(B(u_{\theta_n}(s)) - B(u_\theta(s)), w_{\theta_n}(s)\big)\big| \leq \nu\|w_{\theta_n}(s)\|^2 + \frac{1}{\nu}|w_{\theta_n}(s)|^2|u_\theta(s)|^2. \tag{3.12}$$

The term I_1 can be written as,

$$\begin{aligned}|I_1| &\leq 2\int_0^t \big|\big(\big(\sigma(s,u_{\theta_n}(s)) - \sigma(s,u_\theta(s))\big)\psi_n(s), w_{\theta_n}(s)\big)\big|\,\mathrm{d}s\\
&\quad + 2\left|\int_0^t \big(\sigma(s,u_\theta(s))(\psi_n(s)-\psi(s)), w_{\theta_n}(s)\big)\mathrm{d}s\right|\\
&\quad + 2\int_0^t\int_Z \big|\big((g(u_{\theta_n}(s),z) - g(u_\theta(s),z))\ell(\phi_n(s,z)), w_{\theta_n}(s)\big)\big|\,\lambda(dz)\mathrm{d}s\\
&\quad + 2\left|\int_0^t\int_Z \big(g(u_\theta(s),z)(\ell(\phi_n(s,z)) - \ell(\phi(s,z))), w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|\\
&\leq I_2 + 2\sup_{0\leq t\leq T}\left|\int_0^t \big(\sigma(s,u_\theta(s))(\psi_n(s)-\psi(s)), w_{\theta_n}(s)\big)\mathrm{d}s\right| \qquad (3.13)\\
&\quad + 2\sup_{0\leq t\leq T}\left|\int_0^t\int_Z \big(g(u_\theta(s),z)(\ell(\phi_n(s,z)) - \ell(\phi(s,z))), w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|,\end{aligned}$$

$$\text{where } I_2 = 2\int_0^t \big|\sigma(s,u_{\theta_n}(s)) - \sigma(s,u_\theta(s))\big|_{\mathrm{L_Q}}|\psi_n(s)|_0|w_{\theta_n}(s)|\mathrm{d}s$$
$$+ 2\int_0^t\int_Z \big|g(u_{\theta_n}(s),z) - g(u_\theta(s),z)\big||\ell(\phi_n(s,z)||w_{\theta_n}(s)|\lambda(dz)\mathrm{d}s. \tag{3.14}$$

For I_2, apply $2ab \leq \eta a^2 + \frac{1}{\eta}b^2$ for the first term by taking $\eta = M$ and $2ab \leq a^2+b^2$ for second term by taking $a = \big|g(u_{\theta_n}(s),z) - g(u_\theta(s),z)\big|$ and $b = |w_{\theta_n}(s)|$ to obtain,

$$I_2 \leq M\int_0^t \big|\sigma(s,u_{\theta_n}(s) - \sigma(s,u_\theta(s)\big|^2_{\mathrm{L_Q}}\mathrm{d}s + \frac{1}{M}\int_0^t |\psi_n(s)|_0^2|w_{\theta_n}(s)|^2\mathrm{d}s$$
$$+ \int_0^t\int_Z \big(|g(u_{\theta_n}(s),z) - g(u_\theta(s),z)|^2 + |w_{\theta_n}(s)|^2\big)|\ell(\phi_n(s,z))|\lambda(dz)\mathrm{d}s. \tag{3.15}$$

Let us take the third term, apply Young's inequality, by using the control condition on ϕ and then apply Hypothesis (H.3) to obtain,

$$\begin{aligned}
I_2 &\leq M\int_0^t \left|\sigma(s,u_{\theta_n}(s)-\sigma(s,u_\theta(s)\right|^2_{\mathrm{L_Q}}\mathrm{d}s+\frac{1}{M}\int_0^t|\psi_n(s)|_0^2|w_{\theta_n}(s)|^2\mathrm{d}s\\
&\quad+\left(\int_0^t\int_Z|g(u_{\theta_n}(s),z)-g(u_\theta(s),z)|^2\lambda(dz)\mathrm{d}s\right)\left(\sup_{0\leq t\leq T}\sup_{z\in Z}|\ell(\phi(t,z))|\right)\\
&\quad+\int_0^t\int_Z|w_{\theta_n}(s)|^2|\ell(\phi_n(s,z))|\lambda(dz)\mathrm{d}s\\
&\leq ML\int_0^t|w_{\theta_n}(s)|^2\mathrm{d}s+\int_0^t\left[\frac{1}{M}|\psi_n(s)|_0^2+\int_Z|\ell(\phi_n(s,z))|\lambda(dz)\right]|w_{\theta_n}(s)|^2\mathrm{d}s.
\end{aligned}\tag{3.16}$$

Now let us substitute (3.16) in (3.13) to obtain,

$$\begin{aligned}
&2\left|\int_0^t\big(\sigma(s,u_{\theta_n}(s))\psi_n(s)-\sigma(s,u_\theta(s))\psi(s),w_{\theta_n}(s)\big)\mathrm{d}s\right|\\
&+2\left|\int_0^t\int_Z\big(g(u_{\theta_n}(s),z)\ell(\phi_n(s,z))-g(u_\theta(s),z)\ell(\phi(s,z)),w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|\\
&\leq\int_0^t\left[\frac{1}{M}|\psi_n(s)|_0^2+\int_Z|\ell(\phi_n(s,z))|\lambda(dz)+ML\right]|w_{\theta_n}(s)|^2\mathrm{d}s\\
&+2\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u_\theta(s))(\psi_n(s)-\psi(s)),w_{\theta_n}(s)\big)\mathrm{d}s\right|\\
&+2\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(g(u_\theta(s),z)(\ell(\phi_n(s,z))-\ell(\phi(s,z))),w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|.
\end{aligned}\tag{3.17}$$

By the boundedness of $\{|w_{\theta_n}(s)|^2\}$ in $C(0,T;H)$, and using the Lemma 3.12, the second integral on the right side of (3.17) goes to 0 as $n\to\infty$. Therefore, given any $\epsilon>0$, there exists an integer N_1 large so that for all $n\geq N_1$,

$$\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u_\theta(s))(\psi_n(s)-\psi(s)),w_{\theta_n}(s)\big)ds\right|<\frac{\epsilon}{4}.\tag{3.18}$$

And by applying the dominated convergence theorem, for any given $\epsilon>0$, there exists an integer N_2, large so that for all $n\geq N_2$,

$$\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(g(u_\theta(s),z)(\ell(\phi_n(s,z))-\ell(\phi(s,z))),w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|<\frac{\epsilon}{4}.\tag{3.19}$$

Choose $N=\max(N_1,N_2)$ so that,

$$\begin{aligned}
&\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u_\theta(s))(\psi_n(s)-\psi(s)),w_{\theta_n}(s)\big)ds\right|\\
&+\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(g(u_\theta(s),z)(\ell(\phi_n(s,z))-\ell(\phi(s,z))),w_{\theta_n}(s)\big)\lambda(dz)\mathrm{d}s\right|<\frac{\epsilon}{2}.
\end{aligned}\tag{3.20}$$

Let us define $C_{M,L,\nu} = \max\left\{ML, \frac{1}{M}, \frac{1}{\nu}, 1\right\}$. Applying (3.20), (3.17) and (3.12) in (3.10), one obtains for $n \geq N$,

$$|w_{\theta_n}(t)|^2 + \nu \int_0^t \|w_{\theta_n}(s)\|^2 \mathrm{d}s \tag{3.21}$$
$$\leq C_{M,L,\nu} \int_0^t |w_{\theta_n}(s)|^2 \left(|u_\theta(s)|^2 + |\psi_n(s)|_0^2 + \int_Z |\ell(\phi_n(s,z))|\lambda(dz) + 1 \right) \mathrm{d}s + \epsilon.$$

From the above relation one can get by denoting $C_{M,L,\nu}$ by $\mathbb{C}$,

$$\sup_{0\leq t\leq T} |w_{\theta_n}(t)|^2 + \nu \int_0^T \|w_{\theta_n}(t)\|^2 \mathrm{d}t$$
$$\leq \mathbb{C} \int_0^T \sup_{0\leq s\leq T} |w_{\theta_n}(s)|^2 \left(|u_\theta(s)|^2 + |\psi_n(s)|_0^2 + \int_Z |\ell(\phi_n(s,z))|\lambda(dz) + 1 \right) \mathrm{d}s + \epsilon.$$

Hence by applying Gronwall's inequality we get,

$$\sup_{0\leq t\leq T} |w_{\theta_n}(t)|^2 + \nu \int_0^T \|w_{\theta_n}(t)\|^2 \mathrm{d}t$$
$$\leq \epsilon \exp\left\{ \mathbb{C} \int_0^T \left(|u_\theta(t)|^2 + |\psi_n(t)|_0^2 + \int_Z |\ell(\phi_n(s,z))|\lambda(dz) + 1 \right) \mathrm{d}t \right\}. \tag{3.22}$$

The arbitrariness of ϵ finishes the proof. □

Remark 3.14. From Theorem 3.9 one can see that the equation

$$\mathrm{d}u^\varepsilon_{\theta^\varepsilon}(t) + [\nu A u^\varepsilon_{\theta^\varepsilon}(t) + B(u^\varepsilon_{\theta^\varepsilon}(t), u^\varepsilon_{\theta^\varepsilon}(t))]\, \mathrm{d}t$$
$$= \left[f(t) + \sigma(t, u^\varepsilon_{\theta^\varepsilon}(t))\psi^\varepsilon(t) + \int_Z g(u^\varepsilon_{\theta^\varepsilon}(t), z)\ell(\phi^\varepsilon(t,z))\lambda(dz) \right] \mathrm{d}t$$
$$+ \sqrt{\varepsilon}\sigma(t, u^\varepsilon_{\theta^\varepsilon}(t))\mathrm{d}W(t) + \varepsilon \int_Z g(u^\varepsilon_{\theta^\varepsilon}(t-), z)\tilde{N}(dt, dz), \tag{3.23}$$

with $u^\varepsilon_{\theta^\varepsilon}(0) = \xi \in H$, has unique strong solution in $\mathrm{L}^2(\Omega; X)$.

As we have noted before, the solution of the above equation admits a representation $u^\varepsilon_{\theta^\varepsilon} = \mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot) + \int_0^\cdot \psi^\varepsilon(s)\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\phi^\varepsilon}\right)$ by pathwise uniqueness of the solution, and the Girsanov theorem. For similar type of formulation readers can refer to Budhiraja, Dupuis and Maroulas [7] Section 4.1, where the authors have considered small noise stochastic differential equations(SDE) with finite dimensional jump diffusions.

For all $\theta \in \mathcal{U}$, let u_θ be the solution of the deterministic control equation

$$\mathrm{d}u_\theta(t) + [\nu A u_\theta(t) + B(u_\theta(t), u_\theta(t))]\mathrm{d}t = f(t)\mathrm{d}t + \sigma(t, u_\theta(t))\psi(t)\mathrm{d}t$$
$$+ \int_Z g(u_\theta(t), z)\ell(\phi(s,z))\lambda(dz)\mathrm{d}s,$$

with initial condition $u_\theta(0) = \xi \in H$.

Note that $\int_0^{\cdot} \psi(s)ds \in C([0,T];H_0)$ and $\int_0^{\cdot}\int_Z \phi(s,z)\lambda(dz)\mathrm{d}s \in C([0,T];H)$. Define $\mathcal{G}^0 : C([0,T];H_0)\times C([0,T];H) \to X$ by

$$\mathcal{G}^0(h) = u_\theta \quad \text{if } h = \left(\int_0^{\cdot} \psi(s)\mathrm{d}s, \int_0^{\cdot}\int_Z \phi(s,z)\lambda(dz)\mathrm{d}s\right)$$

for some $\theta = (\psi,\phi) \in \mathcal{U}$. If h cannot be represented as above, then $\mathcal{G}^0(h) = 0$.

Theorem 3.15 (Weak convergence)**.** *For proving weak convergence, let us define the set to be* $\left\{\theta^\varepsilon = (\psi^\varepsilon,\phi^\varepsilon) \in \mathcal{U} : \theta^\varepsilon(\omega) \in \bar{S}^M \ \mathbb{P}- \text{ a.e}\, \omega\ \varepsilon > 0\right\} \subset \mathcal{U}^M$ *converges in distribution to* θ *with respect to the weak topology defined on* $\mathcal{U}$*. Then we have* $\mathcal{G}^\varepsilon\left(\sqrt{\varepsilon}W(\cdot) + \int_0^{\cdot}\psi^\varepsilon(s)\mathrm{d}s, \varepsilon N^{\varepsilon^{-1}\phi^\varepsilon}\right)$ *converges in distribution to* $\mathcal{G}^0(\int_0^{\cdot}\psi(s)\mathrm{d}s, \lambda_T^\phi)$ *in* X*, as* $\varepsilon \to 0$*.*

Proof. Since $\bar{S}^M$ is a Polish space, the Skorokhod representation theorem can be introduced to construct processes $(\tilde{\theta}^\varepsilon, \tilde{\theta}, \tilde{W}^\varepsilon, \tilde{\lambda}_T)$ such that the distribution of $(\tilde{\theta}^\varepsilon, \tilde{\theta}, \tilde{W}^\varepsilon, \tilde{\lambda}_T)$ is same as that of $(\theta^\varepsilon, \theta, W, \lambda_T)$, and $\tilde{\theta}^\varepsilon \to \tilde{\theta}$ a.s. in the weak topology of $\bar{S}^M$. Thus

$$\int_0^t \tilde{\theta}^\varepsilon(s)\mathrm{d}s \to \int_0^t \tilde{\theta}(s)\mathrm{d}s$$

weakly in H a.s. for all $t \in [0,T]$. Without any loss of generality, we will write $(\theta^\varepsilon, \theta, W, \lambda_T)$ in what follows, though strictly speaking, one should write $(\tilde{\theta}^\varepsilon, \tilde{\theta}, \tilde{W}^\varepsilon, \tilde{\lambda}_T)$.

Let $w^\varepsilon_{\theta^\varepsilon}(t) = u^\varepsilon_{\theta^\varepsilon}(t) - u_\theta(t)$. We need to prove, in probability as $\varepsilon \to 0$,

$$\sup_{0\le t\le T} |w^\varepsilon_{\theta^\varepsilon}(t)|^2 + \int_0^T \|w^\varepsilon_{\theta^\varepsilon}(t)\|^2\mathrm{d}t \to 0.$$

For $w^\varepsilon_{\theta^\varepsilon}(t)$ we will get the stochastic differential equation as

$$\begin{aligned}
&\mathrm{d}w^\varepsilon_{\theta^\varepsilon}(t) + [\nu A w^\varepsilon_{\theta^\varepsilon}(t) + B(u^\varepsilon_{\theta^\varepsilon}(t), u^\varepsilon_{\theta^\varepsilon}(t)) - B(u_\theta(t), u_\theta(t))]\mathrm{d}t \\
&= [\sigma(t, u^\varepsilon_{\theta^\varepsilon}(t))\psi^\varepsilon(t) - \sigma(t, u_\theta(t))\psi(t)]\mathrm{d}t \\
&\quad + \int_Z [g(t, u^\varepsilon_{\theta^\varepsilon}(t))\ell(\phi^\varepsilon(t,z)) - g(u_\theta(t), z)\ell(\phi(t,z))]\, \lambda(dz)\mathrm{d}t \\
&\quad + \sqrt{\varepsilon}\sigma(t, u^\varepsilon_{\theta^\varepsilon}(t))\mathrm{d}W(t) + \varepsilon\int_Z g(u^\varepsilon_{\theta^\varepsilon}(t-), z)\tilde{N}(dt,dz). \qquad (3.24)
\end{aligned}$$

By applying Itô's Lemma for the process $|w^\varepsilon_{\theta^\varepsilon}(t)|^2$ and integrating from $0 \le s \le t$,

$$\begin{aligned}
&|w^\varepsilon_{\theta^\varepsilon}(t)|^2 + 2\nu\int_0^t \|w^\varepsilon_{\theta^\varepsilon}(s)\|^2\mathrm{d}s + 2\int_0^t (B(u^\varepsilon_{\theta^\varepsilon}(s)) - B(u_\theta(s)), w^\varepsilon_{\theta^\varepsilon}(s))\mathrm{d}s \\
&= I_3 + I_4 + 2\sqrt{\varepsilon}\int_0^t (\sigma(s, u^\varepsilon_{\theta^\varepsilon}(s)), w^\varepsilon_{\theta^\varepsilon}(s))\mathrm{d}W(s) \\
&\quad + 2\varepsilon\int_0^t\int_Z (w^\varepsilon_{\theta^\varepsilon}(s-), g(u^\varepsilon_{\theta^\varepsilon}(s-), z))\tilde{N}(ds,dz), \qquad (3.25)
\end{aligned}$$

where

$$
\begin{aligned}
I_3 &= 2\int_0^t \big(\sigma(s,u^\varepsilon_{\theta^\varepsilon}(s))\psi^\varepsilon(s) - \sigma(s,u_\theta(s))\psi(s), w^\varepsilon_{\theta^\varepsilon}(s)\big)\mathrm{d}s \\
&\quad + 2\int_0^t\int_Z \big(g(u^\varepsilon_{\theta^\varepsilon}(s),z)\ell(\phi^\varepsilon(s,z)) - g(u_\theta(s),z)\ell(\phi(s,z)), w^\varepsilon_{\theta^\varepsilon}(s)\big)\lambda(dz)\mathrm{d}s, \\
I_4 &= \varepsilon\int_0^t Tr(\sigma(s,u^\varepsilon_{\theta^\varepsilon}(s))Q\sigma(s,u^\varepsilon_{\theta^\varepsilon}(s)))\mathrm{d}s + \varepsilon\int_0^t\int_Z |g(u^\varepsilon_{\theta^\varepsilon}(s-),z)|^2\lambda(dz)\mathrm{d}s.
\end{aligned}
$$

Notice that by applying similar techniques as in Theorem 3.13, one obtains,

$$
\begin{aligned}
|I_3| &\le C_{M,L,\nu}\int_0^t |w^\varepsilon_{\theta^\varepsilon}(s)|^2\left(1 + |\psi^\varepsilon(s)|_0^2 + \int_Z |\ell(\phi^\varepsilon(s,z))|\lambda(dz)\right)\mathrm{d}s \\
&\quad + 2\left|\int_0^t \big(\sigma(s,u_\theta(s))(\psi^\varepsilon(s)-\psi(s)), w^\varepsilon_{\theta^\varepsilon}(s)\big)\mathrm{d}s\right| \\
&\quad + 2\left|\int_0^t\int_Z \big(g(u_\theta(s),z)(\ell(\phi^\varepsilon(s,z)) - \ell(\phi(s,z))), w^\varepsilon_{\theta^\varepsilon}(s)\big)\lambda(dz)\mathrm{d}s\right| \\
&\le C_{M,L,\nu}\int_0^t |w^\varepsilon_{\theta^\varepsilon}(s)|^2\left(1 + |\psi^\varepsilon(s)|_0^2 + \int_Z |\ell(\phi^\varepsilon(s,z))|\lambda(dz)\right)\mathrm{d}s \\
&\quad + \int_0^t \left(|\sigma(s,u_\theta(s))(\psi^\varepsilon(s)-\psi(s))|^2 + |w^\varepsilon_{\theta^\varepsilon}(s)|^2\right)\mathrm{d}s \\
&\quad + \int_0^t\int_Z \left(|g(u_\theta(s),z)(\ell(\phi^\varepsilon(s,z)) - \ell(\phi(s,z)))|^2 + |w^\varepsilon_{\theta^\varepsilon}(s)|^2\right)\lambda(dz)\mathrm{d}s \\
&\le C_{M,L,\nu}\int_0^t |w^\varepsilon_{\theta^\varepsilon}(s)|^2\left(2 + \int_Z\lambda(dz) + |\psi^\varepsilon(s)|_0^2 + \int_Z |\ell(\phi^\varepsilon(s,z))|\lambda(dz)\right)\mathrm{d}s \\
&\quad + \int_0^t |\sigma(s,u_\theta(s))|^2|\psi^\varepsilon(s)-\psi(s)|^2\mathrm{d}s \\
&\quad + \int_0^t\int_Z |g(u_\theta(s),z)|^2|\ell(\phi^\varepsilon(s,z)) - \ell(\phi(s,z))|^2\lambda(dz)\mathrm{d}s, \qquad (3.26)
\end{aligned}
$$

where $C_{M,L,\nu} = \max\{ML, \frac{1}{M}, \frac{1}{\nu}, 1\}$. Now let us take the term I_4 from (3.25) and apply condition (H.2) and Corollary 3.10 to obtain,

$$
\begin{aligned}
&\varepsilon\int_0^t Tr(\sigma(s,u^\varepsilon_{\theta^\varepsilon}(s))Q\sigma(s,u^\varepsilon_{\theta^\varepsilon}(s)))\mathrm{d}s + \varepsilon\int_0^t\int_Z |g(u^\varepsilon_{\theta^\varepsilon}(s),z)|^2\lambda(dz)\mathrm{d}s \\
&\quad \le \varepsilon K\,(T + C)\,. \qquad (3.27)
\end{aligned}
$$

By using the above estimates and denoting $C_{M,L,\nu}$ as $\mathbb{C}$, $\ell(\phi^\varepsilon(t,z)$ as $\ell(\phi^\varepsilon)$ and $\ell(\phi(t,z)$ as $\ell(\phi)$, we can obtain from equation (3.25) by taking supremum from

$0 \leq t \leq T$ and then expectation as before,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2+\nu\int_0^T\|w^{\varepsilon}_{\theta^{\varepsilon}}(t)\|^2\mathrm{d}t\right]\\
&\leq \mathbb{C}\mathbb{E}\left[\int_0^T\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2\left(2+|u_{\theta}(t)|^2+|\psi^{\varepsilon}(t)|_0^2+\int_Z(1+|\ell(\phi^{\varepsilon})|)\lambda(dz)\right)\mathrm{d}t\right]\\
&\quad+\varepsilon K\left(T+C\right)+\int_0^T|\sigma(t,u_{\theta}(t))|^2|\psi^{\varepsilon}(t)-\psi(t)|^2\mathrm{d}t\\
&\quad+\int_0^T\int_Z|g(u_{\theta}(t),z)|^2|\ell(\phi^{\varepsilon})-\ell(\phi)|^2\lambda(dz)\mathrm{d}t\\
&\quad+2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u^{\varepsilon}_{\theta^{\varepsilon}}(s)),w^{\varepsilon}_{\theta^{\varepsilon}}(s)\big)\mathrm{d}W(s)\right|\right]\\
&\quad+2\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(w^{\varepsilon}_{\theta^{\varepsilon}}(s-),g(u^{\varepsilon}_{\theta^{\varepsilon}}(s-),z)\big)\tilde{N}(ds,dz)\right|\right].
\end{aligned}
\tag{3.28}
$$

Let us take the term

$$
2\sqrt{\varepsilon}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u^{\varepsilon}_{\theta^{\varepsilon}}(s)),w^{\varepsilon}_{\theta^{\varepsilon}}(s)\big)\mathrm{d}W(s)\right|\right]
$$

from (3.28) and apply Burkholder-Davis-Gundy inequality, Young's inequality, Hypothesis (H.2) and Corollary 3.10 to obtain,

$$
\begin{aligned}
&2\sqrt{\varepsilon}\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t\big(\sigma(s,u^{\varepsilon}_{\theta^{\varepsilon}}(s)),w^{\varepsilon}_{\theta^{\varepsilon}}(s)\big)\mathrm{d}W(s)\right|\right)\\
&\leq 2\sqrt{2\varepsilon K}\mathbb{E}\left(\int_0^T(1+|u^{\varepsilon}_{\theta^{\varepsilon}}(s)|^2)|w^{\varepsilon}_{\theta^{\varepsilon}}(s)|^2\mathrm{d}s\right)^{1/2}\\
&\leq 2\sqrt{2\varepsilon K}\left[\frac{1}{8\sqrt{2\varepsilon K}}\mathbb{E}\left(\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2\right)+2\sqrt{2\varepsilon K}\mathbb{E}\left(\int_0^T(1+|u^{\varepsilon}_{\theta^{\varepsilon}}(s)|^2)\mathrm{d}s\right)\right]\\
&\leq\frac{1}{4}\mathbb{E}\left(\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2\right)+8\varepsilon K(T+C).
\end{aligned}
\tag{3.29}
$$

Consider the term

$$
2\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(w^{\varepsilon}_{\theta^{\varepsilon}}(s-),g(u^{\varepsilon}_{\theta^{\varepsilon}}(s-),z)\big)\tilde{N}(ds,dz)\right|\right]
$$

from (3.28) and apply the Burkholder-Davis-Gundy inequality in the form given in Lemma 2.4, Hypothesis (H.2) and Corollary 3.10 to obtain,

$$\begin{aligned}
&2\varepsilon\mathbb{E}\left(\sup_{0\leq t\leq T}\left|\int_0^t\int_Z\big(w^\varepsilon_{\theta^\varepsilon}(s-),g(u^\varepsilon_{\theta^\varepsilon}(s-),z)\big)\tilde{N}(ds,dz)\right|\right)\\
&\leq 2\varepsilon\sqrt{2}\mathbb{E}\left(\int_0^T\int_Z|w^\varepsilon_{\theta^\varepsilon}(s)|^2|g(u^\varepsilon_{\theta^\varepsilon}(s),z)|^2\lambda(dz)\mathrm{d}s\right)^{1/2}\\
&\leq 2\varepsilon\sqrt{2K}\mathbb{E}\left[\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|\left(\int_0^T(1+|u^\varepsilon_{\theta^\varepsilon}(s)|^2)\mathrm{d}s\right)^{1/2}\right]\\
&\leq 2\varepsilon\sqrt{2K}\left[\frac{1}{8\varepsilon\sqrt{2K}}\mathbb{E}\left(\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2\right)+2\varepsilon\sqrt{2K}\mathbb{E}\left(\int_0^T(1+|u^\varepsilon_{\theta^\varepsilon}(s)|^2)\mathrm{d}s\right)\right]\\
&\leq \frac{1}{4}\mathbb{E}\left(\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2\right)+8K\varepsilon^2(T+C). \qquad (3.30)
\end{aligned}$$

By using all these estimates, we can reduce the inequality (3.28) as,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2+2\nu\int_0^T\|w^\varepsilon_{\theta^\varepsilon}(t)\|^2\mathrm{d}t\right]\\
&\leq 2\mathbb{C}\mathbb{E}\left[\int_0^T\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2\left(2+|u_\theta(t)|^2+|\psi^\varepsilon(t)|_0^2+\int_Z(1+|\ell(\phi^\varepsilon)|)\lambda(dz)\right)\mathrm{d}t\right]\\
&\quad+2K\varepsilon[(9+\varepsilon)(C+T)]+2\int_0^T|\sigma(t,u_\theta(t))|^2|(\psi^\varepsilon(t)-\psi(t))|^2\mathrm{d}t\\
&\quad+2\int_0^T\int_Z|g(u_\theta(t),z)|^2|\ell(\phi^\varepsilon(t,z))-\ell(\phi(t,z))|^2\lambda(dz)\mathrm{d}t. \qquad (3.31)
\end{aligned}$$

Then the Gronwall's inequality yields,

$$\begin{aligned}
&E\left[\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2+2\nu\int_0^T\|w^\varepsilon_{\theta^\varepsilon}(t)\|^2\mathrm{d}t\right]\\
&\leq\Bigg(2K\varepsilon[(9+\varepsilon)(C+T)]++2\int_0^T|\sigma(t,u_\theta(t))|^2|(\psi^\varepsilon(t)-\psi(t))|^2\mathrm{d}t\\
&\qquad\qquad+2\int_0^T\int_Z|g(u_\theta(t),z)|^2|\ell(\phi^\varepsilon(t,z))-\ell(\phi(t,z))|^2\lambda(dz)\mathrm{d}t\Bigg) \qquad (3.32)\\
&\qquad\times\exp\left\{2\mathbb{C}\int_0^T\left(2+|u_\theta(t)|^2+|\psi^\varepsilon(t)|_0^2+\int_Z(1+|\ell(\phi^\varepsilon(t,z))|)\lambda(dz)\right)\mathrm{d}t\right\}.
\end{aligned}$$

We have given that $\theta^\varepsilon(t)\to\theta(t)$ a.s in the weak topology of $\mathcal{U}^M$. Since $\psi^\varepsilon\to\psi$ a.s. in the weak topology of $\tilde{S}^M$ and $\ell(\phi^\varepsilon(t,z))\to\ell(\phi(t,z))$ in a.s the weak topology of S^M (for further details see Theorem 4.4 of [25]), it is clear from the equation (3.32) that as $\varepsilon\to 0$, $\mathbb{E}\left[\sup_{0\leq t\leq T}|w^\varepsilon_{\theta^\varepsilon}(t)|^2+2\nu\int_0^T\|w^\varepsilon_{\theta^\varepsilon}(t)\|^2\mathrm{d}t\right]\to 0$. Let $\delta>0$

be any arbitrary number. Then by Markov's inequality

$$\mathbb{P}\left\{\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2+2\nu\int_0^T\|w^{\varepsilon}_{\theta^{\varepsilon}}(t)\|^2\mathrm{d}t\geq\delta\right\}$$
$$\leq\frac{1}{\delta}\mathbb{E}\left[\sup_{0\leq t\leq T}|w^{\varepsilon}_{\theta^{\varepsilon}}(t)|^2+2\nu\int_0^T\|w^{\varepsilon}_{\theta^{\varepsilon}}(t)\|^2\mathrm{d}t\right]\to 0\text{ as }\varepsilon\to 0.$$

Thus, $\sup_{0\leq t\leq T}|u^{\varepsilon}_{\theta^{\varepsilon}}(t)-u_{\theta}(t)|^2+\nu\int_0^T\|u^{\varepsilon}_{\theta^{\varepsilon}}(t)-u_{\theta}(t)\|^2\mathrm{d}t\to 0$, in probability as $\varepsilon\to 0$. The proof is now complete. □

Acknowledgements: Manil T. Mohan would like to thank Council of Scientific and Industrial Research (CSIR), India for a Senior Research Fellowship (SRF). The authors would also like to thank Indian Institute of Science Education and Research (IISER)- Thiruvananthapuram for providing stimulating scientific environment and resources. The authors would also like to thank the anonymous referee for his/her valuable comments.

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